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The cumulative hierarchy and the constructible universe of ZFA

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We present two results which shed some more light on the deep connection between ZFA and the standard ZF set theory: First of all we refine a result of Forti and Honsell (see [5]) in order to prove that the universe of ZFA can also be obtained (without appealing to choice) as the least fixed point of a continuous operator and not only as the greatest fixed point of the powerset operator. Next we show that it is possible to define a new absolute Gödel operation in addition to the standard ones in order to obtain the "constructible" model of ZFA as the least fixed point of the continuous operator of Gödel closure with respect to the standard and the new Gödel operations.

1 Introduction

In the following we will conform to the notation of [1] and [3], which we will also use as reference texts; ZF^- stands for the usual ZF set theory where Foundation is dropped, and ZFA stands for ZF where Foundation is replaced by the following axiom[1) :

Definition 1

(X_1) *Every binary relation R on a set A has a unique collapse on a transitive set.*

In ZF the following theorem is provable (see [5, Theorem 3]):

Theorem 2 *Let $\langle M, E_M\rangle$ and $\langle N, E_N\rangle$ be models of* ZFA. *If M^{WF} is isomorphic to N^{WF}, then M is isomorphic to N, where M^{WF} resp. N^{WF} is the universe of well founded sets for the structure $\langle M, E_M\rangle$ resp. $\langle N, E_N\rangle$.*

The theorem above suggests the possibility to extend to the non well founded universe of ZFA all the inner model theory developed so far for the usual ZF universe. In fact we will show:

1. The results of [5] are strong enough to define a cumulative hierarchy for ZFA, in particular it is possible to define a generalized power set operation P^* such that the universe V of ZFA is the union of sets V_α for α ranging over the class Ord of ordinals and

$$V_0 = \emptyset, \quad \text{for all } \alpha \in \mathrm{Ord}, V_{\alpha+1} = P^*(V_\alpha) \cup V_\alpha, \quad \text{for all limits } \gamma, V_\gamma = \textstyle\bigcup_{\alpha<\gamma} V_\alpha.$$

2. It is fairly easy to uncover the basic properties of class models of ZFA and to define the "constructible" model J_{X_1} of ZFA, a model which has all the desirable features that L has for the well-founded universe, for example (a) if $\langle M, \in\rangle \vDash$ ZFA, then $J_{X_1} \subseteq M$, (b) there is a definable well-ordering of J_{X_1}, (c) $J_{X_1} \cap \mathrm{WF} = L$.

The results above can be easily established due to the fact that many notions absolute for WF are absolute also in the universe of X_1, among them the notion of ordinal and the "stability" of definitions by transfinite induction over ordinals.

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1) We are not assuming that Choice is an axiom of ZFA, in contrast with Aczel's notation. Axiom X_1 is in the literature also denoted by AFA (see [1]).

2 The cumulative hierarchy of ZFA

Definition 3 For any set A we define the generalized powerset operation $P^*(A)$ by

$$P^*(A) = \{X : \in_{|\mathrm{TC}(X)} \text{ is bisimilar to } R \text{ for some } R \subseteq \mathrm{TC}(A)^2\}.$$

Remark 4 *For any set A, $P(A) \subset P^*(A)$.*

Proof. If $X \subset A$, then $\mathrm{TC}(X) \subset \mathrm{TC}(A)$ so that $\in_{|\mathrm{TC}(X)} = \in \cap \mathrm{TC}(X)^2 \subset \mathrm{TC}(A)^2$, and the thesis follows. □

Remark 5 *For every set A, $P^*(A)$ is a set.*

Proof. Given $R \subseteq \mathrm{TC}(A)^2$, let $X_R = \{X : \in_{|\mathrm{TC}(X)}$ is bisimilar to $R\}$. It doesn't take long (using axioms Replacement, Power Set, Separation and X_1) to see that X_R is a set definable from the parameter R. Moreover, $P^*(A) = \bigcup_{R \subseteq A^2} X_R$. So $P^*(A)$ is a set. □

Definition 6 In the universe of ZFA we define the V_α hierarchy as follows:

$$V_0 = \emptyset, \quad \text{for all } \alpha \in \mathrm{Ord}, V_{\alpha+1} = V_\alpha \cup P^*(V_\alpha), \quad \text{for all limits } \gamma, V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha.$$

The cumulative hierarchy defined above behave as expected, i. e.,

Lemma 7 $V = \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha$.

Proof. It is just a reelaboration of the proof of [5, Theorem 2]. However we sketch it for the sake of completeness.

It is easy to check by induction over ORD that for all α, $\Pi_\alpha \subseteq V_\alpha$, where Π_α is the α-th element in the usual hierarchy of well founded sets.

Claim *For any set X, $\mathrm{TC}(X)$ is the collapse of some relation R_X on a well founded set A_X.*

In fact, if the claim is true and $A_X \subseteq \Pi_\alpha$, then $\mathrm{TC}(X)$ is the transitive collapse of $R_X \subseteq \Pi_\alpha \subseteq V_\alpha^2$, and so $X \in V_{\alpha+1} \subseteq \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha$, and the thesis follows.

We will prove the claim. First of all we can suppose w. l. o. g. that X is a transitive set. So given the set X let $G^\alpha : X \longrightarrow \mathrm{WF}$ be the following sequence of functions with well founded values:

- G^0 is the constant function with value $\emptyset$;
- for all $x \in X$ and for all α, if $\alpha = \beta + 1$, then $G^\alpha(x) = \{G^\beta(y) : y \in x\}$;
- if $x \in X$ and α is a limit, then $G^\alpha(x)$ is the sequence G^x of length α such that $G^x(\gamma) = G^\gamma(x)$ for all $\gamma \in \alpha$.

Then define the following sequence of sets and relations:

- X^α is the image of G^α;
- for $x, y \in X^\alpha$, $xR^\alpha y$ iff there exist w, z such that $G^\alpha(w) = x$, $G^\alpha(z) = y$ and $w \in z$.

Once the definition are understood one can easily prove by induction that for all α, X^α is a well founded set: Define for each α the equivalence relation $\equiv_\alpha$ on X setting $x \equiv_\alpha y$ if $G^\alpha(x) = G^\alpha(y)$. Observe that if $\alpha > \beta$, $\equiv_\alpha \subset \equiv_\beta$ and define $\equiv$ to be $\bigcap_{\alpha \in \mathrm{ORD}} \equiv_\alpha$. Since $\equiv$ is a set, there must be an α such that $\equiv$ is $\equiv_\alpha$. Consider the set $X/\equiv$ of equivalence classes $[x]$ for $x \in X$, and on this set define the relation $[x] \in^* [y]$ iff there exists $x' \equiv x$ and $y' \equiv y$ such that $x' \in y'$. By the definitions, if α is the least ordinal such that $\equiv$ is $\equiv_\alpha$, then the structure $\langle X/\equiv, \in^*\rangle$ is isomorphic to the structure $\langle X^\alpha, R^\alpha\rangle$. Moreover with some effort it is possible to prove that $\in^*$ is bisimilar to $\in_{|X}$: the bisimilarity associates to any equivalence class $[x]$ of $X/\equiv$ all its elements[2)].

By the above arguments R^α is bisimilar to $\in_{|X}$. □

We notice that the cumulative hierarchy of ZFA allows to obtain the universe of ZFA as the least fixed point of the continuous operator P^*. The usual characterization of ZFA is instead that of being the greatest fixed point of the powerset operator P.

[2)] In fact, it is possible to prove that $\equiv$ is the equality relation (see [5] for more details).

This suggests the idea that in ZFA it might be possible that for a wide class of continuous operators Φ, the greatest fixed point X_Φ of Φ is also the least fixed point of some operator Φ^* which is obtained from Φ appealing to a "uniform" procedure.

3 Absolute notions, transitive models and the constructible universe of ZFA

We will now examine in more detail the elementary properties of standard models of ZFA. We will show that the inner model theory developed for the usual ZF theory can be extended with almost no effort also to ZFA. In particular we first examine notions which are absolute for transitive structures $\langle M, \in\rangle$ in ZFA, then state sufficient conditions for a transitive class in ZF^- to be a model of ZF^- and finally we define the constructible model J_{X_1} and state and prove its main properties.

In general it is easy to see that all Σ_0-formulas $\varphi(x_1, \dots, x_n)$ with free variables among $x_1, \dots, x_n$ are absolute for transitive structures $\langle M, \in\rangle$, i. e., where M is either a transitive class or a transitive set. This is due to the fact that restricted quantifiers behave the same way whenever $M \subseteq N$ and both are transitive. For example the following notions and operations are absolute for such structures in ZF^-:

- R is a function, a relation, an ordered pair,
- $\mathrm{dom}(R)$ (domain of R), $\mathrm{ran}(R)$ (range of R), $\mathrm{ext}(R)$ (extension of R, $\mathrm{ext}(R) = \mathrm{dom}(R) \cup \mathrm{ran}(R)$), etc.
- $X = \bigcup Y$,
- X is transitive, X is strictly ordered by $\in$, ...

In particular all Gödel operations defined in [3, p. 93]) are absolute for transitive models of ZF^- [3]. However the notions of von Neumann ordinal[4] and even of finite ordinal are not absolute in ZF^-. For this reason the operation of Gödel closure (see [3, p. 96]) may not be absolute for transitive structure in ZF^- (contrary to what happens in ZF) because it is defined by induction over natural numbers. This however does not happen if we assume X_1:

Lemma 8 *In* ZFA *the notion of von Neumann ordinal is absolute for transitive structures. In particular* $\omega^M = \omega$, *and the notion of finite ordinal is absolute for transitive structures.*

P r o o f. Let α^* be a set such that for some transitive structure M, $\langle M, \in\rangle \vDash$ "α^* is an ordinal" and such that in the universe α^* is not well founded. Let α be the supremum of all the initial segments of α^* which are really well ordered in the universe. Since being strictly ordered by $\in$ is an absolute property, $\in_{|\alpha^*}$ is a strict linear order whose well ordered initial segment is isomorphic to $\in_{|\alpha}$. Let $\Omega_{\alpha^*} = \alpha \cup \{\Omega_{\alpha^*}\}$, then Ω_{α^*} is a transitive collapse of $\in_{|\alpha^*}$. By X_1 this is the unique transitive collapse of $\in_{|\alpha^*}$, so $\alpha^* = \Omega_{\alpha^*}$. However the latter is not strictly ordered by $\in$, and since this is an absolute notion,

$$\langle M, \in\rangle \vDash \text{“}\alpha^* \text{ is not strictly ordered by } \in\text{”},$$

and this is a contradiction. □

Now we can easily prove that functions defined by induction over ordinals from absolute operations are absolute for transitive models of ZFA.

We can then define the Gödel's operations and the operation of Gödel closure.

Definition 9 The following are Gödel operations:

- All Gödel operations introduced in [3, p. 96] and the operation $\mathrm{ext}(R)$ that for a relation R produces the extension of R[5].
- The operation F defined in the following way: If R is a binary relation and $a \in \mathrm{ext}(R)$, then $F(a, R) = x$ iff x is the set assigned to a in the transitive collapse of R[6].

Remark 10 $\mathrm{ext}(R)$ *is absolute for transitive classes since* $\mathrm{ext}(R) = \mathrm{dom}(R) \cup \mathrm{ran}(R)$.

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[3] Also $\mathrm{ext}(R)$ is an absolute operation for transitive models of ZF^-.

[4] α is a von Neumann ordinal if it is strictly ordered by $\in$ and it is well founded.

[5] In fact, $\mathrm{ext}(R)$ is introduced for the sake of simplicity, and is not essential.

[6] The essential difference between F and the other Gödel's operations is (as we will see below) that F is not absolute for all transitive classes but only for transitive classes that are "closed" enough; in fact F is a Δ_1-operation in ZFA.

Lemma 11 *The new operation F is absolute for transitive M that are models of* ZFA.[7)]

P r o o f. The key point is that the concept "B is a bisimilarity between the relations R and R'" is very simple and can be expressed by the Σ_0-formula[8)]:

$$\begin{aligned}\mathrm{Bis}(B,R,R') \equiv_{df} \text{"}B \text{ is a relation"} \wedge \mathrm{dom}(B) = \mathrm{ext}(R) \wedge \mathrm{ran}(B) = \mathrm{ext}(R')\\ \wedge (\forall t,s)\,(tBs \to tB^+s),\end{aligned}$$

where tB^+s is an abbreviation for the Σ_0-formula[9)]:

$$\forall wRt\, \exists qR's\, wBq \wedge \forall qR's\, \exists wRt\, wBq.$$

$x = F(a,R)$ can be expressed by the Σ_1-formula:

$$\exists B\, (\mathrm{Bis}(B, (\in\cap \mathrm{dom}(B)), R) \wedge xBa),$$

i. e. the formula states that B is a bisimilarity between the $\in$ relation restricted to the transitive set $\mathrm{dom}(B)$ and the relation R, and moreover that xBa.

Claim *For any R there exists a unique B_R such that* $\mathrm{Bis}(B_R, (\in \cap\mathrm{dom}(B_R)), R)$*, moreover B_R has an absolute definition with respect to R. B_R is the function which collapses* $\mathrm{ext}(R)$ *to the transitive set* $\mathrm{dom}(B_R)$ *as prescribed by R.*

If the claim holds and $R \in M$ is a relation, then $F(a,R)$ belongs to M, since $x = F(a,R)$ can be expressed as $x = B_R(a)$ and since B_R is a function with an absolute definition with respect to R, we have $B_R \in M$ and also $x \in \mathrm{TC}(B_R) \subseteq M$.

In ZFA it is easy to prove that for any R, B_R exists and is unique due to axiom X_1. We show that B_R has an absolute definition with respect to R.

Given any M transitive model of ZFA let B_R^M be the unique set in M such that

$$M \models \mathrm{Bis}(B_R^M, (\in\cap\mathrm{dom}(B_R^M)), R);$$

B_R^M exists and is in M, since M is a model of ZFA. Since $\mathrm{Bis}(x,y,z)$ is an absolute formula, we have $\mathrm{Bis}(B_R^M, (\in \cap\mathrm{dom}(B_R^M)), R)$, so B_R^M is also the unique set in the universe which is the collapse of R. This establishes the claim. □

Definition 12 For any set X, $\mathrm{CL}(X) = \bigcup_{n\in\omega} W_n$, where

$$W_0 = X \quad \text{and} \quad W_{n+1} = W_n \cup \{G(x,y) : x,\, y \in W_n \text{ and } G \text{ is a Gödel operation}\}.$$

Clearly $\mathrm{CL}(X)$ is an absolute operation for all transitive classes definable in ZFA that are closed under the Gödel operations, since it is a function defined using absolute operations and induction over natural numbers.

Now we can define the constructible Jensen hierarchy over ZFA:

Definition 13 $J_{X_1} = \bigcup_{\alpha\in\mathrm{ORD}} J_\alpha$, where

$$J_0 = \emptyset, \quad \text{for all } \alpha \in \mathrm{Ord},\ J_{\alpha+1} = \mathrm{CL}(J_\alpha \cup \{J_\alpha \cap \mathrm{WF}\} \cup \{J_\alpha\}), \quad \text{for all limits } \gamma,\ J_\gamma = \bigcup_{\alpha<\gamma} J_\alpha.$$

We notice that the hierarchy cannot be trivial because using induction over ordinals one can easily prove that $J_\alpha \cap \mathrm{WF}$ contains the usual L_α in WF. So J_{X_1} contains L and this means that J is a proper class, while every J_α is a set[10)].

Finally we introduce some auxiliary notions in order to prove that J_{X_1} is the least transitive model of ZFA and that it has a definable well ordering of its elements.

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7) In fact, it is enough that M is transitive and believe to be closed under F and some of the other Gödel's operations.

8) All the quantifiers in the formula below range over the extensions of the relations R and R'.

9) All quantifiers are ranging on the extension of the relations R and R'.

10) It is not hard to see that J_{X_1} is the least fixed point of the continuous operator CL.

Definition 14 M is an *almost universal class* if every set $X \subseteq M$ is contained in some $Y \in M$.

Theorem 15 *Let* $\mathsf{ZF}^- + \mathrm{CP}$ *stands for* ZF^- *enriched with the Collection Principle Schema*[11]. *In* $\mathsf{ZF}^- + \mathrm{CP}$ *it holds: if* M *is a transitive, almost universal, Gödel closed class, then* M *is a model of* ZF^-.

Proof. The proof is standard and can be found in [3, pp. 92 – 99]. Some attention must be drawn to the fact that the theorem in [3] is proved under the assumption that Foundation holds; this is not a serious problem because this assumption is needed only in order to prove the Collection Principle, while we are assuming CP in the hypothesis. □

Theorem 16 *In* ZFA *the Collection Principle holds.*

Proof. This is trivial, because we can apply Scott's trick appealing to the V_α hierarchy [12]. □

Theorem 17 J_{X_1} *is an almost universal, transitive, Gödel closed class.*

Proof. The proof goes trough with minor modifications in the same way it goes for L in the well founded case.

(a) J_{X_1} is almost universal because we have a notion of rank for elements of J_{X_1}: $\varrho(x)$ is the least α such that $x \in J_{\alpha+1}$. For any set X contained in J_{X_1}, $X \subseteq J_\alpha$ for any α bigger than $\sup\{\varrho(x) : x \in X\}$ and by definition $J_\alpha \in J_{\alpha+1} \subseteq J_{X_1}$.

(b) J_{X_1} is transitive because it is the union of transitive sets. In fact, if X is transitive, then $\mathrm{CL}(X)$ is transitive, because $\mathrm{CL}(X) = \bigcup_{n\in\omega} W_n$ and one can easily prove by a suitably chosen induction on the Gödel operations that for all i, if $x \in W_i$, then $x \subseteq \mathrm{CL}(X)$. We show the case relative to F, the others are simpler: Let $y \in F(a, R) \in W_i$, and let i be the least one such that $F(a, R) \in W_i$, we must show that $y \in \mathrm{CL}(X)$.

- If $i = 0$, $W_0 = X$ is transitive and contained in $\mathrm{CL}(X)$, so there is nothing to prove.
- If $i > 0$, there is a $b \in \mathrm{ext}(R)$ such that bRa and $y = F(b, R)$ (i. e. y is assigned to b in the transitive collapse of R); by the choice of i, $R \in W_{i-1}$; this implies $\mathrm{ext}(R) \in W_i$, and by the inductive hypothesis on ext[13] we can suppose that $\mathrm{ext}(R) \subseteq \mathrm{CL}(X)$, so $b \in \mathrm{ext}(R)$ implies $b \in W_j$ for some j, and so $y = F(b, R) \in W_{j+1}$.

Since J_0 is transitive, we can prove by induction on α that every J_α is transitive.

(c) J_α is Gödel closed by definition. □

By the above theorems J_{X_1} is a model of ZF^-. It is immediate to prove that J_{X_1} is also a model of X_1, so that J_{X_1} is a model of ZFA. Moreover exactly as in the case of L the Jensen hierarchy suggests a natural way to define a global absolute well-ordering of J_{X_1}, so that J_{X_1} is also a model of Global Choice.

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[11] By the Collection Principle CP it is meant the axiom schema stating "For every partition of the universe in set many classes there is a set which contains at least a member from each non empty class of the partition".

[12] I.e. For every partition of the universe V in set many non empty classes we define the set whose elements are those which are of least rank in some non empty class of the partition.

[13] We suppose we already dealt with the case relative to ext at stage i; this case can be treated without appealing to F, since ext is a Σ_0-operation for transitive class on which it is defined.