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Forcing axioms and cardinality of the continuum

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Introduzione

Il principale oggetto di studio della teoria degli insiemi è il concetto di infinito, che viene analizzato con gli strumenti della logica matematica; questo studio si fonda sull'analisi del concetto di cardinalità. Cantor definisce la cardinalità di un insieme come la classe di equivalenza formata dagli insiemi in biezione con esso.

Si può notare che questa definizione è una naturale astrazione del concetto di numero. Infatti nel caso finito, questa è la corretta formalizzazione della nostra intuizione del concetto di numero naturale.

Dato un insieme X, $|X|$ indica la sua cardinalità e denotiamo con le lettere greche le classi di equivalenza indotte dalla relazione di biezione. Esiste una naturale relazione d'ordine fra cardinalità: $|X| \leq |Y|$ sse esiste una iniezione di X in Y, mentre $|X| < |Y|$ sse $|X| \leq |Y|$ e non esiste nessuna biezione da $|Y|$ a $|X|$.

Uno dei primi risultati della teoria mostra che lo spazio quoziente indotto dalla relazione di biezione coincide con quello indotto dalla relazione $\leq$ tra cardinalità. Esso va sotto il nome di teorema di Schroeder-Bernstein: $|X| = |Y|$ sse $|X| \leq |Y|$ e $|Y| \leq |X|$. Inoltre, Zermelo ha dimostrato che, assumendo la possibilità di dare un buon ordine ad ogni insieme, per ogni X, Y, $|X| \leq| Y|$ oppure $|Y| \leq |X|$; in questo modo si mostra che i cardinali seguono un un buon ordine lineare che estende quello classico dei numeri naturali. Un altro risultato molto importante è il teorema di Cantor: per ogni insieme $|X|$, $|\mathcal{P}(X)| > |X|$. Questo teorema mostra come il concetto di cardinale infinito non sia banale.

In analogia con l'aritmetica dei numeri naturali è possibile definire le operazioni di somma, prodotto ed elevazione a potenza anche per i numeri cardinali. Dati due cardinali κ e λ la somma viene definita come la cardinalità dell'unione disgiunta dei due insiemi: $|\kappa \sqcup \lambda|$; il prodotto come la cardinalità del prodotto cartesiano: $|\kappa \times \lambda|$; l'elevazione a potenza come la cardinalità dell'insieme di tutte le funzioni dal primo al secondo: $|{}^{\kappa}\lambda|$.

Notiamo che, per quanto riguarda gli insiemi di cardinalità finita, le operazioni appena introdotte coincidono con le operazioni dell'aritmetica elementare;

infatti dati due insiemi X ed Y rispettivamente di n ed m elementi, $n + m = |X \sqcup Y|$, $n \cdot m = |X \times Y|$ ed infine $n^m = |{}^X Y|$. Quindi l'aritmetica cardinale può essere vista come una naturale generalizzazione dell'aritmetica elementare. Essa però, nel caso infinito, ha proprietà differenti, poiché deve tenere conto di proprietà specifiche degli insiemi infiniti, come ad esempio la possibilità di essere in biezione con un sottoinsieme proprio. Per questo motivo non è difficile dimostrare che, dati due cardinali infiniti κ e λ,

$$\kappa + \lambda = \kappa \cdot \lambda = max\{\kappa, \lambda\}.$$

Se da un lato le operazioni di somma e prodotto non creano difficoltà, dall'altro i problemi legati al calcolo della potenza cardinale sono tra i più complicati e profondi dell'intera teoria degli insiemi.

Uno dei primi problemi di aritmetica cardinale che attirò l'interesse di Cantor fu quello della cardinalità del continuo: se nel teorema di Cantor poniamo $X = \mathbb{N}$ otteniamo che

$$|\mathbb{R}| = 2^{|\mathbb{N}|} = |\mathcal{P}(\mathbb{N})| > |\mathbb{N}|.$$

Cantor congetturò che il valore di $2^{|\mathbb{N}|}$ fosse il cardinale immediatamente successivo a $|\mathbb{N}| = \aleph_0$, cioè $\aleph_1$. Tuttavia in più di cento anni di sforzi non si è ancora riusciti a dare una soluzione soddisfacente e definitiva a questo problema.

Dal problema del continuo agli assiomi di forcing

All'inizio dello scorso secolo il problema del continuo assunse la forma oggi nota come CH (Continuum Hypothesis):

$$2^{\aleph_0} = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \aleph_1$$

e la sua ovvia generalizzazione GCH (General Continuum Hypothesis):

$$\forall \alpha \in \mathbb{O}n \;\; 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

Fino agli anni Sessanta e dopo che ZFC si affermò come adeguata formalizzazione della teoria degli insiemi, i più importanti avanzamenti della teoria si ebbero nel tentativo di dimostrare o refutare CH. Dopo che, nel '31, Gödel dimostrò il Teorema di incompletezza si cominciò ad intuire che la soluzione potesse essere diversa da quella sperata. Infatti nel '39 Gödel mostrò che Con(ZFC) $\Rightarrow$ Con(ZFC + CH) e nel '63 Cohen ([3]) mostrò che Con(ZFC) $\Rightarrow$ Con(ZFC + $\neg$CH). In questo modo Cohen riusì a dimostrare l'indipendenza di

CH da ZFC; inoltre dotò la teoria degli insiemi di uno strumento molto potente per le dimostrazioni di indipendenza: il forcing.

Di conseguenza, negli anni Sessanta, la teoria degli insiemi cambiò radicalmente. Molti dei problemi rimasti aperti fino a quel momento furono dimostrati indipendenti da ZFC. Inoltre raffinamenti e generalizzazioni del metodo del forcing mostrarono come il principale ambito della teoria fosse costituito da problemi indecidibili. Questa caratteristica differenzia profondamente la teoria degli insiemi da altre teorie, ugualmente incomplete come la teoria dei numeri, ma che vivono in mondi e si occupano di problemi sufficientemente "semplici" da considerare il fenomeno dell'indipendenza più una curiosità che una delle difficoltà centrali.

Questi risultati di indipendenza oltre all'esistenza di questioni indecidibili ma, come CH, sufficientemente elementari da far sperare nella possibilità di una loro completa soluzione, portarono gli insiemisti a cercare nuovi principi che potessero decidere ciò che con i soli strumenti di ZFC non poteva essere dimostrato.

I primi assiomi che vennero considerati furono quelli dei grandi cardinali. Essi postulano l'esistenza di cardinali infiniti con determinate proprietà combinatoriche. Questi insiemi sono sufficientemente grandi da garantire l'esistenza di un modello transitivo di ZFC; pertanto la loro esistenza non è dimostrabile in ZFC, per via del teorema di incompletezza di Gödel[1].

Questa direzione si inserì nel solco di quello che viene comunemente chiamato il *programma di Gödel*, per come fu enunciato da Gödel in [6].

> *These axioms show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of the series of those set.*[2]

Ciò che preconizzava Gödel era la possibilità che, per esempio, l'esistenza dei grandi cardinali permettesse di risolvere problemi semplici relativi ad i numeri reali (come CH). Infatti grazie alle loro proprietà combinatoriche, i grandi cardinali furono utili per la soluzione di problemi fino ad allora indecidibili. Tuttavia ben presto ci si accorse che per quanto riguardava la soluzione di CH questi nuovi assiomi non erano sufficienti.

Il *"concetto di insieme"* a cui si riferisce Gödel è il concetto iterativo di insieme. Questo si basa sull'idea che si possa iterare in maniera indefinita

[1] Per una trattazione esauriente degli assiomi dei grandi cardinali si veda [11] e [12]

[2] Questi assiomi mostrano con chiarezza non solo che il sistema assiomatico della teoria degli insiemi come usato oggi è incompleto, ma anche che esso può essere integrato in modo non arbitrario mediante nuovi assiomi che si limitino a chiarire il concetto di insieme.

l'operazione di *insieme di*: cioè quella che porta a considerare come insieme una qualunque collezione non contraddittoria di oggetti già definiti in precedenza. Fu il tentativo di estendere questa operazione a spingere per un'estensione "verso l'alto" della gerarchia degli insiemi. Per esempio l'esistenza di un cardinale inaccessibile corrisponde a considerare come insieme la collezione di tutti gli insiemi che è possibile creare grazie agli strumenti offerti dagli assiomi di ZFC.

Una delle giustificazioni teoriche che portarono ad accettare gli assiomi dei grandi cardinali fu dunque quella di massimizzare il dominio di applicazione dell'operazione di *insieme di*; da qui il tentativo di estendenderla oltre l'universo degli insiemi che gli assiomi di ZFC riescono a trattare. Nelle parole di Gödel ([6])

> *Only a maximum property would seem to harmonize with the concept of set.*[3]

Dal momento che gli assiomi dei grandi cardinali hanno un potere di consistenza maggiore di ZFC i criteri che hanno portato ad accettarli, come quello della massimalità, non hanno carattere esclusivamente matematico.

Un buona ragione per credere nella naturalezza degli assiomi dei grandi cardinali è il loro stretto e profondo legame con l'Assioma di Determinatezza (AD). Infatti l'uno giustifica l'altro e insieme completano la teoria di $L(\mathbb{R})$, nella direzione attesa.

Un altro aspetto per il quale i grandi cardinali si sono rivelati molto utili e che depone a favore del loro carattere non arbitrario è il fatto che essi formino un ordine lineare nell' estensione dell'universo. Questo fatto, pur non avendo una certezza dimostrativa, è tuttavia dotato di una solidità empirica.

Dal momento che però molti problemi, tra cui CH, non sono risolti dagli assiomi dei grandi cardinali, dagli anni Ottanta in poi si è allora cercato di trovare nuovi principi, diversi dagli assiomi dei grandi cardinali, che potessero decidere anche la cardinalità del continuo. Quello che il forcing aveva apportato alla teoria era un metodo generale per costruire nuovi modelli di ZFC. Il concetto di massimalità poteva allora essere declinato non solo nel senso di una maggior libertà nell'iterazione dell'operazione *insieme di*, ma anche in quello di una saturazione dell'universo insiemistico rispetto agli oggetti che venivano ad esistere nei modelli costruiti per mezzo del forcing. Fu questa la strada che portò alla definizione degli assiomi di forcing.

Storicamente il primo assioma di forcing fu l'Assioma di Martin (MA), che fu isolato nel tentativo di iterare il metodo del forcing; in un secondo momento si tentò di generalizzare MA. Dopo che furono isolate alcune utili classi di ordini

[3]Solo una proprietà di massimo sembrerebbe essere in armonia con il concetto di insieme.

parziali (in particolare quelle che permettono di non collassare ω_1 nell'estensione generica), agli inizi degli anni Ottanta Baumgartner ([2]) e Shelah introdussero il Proper Forcing Axiom (PFA) e verso la fine del decennio Foreman, Magidor e Shelah ([5]) isolarono il Martin's Maximum (MM).

Gli assiomi di forcing generalizzano il teorema di Categoria di Baire[4] affermando che, per una più ampia classe di spazi topologici, l'intersezione di una quantità più che numerabile di aperti densi è non vuota. Essi hanno delle immediate ricadute sull'aritmetica cardinale: infatti implicano il fallimento di CH. Infatti supponendo che CH valga è possibile esibire un'enumerazione dei reali in tipo d'ordine ω_1: $\mathbb{R} = \{r_\alpha : \alpha \in \omega_1\}$. Ora però se definiamo, per ogni $\alpha \in \omega_1$, $D_\alpha = \{h \in \mathbb{R} : h \neq r_\alpha\}$, D_α è un aperto denso di $\mathbb{R}$ per ogni $\alpha \in \omega_1$ e ogni numero reale che appartiene all'intersezione di tutti questi insiemi è un reale che non appartiene all'enumerazione. Contraddizione.

Gli assiomi di forcing si possono quindi presentare nel seguente modo:

> $FA(\mathcal{A}, \kappa)$ *è vero per una data classe* $\mathcal{A}$ *di spazi topologici, se* $X \in \mathcal{A}$ *e per ogni* $\mathcal{F}$ *famiglia di cardinalità al più* κ *di sottoinsiemi aperti densi di* X*, si ha che* $\bigcap \mathcal{F} \neq \varnothing$.

A differenza degli assiomi dei grandi cardinali, i più potenti assiomi di forcing decidono la cardinalità del continuo e mostrano che $2^{\aleph_0} = \aleph_2$. In questa tesi considereremo gli assiomi PFA: $FA(\mathbb{P}_{Proper}, \aleph_1)$, MM: $FA(\mathbb{P}_{SSP}, \aleph_1)$ e numerose loro varianti; dove $\mathbb{P}_{Proper}$ e $\mathbb{P}_{SSP}$ sono classi di spazi topologici che definiremo in seguito.

Perché accettare nuovi assiomi

La forza di consistenza dei più forti assiomi di forcing è, per quanto finora dimostrato, all'incirca quella di un cardinale supercompatto, uno dei maggiori grandi cardinali conosciuti; di essi tuttavia non si è ancora riusciti a creare un modello interno e vi è quindi una minore sicurezza nell'accettarne l'esistenza.

D'altra parte, a favore degli assiomi di forcing, vi è l'evidenza empirica, sostanziata dal lavoro di Woodin, che essi producano una teoria completa delle struttura $H_{\aleph_2}$, ovvero la collezione degli insiemi di cardinalità ereditariamente inferiore ad $\aleph_2$. Inoltre in più occasioni si è visto che se grazie ad un assioma di forcing è possibile costruire dei controesempi ad un certo enunciato, questi possono essere già dimostrati in ZFC[5]. Il seguente, dovuto a Todorčević, è un

[4] Esso afferma che l'intersezione di una famiglia numerabile di aperti densi di $\mathbb{R}$ è non vuota

[5] Un esempio è il caso dei (κ, λ^*)-gap. In ZFC è possibile dimostrare che esistono solo dei (ω_1, ω_1^*)-gap oppure dei $(\mathfrak{b}, \omega^*)$-gap; assumendo OCA, che è una conseguenza di PFA, questi sono gli unici gaps che possono esistere

teorema ispirato da OCA, una conseguenza di PFA.

Theorem 0.0.1. *Sia $X \subseteq \mathbb{R}$ un insieme Σ_1^1 e $K \subseteq [\mathbb{R}]^2 = \{(x, y) : x > y\}$ un sottoinsieme aperto, allora vale una delle due alternative:*

- *$\exists P \subseteq X$ perfetto omogeneo per K (i.e. $[P]^2 \subseteq K$),*
- *X è ricoperto da una quantità numerabile di insiemi omogenei per $[\mathbb{R}]^2 \setminus K$.*

Questo teorema può essere visto come una generalizzazione a due dimensioni, per gli insiemi Σ_1^1, della proprietà dell'insieme perfetto; basta considerare $K = \mathbb{R}$, per ottenere come corollario del teorema la proprietà dell'insieme perfetto per insiemi analitici.

Questi risultati sono buoni argomenti per credere nella verità degli assiomi di forcing, tuttavia, dal momento che essi hanno un potere di consistenza maggiore di ZFC, sono fondamentalmente di altra natura le ragioni per accettarli. Bagaria in [1] ne ha elencate quattro: *Consistency*, *Maximality*, *Fairness*, *Success.* Essi sono i criteri che dovrebbero guidare la ricerca di nuovi assiomi.

Prima di analizzare questi principi vi è una precisazione da fare. Stando alla definizione di assioma, esso dovrebbe essere un enunciato che trae la sua legittimità dall'autoevidenza della sua verità. Questo presuppone un rapporto intuitivo con gli enti matematici che porta, senza giustificazioni ed argomentazioni, ad accettare una proposizione come assioma sulla base della pura intuizione. Se questo è largamente accettato per l'assiomatica classica, certo non lo è più in questo contesto dove l'intuizione deve essere ammaestrata dall'uso e dalla familiarità con oggetti che spesso hanno proprietà e caratteristiche controintuitive. Ad esempio gli assiomi di forcing non hanno nulla del carattere intuitivo della maggior parte degli assiomi di ZFC. Questo è dovuto non solo ai concetti cui fanno riferimento, che non sono basilari all'interno della teoria, ma anche al loro carattere tecnico. Bisogna quindi trovare altri criteri diversi all'intuizione, che permettano di accettare questi nuovi tipi di assiomi ed aiutino ad avere confidenza nel loro utilizzo.

Vediamo ora i quattro criteri elencati da Bagaria.

Consistency. E' una richiesta necessaria per qualunque sistema formale che aspiri ad essere rappresentativo di una certa realtà matematica. Sebbene una contraddizione renda un sistema banalmente completo, poiché incoerente, e rispetti anche gli altri criteri, essa non può essere considerata una valida estensione. Se da un lato la coerenza di un nuovo assioma rispetto a ZFC è una condizione necessaria, non è tuttavia sufficiente per giustificarne l'adozione.

Maximality. Questo criterio può essere inteso come una generica indicazione a preferire una visione dell'universo degli insiemi la più ricca possibile, e quindi a rifiutare assiomi limitativi come quello di costruibilità. Questo criterio è piuttosto vago e il tentativo più convincente di formalizzarlo è quello di Woodin di cui parleremo in seguito. Nel caso di Woodin esso è inteso come completezza di H_{ω_2} rispetto ad una certa relazione di dimostrabilità.

Fairness viene da Bagaria definito così:

> *One should not discriminate against sentences of the same logical complexity.*[6]

La giustificazione di questo principio è che in mancanza di una corretta intuizione di cosa sia da considerare vero o cosa sia da considerare falso, bisogna accettare come assiomi solo principi che danno una soluzione a tutti i problemi di una data complessità logica.

Success è il criterio che prende in esame non la natura di un assioma ma le sue conseguenze. Maggiori sono le conseguenze che discendono da un nuovo assioma, maggiore è la propensione ad accettarlo. Questo principio non valuta la verità di un nuovo assioma, quanto la sua utilità. L'utilità va qui intesa nel senso di dare soluzione a problemi che ancora non erano stati risolti oppure nel senso di gettare nuove luci su interi ambiti della teoria.

In generale si è portati ad accettare un nuovo assioma nella misura in cui soddisfa questi quattro criteri ed è coerente con i grandi cardinali. Da quanto detto prima si può ritenere che gli assiomi di forcing rispettino, modulo la coerenza relativa, tutti e quattro i criteri sopra elencati.

I risultati prodotti da Woodin negli ultimi anni forniscono nuove evidenze a favore degli assiomi di forcing.

Cenni sul programma di Woodin

Il "programma di Woodin" mira a dare una soluzione al problema del continuo ed allo stesso tempo fornisce solidi argomenti a favore degli assiomi di forcing. Presenteremo qui alcuni cenni di questo programma che, in linea con quello di Gödel, mira ad ampliare ZFC con assiomi che diano una soluzione a problemi indecidibili. Una presentazione accessibile dei risultati di cui ora parleremo si trova in [27], [28], mentre una presentazione completa del lavoro di Woodin è [26].

I presupposti teorici da cui parte il programma di Woodin sono platonisti. Si assume infatti come ambito di ricerca la teoria al primo ordine di $V = (V, \in, =)$:

[6] "Non si dovrebbe discriminare tra proposizioni della medesima complessità logica"

$\mathrm{Th}(V) = \{\phi : V \vDash \phi\}$, considerata come un dato compiuto. Per studiarla, anziché utilizzare la gerarchia dei V_α:

- $V_0 = \varnothing$,
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
- $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ per λ limite,
- $V = \bigcup_{\alpha \in Ord} V_\alpha$,

si considera la stratificazione di V come unione, al variare di λ fra i cardinali, degli H_λ: le strutture date dagli insiemi di cardinalità ereditariamente minore di λ.

$$H_\lambda = \{x \in V : |x| < \lambda \text{ and } \forall y\,(y \in tc(x) \Rightarrow |y| < \lambda)\}.$$

Esse sono una buona approssimazione di V. Infatti la stratificazione degli H_λ permette di considerare segmenti iniziali sufficientemente chiusi rispetto alle operazioni insiemistiche.[7] Inoltre per cardinali fortemente limite le due gerarchie coincidono e $V = \bigcup_{\lambda \in Card} H_\lambda$.

L'obiettivo è quindi quello di ampliare gli assiomi di ZFC risolvendo gradualmente i problemi formulabili nelle $\mathrm{Th}(H_\lambda)$. Questo processo non può ovviamente esaurire tutti i problemi di ZFC, ma permettere di dotarci di assiomi in grado di completare la teoria almeno per segmenti iniziali dell'universo degli insiemi. Il primo passo in questa direzione è il teorema di assolutezza di Schoenfield:

Theorem 0.0.2. ***(Schoenfield)*** *Se ϕ è Σ^1_2 (per esempio una formula aritmetica) allora qualunque modello transitivo M di ZFC soddisfa ϕ, oppure qualunque modello transitivo M di ZFC soddisfa $\neg\phi$.*[8]

L'obiettivo di Woodin è allora quello di ottenere risultati analoghi a questo teorema per H_λ, con λ arbitrario. Come vedremo Woodin è riuscito a dimostrare un analogo del teorema di Schoenfield per H_{ω_1} e in una certa misura anche per H_{ω_2}. Bisogna notare che nel caso si riuscisse a trovare un assioma che decida la teoria di H_{ω_2}, si avrebbe una soluzione di CH.

Nelle parole di Woodin il programma di ricerca da intraprendere è il seguente (vedi [27]):

[7] Un aspetto molto utile è che gli H_λ, a differenza dei V_α, sono chiusi per rimpiazzamento.

[8] Questo teorema ci dice che non possiamo sperare di falsificare un teorema di teoria dei numeri con un modello transitivo di ZFC.

One attempts to understand in turn the structures H_ω, H_{ω_1} and then H_{ω_2}. A little more precisely, one seeks to find the relevant axioms for these structures. Since the Continumm Hypothesis concerns the structure of H_{ω_2}, any reasonably complete collection of axioms for H_{ω_2} will resolve the Continumm Hypothesis.[9]

Woodin sceglie di lavorare su $\langle H_{\omega_2}, \in, =\rangle$ piuttosto che su $\langle \mathcal{P}(\mathcal{P}(\mathbb{N})), \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in\rangle$ (equivalente a $\langle V_{\omega+2}, \in, =\rangle$) e su $\langle H_{\omega_1}, \in, =\rangle$ piuttosto che su $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in\rangle$ (equivalente a $\langle V_{\omega+1}, \in, =\rangle$), come più ovvie generalizzazioni di $\langle \mathbb{N}, +, \cdot, \in \rangle$. Tuttavia se per la struttura $H_{\aleph_1}$ è più intuitivo individuare quali siano le patologie che si vogliono eliminare (come per esempio il paradosso di Banach-Tarski e la conseguente non-misurabilità di alcuni sottoinsiemi dei reali), questo è più complicato per $H_{\aleph_2}$. Come dire che è più complicato addomesticare l'intuizione a riconoscere cosa sia vero e cosa sia falso, man mano che si sale verso il transfinito. Se la presenza di insiemi di reali non misurabili non è certo auspicabile in una corretta teoria di $H_{\aleph_1}$, cosa bisogna evitare in una teoria di $H_{\aleph_2}$ che riesca a comprendere il maggior numero di enunciati veri? Questo rende quindi più complicato trovare un naturale completamento di $H_{\aleph_2}$.

La ricerca di nuovi assiomi, secondo Woodin, deve seguire i seguenti criteri; in accordo con quelli presentati in precedenza. Dato un nuovo assioma ψ,

- ψ deve risolvere molti problemi, dove molti è da intendere secondo il principio di *Fairness* enunciato prima: deve risolvere tutti i problemi di una certa complessità logica,

- ψ deve conciliarsi con un generico criterio di massimalità, cioè l'universo deve essere il più saturo possibile rispetto a qualunque principio di costruzione di insiemi.

L'analisi di Woodin parte dalla considerazione che, dati due enunciati ϕ e ψ di ZFC, la corretta nozione di conseguenza logica non è più quella del primo ordine:

$$\phi \models \psi \iff \forall M \quad M \models ZFC + \phi \Rightarrow M \models \psi,$$

ma, dal momento che V è una classe transitiva, è la seguente, più complessa, relazione:

$$\phi \models_{WF} \psi \iff \forall M \text{ transitivo } \; M \models ZFC + \phi \Rightarrow M \models \psi.$$

[9] "Si cerca di capire in successione le strutture H_ω, H_{ω_1} e poi H_{ω_2}. Un po' più precisamente si cerca di trovare gli assiomi rilevanti per queste strutture. Dal momento che l'Ipotesi del Continuo è formalizzabile nella struttura ha a che fare con la struttura H_{ω_2}, ogni collezione di assiomi per H_{ω_2} ragionevolmente completa risolverà l'Ipotesi del Continuo."

Avendo però solo il metodo del forcing come strumento per generare modelli transitivi di ZFC, Woodin ha definito una nuova nozione di conseguenza logica su ZFC:

$$\phi \models_\Omega \psi \iff \forall B \text{ algebra di Boole completa e } \forall \alpha \ V_\alpha^B \models ZFC + \phi \Rightarrow V_\alpha^B \models \psi.$$

Woodin sostiene che bisogna concentrare lo studio sulla relazione $\models_\Omega$. Facendo così però si assume implicitamente che $\models_{WF}$ sia equivalente a $\models_\Omega$ e quindi che il forcing sia l'unico strumento (non solo l'unico che abbiamo) per produrre modelli transitivi della teoria egli insiemi. Nel caso in cui si riuscisse a dimostrare che il forcing sia effettivamente l'unico metodo per generare modelli transitivi di ZFC allora la strategia di Woodin avrebbe piena legittimità. Ma a tutt'oggi non abbiamo prove del fatto che non si possa inventare un nuovo strumento generale per produrre modelli transitivi della teoria degli insiemi.

Diamo ora una definizione rigorosa di ciò che Woodin intende per una soluzione di una teoria.

Definition 0.0.3. *ψ è detta una soluzione per H_λ rispetto a $\rhd \in \{\models, \models_{WF}, \models_\Omega, \vdash, \ldots\}$ sse per ogni enunciato $\phi \in Th(H_\lambda)$,*

$$ZFC + \psi \rhd \ulcorner H_\lambda \models \phi \urcorner \text{ oppure } ZFC + \psi \rhd \ulcorner H_\lambda \models \neg\phi \urcorner.$$

Dunque il teorema di assolutezza di Schoenfield può essere riletto alla luce della Definizione 0.0.11. Con abuso di notazione confonderemo un insieme ricorsivo di assiomi con un assioma.

Corollary 0.0.4. *ZFC è una soluzione per $H_{\aleph_0}$ rispetto a $\models_{WF}$.*

Il seguente importante teorema dimostra che gli assiomi dei grandi cardinali sono una soluzione per $H_{\aleph_1}$, rispetto a $\models_\Omega$.

Theorem 0.0.5. ***(Woodin e altri)***

- *ZFC + esistono $\omega + 1$ cardinali di Woodin $\vdash \ulcorner L(\mathbb{R}) \models AD \urcorner$,*
- *se esiste una classe propria di cardinali di Woodin in V, allora $Th(L(\mathbb{R})^V) = Th(L(\mathbb{R})^{V^B})$, per ogni B algebra di Boole completa.*

Quindi, dal momento che $H_{\aleph_1} \subseteq L(\mathbb{R})$ e poiché $H_{\aleph_1}^{L(\mathbb{R})} = H_{\aleph_1}$, abbiamo il seguente corollario.

Corollary 0.0.6. *$\phi =$ "esiste una classe propria di cardinali di Woodin" è una soluzione per $H_{\aleph_1}$ rispetto a $\models_\Omega$.*

Dopo aver quindi trovato una soluzione per $H_{\aleph_1}$, in $\models_\Omega$ Woodin sta cercando di estenderla ad una soluzione per $H_{\aleph_2}$. L'idea è quella di definire una relazione di Ω-dimostrabilità $\vdash_\Omega$ in $L(\mathbb{R})$ tale che

$$L(\mathbb{R}) \models \ulcorner \phi \vdash_\Omega \psi \urcorner \Rightarrow \phi \models_\Omega \psi$$

ed inoltre tale che per la più ampia famiglia di enunciati ϕ e ψ:

$$L(\mathbb{R}) \models \ulcorner \phi \vdash_\Omega \psi \urcorner \text{ oppure } L(\mathbb{R}) \models \ulcorner \phi \nvdash_\Omega \psi \urcorner.$$

Woodin dimostra che una relazione con queste caratteristiche esiste.

Theorem 0.0.7. ***(Woodin)*** *Si può definire una nozione di Ω-deduzione $\vdash_\Omega$ tale che:*

1. *se $L(\mathbb{R}) \models \ulcorner \phi \vdash_\Omega \psi \urcorner$, allora in ogni estensione generica V^B si ha che $L(\mathbb{R})^{V^B} \models \ulcorner \phi \vdash_\Omega \psi \urcorner$,*
2. *$\phi \vdash_\Omega \psi \Rightarrow \phi \models_\Omega \psi$,*
3. *esiste ψ (che chiameremo Woodin's Maximum (WM)) soluzione per $H_{\aleph_2}$ rispetto a $\vdash_\Omega$ (e quindi anche rispetto a $\models_\Omega$ per il secondo punto),*
4. *qualunque soluzione ϕ per $H_{\aleph_2}$ rispetto a $\vdash_\Omega$ è tale che $ZFC + \phi \vdash_\Omega \neg CH$.*

Questo teorema, al punto 1), ci dice che la nozione di Ω dimostrabilità è invariante per forcing, al punto 2) che è corretta rispetto alla nozione di Ω-conseguenza logica $\vdash_\Omega$ ed inoltre ci dice che, grazie a 3), nel momento in cui si riuscisse a dimostrarne la completezza rispetto a $\models_\Omega$, allora $\models_\Omega$ sarebbe una naturale nozione di conseguenza logica da utilizzare per decidere ogni problema in $H_{\aleph_2}$. Infine, accettando che il forcing sia l'unico strumento per costruire nuovi modelli transitivi, dal momento che $\vdash_\Omega$ è invariante per forcing, il punto 4) ci direbbe che CH è falsa. Tuttavia sono molte le premesse che si devono ancora accettare per giudicare corretta questa linea di ragionamento; non ultima la completezza della Ω-logica. Questa ha assunto il nome di Ω-congettura:

$$\forall \phi, \psi \in H_{\aleph_2} \text{ di complessità logica } \Pi_2 \quad \phi \models_\Omega \psi \Rightarrow \phi \vdash_\Omega \psi. \tag{1}$$

Dal momento che una vera completezza delle strutture del tipo H_λ non si può avere, a causa del teorema di Gödel, vi è sempre un margine di arbitrarietà nella scelta degli assiomi che decidano la struttura. Di conseguenza, in mancanza di un'adeguata intuizione su ciò che sia da considerare vero nella struttura H_{ω_2}, è più difficile accettare la verità di WM mentre non è problematico accettare l'esistenza di una classe di cardinali di Woodin.

Il seguente risultato mostra come lo studio degli assiomi di forcing ed il programma di Woodin non siano affatto incompatibili, ma anzi paralleli.

Theorem 0.0.8. *WM implica gli assiomi di forcing PFA e MM, ristretti ad ordini parziali di cardinalità inferiore od uguale ad* $\aleph_2$*.*

Struttura della tesi

L'obiettivo di questa tesi è di dare una presentazione degli assiomi di forcing e di mostrare come i più forti tra essi decidano la cardinalità del continuo. Essi infatti implicano che

$$2^{\aleph_0} = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \aleph_2.$$

In questo modo il problema del continuo viene risolto negando CH.

Questo lavoro si struttura in quattro capitoli. Nel primo capitolo verranno introdotti strumenti e definizioni utili per comprendere il seguito. Prima verranno richiamati alcuni risultati sulle nozioni di insiemi club ed insiemi stazionari. Verrà poi presentata una generalizzazione del concetto di insieme club ed insieme stazionario agli spazi $[\kappa]^\lambda$ molto utile nella definizione di properness. Questa verrà introdotta nel paragrafo successivo, insieme alle principali classi di forcing di cui ci occuperemo. Infine verranno presentati alcuni risultati standard sulla combinatorica infinita.

Nel secondo capitolo verrà data la definizione generale degli assiomi di forcing. Si partirà da una discussione sull'assioma di Martin (MA), per poi vedere come gli assiomi di forcing non siano che una sua generalizzazione. Verranno quindi presentati PFA, SPFA e MM e le loro versioni limitate. Infine verrà fatto un breve cenno su FA(σ-closed$*$c.c.c., $\aleph_1$), il più debole assioma di forcing che verrà utilizzato per decidere la cardinalità del continuo.

Nel terzo capitolo verranno presentate le dimostrazione di $2^{\aleph_0} = \aleph_2$ assumendo MM e PFA. Le due dimostrazioni saranno fattorizzate utilizzando due principi di riflessione, rispettivamente lo Strong Reflection Principle (SRP) e il Mapping Refelction Principle (MRP). Inoltre verrà mostrato come assumendo dei principi di riflessione, anche più deboli di SRP, (Reflection Principle (RP) e Weak Reflection Principle (WRP)) la classe dei forcing semiproper e quella dei forcing che preservano gli insiemi stationari su ω_1 vengano a coincidere.

Nel quarto capitolo verrà presentata la dimostrazione di come FA(σ-closed$*$c.c.c., $\aleph_1$), un assioma di forcing più debole di MM e di PFA, decida anch'esso che la cardinalità del continuo è $\aleph_2$. Come nei casi precedenti la dimostrazione verrà portata avanti utilizzando dei principi combinatoriali utili anche per applicazioni diverse: in particolare lo Open Coloring Axiom (OCA).

Introduction

The main object of set theory is the notion of infinity, which is analyzed with the tools of mathematical logic. This study is focused on the analysis of the concept of cardinality. Cantor defines the cardinality of a set as the equivalence class of the sets that are in bijection with it.

We can notice that this definition is a natural abstraction of the notion of number. Indeed, in the finite case, this is the right formalization of the concept of number. Given a set X, $|X|$ means its cardinality, and by Greek letters we denote the equivalence classes induced by the relation of bijection. There is a natural order relation between cardinalities: $|X| \leq| Y|$ iff there is an injection of X into Y, while $|X| < |Y|$ iff $|X| \leq| Y|$ and there is no injection from $|Y|$ to $|X|$.

One of the first results of the theory shows that the quotient space induced by the relation of bijection is the same as the one induced by the relation of $\leq$ between cardinalities. It is named the Schroeder-Bernstein Theorem: $|X| = |Y|$ iff $|X| \leq| Y|$ and $|Y| \leq| X|$. Moreover, Zermelo proved that, assuming we can give a well-order to every set, for every X, Y, $|X| \leq| Y|$ or $|Y| \leq| X|$; in this way it is possible to show that cardinals fall into a linear well-order, extending the classical one of natural numbers. Another very important result is Cantor's Theorem: for every set $|X|$, $|\mathcal{P}(X)| > |X|$. Hence, the notion of infinite cardinal is not trivial.

By analogy with natural numbers it is possible to define the operations of sum, product, and exponentiation for cardinal numbers. Given two cardinals κ and λ, the sum is defined as the cardinality of the disjoint union of the two sets: $|\kappa \sqcup \lambda|$; the product as the cardinality of the cartesian product: $|\kappa \times \lambda|$; the exponentiation as the cardinality of the set of all function from the first one into the second one: $|{}^{\kappa}\lambda|$.

Notice that, as far as finite sets are concerned, the above operations coincide with the operations of elementary arithmetic; as a matter of fact, given two sets X and Y, of cardinality respectively n and m, $n+m = |X \sqcup Y|$, $n \cdot m = |X \times Y|$ and $n^m = |{}^{X}Y|$. Hence cardinal arithmetic can be seen as a natural generalization of elementary arithmetic. Nevertheless, in the infinite case, it has different properties, since there are specific characteristics of infinite sets to be taken into account: for example, the fact that an infinite set can be in bijection with a proper subset. For this reason it is not difficult to show that, given two infinite cardinals κ and λ,

$$\kappa + \lambda = \kappa \cdot \lambda = max\{\kappa, \lambda\}.$$

In one case, the operations of sum and product raise no problems, however, on the other hand the problems related to the calculation of cardinal exponentiation are ones of the deepest and most difficult of all set theory.

One of the first problems of cardinal arithmetic that captured the interest of Cantor was that of the cardinality of the continuum: if we set $X = \mathbb{N}$, in Cantor's Theorem we have that

$$|\mathbb{R}| = 2^{|\mathbb{N}|} = |\mathcal{P}(\mathbb{N})| > |\mathbb{N}|.$$

Cantor conjectured that the value of $2^{|\mathbb{N}|}$ was the successor cardinal after $|\mathbb{N}| = \aleph_0$, that is $\aleph_1$. Nevertheless in more than one hundred years of efforts a satisfactory and fulfilling solution to this problem could not be found.

From the continuum problem to the forcing axioms

At the beginning of the twentieth century the problem developed into the well-known conjecture CH (Continuum Hypothesis):

$$2^{\aleph_0} = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \aleph_1$$

and its subsequent generalization GCH (General Continuum Hypothesis):

$$\forall \alpha \in \mathbb{O}n \;\; 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

Until the Sixties and after ZFC was accepted as the suitable set theory formalisation, the most important progresses had been made attempting to prove or disprove CH. After Gödel proved the Incompleteness Theorem -in 1931- people started gathering that the solution might be different from the expected one. As a matter of fact, in 1939, Gödel showed that Con(ZFC) $\Rightarrow$ Con(ZFC + CH) and in 1963, Cohen ([3]) showed that Con(ZFC) $\Rightarrow$ Con(ZFC + $\neg$CH). In this way Cohen completed the proof of the independence of CH from ZFC. Moreover he endowed set theory with a very powerful tool: the method of forcing.

Thereby, in the Sixties, set theory deeply changed. Most of the unsolved problems were shown to be independent from ZFC. Furthermore, forcing refinements and generalisations showed that the main object of the theory was independent problems. This feature deeply distinguishes set theory from other theories, equally incomplete as number theory, that live in enough "simple" enough universes where the phenomenon of incompleteness is not central to the theory.

Since there were sufficiently elementary problems, as CH, that were expected to have a complete solution, these developments drove set theory scholars to seek for new principles that would allow them to solve problems that ZFC left open.

The first new axioms taken into consideration were the large cardinals axioms. They postulate the existence of infinite cardinals with specific combinatorial properties. These sets are big enough to be a model of ZFC themselves. Therefore, their existence cannot be proved in ZFC because of the second incompleteness theorem.[10].

This direction fell into the path of what is commonly known as the *Gödel program*, as was stated by him in [6].

> *These axioms show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of the series of those set.*

What lead Gödel was the possibility of solving problems related to sets of the lowest hierarchy -of von Neumann- given by the existence of bigger and bigger cardinals. As a matter of fact, thanks to their combinatorial properties, these cardinals were useful to achieve the solution of undecidable problems. However, it was soon realized that these new axioms were not sufficient in finding a solution to CH.

The *"concept of set"* quoted by Gödel is the iterative one. It is based on the idea that it is possible to iterate the operation of *set of* indefinitely: i.e. the operation of putting together some objects and making a non-contradictory set out of them. To extend this operation, the universe of sets was pushed in order to be expanded "upwards". For example, the existence of an inaccessible cardinal was justified by the idea of considering the collection of all the sets -given by admissible operations in ZFC- as a set itself.

Hence, one of the main purposes that led to large cardinal axioms was to maximize the domain of the operation of *set of*, and thus extend the universe of sets, beyond ZFC. In Gödel's words ([6])

[10]For an exhaustive account on the subject of big cardinals see [11] and [12]

Only a maximum property would seem to harmonize with the concept of set.

Since large cardinal axioms have a consistency strength bigger than ZFC, the principles -as maximality- that helped to be confident in them are not simply mathematical arguments.

Another aspect that helps to believe in the naturalness of large cardinal axioms is their deep connection with the Axiom of Determinacy (AD). Moreover, they justify each other and together they give a completion of the theory of $L(\mathbb{R})$ in the expected way.

Another useful feature of large cardinal axioms that speak in favour of their non-arbitrary character is the fact that they fall into a linear order. This fact does not have a demonstrable certainty, but an empirical soundness.

Since there are many problems -as CH- left open by large cardinal axioms, from the Eighties on new principles that could determine the cardinality of the continuum were sought after. What forcing brought in set theory was a general method of constructing new models of ZFC. If the concept of maximality had before been seen as a freer application of the operation of *set of*, it was later meant as a wider saturation of the universe of sets, with respect to new objects that lived in some forcing extension. This way led to the definition of forcing axioms.

Historically, the first forcing axiom was Martin's Axiom (MA), which was discovered in the study of iterated forcing. Soon after MA had been generalized. At the beginning of the Eighties, Baumgartner ([2]) and Shelah defined the Proper Forcing Axiom (PFA) . In the late Eighties, Foreman, Magidor, and Shelah ([5]) discovered Martin's Maximum (MM).

Forcing axioms are a generalization of the Baire Category theorem. For a wider class of topological spaces, they state that an uncountable intersection of open dense sets is non-empty. It is easy to see that they affect cardinal arithmetic. Indeed they imply the failure of CH. To see it, we suppose that CH holds and so we can enumerate all reals in order type ω_1: $\mathbb{R} = \{r_\alpha : \alpha \in \omega_1\}$. Now, for every $\alpha \in \omega_1$, define $D_\alpha = \{h \in \mathbb{R} : \exists n\ h(n) \neq r_\alpha(n)\}$. D_α is dense open of $\mathbb{R}$ for every $\alpha \in \omega_1$ and every real number that is in the intersection of all these sets is a real not belonging to the enumeration. Contradiction.

These axioms can be stated as follows:

$FA(\mathcal{A}, \kappa)$ it is true when, given $\mathcal{A}$ a class of topological spaces, if $X \in \mathcal{A}$ and $\mathcal{F}$ is a family of $\leq \kappa$ open dense subsets of X, then $\bigcap \mathcal{F} \neq \varnothing$.

Unlike large cardinal axioms, the forcing axioms do decide the cardinality of the continuum and show that $2^{\aleph_0} \geq \aleph_2$. In this work we will consider the axioms PFA: $FA(\mathbb{P}_{Proper}, \aleph_1)$, MM: $FA(\mathbb{P}_{SSP}, \aleph_1)$ and many variances of them; where $\mathbb{P}_{Proper}$ and $\mathbb{P}_{SSP}$ are classes of topological spaces that wil will define later on.

Reasons to accept new axioms

The consistency strength of the strongest forcing axiom is -for what we know so far- almost that of a supercompact cardinal: one of the biggest large cardinals. Nevertheless, there has not been an inner model constructed for them yet. Hence, we are less confident about their existence.

By the way, in favour of forcing axioms there is empirical evidence -supported by Woodin's work- that they give a complete theory of the structure $H_{\aleph_2}$: the collection of all sets hereditary of cardinality less than $\aleph_2$. It happens that, if by means of forcing axioms we can build up a counterexample to some sentence, then that counterexample could be proved in ZFC itself.[11]. Finally, thanks to results obtained with forcing axioms, it was possible to find new theorems in ZFC. The following, due to Todorčević, is a theorem inspired by OCA -a consequence of PFA-.

Theorem 0.0.9. *Let $X \subseteq \mathbb{R}$ be a Σ^1_1 set and let $K \subseteq [\mathbb{R}]^2 = \{(x, y) : x > y\}$ be an open subset. Then just one of the following holds:*

- *$\exists P \subseteq X$ perfect set that is homogeneous for K (i.e. $[P]^2 \subseteq K$),*
- *X is covered by countably many sets homogeneous for $[\mathbb{R}]^2 \setminus K$.*

This theorem can be seen as a two-dimensional generalization, for Σ^1_1 sets, of the the Perfect Set Property; to see it, we need to set $K = \mathbb{R}$ to obtain, as a corollary, that the perfect set property holds for analytic sets.

The above arguments are good reasons to believe that forcing axioms are true statements. Nevertheless, since their consistency strength is bigger than ZFC, the reasons we have to accept them are mostly philosophical. Bagaria has listed four principles: *Consistency*, *Maximality*, *Fairness*, *Success*. They are the main criteria that should lead the quest for new axioms.

Before analyzing these principles, we would like to make some comments on the meaning of axioms. By definition, an axiom should be a self-evident statement. Following this definition there should be an intuitive connection between mathematical objects and human mind, that leads, without rational arguments, to accept a sentence as an axiom just on the strength of intuition.

[11]For example, in ZFC we can show that there are just (ω_1, ω_1^*)-gap or $(\mathfrak{b}, \omega^*)$-gap. Under OCA, the former ones are the only gaps that can exist.

This is how the concept of axiom is generally meant in the classical way. As far as forcing axioms are concerned, this is not true. Indeed they are principles for which the intuition has to be trained and has to get familiar with objects that often have properties and characteristics far from being intuitive. Forcing axioms are very different from the other axioms of ZFC; this is due not only to the concepts involved -far from being basic in set theory- but also to their technical nature. Hence, we need to find new criteria different from intuition, to drive us to be more confident with forcing axioms.

We now see the four principles listed by Bagaria.

Consistency. This is the easiest question we would ask a formal system, so that it can be representative of a mathematical context. Even if a contradiction makes a system a trivially complete one, it still cannot be considered a sound extension of the system. In one circumstance, consistency -with ZFC- of a new axiom is a necessary condition, but it is not sufficient to adopt it.

Maximality. This principle can be seen as a generic instruction to prefer a vision of the universe of sets to be as rich as possible. Hence, we need to reject restrictive axioms like the Axiom of Constructability. This rule is rather vague, and the most convincing attempt in formalizing it is that of Woodin; we will discuss it later on. In that case, maximality is intended as completeness of $H_{\aleph_2}$, with respect to a new relation of demonstrability.

Fairness. In Bagaria's words:

> *One should not discriminate against sentences of the same logical complexity.*

The rationale of this principle is that, if we do not have a good intuition of what is true and what is false in a certain domain, then we can just accept axioms that give a solution to all problems of a given logical complexity.

Success. It is the principle that takes into account the consequences of an axiom, not just its nature. The richer the consequences, the more one is inclined to accept it. Success analyzes the utility of an axiom, more that its truth. Utility is here to intended in the sense of giving solutions to problems not yet solved or in the sense of giving new insights in old domains of the theory.

In general, we are confident about new axioms if they satisfy the four principles above, and if they are coherent with big cardinals. Then, as a result, we can believe that forcing axioms -modulo relative coherence- carry all the characteristics shaped by Bagaria's principles.

The results of Woodin are new evidences in favour of forcing axioms.

An outline of Woodin's program

"Woodin's program" aims to give a solution to the problem of the continuum and in the meantime gives good arguments in favour of forcing axioms. We now present an outline of it, which like Gödel's program seeks for new axioms of ZFC, in order to give a solution to undecidable problems. A popular presentation of the following results can be found in [27], [28], while a complete account of the work of Woodin is [26].

The philosophy of mathematics inspiring Woodin's work is platonism. As a matter of fact, the main domain of research is the first order theory of $V = (V, \in, =)$: $\mathrm{Th}(V) = \{\phi : V \vDash \phi\}$, considered as a given object. To study it we do not use the hierarchy of V_α:

- $V_0 = \varnothing$,
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
- $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ per λ limite,
- $V = \bigcup_{\alpha \in Ord} V_\alpha$,

but we consider the stratification of V given by H_λ, where λ is a cardinal: the structures given by the sets of cardinality hereditarily less than λ.

$$H_\lambda = \{x \in V : |x| < \lambda \text{ and } \forall y\, (y \in tc(x) \Rightarrow |y| < \lambda)\}.$$

They are good approximations of V. Indeed the hierarchy of H_λ allows to consider initial segments of the universe, sufficiently close for set operations.[12] Moreover, $V = \bigcup_{\lambda \in \mathbb{O}n} H_\lambda$ and for strongly limit cardinals, the two hierarchies are the same.

The main target is to extend ZFC, solving problems that can be formulated in $\mathrm{Th}(H_\lambda)$, for each λ. This process cannot solve every problem in ZFC, but can give us new axioms that can complete the theory as far as initial segments are concerned. The first step toward this direction is the theorem of absoluteness by Schoenfield:

Theorem 0.0.10. ***(Schoenfield)*** *If ϕ is a Σ^1_2-formula (for example an arithmetical one) then every transitive model M of ZFC satisfy ϕ or every transitive model M of ZFC satisfy $\neg\phi$.*[13]

[12]Note that the sets H_λ, unlike the sets V_α, are closed for replacement.

[13]This theorem tells us that we cannot hope to make false a theorem of number theory with transitive models of ZFC.

The idea of Woodin is to obtain results analogous to this theorem for H_λ, for arbitrary λ. As we shall see Woodin succeeding in doing so for H_{ω_1}, and to a certain extent also for H_{ω_2}. Note that if we could find an axiom that decides the theory of H_{ω_2}, we would have a solution of CH.

In Woodin's words, the program should be as follows:

> *One attempts to understand in turn the structures H_ω, H_{ω_1} and then H_{ω_2}. A little more precisely, one seeks to find the relevant axioms for these structures. Since the Continumm Hypothesis concerns the structure of H_{ω_2}, any reasonably complete collection of axioms for H_{ω_2} will resolve the Continumm Hypothesis.*

Woodin chooses to work on $\langle H_{\omega_2}, \in, =\rangle$ instead of $\langle \mathcal{P}(\mathcal{P}(\mathbb{N})), \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in \rangle$ (equivalent to $\langle V_{\omega+2}, \in, =\rangle$) and on $\langle H_{\omega_1}, \in, =\rangle$ instead of $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in\rangle$ (equivalent to $\langle V_{\omega+1}, \in, =\rangle$) , as more obvious generlizations of $\langle \mathbb{N}, +, \cdot, \in\rangle$. For the structure $H_{\aleph_1}$ we have a good intuition of what the pathological aspects we would like to avoid are (like for example the Banach-Tarski paradox and the consequent non-measurability of some sets of reals), but this is much more difficult for $H_{\aleph_2}$. It is more difficult to train our intuition to see what is true and what is false, as soon as we go on in the hierarchy of the transfinite. If we do not want to find, in $H_{\aleph_1}$, sets of reals that are not measurable, what should we avoid in a right theory of $H_{\aleph_2}$?

Following Woodin, the quest for new axioms should be coherent with the following principles; which agree with the one presented before. Given a new axiom ψ,

- ψ has to provide solutions for many problems, where many is intended with accordance to the principle of *Fairness*: it should solve all problems of a given logical complexity,
- ψ has to be be coherent with a generic principle of maximality, i.e. the universe should be as full as possible with respect to any principle of set construction.

Woodin's analysis starts by considering that, given two sentences ϕ e ψ of ZFC, the right notion of logical consequence is not the classical first order one:

$$\phi \vDash \psi \iff \forall M \quad M \vDash ZFC + \phi \Rightarrow M \vDash \psi,$$

but, since V is a transitive class, is the following more complex one:

$$\phi \vDash_{WF} \psi \iff \forall M \text{ transitive } \; M \vDash ZFC + \phi \Rightarrow M \vDash \psi.$$

Since Woodin only has the method of forcing as a tool to construct transitive models of ZFC, he defined a new notion of logical consequence on ZFC:

$$\phi \models_\Omega \psi \iff \forall B \text{ complete Boolean algebra and } \forall \alpha \; V_\alpha^B \models ZFC{+}\phi \Rightarrow V_\alpha^B \models \psi.$$

Woodin thinks that we need to concentrate the study on the relation $\models_\Omega$. In this way we are implicitly assuming that $\models_{WF}$ is equivalent to $\models_\Omega$ and thus that the method of forcing is the only (not just the only we know) tool to build up transitive models of set theory. If we could show that this is the case, then Woodin's strategy would be fully legitimate. But still we do not have proofs that we could not invent a new method that allows us to construct new transitive models of set theory.

We now give a rigorous definition of what Woodin means by a solution of a theory.

Definition 0.0.11. *ψ is called a solution of the theory H_λ with respect to $\rhd \in \{\models, \models_{WF}, \models_\Omega, \vdash, \ldots\}$ iff for every sentence $\phi \in Th(H_\lambda)$,*

$$ZFC + \psi \rhd \ulcorner H_\lambda \models \phi \urcorner \text{ or } ZFC + \psi \rhd \ulcorner H_\lambda \models \neg\phi \urcorner.$$

Hence, the absoluteness theorem by Schoenfield could be seen in the light of Definition 0.0.11. With abuse of notation we will confuse a recursive set of axioms and an single axiom.

Corollary 0.0.12. *ZFC is a solution for $H_{\aleph_0}$ with respect to $\models_{WF}$.*

The following important theorem shows that large cardinal axioms are a solution for $H_{\aleph_1}$ with respect to $\models_\Omega$.

Theorem 0.0.13. ***(Woodin and others)***

- *ZFC + there exist $\omega + 1$ Woodin's cardinals $\vdash \ulcorner L(\mathbb{R}) \models AD \urcorner$,*
- *if there is a proper class of Woodin's cardinals in V, then $Th(L(\mathbb{R})^V) = Th(L(\mathbb{R})^{V^B})$, for every complete Boolean algebra B.*

Since $H_{\aleph_1} \subseteq L(\mathbb{R})$ and $H_{\aleph_1}^{L(\mathbb{R})} = H_{\aleph_1}$, we have the following corollary.

Corollary 0.0.14. *$\phi =$ "there exist a proper class of Woodin's cardinals" is a solution for $H_{\aleph_1}$ with respect to $\models_\Omega$.*

Now that he has found a solution for $H_{\aleph_1}$, in $\models_\Omega$, Woodin is trying to extend it to a solution for $H_{\aleph_2}$. The main idea is that of defining a relation of Ω-demonstrability $\vdash_\Omega$ in $L(\mathbb{R})$ such that

$$L(\mathbb{R}) \models \ulcorner \phi \vdash_\Omega \psi \urcorner \Rightarrow \phi \models_\Omega \psi$$

and such that for the wider class of sentences ϕ and ψ:

$$L(\mathbb{R}) \models \ulcorner \phi \vdash_\Omega \psi \urcorner \text{ or } L(\mathbb{R}) \models \ulcorner \phi \nvdash_\Omega \psi \urcorner.$$

Woodin shows that such a relation exists.

Theorem 0.0.15. *(**Woodin**) We can define a notion of Ω-deduction $\vdash_\Omega$ such that:*

1. *if $L(\mathbb{R}) \models \ulcorner \phi \vdash_\Omega \psi \urcorner$, then for every generic extension V^B we have that $L(\mathbb{R})^{V^B} \models \ulcorner \phi \vdash_\Omega \psi \urcorner$,*
2. *$\phi \vdash_\Omega \psi \Rightarrow \phi \models_\Omega \psi$,*
3. *there exists ψ (that we shall call Woodin's Maximum (WM)) solution for $H_{\aleph_2}$ with respect to $\vdash_\Omega$ (and so also with respect to $\models_\Omega$ for the second point),*
4. *any solution ϕ for $H_{\aleph_2}$ with respect to $\vdash_\Omega$ is such that $ZFC + \phi \vdash_\Omega \neg CH$.*

This theorem shows, by 1) that the notion of Ω-demonstrability is invariant under forcing; by 2) that it is sound with respect to the notion of $\vdash_\Omega$ logical consequence; by 3) that if we could prove the completeness with respect to $\models_\Omega$, then $\models_\Omega$ would be the natural notion of logical consequence to use in deciding every problem that could be formalized in $H_{\aleph_2}$. If we accept that forcing is the only tool we have to construct transitive models, then, recalling that $\vdash_\Omega$ is invariant under forcing 4) would tell us that CH is false. Thereby there are still many premises to be accepted to believe in this line of reasoning; above all the completeness of the Ω-logica. So this conjecture is called the Ω-conjecture:

$$\forall \phi, \psi \in H_{\aleph_2} \text{ that are } \Pi_2 \quad \phi \models_\Omega \psi \Rightarrow \phi \vdash_\Omega \psi. \tag{2}$$

Since we cannot have a true completeness of the structure like H_λ, because of Gödel's Theorem, there is always a sort of arbitrariness in choosing the axioms that decide the structure. As a consequence, since we do not have a clear intuition of what is true in H_{ω_2}, it is more difficult to accept the truth of WM, while we are more confident in the truth of Woodin's cardinal.

The following result shows that the study of forcing axioms and Woodin's program are not incompatible, but they are very close.

Theorem 0.0.16. *WM implies the forcing axioms PFA and MM, restricted to posets of cardinality less or equal to $\aleph_2$.*

Structure of the thesis

The main target of this thesis is to give an account of forcing axioms and to show that the strongest of them decide the cardinality of the continuum. Indeed they imply that

$$2^{\aleph_0} = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \aleph_2.$$

In this way the problem of the continuum is solved showing that $\neg CH$ holds.

This work is structured in four chapters. In the first one we introduce the tools and the definitions useful to understand the rest. Firstly we recall some results on the notions of club sets and stationary sets. Then we present a generalization of the concepts of stationarity and clubness to the spaces $[\kappa]^\lambda$, which are very useful in the definition of properness. This latter notion will be introduced in a following section, together with the main notions of forcing we will deal with. Finally we will recall some standard results on infinite combinatorics.

In the second chapter we will give the general definition of a forcing axiom. We start with a discussion on Martin's Axiom (MA) and we see that forcing axioms are a generalization of MA. We then present PFA, SPFA e MM and their bounded counterparts. Finally we give the definition of FA(σ-closed$*$c.c.c., $\aleph_1$), the weakest forcing axiom we use to decide the cardinality of the contuinuum.

In the third chapter we present the proof that $2^{\aleph_0} = \aleph_2$ assuming MM and PFA. These proofs are factorized by two reflection principles, respectively the Strong Reflection Principle (SRP) and the Mapping Refelction Principle (MRP). Moreover we show that assuming SRP and some weaker form of reflection, namely the Reflection Principle (RP) and Weak Reflection Principle (WRP) the classes of semiproper forcing and stationary set preserving forcing are the same.

In the fourth chapter we present the proof that FA(σ-closed$*$c.c.c., $\aleph_1$), a forcing axiom weaker than MM and PFA, decides that the cardinality of the continuum is $\aleph_2$. As in the previous cases the proof uses some combinatorial principles: namely the Open Coloring Axiom (OCA).

Chapter 1

Background material

In this chapter we will fix the notation and review some basic facts about stationary sets, forcing and iterated forcing that will be useful for better understanding the meaning of forcing axioms. We assume good knowledge of the method of forcing. We will state some important results without proofs, unless the proofs are essential for the exposition.

1.1 Useful facts about stationary sets

We start with some theorems and definitions we will use later on.

Definition 1.1.1. *Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is a closed unbounded (club) subset of κ if C is unbounded in κ and if it contains all its limit points less than κ.*

A set $S \subseteq \kappa$ is stationary if $S \cap C \neq \varnothing$ for every club C of κ.

The club sets generate the closed unbounded filter on κ, consisting of all the sets $X \subseteq \kappa$ that contain a club.

Definition 1.1.2. *NS_{ω_1} indicates the ideal of non stationary sets, i.e. $\omega_1 \supset X \in NS_{\omega_1}$ iff $\omega_1 \setminus X$ is club.*

NS_{ω_1} is a normal and ω_1-complete ideal.

Lemma 1.1.3. *(**Fodor**) If f is a regressive function on a stationary set $S \subseteq \kappa$ (i.e. $f(\alpha) < \alpha$, for every $\alpha \in S \setminus \{0\}$), then there is a stationary set $T \subseteq S$ and some $\gamma \in \kappa$ such that $f(\alpha) = \gamma$, for every $\alpha \in T$.*

Definition 1.1.4. *By a $(\aleph_0, \aleph_1)$ Ulam matrix we mean a family $\mathcal{A} = \{A_{\alpha,n} : \alpha \in \omega_1, n \in \omega\}$ of sets such that*

- $\forall n \in \omega,\ A_{\alpha,n} \cap A_{\beta,n} = \varnothing$
- $\forall \alpha \in \omega_1,\ \omega_1 \setminus \bigcup_n A_{\alpha,n}$ *is at most countable.*

The next theorem, building up a $(\aleph_0,\ \aleph_1)$ Ulam matrix, shows that there exists a maximal partition of ω_1 in stationary sets.

Theorem 1.1.5. *There exists a family $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$ of disjoint stationary subsets of ω_1 that covers ω_1 (i.e. $\bigcup_{\alpha<\omega_1} S_\alpha = \omega_1$) and is maximal for stationary sets: if $S \subseteq \omega_1$ is stationary, then there is an α such that $S \cap S_\alpha$ is stationary.*

Proof. We begin with building up a $(\aleph_0,\ \aleph_1)$ Ulam matrix; then we will use it to form a maximal antichain of size $\aleph_1$.

For every $\gamma \in \omega_1$ we take a bijection $f_\gamma : \gamma \to \omega$. Define $A_{\alpha,n} = \{\gamma > \alpha : f_\gamma(\alpha) = n\}$. If $\alpha \neq \beta$ we have that $A_{\alpha,n} \cap A_{\beta,n} = \varnothing$, otherwise there was a γ such that $f_\gamma(\alpha) = n = f_\gamma(\beta)$. Hence for each n, the $A_{\alpha,n}$'s are disjoint. Moreover $\bigcup_n A_{\alpha,n} = \{\gamma \in \omega_1 : \gamma > \alpha\}$. So $\omega_1 \setminus \bigcup_n A_{\alpha,n}$ is at most countable. Since NS_{ω_1} is ω_1-complete and $\bigcup_n A_{\alpha,n}$ is club, we have that there is an n such that $A_{\alpha,n} \notin \mathrm{NS}_{\omega_1}$. Thus we can define $h : \omega_1 \to \omega$ such that $A_{\alpha,h(\alpha)} \notin \mathrm{NS}_{\omega_1}$. By the pigeonhole principle we have that exists m such that $|h^{-1}(m)| = \omega_1$. Set $X = \{\alpha : A_{\alpha,m}$ is stationary $\}$. If we define $S = \omega_1 \setminus \bigcup_{\alpha\in X} A_{\alpha,m}$, then $\{S\} \cup \{A_{\alpha,m} : \alpha \in X\}$ is a disjoint partition of ω_1. It is also a stationary partition, since if S is not stationary, we could put it in one of the $A_{\alpha,m}$, for some α and m. Let us renumber the family $\mathcal{S} = \{S\} \cup \{A_{\alpha,m} : \alpha \in X\}$ as $\mathcal{S} = \{T_\alpha : \alpha \in \omega_1\}$.

To make the T_α maximal for the stationary sets, we can use the diagonal union of the family. We call $C = \Sigma_\alpha T_\alpha = \{\beta : \exists \alpha < \beta,\ \beta \in T_\alpha\}$ the diagonal union of the T_α. If C is a club, then $\mathcal{S} = \{T_\alpha : \alpha < \omega_1\}$ is already maximal. Indeed for every stationary set $S \subseteq \omega_1$ we have that $S \cap C$ is stationary. If $\beta \in S \cap C$, then exists an $\alpha < \beta$ such that $\beta \in S \cap T_\alpha$. If we call α_β the minimal such α, we can define a regressive function

$$\begin{aligned} \phi :\ & S \cap C \to \omega_1 \\ & \beta \mapsto \alpha_\beta \end{aligned}$$

and use Fodor lemma to find a stationary subset $S' \subseteq S \cap C$ and α_0 such that $S' \subseteq T_{\alpha_0}$. Thus $S \cap T_{\alpha_0}$ is stationary.

In the case C is not club, we define $T = \omega_1 \setminus C$. Since we want a family of disjoint sets, we need to define $S_\alpha = T_\alpha \setminus T$. It is easy to see that $C = \bigcup_\beta \{T_\beta \setminus (\beta+1)\}$. Hence, since T_α is stationary and α is countable, $T_\alpha \setminus T = T_\alpha \cap C = T_\alpha \cap \bigcup_\beta \{T_\beta \setminus (\beta+1)\} \supseteq T_\alpha \setminus (\alpha+1)$ is stationary. So we can now claim that $\{T\} \cup \{T_\alpha \setminus T : \alpha < \omega_1\}$ is a maximal family of disjoint stationary sets. Let us call $\mathcal{C}' = \{T_\alpha \setminus T : \alpha < \omega_1\}$. Given S a stationary set, we have that $S \cap T$ or $S \cap \bigcup \mathcal{C}'$ are stationary. If we are in the first case, we are done. If $S \cap \bigcup \mathcal{C}'$, as before, we can apply Fodor Lemma and find an α_0 such that $S \cap T_{\alpha_0} \setminus T$ is stationary. $\square$

We end this section by defining a class of stationary sets and some results on them.

Definition 1.1.6. *For κ, a regular and uncountable cardinal and $\lambda < \kappa$ regular, we define $E^\kappa_\lambda = \{\alpha < \kappa : cof(\alpha) = \lambda\}$.*

It is easy to see that E^κ_λ is stationary on κ. We just give, without proof, some classical results related to this definition. For the proofs we refer to [20]

Theorem 1.1.7. *Every stationary subset of E^κ_λ is the union of κ disjoint stationary sets.*

Theorem 1.1.8. ***(Solovay)*** *Let κ be a regular uncountable cardinal. Then every stationary subset of κ is the disjoint union of κ stationary subsets.*

1.2 Generalizing stationarity

It is possible to generalize the concept of clubness and stationarity to $\mathcal{P}(X)$, for a given set X. We will refer to definitions as in [14].

By $[X]^\kappa$, where κ is a regular cardinal and X is a set of cardinality al least κ, we mean the set $\{x \subseteq X : |x| = \kappa\}$; and so $[X]^{<\kappa} = \bigcup\{[X]^\mu : \mu < \kappa\}$.

Definition 1.2.1. *Let $X \neq \varnothing$. A set $C \subseteq \mathcal{P}(X)$ is said to be club in $\mathcal{P}(X)$, if there is a function $F : [X]^{<\omega} \to X$ such that*

$$\{z : F[[z]^{<\omega}] \subseteq z\} \subseteq C$$

i.e. C contains the set of the closure points of F. For this reason we will refer to it as to C_F.

Definition 1.2.2. *Let $X \neq \varnothing$. A set $S \subseteq \mathcal{P}(X)$ is stationary in $\mathcal{P}(X)$ if, for every function $F : [X]^{<\omega} \to X$, $C_F \cap S \neq \varnothing$.*

It is easy to see that if S is a set of ordinals and $\kappa = \sup(S)$, we have that S is stationary in $\mathcal{P}(\kappa)$ iff S is stationary in κ.

Note that in the case of the structure $([X]^{\aleph_0}, \subseteq)$ a club set, as in the definition above, is equivalent to a closed unbounded set in the structure $([X]^{\aleph_0}, \subseteq)$. Hence, for the collection of countable subsets of a given set X, it turns out that the club filter generated by the closed unbounded sets equals the one generated by the sets of the closure points of F, for some $F : [X]^{<\omega} \to X$.

This is not the only generalization of the notion of stationarity and clubness, but it is the most useful for our applications. Indeed if we let $\mathbf{M} = (M, \ldots)$ be a structure for a countable language and we let $F : [M]^{<\omega} \to M$ be a Skolem function, then any $N \subseteq M$ closed under F is, by definition of a Skolem function, the universe of an elementary submodel of $\mathbf{M}$ (i.e. $\mathbf{N} \prec \mathbf{M}$).

By the Lowenheim-Skolem theorem we have that, for a structure $\mathbf{M}$ as before and a $\lambda \leq |M|$, the set

$$\{N \subseteq M : |N| = \lambda\}$$

is stationary in $\mathcal{P}(M)$.

The following lemma shows how stationarity is mantained, passing from sets to subsets.

Lemma 1.2.3. *Suppose that $X \subseteq Y$ and $X \neq \varnothing$.*

- *If $S \subseteq \mathcal{P}(Y)$ is stationary in $\mathcal{P}(Y)$, then $S \restriction X = \{Z \cap X : Z \in S\}$ is stationary in $\mathcal{P}(X)$.*
- *If $S \subseteq \mathcal{P}(X)$ is stationary in $\mathcal{P}(X)$, then $S^Y = \{Z \subseteq Y : Z \cap X \in S\}$ is stationary in $\mathcal{P}(Y)$. S^Y is called the lift of S to Y.*

We have Fodor's lemma, also for thins notion of stationarity.

Lemma 1.2.4. *Let $X \neq \varnothing$ and $S \subseteq \mathcal{P}(X)$ be a stationary set. If $F : S \to X$ is such that $F(Z) \in Z$ for all $Z \in S$, then there is an $a \in X$ such that $\{Z \in S : F(Z) = a\} \subseteq S$ is stationary (i.e. F is costant on a stationary subset of S).*

Definition 1.2.5. *For a cardinal λ, we set $H_\lambda = \{x \in V : |x| < \lambda$ and if $y \in tc(x)$, then $|y| < \lambda\}$, i.e. H_λ is the class of all set hereditarily of cardinality less than λ.*

In what follows we shall be mostly concerned with structures like H_λ, for a sufficiently large cardinal λ, and, thanks to Lemma 1.2.3, with stationary and club sets of $[H_\lambda]^\kappa$. Then we can give the following definition.

Definition 1.2.6. *Given a sufficiently large θ, $X \subseteq H_\theta$ and $h : [H_\theta]^{<\omega} \to H_\theta$ a Skolem function for H_θ, $cl_h(X)$ will be the closure of X under h. Hence if $M \in cl_h(X)$, then $M \prec H_\theta$.*

One can ask, why we need to deal with M elementary substructure of H_λ. The reason is that for λ infinite regular H_λ is a structure closed under enough axioms of ZFC to develop all the relevant arguments.

Moreover, elementary substructures are downward closed for witnesses of existential sentences and if we can define something in H_λ with parameters in M, then this object lives in M. This is the reason why, if λ is uncountable, $\omega \in M$. We can also prove the following theorem.

Theorem 1.2.7. *If λ is regular and uncountable and $M \prec H_\lambda$ is countable, then for any countable set $A \in M$ we have that $A \subseteq M$. Hence $M \cap \omega_1 \in \omega_1$.*

Proof. We can easily define each element of A, since $A, \omega, n \in M$ and, by elementarity, a function f that witnesses the countability of A is in M. Thus if $a \in A$, then $a \in M$. So we have just showed that if $\alpha \in M \cap \omega_1$, then $\alpha \subseteq M$. Hence $M \cap \omega_1$ is an initial segment of ω_1 and it is a proper one because M is countable. $\square$

This is not true for $\kappa > \omega_1$. Hence we will often refer to $\sup(M \cap \kappa)$, instead of $M \cap \kappa$. Because the latter is not an ordinal anymore, but just a countable set of ordinals less then κ.

Corollary 1.2.8. *If $M \prec H_\theta$, then, for all $A \in M \cap \mathcal{P}(\omega_1)$, A is uncountable if and only if $A \cap M$ is unbounded in $\omega_1 \cap M$.*

We end this section with some facts we will use later on.

Fact 1.2.9. *The set of all countable elementary substructures of H_θ is club in $[H_\theta]^\omega$.*

Fact 1.2.10. *If $M \prec H_\theta$ and $X \in H_\theta$ and X is definable from parameters in M, then $X \in M$.*

Fact 1.2.11. *If $X \in H_\theta$ and $A \in [H_\theta]^{\leqslant\omega}$, then $\{M \cap X : A \subseteq M \prec H_\theta\}$ is a club.*

Proof. Take, for every countable elementary submodel M of H_θ, the Skolem hull of $M \cup \{A\} = M^*$, in H_θ. Then the set of all the M^* is club and its trace on X too. $\square$

Fact 1.2.12. *If $S \in M \prec H_\theta$, $X \in H_\theta$, $S \subseteq [X]^\omega$ and $M \cap X \in S$, then S is stationary in $[X]^\omega$.*

Proof. Take $F \in M$ such that $F : [X]^{<\omega} \to X$ and let cl_F be the set of all the closure points of F. We have $X \in H_\theta$ and $M \prec H_\theta$, hence $X \in M$. Now in M we have that $F : [X \cap M]^{<\omega} \to X \cap M$, so $M \cap X \in cl_F$. By hypothesis $M \cap X \in S$, hence, in $S \cap cl_F \neq \varnothing$, so $M \models S \cap cl_F \neq \varnothing$ and using again the fact that $M \prec H_\theta$, we obtain that $S \cap cl_F \neq \varnothing$ is true in H_θ. $\square$

Fact 1.2.13. *Given a continuous increasing* $\in$*-chain* $\langle M_\alpha : \alpha < \omega_1 \rangle$ *(i.e. if* $\alpha < \beta$*,* $M_\alpha \in M_\beta$ *and if* γ *is limit,* $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$*) of countable elementary submodels of* H_θ*, the set*

$$E = \{\xi \in \omega_1 : M_\xi \cap \omega_1 = \xi\}$$

is club in ω_1*.*

1.3 Notions of forcing

We now present the principal classes of notions of forcing, that we will use further on in the discussion.

1.3.1 κ-c.c. and λ-closed forcing

We begin with two basic properties. They are useful in preservation of cofinality and cardinality in generic extensions.

Definition 1.3.1. *We say that a notion of forcing* P *preserves a property* Γ *of an object* X *if* $1_P \Vdash$ "$\check{X}$ *has the property* Γ", *whenever* X *has the property* Γ *in* V*.*

Definition 1.3.2. *For a cardinal* κ*, a notion of forcing* P *is said to be* κ*-c.c if whenever* $A \subseteq P$ *is an antichain, then* $|A| < \kappa$*.*

For example, the forcing that adds a Cohen real in the generic extension is $\aleph_1$-c.c. (also called c.c.c.: countable chain condition). We have the following theorem.

Theorem 1.3.3. *If* P *is* κ*-c.c. and* κ *is a regular cardinal, then* P *preserves cofinalities* $\geq \kappa$ *(i.e. if* $\lambda \geq \kappa$*, then* $cof(\lambda)^V = cof(\lambda)^{V[G]}$*, whenever* G *is a generic filter). Hence it preserves cardinals* $\geq \kappa$*.*

Definition 1.3.4. *A poset* P *is* λ*-closed if whenever* $\gamma < \lambda$ *and* $\langle p_\xi : \xi < \gamma \rangle$ *is a decreasing sequence of elements of* P*, then there is a* $q \in P$ *such that* $q \leq p_\xi$*, for every* $\xi < \gamma$*.*

We have:

Theorem 1.3.5. *If* P *is* λ*-closed, then* P *preserves cofinalities* $\leq \lambda$*. Hence it preserves cardinals* $\leq \lambda$*.*

1.3.2 Properness

We define a wide class of forcing notions, introduced by Shelah: the class of proper posets. There are different, but equivalent definitions of properness. To begin with, we need to define, what is an (M, P)-generic condition

Definition 1.3.6. ***(Shelah)*** *Given a notion of forcing P, an uncountable cardinal $\lambda > 2^{|P|}$, M a countable elementaty substructure of H_λ such that $P \in M$, we say that a condition $q \in P$ is (M, P)-generic iff for every dense $D \subseteq P$, such that $D \in M$, $D \cap M$ is predense below q (i.e. for all $q_1 \leq q$ there is a $d \in D \cap M$ compatible with q_1)*[1]*.*

It is not hard to see that, for q being (M, P)-generic, we could equivalently ask that if $\dot{\alpha} \in M$ is a name of an ordinal, then $q \Vdash \dot{\alpha} \in M$ ($\forall r \leq q\ \exists s \leq r\ \exists \beta \in M\ s \Vdash \dot{\alpha} = \beta$) or, if G denotes a P-generic filter over M, then $q \Vdash$ "$\dot{G} \cap M$ is an M-generic filter$''$.

We can now define what is a proper notion of forcing, see [18] and [19].

Definition 1.3.7. ***(Shelah)*** *A poset P is called proper iff for every regular uncountable cardinal $\lambda > 2^{|P|}$, and countable $M \prec H_\lambda$, with $P \in M$, every $p \in P \cap M$ has an extension $q \leq p$ that is an (M, P)-generic condition.*

It turns out that we can give many alternative definitions of properness. An interesting one involves infinite games. We shall call $\Gamma(P)$ the proper game associated to P. In this game we have two players I and II. At the beginning I chooses $p_0 \in P$ and a dense set $A_0 \subseteq P$. In his turn II chooses a countable $B_0^0 \subseteq A_0$. At the second step I chooses another dense A_1, whereas II answers with $B_0^1 \subseteq A_0$ and $B_1^1 \subseteq A_1$, and so forth for ω steps. At the end of the game, we say that II wins iff $\exists q \leq p_0$ such that $\forall i$, $B_i = \bigcup \{B_i^n : i \leq n\}$ is predense below q.

Theorem 1.3.8. ***(Baumgartner, Jech, Shelah)*** *Given a poset P, the following are equivalent.*

1. *P is proper (as in definition 1.3.7).*
2. *For some regular uncountable cardinal $\lambda_0 > 2^{|P|}$, for every countable $M \prec H_\lambda$, with $P \in M$, every $p \in P \cap M$ has an extension $q \leq p$ that is an (M, P)-generic condition.*
3. *For any uncountable λ, if $S \subseteq [\lambda]^{\leqslant\omega}$ is stationary , $\Vdash_P$ "$\check{S}$ is stationay in $[\lambda]^{\leqslant\omega}$" (i.e. S remains stationary in the generic extension, hence P preserves stationary sets).*

[1] In this definition dense can be replaced by maximal antichain.

4. *For some uncountable λ_0, P preserves stationay subsets of $[\lambda_0]^{\leqslant\omega}$.*

5. *For every regular $\lambda > 2^{|P|}$ the set $T = \{N \prec H_\lambda : N$ countable, $P \in N$, and $\forall p \in P \cap N\ \exists q \leq p$ so that q is (N,P)-generic$\}$ contains a subset club in $[H_\lambda]^{\leqslant\omega}$.*

6. *For some regular $\lambda_0 > 2^{|P|}$ the set $\{N \prec H_{\lambda_0} : N$ countable, $P \in N$, and $\forall p \in P \cap N\ \exists q \leq p$ so that q is (N,P)-generic$\}$ contains a subset club in $[H_{\lambda_0}]^{\leqslant\omega}$.*

7. *Player II has a winning strategy in $\Gamma(P)$.*

Proof. Clearly $1 \Rightarrow 2$, $3 \Rightarrow 4$ and $5 \Rightarrow 6$.

For $1 \Rightarrow 3$, assume that P is proper and that $S \subseteq [\lambda]^{\leqslant\omega}$ is stationary. We need to show that S remains stationary in the generic extension made by P. By the definition of a club set, we just need to find for every $f : [\lambda]^{<\omega} \to \lambda$ in $V[G]$ an $x \in S$ such that x is closed under f. For this purpose let $\dot{f}$ be a forcing name for f.

For a sufficiently large κ, let $M \prec H_\kappa$ be countable with $\dot{f}, \lambda, p_0, P \in M$ and such that $M \cap \lambda \in S$. This is possible because the set $\{M \cap \lambda : M \in [\kappa]^\omega$, $M \prec H_\kappa$ and $\dot{f}, \lambda, p_0, P \in M\}$ is club in $[\lambda]^{\leqslant\omega}$. As P is assumed to be proper, there is a $q \leq p_0$ that is (M,P)-generic.

We claim that $q \Vdash$ "$M \cap \lambda$ is closed under f". Let $x \in [M \cap \lambda]^{<\omega}$ (note that $x \in M$) and consider the dense set $D = \{p \in P : \exists \alpha < \lambda$ and $p \Vdash \dot{f}(x) = \alpha\}$. By (M,P)-genericity of q, $D \cap M$ is predense below q. Thus, if there is a $q' \leq q$ that decides $\dot{f}(x) = \alpha$, there must be a compatible $r \in D \cap M$. So $r \Vdash \dot{f}(x) = \alpha$. Now we see that α is defined by $x, \dot{f}$ and r, thus $\alpha \in M$. Hence $q' \Vdash \dot{f}(x) \in M \cap \lambda$ and by genericity of q', $q \Vdash \dot{f}(x) \in M \cap \lambda$. Since x was arbitrary $q \Vdash$ "$M \cap \lambda$ is closed under f".

For $3 \Rightarrow 5$, let $\lambda > 2^{|P|}$ be a regular uncountable cardinal and P a notion of forcing that preserves stationary sets. Set $C = \{M \prec H_\lambda : M$ is countable and $P \in M\}$. C is club in $[H_\lambda]^{\leqslant\omega}$. For $p \in P$ define $S_p = \{M \in C : p \in M$ and $\neg\exists q \leq p$ such that q is (M,P)-generic $\} \in V$. We claim that each S_p is not stationay, so $S = \{M \in C : \exists p \in M$ and $M \in S_p\}$ is not stationary, since I_{NS} (the ideal of not stationary sets on $[H_\lambda]^{\leqslant\omega}$) is normal. Note that $T = C \setminus S$ and so the claim proves that T contains a club.

Let G_p be a P-generic filter over V, with $p \in G_p$. In $V[G_p]$ we define a function f as follows

$$\begin{gathered} f : \{D \subseteq P : D \text{ dense }\} \cap V \to G_p \\ D \mapsto f(D) = q \in G_p \cap D \end{gathered}$$

Set $C_p = \{M \prec H^V_\lambda : p \in M$ and $\forall D \in M$ if D is a dense subset of P, then $f(D) \in M\}$. Clearly C_p is a club in $[H_\lambda]^{\leqslant\omega}$. We now prove in $V[G]$ that $S_p \cap C_p = \varnothing$. Let $M \in S_p$ and $q \leq p$. Since q cannot be (M, P)-generic, there exists a $D \in M$ dense in P, such that $D \cap M$ is not predense below q. Hence there is a $q' \leq q$ incompatible with every element of $D \cap M$. If we call

$E = \{q' \leq p :$ there exists a $D \in M$ dense, such that q' is incompatible with every element of $D \cap M\}$,

we have just shown that E is dense in P. By genericity of q, $E \cap G_p \neq \varnothing$. Let q_0 be in $E \cap G_p$. By way of contradiction, we suppose that M is also in C_p. Then $f(D) \in M \cap G_p$ for all $D \in M$ and so $f(D)$ and q_0 must be compatible for all $D \in M$, since they both lie in G_p: a contradiction. This argument took place in $V[G_p]$, but P preserves stationary sets and so S_p must not be non stationary in V as well.

Similary we can prove that $2 \Rightarrow 4$ and $4 \Rightarrow 6$.

For $5 \Rightarrow 7$, let λ be as in 5) and let $C \subseteq T$ be club in $[H_\lambda]^{\leqslant\omega}$. We show how to build up a winning strategy for II in $\Gamma(P)$. At the first move player I chooses a $p_0 \in P$ and A_0 and II finds a $N_0 \in C$ such that $p_0, A_0 \in N_0$ and plays $B^0_0 = N_0 \cap A_0$. In the second turn I chooses A_1 and II finds N_1 such that $N_0 \subseteq N_1$ and $A_1 \in N_1$, then he or she plays $B^1_0 = N_1 \cap A_0$ and $B^1_1 = N_1 \cap A_1$, and so forth for the rest of the game. II can always find an N_i, because C is unbounded. At the end $N = \bigcup\{N_n : n \in \omega\} \in C$, since C is closed. So $p_0 \in N$ and there is a $q \leq p$ that is (N, P)-generic. But for all i we have that $B_i = \bigcup\{B^n_i : i \leq n\} = \bigcup\{N_n \cap A_i : i \leq n\} = N \cap A_i$, for (M, P)-genericity of q, is predense below q. So II has a winning strategy.

Finally we prove that $7 \Rightarrow 1$. Let σ be a winning strategy for II in $\Gamma(P)$. So $(B^n_0, B^n_1, \ldots, B^n_n) = \sigma(p_0, A_0, A_1, \ldots, A_n)$. Let λ be sufficiently large and $M \prec H_\lambda$, countable and such that $P, \sigma \in M$ (note that $\sigma \in H_\lambda$). If $p_0 \in M \cap P$, let $A_0, A_1, \ldots, A_n, \ldots$ be an enumeration of all dense subset of P that lie in M. Without loss of generality we can suppose that I plays $p_0, A_0, A_1, \ldots, A_n, \ldots$ By elementarity $\sigma \in M$ and, since $p_0, A_0, A_1, \ldots, A_n \in M$, $\sigma(p_0, A_0, A_1, \ldots, A_n) = (B^n_0, B^n_1, \ldots, B^n_n) \in M$. Since each B^n_i is countable, $B^n_i \subseteq M$. Thus $B_i = \bigcup\{B^n_i : i \leq n\} \subseteq A_i \cap N$. Since σ is a winning strategy for II, there is a $q \leq p_0$ such that, for all i, B_i is predense below q. Then so is $A_i \cap N$. This means that q is (M, P)-generic.

□

Associating names for ordinals to dense sets that decide the names, we can easily see that the proper game can also be played in the following way: I chooses

$p \in P$ and plays $\dot{\alpha_n} \in V^P$, names for ordinals, whereas II plays real ordinals β_n. II wins the game if there is a $q \leq p$ such that $q \Vdash \forall n \exists k\ \dot{\alpha_n} = \beta_k$.

A useful fact about proper notions of forcing is that we can "control" the countable sets of ordinals in the generic extension.

Fact 1.3.9. *If P is proper and A is a countable set of ordinal in $V[G]$ (where G is a P-generic filter) then there exists a countable B in V such that $A \subseteq B$*

In particular there is no ordinal $\alpha < \omega_1^{V[G]}$ such that $\forall \beta \in V(\beta < \omega_1^V \to \beta < \alpha)$. So a proper forcing notion does not collapse $\aleph_1$.

Fact 1.3.10. *If P and Q are two proper notions of forcing we have that the two step iteration $P * Q$ is still proper.*

Proof. For an uncountable λ, let $X \subseteq [\lambda_0]^{\leqslant\omega}$ be stationary. Let G be a P-generic filter, hence, by properness, X is still stationary in $V[G]$. We now force with Q over $V[G]$. Let H be a Q-generic filter. Since Q is proper X is stationary in $V[G][H] = V[G * H]$. Hence $P * Q$ is proper. $\square$

1.3.3 Semiproperness

We now turn to another class of notions of forcing: the semiproper ones. Define $M^P = V^P \cap M$. If q is (M, P)-generic and if $\dot{\alpha} \in M^P$ is a name for an ordinal, then q forces $\dot{\alpha}$ to be in M. That is, if the generic extension thinks that some object in M^P is a name of an ordinal, then q forces it to be interpreted as an ordinal belonging to M. So we have that

$$q \Vdash M[G] \cap Ord = M \cap Ord.$$

Since we have a predicate for V

$$V = \{\check{x}_G : x \in V\},$$

where $\check{x}$ is the canonical name for x, we can see that $q \Vdash M[G] \cap V = M \cap V$.

If we ask the above properties only for countable ordinals, we have semiproperness.

Definition 1.3.11. ***(Shelah)*** *Given a sufficiently large regular cardinal λ, $M \in [H_\lambda]^\omega$ and $M \prec H_\lambda$, $P \in M$, we say that $q \in P$ is (M, P)-semigeneric if $\dot{\alpha} \in M^P$, a P-name for an ordinal, and $\Vdash_P \dot{\alpha}$ is a countable ordinal imply $q \Vdash \dot{\alpha} \in M$.*

And so:

Definition 1.3.12. ***(Shelah)*** *A poset P is called semiproper iff for every regular uncountable cardinal $\lambda > 2^{|P|}$, and countable $M \prec H_\lambda$, with $P \in M$, every $p \in P \cap M$ has an extension $q \leq p$ that is an (M, P)-semigeneric condition.*

As for properness we have equivalent formulations, but restricted to countable ordinals. We also have the notion of a semiproper game, where, instead of ordinals and name for ordinals, I and II play countable ordinals and names for countable ordinals.

Theorem 1.3.13. ***(Shelah)*** *The following are equivalent.*

1. *P is semiproper (as in definition 1.3.12).*
2. *For some regular uncountable cardinal $\lambda_0 > 2^{|P|}$, for every countable $M \prec H_{\lambda_0}$, with $P \in M$, every $p \in P \cap M$ has an extension $q \leq p$ that is an (M, P)-semigeneric condition.*
3. *For every regular $\lambda > 2^{|P|}$ the set $T = \{N \prec H_\lambda : N$ countable, $P \in N$, and $\forall p \in P \cap N\ \exists q \leq p$ such that q is (N, P)-semigeneric$\}$ contains a subset club in $[H_\lambda]^{\leqslant\omega}$.*
4. *For some regular $\lambda_0 > 2^{|P|}$ the set $\{N \prec H_{\lambda_0} : N$ countable, $P \in N$, and $\forall p \in P \cap N\ \exists q \leq p$ so that q is (N, P)-semigeneric$\}$ contains a subset club in $[H_{\lambda_0}]^{\leqslant\omega}$.*
5. *Player II has a winning strategy in the semiproper game.*

Note that the equivalent formulation of properness in terms of preserving stationarity on $[\lambda]^\omega$ does not generalize to semiproperness, in preserving stationarity on $[\omega_1]^\omega$. We just have one direction, summarized in the following theorem. The proof is the same as in the case of properness.

Theorem 1.3.14. ***(Jech)*** *If P is semiproper and if $S \subseteq [\omega_1]^{\leqslant\omega}$ is stationary, then $\Vdash_P S$ is stationary in $[\omega_1]^{\leqslant\omega}$ (since S is stationary in $[\omega_1]^{\leqslant\omega}$ iff S is stationary in ω_1, this means that P preserves stationary subsets of ω_1).*

Under strong assumptions, namely the strongest forcing axioms we will introduce in the next chapter, we can have a converse of theorem 1.3.14. Then, in that context, we will see that the class of semiproper notions of forcing and the class of the ones that preserves stationary sets of ω_1 are the same.

Definition 1.3.15. *We say that a notion of forcing P is SSP (Stationary Set Preseving) if $1_P \Vdash$ "$\check{S}$ is stationary", whenever S is a stationary set on ω_1.*

1.4 Infinitary combinatorics

We recall some standard facts, useful in the applications. For the proofs or more on the subject see [13].

Definition 1.4.1. *A family $\mathcal{F}$ of sets is called a Δ-system if there is a fixed r, called the root of the Δ-system, such that if $a, b \in \mathcal{F}$, then $a \cap b = r$.*

We will refer to the following Lemma as the Δ-system Lemma.

Lemma 1.4.2. *Let κ be an infinite cardinal. Let $\theta > \kappa$ be regular and satisfy $\forall \alpha < \theta\,(|\alpha|^{<\kappa} < \theta)$. Assume $|\mathcal{F}| \geq \theta$ and $\forall x \in \mathcal{F}\,(|x| < \kappa)$, then there is an $\mathcal{A} \subseteq \mathcal{F}$, such that $|\mathcal{A}| = \theta$ and $\mathcal{A}$ forms a Δ-system. Moreover, if $\mathcal{F} \subseteq [\theta]^{\leq\kappa}$ we can assume that for all $x, y \in \mathcal{A}$, $min(x \backslash r) > sup(y \backslash r)$ or $min(y \backslash r) > sup(x \backslash r)$.*

We will often use the Δ-system Lemma in the following form.

Corollary 1.4.3. *If $\mathcal{F}$ is any uncountable family of finite sets, there is an uncountable $\mathcal{A} \subseteq \mathcal{F}$ which forms a Δ-system.*

The following theorem is a useful application of the Δ-system Lemma.

Definition 1.4.4. *A topological space X has the countable chain condition (c.c.c.) if there are no uncountable family of pairwise disjoint non empty open subsets of X.*

Theorem 1.4.5. *Suppose that $X_i (i \in I)$ are spaces such that every finite $r \subseteq I$, $\prod_{i \in r} X_i$ is c.c.c. Then $\prod_{i \in I} X_i$ is c.c.c.*

It is not difficult to see that what can be shown for c.c.c. spaces can be translated for c.c.c. posets and vice-versa. Hence Theorem 1.4.5 can be restated as follows: *suppose that $P_i (i \in I)$ are posets such that every finite $r \subseteq I$, $\prod_{i \in r} P_i$ is c.c.c. Then $\prod_{i \in I} P_i$ is c.c.c.*

1.4.1 Combinatorial principles

We now recall some combinatorial principles and their consequences on cardinal arithmetic.

Definition 1.4.6. ***(Jensen)*** *(**$\diamondsuit$ principle**) $\diamondsuit$ is the following statement: there is a sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ of function $f_\alpha : \alpha \to 2 = \{0, 1\}$ such that for every function $f : \omega_1 \to 2$ the set $\{\alpha \in \omega_1 : f \restriction \alpha = f_\alpha\}$ is stationary in ω_1.*

$\langle f_\alpha : \alpha < \omega_1 \rangle$ is called a $\diamondsuit$-sequence. Jensen showed that $\diamondsuit$ holds in L. The following proposition is easy to prove.

Proposition 1.4.7. $\diamondsuit$ *implies CH.*

We now define a weaker form of $\diamondsuit$.

Definition 1.4.8. ***(Devlin, Shelah) (weak $\diamondsuit$ principle)*** *$w\diamondsuit$ is the following statement: for each function $F : 2^{<\omega_1} \to 2$ there is a $g : \omega_1 \to 2$ such that for any $f : \omega_1 \to 2$ the set $\{\alpha \in \omega_1 : F(f \restriction \alpha) = g(\alpha)\}$ is stationary in ω_1.*

To see that $w\diamondsuit$ follows from $\diamondsuit$, we just need to define, $g(\alpha) = F(f_\alpha)$, for a given F; where f_α is an element of a $\diamondsuit$-sequence.

An interesting characterisation of $w\diamondsuit$, we will use later on, is the following; see [4].

Theorem 1.4.9. ***(Devlin, Shelah)*** *$2^{\aleph_0} < 2^{\aleph_1}$ iff $w\diamondsuit$.*

Hence if $w\diamondsuit$ fails we have that $2^{\aleph_0} = 2^{\aleph_1}$.

Chapter 2

Forcing Axioms

In this chapter we introduce the principal forcing axioms. They are topological principles that assure the existence, in V, of objects, whose existence cannot be proved nor disproved in ZFC only.

We start with the general definition of a forcing axiom.

Definition 2.0.10. *FA*(Γ, κ)*: For every notion of forcing P with the property Γ and $\mathcal{D}$ a collection of κ-many dense subsets of P, there is a $\mathcal{D}$-generic filter G that intersects every $D \in \mathcal{D}$.*

2.1 Martin's Axiom

Historically the forcing axioms were defined trying to generalize Martin's axiom. This axiom was formulated for the first time by Martin and it has its origin in the study of iterated forcing, whose first application was to show that Suslin's Hypothesis is independent from ZFC.

Definition 2.1.1. ***(Martin's Axiom)*** *MA(κ): whenever P is a non-empty c.c.c. partial order and $\mathcal{D}$ is a family of $\leq \kappa$ dense subsets of P, there is a filter G in P such that $G \cap D \neq \varnothing$, for all $D \in \mathcal{D}$. MA is the statement $\forall \kappa < 2^{\aleph_0}$ MA(κ).*

Thus MA(κ) =FA(c.c.c., κ). Some consequences of MA show that the cardinals between ω and 2^ω share many properties with ω. For example, under MA, we have that for $\kappa < 2^\omega$, $2^\kappa = 2^\omega$ and that the union of κ subsets of $\mathbb{R}$, each of Lebesgue measure zero, has Lebesgue measure zero.

MA however does not say much about the cardinality of the continuum. By one of the first application of an iterated forcing, it was shown that MA and CH are independent. Indeed MA(κ) is consistent with $2^{\aleph_0} = \lambda$, for any λ such that $cof(\lambda) > \kappa$.

Theorem 2.1.2. ***(Solovay and Tennenbaum)*** *Assume that $\kappa \geq \omega_1$, κ is regular and $2^{<\kappa} = \kappa$. Then there is a generic extension in which MA holds and $2^\omega = \kappa$.*

The following theorem shows that we have an upper bound for MA(κ).

Fact 2.1.3. *MA($2^{\aleph_0}$) is inconsistent.*

Proof. Let P be the forcing of all finite partial functions from ω to 2, ordered by end-extension. Define the countable family of dense sets

$$D_n = \{p \in P : n \in \mathtt{dom}(p)\}.$$

Moreover for every total function $h : \omega \to 2$ define the dense set

$$E_h = \{p \in P : \exists n \in \mathtt{dom}(p)(p(n) \neq h(n))\}$$

If G is a $\{D_n : n \in \omega\}$-generic filter and $G \cap E_h \neq \varnothing$, then $\bigcup G \neq h$.

Now if we set $\mathcal{D} = \{D_n : n \in \omega\} \cup \{ E_h : h \in 2^\omega\}$, we have $|\mathcal{D}| = 2^{\aleph_0}$. Assuming MA($2^{\aleph_0}$), we obtain a generic filter G such that if $A \in \mathcal{D}$, then $G \cap A \neq \varnothing$. Hence $\bigcup G$ is a real that differs from all the other reals: a contradiction. $\square$

The following theorem gives a topological equivalent formulation of MA. See [13] for the proof.

Theorem 2.1.4. *The following are equivalent:*

- *MA(κ)*
- *Let X be a compact c.c.c. (i.e. a space with no uncountable family of pairwise disjoint, non-empty open subsets) Hausdorff space and U_α be dense open subsets of X, for $\alpha < \kappa$. Then $\bigcap_{\alpha<\kappa} U_\alpha \neq \varnothing$.*

Observe that for $\kappa = \omega$ we have stated the Baire category theorem that holds for any compact topological space. For $\kappa > \omega$ the countable chain condition hypothesis is essential.

Definition 2.1.5. *MA_κ: MA restricted to posets of size strictly less than κ.*

The following theorem is useful in proving that MA is consistent with ZFC.

Lemma 2.1.6. *MA is equivalent to $MA_{2^{\aleph_0}}$.*

Hence it is sufficient to force MA in the generic extension for notions of forcing of cardinality less than $2^{\aleph_0}$. We will see that this is not the case for stronger forcing axioms.

We can give a more general definition in the line of Definition 2.1.5.

Definition 2.1.7. *FA(Γ_λ, κ): FA(Γ, κ) restricted to posets with the property* Γ *and of size* $< \lambda$.

We can restate Lemma 2.1.6 as FA($c.c.c., \kappa$) $=$ FA($c.c.c._{2^{\aleph_0}}, \kappa$).

2.2 PFA, SPFA and MM

If we want to generalize MA to obtain stronger forcing axioms we could widen the class of forcing notions involved and change the cardinality of the family of the dense sets.

The following facts show one of the reasons why we use the class of proper notions of forcing.

Fact 2.2.1. *If a notion of forcing P is c.c.c. then it is proper.*

Proof. Every condition is (M, P)-generic, for a suitable $M \prec H_\lambda$. This is because in the definition of properness we could equivalently use dense sets or maximal antichains and if A is a maximal antichain, then by Theorem 1.2.7 since $A \in M$ and by the c.c.c. condition is countable, then $A \subseteq M$. Hence for every $p \in P$ there is a compatible extension of p in $A \cap M$. $\square$

Fact 2.2.2. *Every ω-closed forcing P is proper.*

Proof. Set $p_0 = p \in P$ and find, once given an enumeration $\{D_n\}_{n\in\omega}$ of all the dense sets of P in M, a decreasing sequence $\{p_n\}$ of conditions in $P \cap M$ such that $p_n \in D_n$. Then a common extension of all the p_n's is (M, P)-generic and it exists because P is ω-closed. $\square$

So we have that $\mathbb{P}_{c.c.c.} \cup \mathbb{P}_{\omega\text{-}closed} \subseteq \mathbb{P}_{Proper}$, where

$$\mathbb{P}_\Gamma = \{P : P \text{ is a notion of forcing with the property } \Gamma\}.$$

This is one of the reasons why, for a natural generalization of MA, we consider a forcing axiom where the class of forcing notions is the proper one. This is called Proper Forcing Axiom (PFA):

Definition 2.2.3. ***(Proper Forcing Axiom)*** *PFA: FA($\mathbb{P}_{Proper}, \aleph_1$) holds.*

If we replace proper with semiproper we have a stronger forcing axiom: the SemiProper Forcing Axiom. This is because $\mathbb{P}_{Proper} \subseteq \mathbb{P}_{SP}$. Indeed, if for some countable model M all the names for ordinals in M^P are forced to be ordinals in M, then the names for countable ordinals are forced too.

Definition 2.2.4. ***(SemiProper Forcing Axiom)*** *SPFA: FA($\mathbb{P}_{SP}, \aleph_1$) holds.*

A property shared by all the classes of forcing introduced so far is that of preserving stationary subsets of ω_1. Recalling definition 1.3.15, $\mathbb{P}_{SSP}$ is the class of forcing notions that preserve stationary subsets of ω_1. The corresponding forcing axiom is called Martin's Maximum, MM.

Definition 2.2.5. ***(Martin's Maximum)*** *MM: FA($\mathbb{P}_{SSP}, \aleph_1$) holds.*

So far we have that

$$\mathbb{P}_{c.c.c.} \cup \mathbb{P}_{\omega\text{-}closed} \subseteq \mathbb{P}_{Proper} \subseteq \mathbb{P}_{SP} \subseteq \mathbb{P}_{SSP}$$

which implies that

$$MM \Rightarrow SPFA \Rightarrow PFA \Rightarrow MA(\omega_1).$$

In the next chapter we will see that assuming SPFA the notions of semiproperness and of stationary set preservation coincide (i.e. $\mathbb{P}_{SP} = \mathbb{P}_{SSP}$) and so MM and SPFA are equivalent.

One may try to improve MM, but next fact shows that Martin's Maximum is the strongest forcing axiom of the kind FA(Γ, $\aleph_1$). This is the reason of its name.

Theorem 2.2.6. *Let P be a notion of forcing such that there is a stationary set $S \subseteq \omega_1$ and $\Vdash_P$ "S is not stationary". Then there is a family $\mathcal{D}$ of size $\aleph_1$ of dense subsets of P such that there is no generic filter G that intersects every $D \in \mathcal{D}$.*

Proof. Since S in not stationary in $V[G]$, whenever G is a P-generic filter, there is a club $\dot{C} \in V^P$ such that $\Vdash_P S \cap \dot{C} = \varnothing$. Define for every $\alpha \in \omega_1$

$$D_\alpha = \{p \in P : \exists \beta > \alpha \text{ such that } p \Vdash \beta \in \dot{C}\}.$$

It is easy to check that every D_α is dense. For each $\alpha \in \omega_1$ define

$$E_\alpha = \{p \in P : \text{ either } p \Vdash \alpha \in \dot{C} \text{ or } \exists \gamma < \alpha \, \forall \xi \in (\gamma, \alpha) \, p \Vdash \xi \notin \dot{C}\}.$$

All the E_α's are dense as well. Suppose that there is a G such that $\forall\alpha\,(G\cap D_\alpha \neq \varnothing)$ and $\forall\alpha\,(G\cap E_\alpha \neq \varnothing)$ and define

$$C = \{\alpha \in \omega_1 : \exists p \in G\, p \Vdash \alpha \in \dot{C}\}.$$

Then C is a club in ω_1: we show it in the same way we would show that the D_α and the E_α are dense sets. Indeed if $\alpha \in \omega_1$, then $G \cap D_\alpha \neq \varnothing$ and so there is a $p \in G$ such that $p \Vdash \beta \in \dot{C}$, for some $\beta > \alpha$. Hence C is unbounded. To see that it is closed we use the sets E_α. Let γ be a limit point such that there is a $p \in G$ that force $\gamma \notin \dot{C}$, but is a limit of elements of C, say $(\gamma_n)_n$. Then $p \in E_\alpha$ and so there is a ξ such that $\dot{C} \cap \gamma = \xi$. Take a sufficiently big m such that there is a $q \in G$ with $q \Vdash \gamma_m \in \dot{C}$ and $\xi < \gamma_m < \gamma$. Since G is a filter there is a common extension of both p and q, say r that forces γ_m to be and not to be in $\dot{C}$: impossible. Thus $C \cap S = \varnothing$, contradicting the stationarity of S. □

The next theorems gives respectively, an upper and a lower bound for the consistency strength of MM in terms of large cardinals.

Theorem 2.2.7. ***(Foreman, Magidor, Shelah)*** *If there is a supercompact cardinal, then there is a generic model that satisfies MM (hence PFA and SPFA).*

Theorem 2.2.8. ***(Foreman, Magidor, Shelah)*** *Con(ZFC + MM) implies Con(ZFC + $\exists$ class many Woodin cardinals).*

2.3 Bounded Forcing Axioms

We now present another interesting class of forcing axioms, the Bounded Forcing Axioms: $\mathrm{FA}(\Gamma, \kappa, \lambda)$, where Γ is a property that defines a class of posets and κ and λ are cardinals.

Definition 2.3.1. ***(Bounded Forcing Axioms)*** *FA$(\Gamma, \kappa, \lambda)$: for every notion of forcing P with the property Γ and $\mathcal{I}$ a collection of κ maximal antichains of $\mathbb{B} = r.o.(P) \setminus \{0\}$ (where r.o.(P) is the regular open boolean algebra associated to P), such that $|I| \leq \lambda$, for all $I \in \mathcal{I}$, there is a generic filter G that intersects every $I \in \mathcal{I}$.*

If $\phi \in \mathrm{FA}(\Gamma_\lambda, \kappa)$, then $\phi \in \mathrm{FA}(\Gamma, \kappa, \lambda)$, because there are no maximal antichains in P of cardinality greater than λ, if P itself has a cardinality at most λ. Then $\mathrm{MA}(\aleph_1)$ can be seen as the first bounded forcing axiom.

Note that in the previous definition we are dealing with antichains instead of dense subsets. In the definition of $\mathrm{FA}(\Gamma, \kappa)$ there was no difference in using dense sets or antichains, but here we need to stress the difference. The reason is that if a poset P has cardinality κ, where κ is regular, then the dense subsets of P may have cardinality κ.

It is also useful to remark that we are using the antichains of $\mathbb{B} = r.o.(P) \setminus \{0\}$ and not of P just for technical reasons, because the generic extensions yield by a poset P or by $\mathbb{B} = r.o.(P) \setminus \{0\}$ are exactly the same.

The bounded forcing axioms we will be interested in are the FA$(\Gamma, \omega_1, \omega_1)$, where Γ is proper, semiproper or stationary set preserving. The corresponding axioms are respectively BPFA (Bounded Proper Forcing Axiom), BSPFA (Bounded SemiProper Forcing Axiom) and BMM (Bounded Martin's Maximum). As before we have that BMM $\Rightarrow$ BSPFA $\Rightarrow$ BPFA $\Rightarrow$ MA$(\aleph_1)$.

Another equivalent formulation of the bounded forcing axioms is in terms of absoluteness. Recall that by the Levy-Schoenfield absoluteness theorem we have that given a Σ_1-formula of the language of set theory $\phi = \exists x\, \psi(x, a)$, where a is a parameters in H_{ω_2}, if there is a witness of the existence of such an x, then there is a witness of it in H_{ω_2}. In symbols

$$(H_{\omega_2}, \in) \prec_1 (V, \in).$$

The characterization of the bounded forcing axioms is then in terms of generic absoluteness. See [1].

Theorem 2.3.2. ***(Bagaria)*** *Let Γ be a class of posets and κ an infinite cardinal of uncountable cofinality, then the following are equivalent:*

1. *FA(Γ, κ, κ)*
2. *$(H_{\kappa^+}, \in) \prec_1 (V^{\mathbb{P}}, \in)$, where $P \in \Gamma$.*

In other words FA(Γ, κ, κ) is equivalent with having Σ_1-absoluteness for formulas with parameters in H_{κ^+} between the universe and a generic extension performed with a poset in $\mathbb{P}_\Gamma$.

As an easy corollary we have the following.

Corollary 2.3.3. *Let Γ be a class of posets and κ an infinite cardinal of uncountable cofinality, then FA(Γ, κ) implies that $(H_{\kappa^+}, \in) \prec_1 (V^{\mathbb{P}}, \in)$, where $P \in \Gamma$.*

The converse also holds if Γ is the class of κ^+-cc posets, because if P is κ^+-cc, then FA(Γ, κ) and FA(Γ, κ, κ) are equivalent.

Of course there is a similar theorem for MA, since MA can be seen as a bounded forcing axioms. It says that every Σ_1-sentence of the language of set theory with parameters in κ, $\kappa < 2^{\aleph_0}$, forced by a c.c.c. poset is true.

There is a question that arises naturally: is there an equivalent characterization for FA(Γ, κ), in terms of absoluteness? One can expect to find it increasing the size of the parameters, but this is not the case. Indeed PFA does not imply Σ_1-absoluteness for formulas with parameters in H_{ω_3}, forced by a proper forcing, because, as we will see in the next chapter, it implies that the continuum is $\aleph_2$. So we would have an enumeration or the reals $A = \{r_\alpha : \alpha \in \omega_2\}$ belonging to H_{ω_3}. Now if we force to have a new Cohen real, then the following sentence: "There is a real r that is not coded by A" is a Σ_1 sentence with parameter A, hence in H_{ω_3}, that is true in the generic extension but not in the ground model.

Also increasing the logic complexity of the sentences involved we cannot get more, since PFA does not implies Σ_2-absoluteness either. Indeed using a proper forcing, we can collapse the continuum to be ω_1, so that CH holds in the generic extension but not in the ground model. But CH is a Σ_2-statement and so we cannot have Σ_2-absoluteness using PFA.

2.4 FA(σ-closed$*$c.c.c., $\aleph_1$)

We now present the last forcing axiom that we will use in the last chapter to prove that the continuum is $\aleph_2$.

Definition 2.4.1. *FA(σ-closed$*$c.c.c., $\aleph_1$) is the forcing axiom obtained considering the notions of forcing that are the two step iteration of a σ-closed forcing, followed by a c.c.c. forcing*

Next theorem shows that FA(σ-closed$*$c.c.c., $\aleph_1$) is weaker than PFA.

Theorem 2.4.2. *PFA implies FA(σ-closed$*$c.c.c., $\aleph_1$)*

Proof. Fact 2.2.1 tells us that a c.c.c. forcing is proper and Fact 2.2.2 says that a σ-closed forcing is proper. Since the iteration of proper forcing is also proper, we have that $\mathbb{P}_{\sigma\text{-}closed*c.c.c.} \subseteq \mathbb{P}_{Proper}$, hence PFA $=$ FA$(\mathbb{P}_{Proper}, \aleph_1)$ implies FA(σ-closed$*$c.c.c., $\aleph_1$). $\square$

Chapter 3

MM and PFA imply that $2^{\aleph_0} = \aleph_2$

In this chapter we will present some of the proofs that $2^{\aleph_0} = \aleph_2$ using some forcing axioms: namely MM and PFA.

3.1 MM and SRP

We begin by the direct proof that MM decides the value of the continuum. This is maybe the easiest proof, because MM is the strongest forcing axiom. Indeed we can demonstrate something stronger.

Note before that

$$|[\aleph_1]^{\aleph_1}| \leq |[\kappa]^{\aleph_1}| \leq | {}^{\omega_1}\kappa| \leq | [\kappa]^{\aleph_1}| \times 2^{\aleph_1} = |[\kappa]^{\aleph_1}|.$$

We argue for the equality $|{}^{\omega_1}\kappa| = |[\kappa]^{\aleph_1}| \times 2^{\aleph_1}$ as follows. Given $X \in [\kappa]^{\aleph_1}$ we can define $\bar{X} = \{f \in {}^{\omega_1}\kappa : ran(f) = X\}$. Let now $\lessdot_X$ be a well order of $\bar{X}$ in order type $2^{\aleph_1}$. Now define

$$\begin{aligned} \Psi : {}^{\omega_1}\kappa &\to [\kappa]^{\aleph_1} \times 2^{\aleph_1} \\ f &\mapsto (ran(f), \alpha) \end{aligned}$$

where α indicates that f is the α-th function in $(\bar{X}, \lessdot_X)$ and $ran(f) = X$. It is clear that Ψ is an injection. On the other hand the map sanding $X \in [\kappa]^{\aleph_1}$ to some $f_X : \omega_1 \to \kappa$ such that $\mathtt{ran}(f_X) = X$ is injective so $|[\kappa]^{\aleph_1}| \leq | {}^{\omega_1}\kappa| \leq |[\kappa]^{\aleph_1}| \times 2^{\aleph_1}$ and $2^{\aleph_1} \leq |{}^{\omega_1}\kappa| \leq | [\kappa]^{\aleph_1}| \times 2^{\aleph_1}$. Hence $|[\kappa]^{\aleph_1}| \times 2^{\aleph_1} = |{}^{\omega_1}\kappa|$.

Theorem 3.1.1. ***(Foreman, Magidor, Shelah)*** *Assume MM. Than for every regular cardinal $\kappa \geq \aleph_2$, we have that $\kappa^{\aleph_1} = \kappa$.*

Proof. Let $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ be a family of pairwise disjoint stationary subsets of E^κ_ω. $\mathcal{A}$ exists due to theorem 1.1.5. The following claim is enough to prove the theorem.

Claim 3.1.2. *Assume MM. Then for every function $f : \omega_1 \to \kappa$, there exists an ordinal $\gamma_f < \kappa$ of cofinality ω_1 such that for all $\alpha < \kappa$ we have that*

$$\alpha \in ran(f) \text{ iff } A_\alpha \cap \gamma_f \text{ is stationary.} \tag{3.1}$$

Assume that the claim holds and let f and g be functions with distinct ranges. We first notice that γ_f and γ_g cannot be equal. By hypothesis there is $\alpha \in ran(f)\Delta ran(g)$. We may assume that $\alpha \in ran(f) \setminus ran(g)$. Now by the claim we have that $A_\alpha \cap \gamma_f$ is a stationary subset of γ_f, while $A_\alpha \cap \gamma_g$ is a non-stationary subset of γ_g. So $\gamma_f \neq \gamma_g$.

It is clear that

$$\begin{aligned} \Phi :& [\kappa]^{\aleph_1} \to \kappa \\ & ran(f) \mapsto \gamma_f \end{aligned}$$

is an injection of $[\kappa]^{\aleph_1}$ into κ. Hence the theorem follows.

We now turn to the proof of the claim 3.1.2. Take $f : \omega_1 \to \kappa$ and fix a family $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$ of pairwise disjoint subsets of ω_1 that covers ω_1 and such that for every $S \subseteq \omega_1$ stationary, there exists α so that $S \cap S_\alpha$ is stationary (this is possible due to theorem 1.1.5).

We show that using MM we can construct a continuos increasing $F : \omega_1 \to \kappa$ such that, for every $\delta < \omega_1$

$$\delta \in S_\alpha \text{ iff } F(\delta) \in A_{f(\alpha)}.$$

To show that such a function F suffices for the proof of the claim, we let $\gamma_f = \sup_{\delta < \omega_1}(F(\delta))$.

Now set $E = \{F(\alpha) : \alpha < \omega_1\}$. E is a club subset of γ_f as it is a continuos and injective image of a club subset of ω_1. Hence $cof(\gamma_f) = \omega_1$. For the left to right direction of 3.1, let $\alpha < \omega_1$ be in $\mathtt{dom}(f)$ and C be a club in γ_f. We need to find a $\delta \in C \cap A_{f(\alpha)}$. Since F is continuos and injective, we have that $F^{-1}[C \cap E]$ is club in ω_1. By the choice of $\mathcal{S}$, we have that exists $\gamma \in S_\alpha \cap F^{-1}(C)$. Thus, by the definition of F we have that $F(\gamma) = \delta \in C \cap A_{f(\alpha)}$. For the other direction of 3.1, we suppose $A_\xi \cap \gamma_f$ is a stationary subset of γ_f and we have to show that $\xi \in \mathtt{ran}(f)$. By the property of F we have that $F^{-1}[A_\xi \cap E]$ is stationary in ω_1. By the maximality of the stationary partition of ω_1, we have that exists α such that $F^{-1}[A_\xi \cap E] \cap S_\alpha$ is stationary and so is not empty. Pick a $\gamma \in F^{-1}[A_\xi \cap E] \cap S_\alpha$ so $F(\gamma) \in A_\xi$, but, by definition of F, $F(\delta) \in A_{f(\alpha)}$. Since we required the A_η to be pairwise disjoint, we have that $A_{f(\alpha)} = A_\xi$. Thus $\xi = f(\alpha)$.

We now proceed to exhibit the F by an application of MM. For this purpose, we define a stationary set preserving notion of forcing P that approximate the F. Then, by a standard application of MM, we can construct such an F in V.

Let $p \in P$ iff $p : \gamma + 1 \to \kappa$ is an increasing continuos function with domain a countable successor ordinal and such that $\delta \in S_\alpha$ iff $p(\delta) \in A_{f(\alpha)}$ for all $\delta \leq \gamma$.

Let $q \leq p$ iff q is an end-extention of p. We will show that $D_\alpha = \{p \in P : \alpha \in \mathtt{dom}(p)\}$ is dense for all $\alpha < \omega_1$ and that P is stationary set preserving. It is clear that for any filter G intersecting all the dense sets D_α, $\bigcup G$ is a function which satisfies 3.1.

We prove the density part by induction. The base case as well as the successor step are trivial. To prove the limit case, first of all notice that

$$E_\xi = \{p : \mathtt{ran}(p) \setminus \xi \neq \emptyset\}$$

is dense for all $\xi < \kappa$.

Now assume that for some limit $\alpha < \omega_1$, D_β is dense for all $\beta < \alpha$. Since the family $\mathcal{S}$ is a partition of ω_1, there is a γ such that $\alpha \in S_\gamma$. Now for a sufficiently large cardinal λ,

$$C = \{M \in [H_\lambda]^\omega : M \prec H_\lambda \text{ and } p, P, \alpha \in M\}$$

is a club in $[H_\lambda]^\omega$, thus $C_\kappa = \{\xi : \exists M \in C \text{ and } \sup(M \cap \kappa) = \xi\}$ is a club subset of E^κ_ω. Now $A_{f(\alpha)}$ is stationary in E^κ_ω, so there is $M \in C$ such that $\sup(M \cap \kappa) = \eta \in A_{f(\gamma)}$. Let $(\alpha_n)_n$ and $(\eta_n)_n$ be two sequences contained in M with limit respectively α and η. By induction we define $(p_n)_n$, a decreasing sequence of conditions in M extending p with $p_0 = p$. Given $p_n \in M$ we want to construct $p_{n+1} \in M$. Note that D_{α_n} and E_{η_n} are definable by P, α_n and η_n which are all in M, hence $D_{\alpha_n}, E_{\eta_n} \in M$. By inductive hypotesis D_{α_n} is dense. Then, by a density argument applied in M to the dense open set $E_{\eta_n} \cap D_{\alpha_n}$, we can define $p_{n+1} \in M$ such that $\alpha_n \in \mathtt{dom}(p_{n+1})$ and $p_{n+1}(\xi) \geq \eta_n$ for some $\xi \geq \alpha_n$. If for some n, $\alpha \in \mathtt{dom}(p_n)$ we are done since p_n is a condition in $P \cap D_\alpha$. Otherwise notice that the choice of the conditions p_n is such that $\alpha = \bigcup_n \mathtt{dom}(p_n)$ and $\eta = \sup(\bigcup_n \mathtt{ran}(p_n))$. So if we set $q = \bigcup_n p_n \cup \{\langle \alpha, \eta \rangle\}$, we have that q is a condition in D_α below p since $\eta \in A_{f(\gamma)}$ and any other pair $\langle \beta, \xi \rangle \in q$ appears already in some p_n, now since p_n is a condition in P we must have $\xi \in A_{f(\beta)}$.

We now show that P is stationary set preserving. To this aim let $S \subseteq \omega_1$ be stationary, $p \in P$ and $\dot{C}$ a name for a club in V^P. We must find $\delta \in S$ and $q \leq p$ such that $q \Vdash \delta \in \dot{C}$. First of all by maximality of $\mathcal{S}$ there is α such that $S \cap S_\alpha$ is stationary.

Let λ be a sufficiently large regular cardinal, such that $p, P, \dot{C} \in H_\lambda$, then

$$E = \{N \in [H_\lambda]^{\omega_1} : \dot{C} \in N \prec H_\lambda \text{ such that } \omega_1 \subseteq N \text{ and } \sup(N \cap \kappa) \in A_{f(\alpha)}\}$$

is a stationary subset of $[H_\lambda]^{\aleph_1}$ such that

$$E_\kappa = \{\sup(N \cap \kappa) : N \in E\}$$

is a stationary subset of E^κ_ω.

We show that E is stationary in $[H_\lambda]^{\aleph_1}$ and that E_κ is stationary in E^κ_ω at the same time . Fix a function $F : [H_\lambda]^{<\omega} \to H_\lambda$, then it is easy to see that $H = \{\eta < \kappa : \kappa \cap F\text{"}[\eta]^{<\omega} \subseteq \eta\}$ is club. Let $S \subseteq E^\kappa_\omega$ be a stationary set, then $S \cap H \neq \varnothing$. Now pick $\delta \in A_{f(\alpha)} \cap H$ and fix a sequence $(\delta_n)_n$ with limit δ. We set $M = cl_F(\omega_1 \cup \{\delta_n\}_n)$, the closure of $\omega_1 \cup \{\delta_n\}_n$ under F. Hence $M \in E$, because δ has countable cofinality and is closed for F. This shows that E is stationary. Moreover $sup(M \cap \kappa) = \delta$, because we did not get new bigger ordinals, since $\kappa \cap F\text{"}[\delta]^{<\omega} \subseteq \delta$ and so E_κ is stationay in E^κ_ω.

Moreover

$$D = \{M \in [N_0]^\omega : \dot{C} \in M \prec N_0 \text{ is such that } \sup(M \cap \kappa) = \eta\}$$

is a club subset of $[N_0]^\omega$. To see it let $H : [N_0]^{<\omega} \to N_0$ and let X be closed for H such that $sup(H) = \gamma$ an $\gamma \notin M$.

Then we have that

$$D_{\omega_1} = \{\delta : \exists M \in D \text{ such that } M \cap \omega_1 = \delta\}$$

is club in ω_1.

Hence, by stationarity of $S \cap S_\alpha$, there exists M countable elementary substructure of N_0, such that $P, p, \dot{C} \in M$, $\sup(M \cap \kappa) = \eta \in A_{f(\alpha)}$ and $M \cap \omega_1 = \delta \in S \cap S_\alpha$. As before we fix $(\delta_n)_n$ and $(\eta_n)_n$ two sequences with limit respectively δ and η and we find an increasing sequence of $(p_n)_n$, such that each $p_n \in M$, $\delta_n \in \mathtt{dom}(p_{n+1})$ and $p_{n+1}(\xi) \geq \eta_n$ for some $\xi \geq \delta_n$. Now we also ask that at stage n there is a $\beta_n \geq \delta_n$ in M such that $p_{n+1} \Vdash \beta_n \in \dot{C}$, since $\dot{C}$ is a name for a club. Then as before we can argue that $q = \bigcup_n p_n \cup \{\langle \delta, \eta \rangle\} \in P$, moreover we have that $\delta = \lim_n \beta_n$ and $q \Vdash \delta \in \dot{C}$ since $q \Vdash \beta_n \in \dot{C}$ for all n and $\delta = \sup_n \beta_n$. This is true because $\beta_n < \delta$ for every n, since β_n is definable by parameters in M, hence $\beta_n \in M$ and so $\beta_n < M \cap \omega_1 = \delta$. This completes the proof of the claim and proves the theorem. $\square$

We then have our theorem as a corollary.

Corollary 3.1.3. *MM implies that* $2^{\aleph_0} = \aleph_2$.

Proof. By the above theorem we have that $2^{\aleph_0} \leq 2^{\aleph_1} \leq \aleph_2^{\aleph_1} = \aleph_2$. But MM implies MA($\aleph_1$) so $\aleph_1 < 2^{\aleph_0}$. So we have that $2^{\aleph_0} = \aleph_2$. $\square$

A careful reading of the proof shows that we can divide it into two parts. In the first one we prove that MM gives us an object in V (in this case a function) with the desired properties. In the other part we see that the existence of such an object implies some reflecting properties involving stationary sets.

This is the reason why, soon after this proof, some reflection principles have been isolated. The strategy is to show that MM implies them, using MM just once, and then, without reference to forcing, use these principles to infer statements about the universe of sets.

We need the definition of a projective stationary set.

Definition 3.1.4. *We say that $S \subseteq [\lambda]^\omega$ is projective stationary if for every stationary set $T \subseteq \omega_1$, the set $\{X \in S : X \cap \omega_1 \in T\}$ is stationary in $[\lambda]^\omega$.*

The original definition of SRP is due to Todorčević, but the following definition is an equivalent version due to Jech. The concepts we are going to introduce are been extensively studied by Jech, Feng and Zapletal, see [8], [9], [10].

Definition 3.1.5. ***(Jech-Todorčević)(Strong Reflection Principle (SRP))*** *For every regular $\lambda \geq \aleph_2$ the following principle SRP(λ) holds: if S is projective stationary in $[H_\lambda]^\omega$, then there is a continuous increasing $\in$-chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ (i.e. if $\alpha < \beta$, $M_\alpha \in M_\beta$ and if γ is limit, $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$) of countable elementary submodels of H_λ, such that, for every $\alpha < \omega_1$, $M_\alpha \in S$.*

We now define another reflection priciple that follows from SRP and that is simpler to use in the applications.

Definition 3.1.6. ***(Jech)(Reflection Principle (RP))*** *For every regular $\lambda \geq \aleph_2$ the following principle RP(λ) holds: if S is stationary in $[H_\lambda]^\omega$, then there is a continuous increasing $\in$-chain $(M_\alpha : \alpha < \omega_1)$ of countable elementary submodels of H_λ, such that $\{\alpha < \omega_1 : M_\alpha \in S\}$ is stationary.*

Before, we need a useful consequence of SRP.

Definition 3.1.7. *Given κ a regular uncountable cardinal, we say that an ideal I on κ is λ-saturated if there exists no collection W of size λ of subsets of κ such that $X \notin I$ for all $X \in W$ and if X, Y are distict members of W, then $X \cap Y \in I$.*

Theorem 3.1.8. *Assuming SRP we have that NS_{ω_1}, the ideal of the non stationary sets on ω_1, is $\aleph_2$-saturated (i.e. every maximal stationary antichain in NS_{ω_1} has size at most $\aleph_1$).*

Proof. Let W be a maximal family of stationary sets such that if $X, Y \in W$, then $X \cap Y$ is not stationary. We call such a family a maximal stationary antichain. We shall show that $|W| \leq \aleph_1$. For this purpose we claim that the set

$$S = \{M \in [H_{\omega_2}]^\omega : M \prec H_{\omega_2}, W \in M \text{ and } \exists A \in W \cap M \, (M \cap \omega_1 \in A)\}$$

is projective stationary.

For a stationary $T \subseteq \omega_1$, since W is a maximal stationary antichain, we have that there exists $A \in W$ such that $T \cap A$ is stationary. Let $C \subseteq \{M \in [H_{\omega_2}]^\omega : A \in M \prec H_{\omega_2}\}$, C is club, so $\{M \cap \omega_1 : M \in C\}$ is club in ω_1. By stationarity of $T \cap A$, there is a $M \in C$ such that $M \cap \omega_1 \in T \cap A$. Hence $\{M \in S : M \cap \omega_1 \in T\}$ is stationary in $[H_{\omega_2}]^\omega$ and so S is projective stationary.

So we can apply SRP and find an $\in$-increasing continuous chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countably elementary substructures of H_{ω_2}, such that, for every $\alpha \in \omega_1$, $M_\alpha \in S$. If we call $M = \bigcup_{\alpha<\omega_1} M_\alpha$ and we show that $W \subseteq M$, we are done.

By way of contradiction we suppose that there is a $A \in W \setminus M$. Now consider the elementary chain $\langle N_\alpha : \alpha < \omega_1\rangle$, where N_α is the Skolem hull of $M_\alpha \cup \{A\} \cup \alpha$ and $N = \bigcup_{\alpha<\omega_1} N_\alpha$. By a fixed point argument we can see that set $\{\alpha \in \omega_1 : M_\alpha \cap \omega_1 = \alpha\}$ is club. So, by definition of N_α, the set

$$C = \{\alpha \in \omega_1 : M_\alpha \cap \omega_1 = N_\alpha \cap \omega_1 = \alpha\}.$$

is a club.

Take $\alpha \in C \cap A$. Since S is projective stationary, $M_\alpha \in S$ and, by definition of S, there is a $B \in W \cap M_\alpha$ such that $\alpha \in B$. But W is a maximal stationary antichain and $A, B \in W$, so we have that $A \cap B$ is non stationary. Then we can find a club set D such that $A \cap B \cap D = \varnothing$. By definition of N_α, we have that $A, B \in N_\alpha$, so we have that $D \in N_\alpha$. But $\alpha \in D$, because $D \in N_\alpha$ is club and, by en elementary argument, $\alpha = N_\alpha \cap \omega_1$ is a limit point of D; indeed D is unbounded in $N_\alpha \cap \omega_1$. This is a contradiction, since $\alpha \in A$, $\alpha \in B$ and $\alpha \in D$, but $A \cap B \cap D = \varnothing$. $\square$

Thanks to theorem 3.1.8, we have the following fact.

Fact 3.1.9. *If NS_{ω_1} is $\aleph_2$ saturated and λ is a regular cardinal, then for every $S \subseteq [\lambda]^\omega$ there is a stationary $A \subseteq \omega_1$ such that for every stationary $B \subseteq A$, the set $\{X \in S : X \cap \omega_1 \in B\}$ is stationary.*

Proof. For every stationary set $A \subseteq \omega_1$ define

$$S_A = \{X \in S : X \cap \omega_1 \in A\}.$$

Now let $W = \{A_\xi : \xi < \theta\}$ be a maximal antichain of stationary sets such that S_{A_ξ} is non stationary. Since NS_{ω_1} is $\aleph_2$-saturated, we have $\theta \leq \omega_1$. Without loss of generality we can suppose $\theta = \omega_1$.

Let $C_\xi \subseteq [\lambda]^\omega$ be the club that witnesses that S_{A_ξ} is non stationary, i.e. $C_\xi \cap S_{A_\xi} = \varnothing$. Then define $A = \triangle_\xi(\omega_1 \setminus A_\xi)$ and

$$C = \triangle_\xi C_\xi = \{X \in [\lambda]^\omega : X \in \bigcap_{\alpha \in X} C_\alpha\}$$

respectively the diagonal intersection of the complements of the A_ξ and of the C_ξ.

By normality of the club filter, C is still a club. Hence $C \cap S$ is stationary. Explicitly $C \cap S = \{X \in S : \forall \alpha \in X \cap \omega_1 \, X \in C_\alpha\}$. If we set $X \cap \omega_1 = \beta$, we can read it as: there are stationarily many $X \in S$ such that their traces avoid the sets A_α, for $\alpha < \beta$. Then it follows that $A = \{\beta \in \omega_1 : \beta \notin \bigcup_{\alpha<\beta} A_\beta\}$ is stationary. Hence A is the stationary set we were looking for. The stationarity of S_A lies on the stationarity of $C \cap S$. □

We can now prove the following chain of implications: MM ⇒ SRP ⇒ RP.

Theorem 3.1.10. *SRP implies RP.*

Proof. Let $S \subseteq [H_\lambda]^\omega$ be stationary, we need to find a continuous increasing $\in$-chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable elementary submodels of H_λ, such that the set $\{\alpha < \omega_1 : M_\alpha \in S\}$ is stationary. By theorem 3.1.8 and fact 3.1.9, we have that there is an $A \subseteq \omega_1$ such that for every stationary $B \subseteq A$, the set $\{X \in S : X \cap \omega_1 \in B\}$ is stationary. We claim that the set

$$T = \{M : M \in S \text{ or } M \cap \omega_1 \notin A\}$$

is projective stationary. Indeed, if $R \subseteq \omega_1$ is stationary we have $R \subseteq A$ or $R \cap A \notin \mathrm{NS}_{\omega_1}$ (or both, as well). In the first case the fact 3.1.9 tells us that $\{M \in T : M \cap \omega_1 \in R\}$ is stationary. Otherwise, if $R \cap A \notin \mathrm{NS}_{\omega_1}$, for stationary many M we have that $M \cap \omega_1 \in R \setminus A$, since $\{\alpha \in R : \alpha \notin A\}$ is stationary.

Then we can use SRP and find a continuous increasing $\in$-chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable elementary submodels of H_λ, such that $M_\alpha \in T$ for all α. So, for stationary many α, $M_\alpha \cap \omega_1 \in A$, hence, for stationary many α, $M_\alpha \in S$. □

Theorem 3.1.11. *MM implies SRP.*

Proof. Given an $S \subseteq [H_\kappa]^\omega$ projective stationary, we shall force an $\in$-increasing continuous chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable elementary substructures to be in

S, using a stationary set preserving notion of forcing P. Then, by a standard application of MM, this chain will live in V.

We say that $p \in P$ iff $p = \langle M_\alpha : \alpha \leq \gamma \rangle$ and, for all $\alpha \leq \gamma$, $M_\alpha \in S$. We can see conditions as sets of pairs $\{(\alpha, M_\alpha) : \alpha \leq \gamma\}$. Then, $q \leq p$ iff q is an end-extension of p. By MM, there is a filter G that intersects the dense sets $D_\alpha = \{p \in P : \alpha \in \mathtt{dom}(p)\}$ for all $\alpha \in \omega_1$. So clearly $\bigcup G$ is the chain with the desired property, provided that D_α is dense for all α.

We still need to prove that P is a stationary set preserving notion of forcing. To this aim let $T \subseteq \omega_1$ be stationary and $\dot{C} \in V^P$ a name for a club. For $p \in P$ we need to find a $q \leq p$ and a $\delta \in T$ such that $q \Vdash \delta \in \dot{C}$. We fix a sufficiently large cardinal λ such that $P, \dot{C}, S, T, p \in H_\lambda$. Since S is projective stationary $S_T = \{M \in S : M \cap \omega_1 \in T\}$ is stationary in $[H_\kappa]^\omega$. Then, by lemma 1.2.3, $S_T^{[H_\lambda]^\omega}$ (i.e. the lift of S_T to $[H_\lambda]^\omega$) is stationary in $[H_\lambda]^\omega$. Moreover we have that

$$\{M \in [H_\lambda]^\omega : M \prec H_\lambda\}.$$

is club in $[H_\lambda]^\omega$. Hence there is an $M \prec H_\lambda$ such that $M \cap H_\kappa \in S$ and $M \cap \omega_1 = \delta \in T$.

Since M is countable we can enumerate all its dense subsets $\{D_n\}_{n \in \omega}$. Now we define a sequence of decreasing conditions $p_{n+1} \leq p_n$ such that

- $p_0 = p$,
- for all n, $p_{n+1} \in D_n$,
- for all n, $p_n \in M$.

Fix a sequence $(\delta_n)_n \subseteq M$ that converges to δ. If $p_n = \langle M_\alpha : \alpha \leq \gamma_n \rangle$, by a density argument applied in M and since $\dot{C}$ is club, we can find a sequence $(\eta_n)_n \subseteq M$ with limit δ such that, for all n, $\gamma_{n+1} > \eta_{n+1} \geq \delta_n$ and $p_{n+1} \Vdash \eta_{n+1} \in \dot{C}$. So, $\lim \gamma_n = \delta = \lim \eta_n$. Since, for every n, $M_{\gamma_n} \in S$ and $p_n \in M$, we have that $\bigcup_{n \in \omega} M_{\gamma_n} = M \cap H_\kappa$. Thus, if we set $q = \bigcup_{n \in \omega} p_n \cup \{(\delta, M \cap H_\kappa)\}$, then $q \in P$ and $q \Vdash \delta \in \dot{C}$. $\square$

We now see how to factorize the proof of MM via SRP. We just need to show that the claim of theorem 3.1.1 follows from SRP.

Claim 3.1.12. *Given a maximal family $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$ of disjoint stationary subsets of ω_1, a family $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ of disjoint stationary subsets of E^κ_ω, a function $f : \omega_1 \to \kappa$ and λ, a sufficiently large cardinal, we have that the following set*

$$R = \{M \in [H_\lambda]^\omega : M \prec H_\lambda,\ \textit{for some } \alpha \in M \cap \omega_1, M \cap \omega_1 \in S_\alpha \textit{ and } sup(M \cap \kappa) \in A_{f(\alpha)}\}$$

is projective stationary.

Proof. We need to show that if $T \subseteq \omega_1$ is stationary; then the set $\{M \in R : M \cap \omega_1 \in T\}$ is stationary. Since the family $\mathcal{S}$ is maximal, there is α such that $T \cap S_\alpha$ is stationary. Then a careful reading of the proof that the forcing P, in theorem 3.1.1, is stationary set preserving shows that there are stationary many models in R with the desired property. $\square$

Since R is projective stationary we can use SRP to exibit a a continuous increasing $\in$-chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ such that for every $\alpha < \omega_1$, $M_\alpha \in R$. Set $F : \omega_1 \to \kappa$ such that, for $\delta \in \omega_1$, $F(\delta) = \sup(M_\beta \cap \kappa)$.

3.1.1 Reflection implies $\mathbb{P}_{SP} = \mathbb{P}_{SSP}$

We end this section showing that, assuming SRP, the class of semiproper forcing and the class of stationary set preserving ones are the same.

Theorem 3.1.13. *If we call $\mathbb{P}_{SP}$ the class of semiproper forcing and $\mathbb{P}_{SSP}$ the class of stationary set preserving forcing, we have that SRP implies $\mathbb{P}_{SP} = \mathbb{P}_{SSP}$.*

Proof. We first note that a semiproper forcing P is stationary set preserving. So we just need to show that, under SRP, every stationary set preserving forcing is also semiproper.

By way of contradiction, suppose that there is a forcing $Q \in \mathbb{P}_{SSP} \setminus \mathbb{P}_{SP}$ and fix a sufficiently large cardinal κ such that every Q-name for a countable ordinal is in H_κ. Then, by theorem 1.3.13, there is a $p \in Q$ such that

$$X = \{M \in [H_\kappa]^\omega : M \prec H_\kappa \text{ and } \neg\exists q \leq p \text{ such that } q \text{ is } (M, Q)\text{-semigeneric } \}$$

is stationary in $[H_\kappa]^\omega$.

By $X^\perp$ we denote the set

$$X^\perp = \{M \in [H_\kappa]^\omega : M \prec H_\kappa, \forall N (M \prec N \prec H_\kappa \text{ and } N \cap \omega_1 = M \cap \omega_1 \Rightarrow N \notin X)\}$$

and we claim that $S = X \cup X^\perp$ is projective stationary.

Claim 3.1.14. $S = X \cup X^\perp$ *is projective stationary*

Proof. Let $W \subseteq \mathcal{P}(\omega_1)$ be the collection of all stationary sets $S \subseteq \omega_1$ such that

$$X_S = \{M \in X : M \cap \omega_1 \notin S\}$$

is a club subset of X, i.e. the sets that witness that X is not projective stationary. Now define $\bar{X} = \Delta_{S \in W} X_S$, that is

$$\bar{X} = \{M \in X : \forall S \in W \cap M \text{ we have that } M \cap \omega_1 \notin S\}$$

Note that $X \triangle \bar{X}$ is non stationary. Actually $\bar{X} \subseteq X$, but if $X \setminus \bar{X}$ were stationary, then for every $M \in X \setminus \bar{X}$ we would have that there is an $S \in W \cap M$ such that $M \cap \omega_1 \in S$ and so by Fodor's Lemma for some $S \in W$ the set $\{M \in X \setminus \bar{X} : M \cap \omega_1 \in S\}$ would be stationary, contradicting that $S \in W$.

Now define

$$Y_S = \{M \in [H_\kappa]^\omega : M \notin X, W, S \in M \text{ and } M \cap \omega_1 \in S\}$$

and note that for every $S \in W$ we have that $Y_S \subseteq X^\perp$. Thus $\bar{X} \cup \bigcup_{S \in W} Y_S \subseteq \bar{X} \cup X^\perp$.

It is then sufficient to show that $\bar{X} \cup \bigcup_{S \in W} Y_S$ is projective stationary. This is because, if $S \notin W$ then, by definition of W, there are stationary many elements of X that have a trace on S. Otherwise, suppose $S \in W$ and note that Y_S is the lift of S, modulo two club conditions: $M \notin X$ and $W, S \in M$. Hence it is stationary and it witnesses the projective stationarity of $\bar{X} \cup \bigcup_{S \in W} Y_S$ relative to S. □

By SRP we can find an $\in$-increasing continuous chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of contable elementary substructures of H_κ.

Claim 3.1.15. *The set $S_X = \{\alpha < \omega_1 : M_\alpha \in X\}$ is not stationary.*

Proof. Let $G \subseteq P$ be a filter and $V[G]$ the generic extension by G. Since $P \in \mathbb{P}_{SSP}$, if we suppose that S_X is stationary, then it remains stationary in $V[G]$. Since every M_α is countable, let, in $V[G]$, $\{\dot{\delta}_\xi : \xi < \omega_1\}$ be an enumeration of all names for countable ordinals that are in $\bigcup_{\alpha \in \omega_1} M_\alpha$.

We now work in $V[G]$. Let

$$C = \{\alpha < \omega_1 : M_\alpha \cap \omega_1 = \alpha \text{ and } \forall \xi < \alpha (\dot{\delta}_\xi \in M_\alpha \text{ and } \dot{\delta}_\xi^G < \alpha)\}.$$

It is easy to see that, by definition, C is club. If $\alpha \in C$, then there is a $q \in G$ such that, for every $\dot{\delta}_\xi \in M_\alpha$, $q \Vdash \exists \beta \in M_\alpha \, (\dot{\delta}_\xi = \beta)$. Thus, q is (M_α, Q)-semigeneric: a contradiction, since $M_\alpha \in X$. So S in not stationary in $V[G]$ and hence in V. □

Now we come back to the proof of theorem 3.1.13. Thanks to the claim, we can say that, modulo eliminating non stationary many M_α, the elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ is in $X^\perp$. But now we can get to a contradiction again.

Let $\lambda > \kappa$ be sufficiently large, such that Q and $\langle M_\alpha : \alpha < \omega_1 \rangle$ are in H_λ. Let $M \prec H_\lambda$ be countable. If we call $\delta = M \cap \omega_1$, then, by elementarity, we have that $M_\delta \subseteq M \cap H_\kappa$. Note that, for every α, $\alpha \subseteq M_\alpha$. Then $\delta = M_\delta \cap \omega_1$; otherwise we could not have $M_\delta \subseteq M \cap H_\kappa$. By definition of $X^\perp$ and since $M_\delta \in X^\perp$, we have that $M \cap H_\kappa \notin X$. Hence there is a $q \leq p$ that is (M, Q)-semigeneric. The arbitrary choice of M contradicts the stationarity of $X^{[H_\lambda]^\omega}$.

$\square$

Actually we can use just RP to see that $\mathbb{P}_{SP} = \mathbb{P}_{SSP}$. We can even use a weaker form of reflection that we can call WRP.

Definition 3.1.16. *(**Weak Reflection Principle (WRP)**) WRP is the following statement: given a cardinal $\lambda \geq \aleph_2$ and $S \subseteq [\lambda]^\omega$ a stationary set, there exists a set $X \subseteq \lambda$ of size $\aleph_1$ such that $\omega_1 \subseteq X$ and $S \cap [X]^\omega$ is stationary in $[X]^\omega$. We say that S reflects at X.*

Theorem 3.1.17. *RP implies WRP*

Proof. Suppose that S is a stationary subset of $[\lambda]^\omega$, we need to find a subset X of λ of size $\aleph_1$ such that $S \cap [X]^\omega$ is stationary in $[X]^\omega$. We define $S^* = \{M \prec H_\lambda : M \in [H_\lambda]^\omega \text{ and } M \cap \lambda \in S\}$. By stationarity of S, S^* is stationary in $[H_\lambda]^\omega$. We can then apply RP to S^* and get a continuous increasing $\in$-chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable elementary submodels of H_λ, such that $\{\alpha < \omega_1 : M_\alpha \in S^*\}$ is stationary. If we call $M = \bigcup_{\alpha \in \omega_1} M_\alpha$, we have that M is an elementary substruture of H_λ of size $\aleph_1$ and that $M \cap \lambda$ is the X we were looking for. If we call $X_\alpha = M_\alpha \cap X = M_\alpha \cap M \cap \lambda = M_\alpha \cap \lambda$, it is easy to see that by constuction of X the set $\{X_\alpha : \alpha \in \omega_1\}$ is club in $[X]^\omega$. On the other hand, we know by RP that $\{\alpha < \omega_1 : M_\alpha \in S^*\} = \{\alpha < \omega_1 : M_\alpha \cap \lambda \in S\} = \{\alpha < \omega_1 : X_\alpha \in S\}$ is stationary. It follows that $S \cap [X]^\omega$ is stationary in $[X]^\omega$, because if C is club in $[X]^\omega$ we can just consider its intersection with the set of all the X_α, that is still a club, and for stationary many index α we have that $X_\alpha \in S$. $\square$

Definition 3.1.18. *A set $S \subseteq [\kappa]^\omega$ is a local club if the set*

$$\{X \in [\kappa]^{\omega_1} : S \cap [X]^\omega \textit{ contains a club in } [X]^\omega\}$$

contains a club in $[\kappa]^{\omega_1}$.

It is not difficult to see that WRP implies that for $\kappa \geq \aleph_2$ and $S \subseteq [\kappa]^\omega$ a stationary set, the set

$$\{X \in [\kappa]^{\omega_1} : S \text{ reflects at } X\}$$

is stationary in $[\kappa]^{\omega_1}$. This means that every local club in $[\kappa]^\omega$ contains a club.

By theorem 1.3.13 we have that, if we set, for a sufficiently large θ

$$S_P = \{M \in [H_\theta]^\omega : P \in M \prec H_\theta, \forall p \in M\, \exists q \leq p \text{ and } q \text{ is semigeneric for } M\}$$

$P \in \mathbb{P}_{SP}$ iff S_P contains a club.

Jech in [8] showed that $P \in \mathbb{P}_{SSP}$ iff S_P is a local club. Hence, since WRP implies that every local club contains a club, we have that $\mathbb{P}_{SP} = \mathbb{P}_{SSP}$.

3.2 PFA and MRP

We now turn to PFA and how it decides the cardinality of the continuum. As for MM, we present a proof that PFA implies $2^{\aleph_0} = \aleph_2$ which uses a reflection principle that can be stated without any reference to the machinery of forcing: the Mapping Reflection Principle. The results in this section are taken from [15] and [16].

Before we state MRP we need some definitions.

Definition 3.2.1. *Let $X \neq \varnothing$ be an uncountable set, λ a sufficiently large regular cardinal and M a countable elementary submodel of H_λ such that $[X]^\omega \in M$. We say that $\Sigma \subseteq [X]^\omega$ is M-stationary if whenever $E \subseteq [X]^\omega$ is club and $E \in M$, then $E \cap \Sigma \cap M \neq \varnothing$.*

It is possible to give a topology on $[X]^\omega$, called the Ellentuck topology, declearing the sets

$$[x, N] = \{Y \in [X]^\omega : x \subseteq Y \subseteq N\}$$

to be open; where $N \in [X]^\omega$ and $x \subseteq N$ is finite. This topology is useful for our purpose, since the closed sets cofinal in the order structure generate the club filter on $[X]^\omega$.

Definition 3.2.2. *A set mapping Σ is said to be open stationary if for an uncountable $X = X_\Sigma$ and a regular uncountable cardinal $\lambda = \lambda_\Sigma$ sufficiently large such that $X \in H_\lambda$ and $[X]^\omega \in H_\lambda$, we have that $\mathtt{dom}(\Sigma) \subseteq \{M \in [H_\lambda]^\omega : M \prec H_\lambda\}$ is club and, for every $M \in \mathtt{dom}(\Sigma)$ with $X \in M$, $\Sigma(M) \subseteq [X]^\omega$ is open in the Ellentuck topology on $[X]^\omega$ and M-stationary.*

We can now state the Mapping Reflection Principle.

Definition 3.2.3. ***(Mapping Reflection Principle (MRP))*** *If Σ is an open stationary set mapping, there is a continuous $\in$-increasing chain $\langle N_\xi : \xi < \omega_1 \rangle$ of elements in the domain of Σ such that for all limit ordinal $0 < \nu < \omega_1$ there is a $\nu_0 < \nu$ such that, for all $\eta \in (\nu_0, \nu)$, if $\nu_0 \in N_\eta$, then $N_\eta \cap X_\Sigma \in \Sigma(N_\nu)$.*

We now prove that MRP is a consequence of PFA.

Theorem 3.2.4. ***(Moore)*** *PFA implies MRP*

Proof. Let Σ be an open stationary set mapping defined on a club set C of elementary models of $H_\lambda$and let $X = X_\Sigma$ and $\lambda = \lambda_\Sigma$ be the assiocated parameters. The strategy will be of using a proper notion of forcing P_Σ to shoot an elementary chain with the desired property through C. Then, applying PFA, the desired chain will be found in V.

To this aim, take $p \in P_\Sigma$ iff $p : \alpha+1 \to \mathtt{dom}(\Sigma)$. p is a continuous $\in$-increasing map such that for all $0 < \nu \leq \alpha$ there is an $\nu_0 < \nu$ with $p(\xi) \cap X \in \Sigma(p(\nu))$, whenever $\nu_0 \in p(\xi)$ and $\xi \in (\nu_0, \nu)$. We say that $q \leq p$ iff q is an end-extension of p.

If we show that the sets $E_\alpha = \{p \in P_\Sigma : \alpha \in \mathtt{dom}(p)\}$ are dense for all $\alpha \in \omega_1$, then, by PFA, $\bigcup G$ will be the desired $\in$-chain, whenever G is a generic filter.

Since $\mathtt{dom}(\Sigma)$ is a club of elementary countable model of H_λ, we have that for every $x \in X$ the following set

$$E^*_x = \{p \in P_\Sigma : \exists \nu_x \in \mathtt{dom}(p) \text{ such that } x \in p(\nu)\}$$

is dense. We now work in $V[G]$. We can define

$$\begin{aligned} \Phi : X &\to \omega_1 \\ x &\mapsto \nu_x \end{aligned}$$

(where $\nu_x =$ some α such that $x \in p(\alpha)$) a surjection: $X = \bigcup_{\nu \in \omega_1} p(\nu) \cap X$, hence $\Phi[X]$ is unbounded in ω_1. Since X is uncountable and since $p(\nu)$ is countable, then Φ is ω to one. We fix a countable model $M \prec H_{2^{|\mathbb{P}_\Sigma|^+}}$ such that $\dot\Phi, X, \Sigma, \mathbb{P}_\Sigma, \alpha, x \in M$. If we set $D_\alpha = \{p \in \mathbb{P}_\Sigma : \exists x \in X \, \exists \eta \in \mathtt{dom}(p), x \in p(\eta)$ and $p \Vdash \dot\Phi(x) = \check\alpha\}$, then, assuming P_Σ proper, we can find a (M, P_Σ)-generic condition q such that $q \Vdash \Phi[X \cap M] = M \cap \omega_1$. We now extend q to decide α and so there is a q' that forces $\Phi(x) = \alpha \in M \cap \omega_1$ (by properness). Hence, if P_Σ is proper, then D_α is dense for every $\alpha \in \omega_1$. But $D_\alpha \subseteq E_\alpha$, hence E_α is dense.

Finally we show that P_Σ is proper. For $p \in P_\Sigma$, let $\theta > |P_\Sigma|^+$ be a sufficiently large regular cardinal and M a countable elementary submodel of H_θ such that $p, \Sigma, P_\Sigma, H_{|P_\Sigma|^+} \in M$ and $M \cap H_\lambda \in C$. This is possible since C is club. We will find an extension of p that is (M, P_Σ)-generic.

Let $\{D_i \subseteq P_\Sigma : i \in \omega\}$ be an enumeration of the dense subsets of P_Σ in M. By induction we define a sequence $(p_n)_n$ of decreasing conditions, with $p_0 = p$ such that $p_n \in M \cap D_n$ for all n and such that for every $sup(\mathtt{dom}(p_n)) > \xi > max(\mathtt{dom}(p_0))$ we have $p_n(\xi) \cap X \in \Sigma(M \cap H_\lambda)$. We will find a $q \leq p_n$ for all n such that $q(M \cap \omega_1) = M \cap H_\lambda$ and q will be (M, P_Σ)-generic. Then $\nu = max(p_0)$ will witness that $\forall \xi \in (\nu, M \cap \omega_1)$ $q(\xi) \cap X \in \Sigma(M \cap H_\lambda)$. To this aim, fix a sequence of countable ordinals $\eta_n \to \eta = M \cap \omega_1$.

Suppose that p_i has been defined such that $\forall\xi \in (max(\mathtt{dom}(p_0)), sup(\mathtt{dom}(p_i)))$ $p_i(\xi) \cap X \in \Sigma(M \cap H_\lambda)$. We now want to define p_{i+1}. We set

$$E^i = \{N^* \in [H_{|P_\Sigma|^+}]^\omega : N^* \prec H_{|P_\Sigma|^+} \text{ such that } H_\lambda, D_i, p_i, P_\Sigma, \eta_i \in N^*\}.$$

Notice that $E^i \in M$ since is defined with parameters in M. Since E^i is club in $[H_{|P_\Sigma|^+}]^\omega$, so is

$$E^i_X = \{N : N = N^* \cap X \text{ and } N^* \in E^i\}$$

in $[X]^\omega$. Also $E^i_X \in M$.

Since $\Sigma(M \cap H_\lambda)$ is open and $M \cap H_\lambda$- stationary, there is an $N_i \in E^i_X \cap \Sigma(M \cap H_\lambda) \cap M$ and an $x_i \subseteq N_i$ finite, such that $[x_i, N_i] \subseteq \Sigma(M \cap H_\lambda)$.

Now we extend p_i to

$$q_i = p_i \cup \{(\zeta_i + 1, \mathrm{hull}(\{p_i\} \cup x_i))\}$$

where ζ_i is the biggest element of $\mathtt{dom}(p_i)$ and $\mathrm{hull}(\{p_i\} \cup x_i))$ is the Skolem hull of $\{p_i\} \cup x_i$ taken in H_λ. Notice that $q_i \in M$. Since $\mathrm{hull}(\{p_i\} \cup x_i))$ contains the range of p_i, the function $q_i : \zeta_i + 1 \to \mathtt{dom}(\Sigma)$ is $\in$-increasing, and since $\zeta_i + 1$ is a successor and $\mathtt{dom}(q_i)$ does not contain new limit ordinal, q_i is also continuous. Thus $q_i \in P_\Sigma \cap M$.

Moreover, since $N^*_i \in E^i$, we have that N^*_i contains p_i, x_i and H_λ. Hence $q_i \in N^*_i$. So in N^*_i we can find an extension p_{i+1} of q_i that is in $N^*_i \cap D_i$, with $sup(\mathtt{dom}(p_{i+1})) \geq \eta_i$. To see that for all $\xi \in \mathtt{dom}(p_{i+1}) \setminus \mathtt{dom}(p_i)$ $p_{i+1}(\xi) \cap X \in \Sigma(M \cap H_\lambda)$, we observe that if $z \in \mathtt{ran}(p_{i+1}) \setminus \mathtt{ran}(p_i)$, then $z \cap X \in [x_i, N_i]$, because $p_{i+1} \in N^*_i$ and so $z \cap X \in [x_i, N^*_i \cap X] = [x_i, N_i]$. Hence, since $\Sigma(M \cap H_\lambda)$ contains a neighborhood of the basic open $[x_i, N_i]$, $z \in \Sigma(M \cap H_\lambda)$.

Now we set $q = \bigcup_i p_i \cup \{(\sup_i \zeta_i, M \cap H_\lambda)\}$, our argument shows that $sup\{\zeta_i : i \in \omega\} = sup\{\eta_i : i \in \omega\} = M \cap \omega_1$, $q(M \cap \omega_1) = M \cap H_\lambda$, $q \in P_\Sigma$ and q is (M, P_Σ)-generic.

$\square$

We now present a priciple that follows from MRP and that we will use to show that PFA decides the cardinality of the continuum. We need to fix some notation.

Definition 3.2.5. *A sequence $\vec{C} = \langle C_\xi : \xi < \omega_1 \text{ and } lim(\xi)\rangle$ is called a C-sequence (or a ladder system) if C_ξ is an unbounded subset of ξ of order type ω, for all limit ordinals $\xi < \omega_1$.*

For a fixed C-sequence $\vec{C}$ and $N \subseteq M$ countable sets such that $\mathrm{ot}(M \cap Ord)$ $= \alpha$ is a limit ordinal and $sup(N \cap Ord) < sup(M \cap Ord)$, we define

$$w(N, M) = |sup(N \cap Ord) \cap \pi^{-1}[C_\alpha]|$$

where $\pi : M \to \alpha$ is the transitive collapse of M. Is useful to note that w is left monotonic in the sense that if $N_1 \subseteq N_2 \subseteq M$ and $sup(N_2 \cap Ord) < sup(M \cap Ord)$, then $w(N_1, M) \leq w(N_2, M)$.

Definition 3.2.6. ***(Moore)*** *(v_{AC}) For every $A \subseteq \omega_1$, $v_{AC}(A)$ holds, where $v_{AC}(A)$ is the following statement. There is an uncountable $\delta < \omega_2$ and an increasing sequence $\vec{N} = \langle N_\xi : \xi < \omega_1 \rangle$ which is club in $[\delta]^\omega$ such that for all limit $\nu < \omega_1$ there is a $\nu_0 < \nu$ such that if $\xi \in (\nu_0, \nu)$, then*

$$N_\nu \cap \omega_1 \in A \iff w(N_\xi \cap \omega_1, N_\nu \cap \omega_1) < w(N_\xi, N_\nu).$$

Note that if A is stationary and B is a club set and $\vec{N}_A$ and $\vec{N}_B$ are witness for $v_{AC}(A)$, respectively $v_{AC}(B)$, such that $\bigcup \vec{N}_A = \bigcup \vec{N}_B$, then $\langle N_\xi : \xi < \omega_1, N_\xi \in \vec{N}_A$ and $N_\xi \in \vec{N}_B \rangle$ witnesses $v_{AC}(A \cap B)$.

We now show how v_{AC} follows from MRP and that v_{AC} implies that $2^{\aleph_1} = \aleph_2$. Then, by a standard application of PFA, we will see that $\aleph_1 < 2^{\aleph_0} \leq 2^{\aleph_1} = \aleph_2$ and this will conclude the proof that PFA implies $2^{\aleph_0} = \aleph_2$.

Now we need a lemma.

Lemma 3.2.7. *Take $M \in \{N \in [H_{(2^{\aleph_1})^+}]^\omega : N \prec H_{(2^{\aleph_1})^+}\}$. Then $\Sigma_<(M)$ and $\Sigma_\geq(M)$ are open in the Ellentuck topology on $[\omega_2]^\omega$ and M-stationary, where*

$$\Sigma_<(M) = \{N \in [M \cap \omega_2]^\omega : w(N \cap \omega_1, M \cap \omega_1) < w(N, M \cap \omega_2)\}$$

and

$$\Sigma_\geq(M) = \{N \in [M \cap \omega_2]^\omega : w(N \cap \omega_1, M \cap \omega_1) \geq w(N, M \cap \omega_2)\}.$$

Proof. We begin showing that $\Sigma_<(M)$ is M-stationary. Let $E \subseteq [\omega_2]^\omega$ be club and in M. We need to find an $N \in E \cap \Sigma_<(M) \cap M$. Since $sup\{sup(Y) : Y \in E\} = \aleph_2 > \aleph_1$, by the pigeonhole principle, there is a $\gamma < \omega_1$ such that

$$\{sup(N) : N \in E \text{ and } N \cap \omega_1 \subseteq \gamma\}$$

is unbounded in ω_2. By elementarity of M, there is γ in $M \cap \omega_1$ such that

$$\{sup(N) : N \in E \cap M \text{ and } N \cap \omega_1 \subseteq \gamma\}$$

is unbounded in $M \cap \omega_2$. Then we can take $N \in E \cap M$ such that $N \cap \omega \subseteq \gamma$ such that

$$w(N, M \cap \omega_2) = |sup(N) \cap \pi^{-1}[C_{ot(M \cap \omega_2)}]| > |C_{M \cap \omega_1} \cap \gamma|.$$

By construction of N, we have that $|C_{M\cap\omega_1}\cap\gamma|\ \geq|\ sup(N\cap\omega_1)\cap C_{M\cap\omega_1}| = w(N\cap\omega_1, M\cap\omega_1)$, then $N\in E\cap\Sigma_<(M)\cap M$.

For $\Sigma_\geq(M)$, fix again an $E\subseteq[\omega_2]^\omega$, club and in M. Now let $\gamma<\omega_2$ be uncountable such that $E\cap[\gamma]^\omega$ is club in $[\gamma]^\omega$. This is possible since E is club and so there is an $F:\ [\omega_2]^{<\omega}\to\omega_2$ such that $cl_F\subseteq E$. Then $C=\{\delta\in\omega_2:\ F[\delta]^{<\omega}\subseteq\delta\}$ is club in ω_1. Now it is sufficient to take a $\gamma\in C$ and $F\restriction\gamma:\ [\gamma]^{<\omega}\to\gamma$ witnesses that $E\cap[\gamma]^\omega$ is club, since $cl_{F\restriction\gamma}\subseteq E\cap[\gamma]^\omega$. By elementarity we can find such a γ in M. Then, since $\{sup(N\cap\omega_1):\ N\in E\cap[\gamma]^\omega\}$ is club in ω_1, we can find $N\in E\cap[\gamma]^\omega\cap M$ such that

$$w(N\cap\omega_1, M\cap\omega_1)=|sup(N\cap\omega_1)\cap C_{M\cap\omega_1}|\geq|\gamma\cap\pi^{-1}_{M\cap\omega_2}[C_{ot(M\cap\omega_2)}]|.$$

and $\gamma\cap\pi^{-1}_{M\cap\omega_2}[C_{ot(M\cap\omega_2)}]\subseteq N$. Since by definition of γ, $|\gamma\cap\pi^{-1}_{M\cap\omega_2}[C_{ot(M\cap\omega_2)}]|\geq|N\cap\omega_2\cap\pi^{-1}_{M\cap\omega_2}[C_{ot(M\cap\omega_2)}]|$ then $N\in E\cap\Sigma_\geq(M)\cap M$.

Finally we see that $\Sigma_<(M)$ is open; the same argument works equally well for $\Sigma_\geq(M)$.

We show that for every point $x\in\Sigma_<(M)$ there is a neighborhood of x, in the Ellentuck topology, contained in $\Sigma_<(M)$. Take $N\in\Sigma_<(M)$. $sup(N)<sup(M\cap\omega_2)$ and so there is a β such that

$$sup(N)\cap\pi^{-1}[C_{ot(M\cap\omega_2)}]\subseteq\beta$$

Note that $|sup(N)\cap\pi^{-1}[C_{ot(M\cap\omega_2)}]|$ is always finite, since C_α is an ω-sequence. Hence there is always a $\beta\in N$ in the interval $[(sup(N)\cap\pi^{-1}[C_{ot(M\cap\omega_2)}]), sup(N)]$.

Since we also have $sup(N\cap\omega_1)<M\cap\omega_1$ there is a γ such that

$$sup(N\cup\omega_1)\cap\pi^{-1}[C_{ot(M\cap\omega_1)}]\subseteq\gamma$$

As before we can find such a γ in N.

Then, since w is left monotonic, $[\{\beta,\gamma\},N]\subseteq\Sigma_<(M)$. And this concludes the proof of the lemma. $\square$

Now we can prove what we claimed.

Theorem 3.2.8. *(**Moore**) MRP implies v_{AC}*

Proof. Take $A\subseteq\omega_1$ and $M\in\{N\in[H_{(2^{\aleph_1})^+}]^\omega:\ N\prec H_{(2^{\aleph_1})^+}\}$, if we define Σ_A as follows

$$\Sigma_A(M)=\begin{cases}\Sigma_<(M) & \text{if } M\cap\omega_1\in A\\ \Sigma_\geq(M) & \text{is } M\cap\omega_1\notin A,\end{cases}$$

lemma 3.2.7 implies that Σ_A is open stationary. Then we can apply MRP, with $X = \omega_2$ and $\lambda = (2^{\aleph_1})^+$, and find a reflecting sequence $\vec{N} = \langle N_\xi : \xi < \omega_1 \rangle$ for Σ_A. If we let $\delta = \omega_2 \cap \bigcup\{N_\xi : \xi < \omega_1\}$, then, since $\vec{N}$ is a continuous $\in$ chain, δ is an ordinal. Hence δ and $\langle N_\xi \cap \omega_2 : \xi < \omega_1 \rangle$ witness $v_{AC}(A)$. $\square$

We now present a consequence of v_{AC} on cardinal arithmetic, namely that $2^{\aleph_1} = \aleph_2$. This will conclude the following chain of implications: $PFA \Rightarrow MRP \Rightarrow v_{AC} \Rightarrow 2^{\aleph_1} = \aleph_2$. This result will be the first part of the proof that PFA implies $2^{\aleph_0} = \aleph_2$.

Theorem 3.2.9. *(**Moore**) v_{AC} implies that $2^{\aleph_1} = \aleph_2$.*

Proof. Our target will be to show that under v_{AC} we can find a well ordering of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ in ordertype ω_2. This is sufficient since $|\mathcal{P}(\omega_1)| = |\mathcal{P}(\omega_1)/NS_{\omega_1}|$. Indeed, let $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$ be a partition of ω_1 in stationary sets, then the map

$$\begin{gathered}\phi : \mathcal{P}(\omega_1) \to \mathcal{P}(\omega_1)/NS_{\omega_1}\\ A \mapsto \textstyle\bigcup_{\alpha\in A} S_\alpha\end{gathered}$$

is injective. So $|\mathcal{P}(\omega_1)| = |\mathcal{P}(\omega_1)/NS_{\omega_1}|$.

Fix $A \subseteq \omega_1$, by an application of $v_{AC}(A)$, we have an uncountable $\delta_A < \omega_2$ and a continuous $\in$-chain $\vec{N_A} = \langle N^A_\xi : \xi < \omega_1 \rangle$ club in $[\delta_A]^\omega$. We claim that the following function

$$\begin{gathered}\psi : \mathcal{P}(\omega_1)/NS_{\omega_1} \to \omega_2\\ A \mapsto min\{\delta : \delta \text{ witnesses } v_{AC}(A)\} \;=\; \delta_A\end{gathered}$$

is well defined and injective.

To see that $\delta_A = \delta_B$, whenever $A =_{NS_{\omega_1}} B$, let C be the complement, in ω_1 of $A \triangle B$. By hypothesis $A \triangle B$ is nonstationary, so C is club and $A \cap C = B \cap C$.

If $\vec{N_A}$ witnesses $v_{AC}(A)$, then, since C is club, the set

$$E = \{\xi : N_\xi \cap \omega_1 \in C\}$$

is club in ω_1. So $\vec{N} = \langle N_\xi : \xi \in C \rangle$ is a witness for $v_{AC}(B)$ and $\delta_B \leq \delta_A$. With the same argument and assuming $\vec{N_B}$, we show that $\delta_A \leq \delta_B$.

For the injectivity we suppose that $\delta_A = \delta_B = \delta$ and we show that $A =_{NS_{\omega_1}} B$. The key observation here is that the $\in$-chain $\vec{N_A}$ and $\vec{N_B}$ coincide on club many elements. Indeed

$$F = \{\xi \in \omega_1 : N^A_\xi = N^B_\xi \text{ and } \xi = M_\xi \cap \omega_1\}$$

is club in ω_1, because both $\vec{N_A}$ and $\vec{N_B}$ are club in $[\delta]^\omega$. Let us consider $lim(F)$ the set of the limit points of F. If we show that $lim(F) \cap A = lim(F) \cap B$. It follows that $A =_{NS_{\omega_1}} B$.

Suppose that γ is the first ordinal in $(lim(F) \cap A) \setminus B$. Since $\gamma \in lim(F)$, we have that $\gamma = N^A_\gamma \cap \omega_1$. By definition of the $\in$-chains we have that there is a $\eta_B < \gamma$ such that for every $\mu \in (\eta_B, \gamma)$

$$w(N^B_\mu \cap \omega_1, N^B_\gamma \cap \omega_1) \geq w(N^B_\mu, N^B_\gamma),$$

since $\gamma = N^A_\gamma \cap \omega_1$ was chosen not to be in B. On the other hand there is $\eta_A < \gamma$ such that for every $\mu \in (\eta_A, \gamma)$

$$w(N^A_\mu \cap \omega_1, N^A_\gamma \cap \omega_1) < w(N^A_\mu, N^A_\gamma),$$

since $\gamma \in A$.

Set $\eta = max\{\eta_A, \eta_B\}$, since γ is a limit ordinal, there is $\nu \in (\eta, \gamma) \cap F$. Hence we have that $N^A_\gamma = N^B_\gamma$ and $N^A_\nu = N^B_\nu$. This is a contradiction, because at the same time we have that $w(N^B_\nu \cap \omega_1, N^B_\gamma \cap \omega_1) \geq w(N^B_\nu, N^B_\gamma)$ and $w(N^B_\nu \cap \omega_1, N^B_\gamma \cap \omega_1) < w(N^B_\nu, N^B_\gamma)$.

Hence ψ is injective and so $2^{\aleph_1} = |\mathcal{P}(\omega_1)| = |\mathcal{P}(\omega_1)/NS_{\omega_1}| = \aleph_2$.

□

To conclude this section we still have to show that, assuming PFA, $2^{\aleph_0} = 2^{\aleph_1}$. The following theorem will suffice.

Theorem 3.2.10. *Assuming PFA we have that* $|\mathcal{P}(\omega)| \neq \aleph_1$.

Proof. Assuming the contrary, let $\{r_\alpha : \alpha < \omega_1\}$ be an enumeration of 2^ω (i.e. of the reals). Now let $P_{2^{<\omega}}$ be the Cohen forcing that adds a real to the generic extension $V[G]$, whenever G is a generic filter over $P_{2^{<\omega}}$.

For every r_α, define

$$D_{r_\alpha} = \{p \in P_{2^{<\omega}} : p \Vdash \bigcup G \neq r_\alpha\}$$

Since the Cohen forcing is c.c.c., it is proper. Then we can apply PFA to find a filter G in V such that $G \cap D_{r_\alpha} \neq \varnothing$ for every $\alpha \in \omega_1$. Then we have that $\bigcup G = r \notin \{r_\alpha : \alpha < \omega_1\}$. A contradiction. □

Corollary 3.2.11. *PFA implies that* $2^{\aleph_0} = \aleph_2$.

Proof. PFA $\Rightarrow$ v_{AC} $\Rightarrow$ $2^{\aleph_1} = \aleph_2$. Moreover PFA $\Rightarrow$ $2^{\aleph_0} \neq \aleph_1$. Then, since $2^{\aleph_0} \leq 2^{\aleph_1}$, it follows that $2^{\aleph_0} = 2^{\aleph_1}$. Hence $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. □

Note that in theorem 3.2.10 we could just assume BPFA (or even MA(ω_1)), because we just have a forcing c.c.c. and $\aleph_1$ dense set to be intersected.

Actually BPFA is also sufficient to have $v_{AC}(A)$, for a fixed A. Indeed, the sentence expressed by $v_{AC}(A)$ is a Σ_1-sentence in the parameters $\vec{C} \in H_{\omega_2}$, the ladder system and $A \subseteq \omega_1$. Moreover for some A, $v_{AC}(A)$ can be forced in the extension by a proper notion of forcing. Hence by the absoluteness given by BPFA we have $v_{AC}(A)$. Thus we have the following corollary.

Corollary 3.2.12. *(**Moore**) BPFA implies that* $2^{\aleph_0} = \aleph_2$.

We remark that BPFA is actually weaker than PFA, indeed we have that MRP does not follow from BPFA (since they have a different consistency strength), but $v_{AC}(A)$ does. The notion of forcing we need to use to force $v_{AC}(A)$ is a forcing that shoots a continuous increasing chain in the open and M- stationary map we defined in lemma 3.2.7 and this forcing is proper. Note that we are not forcing MRP, but just an instance of it: the one for that particular Σ.

We conclude this section with the following remark. It is possible to show directly that v_{AC} decides the cardinality of the continuum. Recall that $w\Diamond$ is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$; hence if $w\Diamond$ fails, then $2^{\aleph_0} = 2^{\aleph_1}$.

Theorem 3.2.13. *v_{AC} implies the failure of $w\Diamond$.*

We conclude this chapter noting that, since BMM implies BPFA, also BMM decides that the cardinality of the continuum is $\aleph_2$. Historically the proof that BMM implies $2^{\aleph_0} = \aleph_2$ was the first one using a bounded forcing axiom. It used a general combinatorial principle, called θ_{AC}. This principle is similar to v_{AC} and can be stated with a Σ_1-sentence forcable by a stationary set preserving notion of forcing. For more on this subject see [23].

Chapter 4

FA(σ-closed$*$c.c.c., $\aleph_1$) and the continuum

In this chapter we see how FA(σ-closed$*$c.c.c., $\aleph_1$) affects the cardinality of the continuum. This is the weaker unbounded forcing axiom we consider and it follows from PFA. From now on we will abbreviate σ-closed$*$c.c.c. with $\sigma*$c.

In the same fashion as with the other forcing axioms we will use some combinatorial principles to factorize the proof that FA($\sigma*$c,$\aleph_1$) implies $2^{\aleph_0} = \aleph_2$. On one hand we will show that the Open Coloring Axiom (OCA), introduced by Todorčević in [22], gives $\aleph_2$ as a lower bound to the continuum. On the other hand we will see that by means of a certain oscillation map FA($\sigma*$c,$\aleph_1$) also gives $\aleph_2$ as an upper bound for $2^{\aleph_0}$.

4.1 FA($\sigma*$c,$\aleph_1$) and OCA

OCA can be see as a two-dimensional version of the Perfect Set Property (PSP). We recall that the PSP says that, for every $X \subseteq \mathbb{R}$, either X is countable, or it contains a perfect set and that it follows from AD.

Every separable metric space is homeomorphic to a subset of $\mathbb{R}$ with the relative topology. We shall thus concentrate on subsets of $\mathbb{R}$.

Definition 4.1.1. *Given $Y \subseteq \mathbb{R}$, let $[Y]^2 = \{(x, y) \in Y^2 : x > y\}$ (the half of the plane below the bisector) and let $K \subseteq [Y]^2$ be a coloring.*

- *We say that K is an open coloring, if K is open in $[Y]^2$ with the relative topology induced by $\mathbb{R}^2$ over $[Y]^2$.*
- *Y is said to be K-countable if $Y = \bigcup\{Y_n : n \in \omega\}$, where, for every n, $[Y_n]^2 \subseteq [Y]^2 \setminus K$ i.e. if Y_n is homogeneous for $K^c = [Y]^2 \setminus K$.*

We state two different axioms. The first one OCA_P is a natural consequence of AD and we state it to justify the second one OCA, that widen the class of sets for which the relevant dichotomy holds.

Definition 4.1.2. *Given $Y \subseteq \mathbb{R}$, $OCA_P(Y)$ holds if for any K open coloring of Y exactly one of the following holds:*

- *Y is K-countable,*
- *there is a perfect set $P \subseteq Y$, homogeneous for K.*

OCA_P holds iff $OCA_P(Y)$ holds, for every Y.

We have the following important theorems.

Theorem 4.1.3. ***(Todorčević)*** *If $X \subseteq \mathbb{R}$ is Σ^1_1, then $OCA_P(X)$.*

Theorem 4.1.4. ***(Feng)*** *AD(X) implies $OCA_P(X)$.*

To see $\mathrm{OCA}(X)$ as a two dimensional version of the PSP(X), we have to consider the coloring $K = [X]^2$, for X as in the above theorems.

We now present the axiom OCA.

Definition 4.1.5. ***(Todorčević) (Open Coloring Axiom (OCA))*** *Given $Y \subseteq \mathbb{R}$, $OCA(Y)$ holds if for any K open coloring of Y, exactly one of the following holds:*

- *Y is K-countable,*
- *there is an uncountable set $X \subseteq Y$, homogeneous for K.*

OCA holds iff $OCA(Y)$ holds, for every Y.

The next fact shows that we cannot have the same principle for closed colorings.

Fact 4.1.6. *There is a closed coloring K of $[\mathbb{R}]^2$ such that neither $\mathbb{R}$ is K-countable nor K has a perfect homogeneous subset.*

Proof. Consider the lines $l_n = \{(x, y) : y = x - 1/n\}$ that accumulate on the bisector and define $K = \bigcup_{n<\omega} l_n$. Clearly K is closed in $[\mathbb{R}]^2$. We can then define the fiber $K(x) = \{y : (x, y) \in K\} = \{y : \exists n \in \omega\, y = x - 1/n\}$; then $K(x)$ is countable. But if Y is homogeneous for K and $x \in Y$, then $Y \subseteq K(x)$. Hence there cannot be a perfect set homogeneous for K.

For the other part, suppose that $\mathbb{R} = \bigcup_{n\in\omega} X_n$. Since $\mathbb{R}$ has the Baire Property there must be an n such that X_n is not meager. We claim that if Z is homogeneous for K^c, then Z is meager, hence a contradiction. To see it, suppose the contrary and let I be an in interval (it can also be $\mathbb{R}$) on which Z is comeager and let $\bigcap_{n\in\omega} D_n \subseteq Z$, where D_n are open dense in I. Hence $K\cap[D_n]^2$ is open dense in $K \cap [I]^2$, with the relative topology induced by $[I]^2$. Since $\mathbb{R}$ is a Baire space, $K \cap \bigcap_{n\in\omega}[D_n]^2$ is dense in $K \cap [I]^2$ and hence nonempty. But this contradicts the fact that Z is homogeneous for K^c. □

We now show that OCA is a consequence of FA(σ∗c,$\aleph_1$). We will follow [24].

Given an open coloring K, we will use the convention to set $K = K_0$ and $K^c = K_1$.

Definition 4.1.7. *Given $Y \subseteq \mathbb{R}$ and an open coloring K of $[Y]^2$, we say that a subset $X \subseteq Y$ is i-homogeneous if $[X]^2 \subseteq K_i$; where $i \in \{0, 1\}$.*

We now present the proof of Veličković of a theorem by Todorčević.

Theorem 4.1.8. ***(Todorčević, Veličković)*** *Let S be a set of reals and suppose that K is a given open coloring (i.e. $[S]^2 = K_0 \cup K_1$). Assume that S is not the union of $< 2^{\aleph_0}$ 1-homogeneous sets. Then there is $Y \subseteq S$ of size $2^{\aleph_0}$ such that $P_Y = (\{X \in [Y]^{<\omega} : [X]^2 \subseteq K_0\}, \supseteq)$, the poset of finite 0-homogeneous subsets of Y ordered by reverse inclusion, has the $2^{\aleph_0}$-chain condition.*

Proof. Let $p \in S^n$ and $U \subseteq S^n$ be open in the product topology with $p \in U$; p_i will denote the ith element of p. Define

$$U_p = \{q \in U : q_i \neq p_i \text{ and } \{p_i, q_i\} \in K_0, \text{ for all } i < n\}.$$

Give $A \subseteq S^n$, $f : A \to S$, and p arbitrary in S^n (p not in $\mathtt{dom}(f)$ is also possible), let

$$\omega_f(p) = \bigcap\{cl(f[U_p \cap A]) : U \subseteq S^n \text{ open and } p \in U\}.$$

$\omega_f(p)$ is the collection of the accumulation points x of f such that for every $\epsilon > 0$ we can find $q \in S^n$ such that

- $|q - p| < \epsilon$,
- $\forall i < n \ \{q_i, p_i\} \in K_0$,
- $|f(q) - x| < \epsilon$.

It is useful to observe here that, once we fixed p, $\omega_f(p)$ depends just on a countable dense subfuntion of f. Let $g \subseteq f$ be so, then $\omega_g(p) \subseteq \omega_f(p)$, because $g \subseteq f$, but if we choose $x \in \omega_f(p)$ by the definition of $\omega_f(p)$ we can find $(q_n)_n$ such that $q_n \to p$ and $f(q_n) \to x$, then using the fact that g is dense we can find, $\forall \epsilon > 0$, $(t_n)_n$ such that

$$\forall t \in \mathtt{dom}(g)\, \forall n\, |t_n - q_n| < \epsilon$$

and

$$\forall n\, |g(t_n) - f(q_n)| < \epsilon.$$

Since ϵ is arbitrary $x \in \omega_g(p)$. This shows that $\omega_f(p) \subseteq \omega_g(p)$.

We also observe that a priori we could have more than $2^{\aleph_0}$ 1-homogeneous sets. Note that since K_1 is closed, if a set T is 1-homogeneous, then its closure is still 1-homogeneous. Since $\mathbb{R}$ has a countable basis (namely that given by the rational intervals) we can code a closed set by a real.[1] Thus we can enumerate the closed 1-homogeneous subsets in order type the continuum. It is then this sufficient for our propose that $\{T_\xi : \xi < 2^{\aleph_0}\}$ is an enumeration of all the closed 1-homogeneous subsets of S. Let $\{f_\xi : \xi < 2^{\aleph_0}\}$ be an enumeration of all countable functions from a finite power of S to S.

We can now define $Y = \{x_\xi : \xi < 2^{\aleph_0}\}$ in the following way:

1. $x_\alpha \in S \setminus \{x_\xi : \xi < \alpha\}$ (hence Y has size $2^{\aleph_0}$),
2. $x_\alpha \notin T_\xi$, for $\xi < \alpha$,
3. x_α does not belong to any 1-homogeneous set of the form $\omega_{f_\xi}(p) \cap S$, where $\xi < \alpha$ and $p \in [\{x_\xi : \xi < \alpha\}]^{<\omega}$.

We now show that P_Y has the $2^{\aleph_0}$-c.c. Suppose not and let $\mathcal{F} = \{s_\alpha : \alpha \in 2^{\aleph_0}\}$ be a family of size $2^{\aleph_0}$ of pairwise incompatible conditions of P_Y. Without loss of generality we can assume that every element in $\mathcal{F}$ has the same size $n \geq 1$. We prove, by induction, that for every n we can find two conditions such that their union is still a finite 0-homogeneous subset of Y, hence they are compatible.

For $n = 1$, $\mathcal{F} = \{\{x_{\alpha_\xi}\} : \xi < 2^{\aleph_0}\}$, $\bigcup \mathcal{F} \subseteq Y$ and we argue as follows. If there are $\{x_\alpha\}, \{x_\beta\} \in \mathcal{F}$ such that $\{x_\alpha, x_\beta\} \in K_0$ we are done, otherwise we have that $\bigcup \mathcal{F} = \{x_{\alpha_\xi} : \xi < 2^{\aleph_0}\}$ is 1-homogeneous and so is its closure. Thus we have that there is a ξ such that $cl(\bigcup \mathcal{F}) = T_\xi$, but now for every $\alpha > \xi$ $\{x_\alpha\} \in \mathcal{F}$, but $x_\alpha \notin Y$, because of 2): a contradiction.

[1] For example we can consider a real that enumerate finite unions of rational intervals, whose intersection is the given closed set.

Suppose $n > 1$. We can refine $\mathcal{F}$ with an application of the Δ-system lemma and find a root r such that $\forall s, t \in \mathcal{F}\ s \cap t = r$. Note that given $s, t \in \mathcal{F}$ $(s \setminus r) \cup (t \cap r) = s$ is 0-homogeneous and if $s \cup t$ is not 0-homogeneous, this is witnessed by some $x_\alpha \in s \setminus r$ and $x_\beta int \subseteq r$. We can thus concentrate on the family $\{s \setminus r : s \in \mathcal{F}\}$; let us call this new family still $\mathcal{F}$. We can assume that $r = \varnothing$. Again, by the Δ-system lemma we can have that $\forall s, t \in \mathcal{F}$, $s = \{x_{\alpha_0^s}, \ldots, x_{\alpha_{n-1}^s}\}$, $t = \{x_{\alpha_0^t}, \ldots, x_{\alpha_{n-1}^t}\}$ are listed in increasing order with respect to the enumeration and $\alpha_0^s > \alpha_{n-1}^t$ or $\alpha_0^t > \alpha_{n-1}^s$. Hence if $s \in \mathcal{F}$, then $s = \{s(0), \ldots, s(n-1)\}$, where the elements of s are listed in order: $s(0) < \ldots < s(n-1) < 2^{\aleph_0}$. Moreover if $s \neq t$ then there are no $i, j \leq n-1$ such that $s(i) = t(j)$.

Since, for every α, s_α is 0-homogeneous and K_0 is open, we can find an open $U_\alpha = I_0^\alpha \times \ldots \times I_{n-1}^\alpha \subseteq S^n$ such that,

- for every i, I_i^α has rational coordinates ,
- $s_\alpha(i) \in I_i^\alpha$,
- $I_i^\alpha \times I_j^\alpha \subseteq K_0$, for $i \neq j$.

Refining $\mathcal{F}$, using the pigeonhole principle we can find a basic open U such that $U = U_\alpha$, for avery α. Our goal is to find two conditions $s, t \in \mathcal{F}$ whose union is 0-omogeneous. The couples with different indices raise no problems as $s_\alpha(i) \in I_i$, $s_\beta(j) \in I_j$ and $\forall i, j < n-1\ I_i \times I_j \subseteq K_0$. It remains to check the homogeneity of the couples $\{s(i), t(i)\}$, for all $i < n$.

Since for $s, t \in \mathcal{F}$, $s \cap t = \varnothing$, we can think of $\mathcal{F}$ as the graph of an $(n-1)$-ary injective function g, such that, for all $s \in \mathcal{F}$, $g(s \restriction (n-1)) = s(n-1)$ (i.e. the function that, given the $n-1$ first elements of s, gives back the last one). Note that

> $\omega_g(s \restriction (n-1)) = \bigcap_m cl\{t(n-1) : t \restriction (n-1) \in \prod_{i=0}^{n-2}(s(i) - 1/m, s(i) + 1/m), t \restriction (n-1)$ and $s \restriction (n-1)$ are 0-homogeneous $\}$.

As we remarked above $\omega_g(s \restriction (n-1))$ is a closed set made of the points that are limits of sequences $(t_j(n-1))_j$ such that $t_j \restriction (n-1) \to s \restriction (n-1)$ and $t_j \restriction (n-1) \cup s \restriction (n-1)$ is 0-homogeneous. It is here that we use the inductive hypothesis: to find $t_j \restriction (n-1)$ such that $t_j \restriction (n-1) \cup s \restriction (n-1)$ is 0-homogeneous.

Now we can define

$$\mathcal{F}_0 = \{s \in \mathcal{F} : s(n-1) \in \omega_g(s \restriction (n-1))\}.$$

The next claim shows that for almost all elements $s \in \mathcal{F}$ it is the case that $s(n-1) \in \omega_g(s \restriction (n-1))$.

Claim 4.1.9. *$\mathcal{F} \setminus \mathcal{F}_0$ has size $< 2^{\aleph_0}$*

Proof. By way of contradiction, suppose the contrary. For each $s \in \mathcal{F} \setminus \mathcal{F}_0$ pick a rational open interval I^s such that

- $s(n-1) \in I^s$,
- $I^s \cap \omega_g(s \restriction (n-1)) = \varnothing$.

Fix also a basic open set $U^s \subseteq S^{n-1}$ (given by a product of rational intervals) that contains $s \restriction (n-1)$ and such that, if $q \in U^s_{s\restriction(n-1)}$, then $g(q) \notin I^s$; this is possible by definition of $\omega_g(s \restriction (n-1))$ Since we have countable many rational intervals, there is a $Z \subseteq \mathcal{F} \setminus \mathcal{F}_0$ of size $2^{\aleph_0}$ such that all the I^s equals a fixed I and all the U^s are equal to U. By inductive assumption pick $s, t \in Z$ such that $s \cup t$ is 0-homogeneous. Since I and U are all the same, we get $t \restriction (n-1) \in U_{s\restriction(n-1)}$ and $g(t \restriction (n-1)) = t(n-1) \in I$. A contradiction with the fact that $g(t \restriction (n-1)) \notin I$. $\square$

Back to the proof of the theorem, we refine g and define g_0 to be a countable dense subfunction of g. Then there is a ξ such that $g_0 = f_\xi$. By the above claim we can pick $s \in \mathcal{F}_0$ such that, for all i, $s(i)$ is above ξ and above any element of g_0. Since $s \in \mathcal{F}_0$, $s(n-1) \in \omega_g(s \restriction (n-1))$. For what we remarked above $\omega_g(s \restriction (n-1)) = \omega_{g_0}(s \restriction (n-1)) = \omega_{f_\xi}(s \restriction (n-1))$ and so, by 3), $\omega_{f_\xi}(s \restriction (n-1))$ is not 1-homogeneous, since $s(n-1) \in \omega_{f_\xi}(s \restriction (n-1))$. Recall that K_0 is open; so we can take $u, v \in \omega_{f_\xi}(s \restriction (n-1))$ such that $\{u, v\} \in K_0$ and two open intervals I and J, such that $u \in I$, $v \in J$ and $I \times J \subseteq K_0$. By the definition of $\omega_{g_0}(s \restriction (n-1))$, there is a $p \in \mathtt{dom} g_0$ such that $p \cup s \restriction (n-1)$ is 0-homogeneous and $g_0(p) \in I$. Now take a sufficiently small $U \subseteq S^{n-1}$ such that, $s \restriction (n-1) \in U$ and for every $q \in U$, $p \cup q$ is 0-homogeneous. This is possible since K_0 is open. Pick a $q \in U$ such that $g_0(q) \in J$. Thus $p \cup g_0(p)$ and $q \cup g_0(q)$ are two elements of $\mathcal{F}$ whose union is 0-homogeneous.

$\square$

Theorem 4.1.10. *FA($\sigma{*}c$,$\aleph_1$) implies OCA.*

Proof. Fix a set of reals S and an open coloring K. Assume that S cannot be covered by countably many 1-homogeneous sets. Let P be the σ-closed poset that collapse $2^{\aleph_0}$ to $\aleph_1$. We now work in $V[G]$, where G is a P-generic filter. Since P is σ-closed, in the generic extension, it is still true that S cannot be covered by countably many 1-homogeneous sets. By Theorem 4.1.8, since in

$V[G]$ holds CH, there is a $Y = \{x_\xi : \xi < 2^{\aleph_0}\} \subseteq S$ of size $\aleph_1$ such that P_Y is c.c.c. We force over $V[G]$ with P_Y. Let $\dot{f} \in V^P$, $\dot{f} : \omega_1 \to 2^{\aleph_0}$ be a bijection and choose a $\dot{Z} \in V^{P * P_Y}$ such that $1_{P * P_Y} \Vdash$ "$\dot{Z} = \{x_{\alpha_\xi} : \xi < \omega_1\} \subseteq Y$ is 0-homogeneous".

Since, for every $\alpha \in \omega_1$,

$$D_\alpha = \{\langle p, K\rangle : \exists \xi\, \dot{f}(\xi) \geq \dot{f}(\alpha) \text{ and } \langle p, K\rangle \Vdash x_{f(\xi)} \in \dot{Z}\}$$

is dense. Let H be a $\{D_\alpha : \alpha \in \omega_1\}$-generic filter, then $\{x : \exists p \in H\, p \vDash x \in \dot{Z}\}$ is an uncountable 0-homogeneous subset of S.

Finally assuming FA(σ*c,$\aleph_1$) we can have such a $\{D_\alpha : \alpha \in \omega_1\}$-generic filter in V. Hence $Z \subseteq S$ is a 0-homogeneous set of size $\aleph_1$; thus OCA holds. $\square$

4.2 OCA, gaps and the continuum

We now see how OCA affects the continuum. We will show that it decides an important cardinal invariant of the continuum; see [7].

We need to recall some theorems and definitions relative to objects that live in the space $\omega^\omega = \{f : f : \omega \to \omega\}$.

Definition 4.2.1. *Given $u \in \omega^n$ for some $n \in \omega$, we define*

$$[u] = \{f \in \omega^\omega : f \restriction n = u\}.$$

Definition 4.2.2. *Given $u, v \in \omega^n$ for some $n \in \omega$, we define*

$$[u] \otimes [v] = \{(x, y) : (x \in [u] \wedge y \in [v]) \vee (x \in [v] \wedge y \in [u])\}.$$

Definition 4.2.3. *Given $f, g \in \omega^\omega$ we say that f eventually dominates g and we write $g <^* f$ if $|\{n : f(n) \geq g(n)\}|$ is finite.*

A set $A \subseteq \omega^\omega$ is said to be bounded if there is a $f \in \omega^\omega$ such that $g <^* f$, whenever $g \in A$. Otherwise is said to be unbounded.

Definition 4.2.4. *The cardinal $\mathfrak{b}$ is the minimal size of an unbounded family on ω^ω.*

Our goal is to show that, under OCA, we can prove that $\mathfrak{b} = \aleph_2$. Before we need to develop the theory of gaps on ω^ω.

Definition 4.2.5. *(A, B) is called a (κ, λ^*)-pregap in ω^ω if $A = \{f_\alpha : \alpha \in \kappa\}$, $B = \{g_\beta : \beta \in \lambda\}$ and the following conditions are respected:*

1. *for all* $\alpha < \gamma < \kappa$*, we have* $f_\alpha <^* f_\gamma$*,*

2. *for all* $\beta < \rho < \lambda$*, we have* $g_\beta >^* g_\rho$*,*

3. *for all* $\alpha < \kappa$ *and* $\beta < \lambda$*, we have* $f_\alpha <^* g_\beta$*.*

We say that a pregap is filled if there is an $h \in \omega^\omega$ *such that for all* $f \in A$ *and* $g \in B$

$$f <^* h <^* g.$$

By a gap we will mean an unfilled pregap.

Given a (κ, λ^*)-gap it is possible to build up a (λ, κ^*)-gap; thus we can assume that $\kappa \geq \lambda$. Also we can assume that κ and λ are regular since, given a (κ, λ^*)-gap, we can construct in an obvious way a $(cf(\kappa), cf(\lambda)^*)$-gap.

In ZFC the best results we can have is that there are no (ω, ω^*)-gaps and the following theorems. For more on the subject see [17].

Theorem 4.2.6. ***(Hausdorff)*** *There exists a* (ω_1, ω_1^*)*-gap on* ω^ω*.*

Theorem 4.2.7. *There exists a* $(\mathfrak{b}, \omega^*)$*-gap on* ω^ω*.*

Under OCA, these gaps are the only ones that provably exist.

Theorem 4.2.8. ***(Todorčević)*** *Assuming OCA, the only kind of gap that exists are either* (ω_1, ω_1^*)*, or* (κ, ω^*)*, where* $\kappa \geq \mathfrak{b}$*.*

Proof. To get to a contradiction suppose that $A = \{f_\alpha : \alpha < \kappa\}$, $B = \{g_\beta : \beta < \lambda\}$ and (A, B) is a (κ, λ^*)-gap, where κ and λ are uncountable regular cardinals and $\kappa > \omega_1$.

We now refine the gap in the following way. Note that for every α there is m_α such that $|\{\beta : f_\alpha(n) < g_\beta(n) \, \forall n \geq m_\alpha\}| = \lambda$. By the pigeonhole principle, for κ-many α, m_α is the same number. Rescaling the f_α and the g_β and removing the α that are not useful we can assume that for all $\alpha < \kappa$ $m_\alpha = 0$. Hence we can define

$$X = \{(f_\alpha, g_\beta) : \forall n \, f_\alpha(n) < g_\beta(n), \, \alpha < \kappa, \, \beta < \lambda\}.$$

We now give a coloring on X.

$$K = \{(f_\alpha, g_\beta), (f_\xi, g_\eta) : \exists n \, f_\alpha(n) \geq g_\eta(n) \text{ or } \exists n \, f_\xi(n) \geq g_\beta(n)\}.$$

Note that K is open since if $\{(f_\alpha, g_\beta), (f_\xi, g_\eta)\} \in K$, then there is a n that witnesses it. The following open

$U_{\{(f_\alpha,g_\beta),(f_\xi,g_\eta)\}} = \{\{(t,s),(t',s')\} \in [X]^2 : t \upharpoonright (n+1) = f_\alpha \upharpoonright (n+1),\, s \upharpoonright (n+1) = g_\beta \upharpoonright (n+1),\, t' \upharpoonright (n+1) = f_\xi \upharpoonright (n+1), s' \upharpoonright (n+1) = g_\eta \upharpoonright (n+1)\}$

is an open neighborhood of $\{(f_\alpha, g_\beta), (f_\xi, g_\eta)\}$ contained in K.

To prove the theorem it is sufficient to show that X is neither K-countable, nor has an uncountable subset homogeneous for K.

Suppose that X is K-countable. Hence $X = \bigcup_n X_n$, with each X_n 1-homogeneous. Set

$$B_n = \{\beta : \exists \alpha\, (f_\alpha, g_\beta) \in X_n\},$$

and

$$A_n = \{\alpha : \exists \beta\, (f_\alpha, g_\beta) \in X_n\}.$$

By the definition of K and since X_n is 1-homogeneous, given $\alpha \in A_n$ and $\beta \in B_n$, for every m, $f_\alpha(m) < g_\beta(m)$. There are two cases: if there is n such that $|A_n| = \kappa$ and $|B_n| = \lambda$, then we can define a function g setting $g(m) = min\{g_\beta(m) : \beta \in B_n\}$. For all $\alpha \in A_n$ and $m \in \omega$, $f_\alpha(m) < g(m)$ and for all $\beta \in B_n$ $g <^* g_\beta$; then g fills the (κ, λ^*)-gap (A_n, B_n).

The other possibility is that, for all n, either $|B_n| < \lambda$ or $|A_n| < \kappa$; then, by regularity,

$$\bigcup_n \{A_n : |A_n| < \kappa\} = \alpha_0 < \kappa$$

and

$$\bigcup_n \{B_n : |B_n| < \lambda\} = \beta_0 < \lambda.$$

Choose $\beta \geq \beta_0$ and find n_0 such that $(f_{\alpha_0}, g_\beta) \in X_{n_0}$. Notice that $X_{n_0} \subseteq A_{n_0} \times B_{n_0}$. We get a contradiction noting that if $|A_{n_0}| < \kappa$, then $\alpha_0 \notin A_{n_0}$ while if $|B_{n_0}| < \lambda$, then $\beta \notin B_{n_0}$; but we supposed that, for every n, either $|B_n| < \lambda$ or $|A_n| < \kappa$.

Since X is not K-countable, by OCA it should have an uncountable subset 0-homogeneous. We now show that such a subset cannot exist.

Again suppose the contrary and let Y be uncountable and such that $[Y]^2 \subseteq K$. Note that for all $(f_\alpha, g_\beta), (f_\xi, g_\eta) \in Y$, $\alpha \neq \xi$ and $\beta \neq \eta$; else, since Y is 0-homogeneous, there is a n such that $f_\alpha(n) \geq g_\eta(n)$ and so $f_\xi(n) \geq g_\eta(n)$, contradicting the fact that $(f_\xi, g_\eta) \in X$.

With a bit of work we can build inductively an ω_1-sequence $\{(f_{\alpha_\nu}, g_{\beta_\nu}) : (f_{\alpha_\nu}, g_{\beta_\nu}) \in Y, \nu \in \omega_1\}$ such that $f_{\alpha_\rho} <^* f_{\alpha_\gamma} <^* g_{\beta\gamma} <^* g_{\beta\rho}$, for $\rho < \gamma$. Recall that we supposed that $\kappa > \omega_1$; then there is a η such that, for all ν, $f_{\alpha_\nu} <^* f_\eta$.

By the pigeonhole principle we can find a n_0 such that

$$A = \{\nu : \forall n \geq n_0\, f_{\alpha_\nu}(n) < f_\eta(n)\}$$

is uncountable and we can find a $n_1 \geq n_0$ such that

$$B = \{\nu \in A : \forall n \geq n_1\, g_{\beta\nu}(n) > f_\eta(n)\}$$

is uncountable. If we define $B_Y = \{f_{\alpha_\nu} : \nu \in B\}$ we have that $B_Y = \bigcup_{u \in \omega^{<\omega}} B_Y \cap [u]$. Hence, since $|\omega^{<\omega}| = \aleph_0$ we can find $u_0 \in \omega^{n_1}$ such that

$$C = \{\nu \in B : f_{\alpha_\nu} \in [u_0]\}$$

is uncountable and for the same reason we can find $u_1 \in \omega^{n_1}$ such that

$$D = \{\nu \in C : g_{\beta\nu} \in [u_1]\}$$

is uncountable.

Observe that for all $\rho, \gamma \in D$, if $k < n_1$, $f_{\alpha_\rho}(k) = f_{\alpha_\gamma}(k) < g_{\beta\gamma}(k)$, while if $k \geq n_1$, then $f_{\alpha_\rho}(k) < f_\eta(k) < g_{\beta\gamma}(k)$. This means, for all $\rho, \gamma \in D$ and for all $k \in \omega$, $f_{\alpha_\rho}(k) < g_{\beta\gamma}(k)$. Hence $\{(f_{\alpha_\rho}, g_{\beta\rho}), (f_{\alpha_\gamma}, g_{\beta\gamma})\} \notin K$: a contradiction.

This conclude the proof of the theorem. □

We now come back to our problem: how OCA effects the continuum.

Theorem 4.2.9. *(**Todorčević**) OCA implies that every family $\mathcal{F} \subseteq \omega^\omega$ of size $\aleph_1$ is bounded (Hence $\mathfrak{b} > \aleph_1$).*

Proof. Let $\mathcal{A} = \{f_\alpha \in \omega^\omega : \alpha < \mathfrak{b}\}$ an unbounded family, without loss of generality we can assume that $\mathcal{A}$ is a family of strictly increasing functions. We define the following coloring on $[\mathcal{A}]^2$

$K = \{(f_\alpha, f_\beta) : \exists n, m\, (f_\alpha(m) < f_\beta(m) \wedge f_\alpha(n) > f_\beta(n)) \vee (f_\alpha(m) > f_\beta(m) \wedge f_\alpha(n) < f_\beta(n))\}$.

K is open: given $(f_\alpha, f_\beta) \in K$ it is sufficient to fix the first $k > n, m$ coordinates, then the set of couples that coincide with (f_α, f_β) on the first k values is a 0-homogeneous neighborhood of the point:

$$[f_\alpha \restriction k] \otimes [f_\beta \restriction k] \subseteq K.$$

Observe that $\mathcal{A}$ cannot be K-countable; else $\mathcal{A} = \bigcup_n A_n$, with each A_n 1-homogeneous, and so there would be an n_0 such that A_{n_0} is uncountable. Hence, by definition of the coloring, $(A_{n_0}, <_{lex})$ would be an uncountable well order inside ω^ω, which is impossible: We can see ω^ω as the irrational in $\mathbb{R}$. Hence if we let $\{x_\xi : \xi \in \omega_1\}$ be an increasing sequence of reals in order type ω_1 under $<_{lex}$ we have that $\{(x_\xi, x_{\xi+1}) : \xi \in \omega_1\}$ is an uncountable family of pairwise disjoint non empty open subsetes of $\mathbb{R}$; contradicting the fact that $(\mathbb{R}, <)$ has a countable dense set.

So, by OCA, there is an uncountable $\mathcal{F} \subseteq \mathcal{A}$, that is 0-homogeneous. We now show that $\mathcal{F}$ is bounded and, since $|A| > |\mathcal{F}| \geq \omega_1$, this will conclude the theorem.

Suppose that $\mathcal{F}$ is unbounded. To each $t \in \omega^{<\omega}$, such that $[t] \cap \mathcal{F} = \varnothing$, we associate α_t such that $f_{\alpha_t} \in \mathcal{F}$ and $f_{\alpha_t} <^* f_{\alpha_s}$, whenever $\alpha_t < \alpha_s$. Choose $\gamma > sup\{\alpha_t : t \in \omega^{<\omega}\}$ with $f_\gamma \in \mathcal{F}$ such that for all $t \in \omega^{<\omega}$ $f_{\alpha_t} <^* f_\gamma$. This is possible since $|\omega^{<\omega}| = \aleph_0$ and $\mathcal{F}$ is uncountable. Pick now $k_0 \in \omega$ such that

$$Z = \{f \in \mathcal{F} : \forall k \geq k_0\, f(k) > f_\gamma(k)\}$$

is still unbounded. Such a k_0 exists, because if we call $Z_n = \{f \in \mathcal{F} : \forall k \geq n\, f(k) > f_\gamma(k)\}$, we have that $\bigcup_n Z_n$ is a final segment in $\mathcal{A}$. Hence there must be an n such that Z_n is uncountable. Let then $n = k_0$ and $Z = Z_n$.

For our purposes we say that $u \in \omega^{<\omega}$ is good if $Z \cap [u]$ is unbounded.

Claim 4.2.10. *There is a good u, $|u| \geq k_0$ such that $\{n : u^\frown n$ is good $\}$ is infinite.*

Proof. Suppose the contrary, then for every u that is not good define $g_u \in \omega^\omega$ such that bounds every element of $Z \cap [u]$. Since there are at most countably many such u, let g be an element of ω^ω such that $g_u <^* g$ for every u that is not good. Set now

$$T = \{u : \forall s \preccurlyeq u\, s \text{ is good }\}$$

and note that $Z = Z \cap [T] \cup \bigcup\{Z \cap [s] : s \text{ is not good }\}$. We assume that T is a finite splitting tree, else the claim was proved. For every $u \in T$ define $f_u \in \omega^\omega$ such that at the step n, $f_u(n)$ is bigger then the value at n of any extension of u: this is possible since T is finitely splitting. Hence if $x \in [u] \cap T$, then $x <^* f_u$. Again, since there are countably many $u \in T$, it is possible to define an f that dominates (with respect to $<^*$) every f_u. But this is a contradiction, because for $f \in Z$ either it is dominated by g, or by f, hence Z should be bounded. □

Thanks to the claim we can find $k_1 \geq k_0$ and $u \in \omega^{k_1}$ such that $Z \cap [u]$ is unbounded and such that the set $H = \{f(k_1) : f \in Z \cap [u]\}$ is infinite.

By the definition of $f_\gamma(k)$ there is a $k_2 \geq k_1$ such that

$$\forall k \geq k_2 \;\; f_{\alpha_u}(k) < f_\gamma(k).$$

and there is $f \in Z \cap [u]$ such that $f(k_1) > f_\gamma(k_2)$, because H is infinite.

Note that

- if $k < k_1$, then $f_{\alpha_u}(k) = f(k)$ (by the choice of k_1),
- if $k_1 \leq k \leq k_2$, then $f_{\alpha_u}(k) \leq f_{\alpha_u}(k_2) < f_\gamma(k_2) < f(k_1) \leq f(k)$,
- if $k > k_2$, then $f_{\alpha_u}(k) < f_\gamma(k) < f(k)$ (because $f \in Z$).

We finally got to a contradiction, because both f and f_{α_u} are in $\mathcal{F}$, but the inequalities above show that $\{f, f_{\alpha_u}\} \notin K$.

$\square$

It is still an open question whether OCA decides the cardinality of the continuum. The following result is optimal up to now and gives a lower bound to $2^{\aleph_0}$.

Theorem 4.2.11. ***(Todorčević)*** *OCA implies that* $\mathfrak{b} = \aleph_2$.

Proof. We show that $\mathfrak{b} > \aleph_2$ implies that there is a an (ω_2, λ^*) gap for some uncountable λ. By Theorem 4.2.8 this contradicts OCA. Moreover, by Theorem 4.2.9 $\mathfrak{b} \neq \aleph_1$. We conclude that $\mathfrak{b} = \aleph_2$.

Let $A = \{f_\alpha : \alpha \in \omega_2\}$ be a family of strictly increasing functions in ω^ω. Since we supposed $\mathfrak{b} > \omega_2$, the set $\mathcal{F} = \{g \in \omega^\omega : \forall \alpha \in \omega_2 \, f_\alpha <^* g\} \neq \varnothing$. Now let $B = \{g_\alpha : \alpha < \lambda\} \subseteq \mathcal{F}$ be a maximal chain with respect to the reverse eventual domination (i.e. $>^*$).

We claim that $cof(\lambda) > \omega$. The theorem will follow once we note that (A, B) is a (ω_2, λ^*) gap with $cof(\lambda)$ an uncountable cardinal. Hence OCA is contradicted.

To see that $cof(\lambda) > \omega$ we will see that for every $\{g_n : n \in \omega\} \subseteq \mathcal{F}$ decreasing chain under $<^*$, it is possible to find a $g \in \mathcal{F}$ such that $g <^* g_n$, for every n.

Fix $f_\alpha \in A$. Note that, for all i, $g_i \in \mathcal{F}$, hence $f_\alpha <^* g_i$. Hence, for each $n \in \omega$, there is a k_α^n such that $\forall k \geq k_\alpha^n$

$$f_\alpha(k) < min\{g_i(k) : i \leq n\}.$$

Thus we can define a function $m_\alpha \in \omega^\omega$ such that for every n

$$m_\alpha(n) = k_\alpha^n.$$

Since $\mathfrak{b} > \omega_2$, there is $m >^* m_\alpha$, for all $\alpha \in \omega_2$. We can now define the function g as follows, for any $k \in [m(n), m(n+1))$

$$g(k) = min\{g_i(k) : i \leq n\}.$$

Given $f_\alpha \in A$, let n be sufficiently large such that $\forall k \geq n$, $m(k) > m_\alpha(k)$; then for all $l > n$, if $j \in [m(l), m(l+1))$, then $j \in [m_\alpha(l'), m_\alpha(l'+1))$, for some $l' \geq l$. So

$$f_\alpha(j) < min\{g_i(j) : i \leq l'\} \leq min\{g_i(j) : i \leq l\} = g(j)$$

Hence $g \in \mathcal{F}$ and $g <^* g_n$, for all n.

This conclude the proof of the theorem. $\square$

Since $|\omega^\omega| = 2^{\aleph_0}$ we have the following corollary, that is the first part of the proof that FA(σ*c,$\aleph_1$) decides the cardinality of the continuum.

Corollary 4.2.12. *OCA implies that* $2^{\aleph_0} \geq \aleph_2$.

Proof. OCA implies that $\mathfrak{b} = \aleph_2$. Hence $\aleph_2 = \mathfrak{b} \leq |\omega^\omega| = 2^{\aleph_0}$. $\square$

4.3 FA(σ*c,$\aleph_1$) implies that $2^{\aleph_1} = \aleph_2$

In this section we will see that, by means of to FA(σ*c,$\aleph_1$), it is possible to define a coding of the subsets of ω_1 in order type ω_2. The results in this section are due to Veličković. See [25].

4.3.1 Colorings and coding

Definition 4.3.1. *Let $\kappa > \omega_1$ be a regular cardinal and suppose to have a sequence of ω_1 colorings*

$$[\kappa]^2 = K_0^\xi \cup K_1^\xi$$

for $\xi \in \omega_1$.

We say that $\alpha \leq \kappa$, with $cof(\alpha) = \omega_1$, is good if for every $\xi \in \omega_1$, there is a club $C \subseteq \alpha$, which is either K_0^ξ-countable or K_1^ξ-countable.

Moreover define, for a good α,

$$A_\alpha = \{\xi \in \omega_1 : \exists C \subseteq \alpha, C \text{ club and } K_1\text{-countable}\}.$$

Note that, with an abuse of notation, we ask $\alpha \leq \kappa$, even if we required $\kappa > \omega_1$ regular and $cof(\alpha) = \omega_1$. This is because if we collapse κ to ω_1 with a σ-closed poset, then $cof(\kappa)$ will be ω_1 and so we could have κ good. The main reason for this definition is to show, in the next theorem, that it is possible to use the good α's to code the subsets of ω_1.

Note also that if $C \subseteq \kappa$ is uncountable it cannot be both K_0-countable and K_1-countable; else we would have an uncountable subset both 0-homogeneous and 1-homogeneous. So the above definition makes sense.

Definition 4.3.2. *Let $\kappa > \omega_1$ be regular and let $P = Coll(\aleph_1, \kappa)$ the collapse of κ to $\aleph_1$ by countable conditions. Let ω_1 colorings be given as in Definition 4.3.8. In $V[G]$ (a generic extension made via P) define the poset of finite K_i homogeneous subsets of $C \subseteq \kappa$, ordered by reverse inclusion*

$$\bar{Q}_i^\xi(C) = (\{F \in [C]^{<\omega} : [F]^2 \subseteq K_i^\xi\}, \supseteq),$$

where $i \in \{0, 1\}$ and $\xi \in \omega_1$.

$\bar{Q}_i^\xi(C)$ will be used to force a subset $X \subseteq C$, that is K_i homogeneous for the ξth coloring.

Definition 4.3.3. *For $i \in \{0,1\}$, $\xi \in \omega_1$ and $C \subseteq \kappa$, let*

$$Q_i^\xi(C) = \prod_{n\in\omega} \bar{Q}_i^\xi(C)$$

be the finite support product of ω copies of $\bar{Q}_i^\xi(C)$.

Note that, forcing with $Q_i^\xi(C)$, we make C K_i-countable for the ξth coloring. When $C = \kappa$, we write $\bar{Q}_i^\xi$ instead of $\bar{Q}_i^\xi(C)$ and Q_i^ξ instead of $Q_i^\xi(C)$.

Definition 4.3.4. *Let $Fn(\omega_1, 2)$ be the collection of all finite partial functions from ω_1 to 2.*

From now on we will fix a σ-closed poset P that collapses κ to ω_1, a P-generic filter G and a P-name $\dot{C}$ for a club set of κ of order type ω_1.

Consider the following statement

$$\Vdash_P \forall s \in Fn(\omega_1, 2) \prod_{\xi\in\mathtt{dom}(s)} Q_{s(\xi)}^\xi(\dot{C}) \text{ is c.c.c.} \tag{4.1}$$

Next theorem will use the strength of FA(σ∗c,$\aleph_1$) to show that it is possible to obtain, in $V[G]$, every $A \subseteq \omega_1$ as A_α, for some good α.

Theorem 4.3.5. ***(Veličković)*** *Assume FA(σ∗c,$\aleph_1$) and let, for $\xi \in \omega_1$,*

$$[\kappa]^2 = K_0^\xi \cup K_1^\xi$$

be a sequence of colorings such that 4.1 holds. Then, for every $A \subseteq \omega_1$ there is a good $\alpha \leq \kappa$ such that $A_\alpha = A$.

Proof. Fix $A \subseteq \omega_1$ and letχ_A be the characteristic function of A. We now work in $V[G]$. Let

$$Q_{\chi_A} = \prod_{\xi\in\omega_1} Q_{\chi_A(\xi)}^\xi(\dot{C}),$$

be the product with finite support of the $Q_{\chi_A(\xi)}^\xi(\dot{C})$.

Using 4.1, a standard application of Theorem 1.4.5 guarantees that Q_{χ_A} is c.c.c., provided that all its factors are c.c.c. Fix now in $V[G]$ a Q_{χ_A}-generic filter H. In $V[G * H]$ we have that $\dot{C}$ is $K_{\chi_A(\xi)}$ countable for the ξth partition. Thus there are $\dot{H}_n^\xi$, for $\xi \in \omega_1$ and $n \in \omega$, such that

$$\Vdash_{P*Q_{\chi_A}} [\dot{H}_n^\xi]^2 \subseteq K_{\chi_A(\xi)}^\xi \text{ and } \dot{C} = \bigcup_{n\in\omega} \dot{H}_n^\xi.$$

Now for every $\alpha, \xi \in \omega_1$, the following sets

$$D^\xi_\alpha = \{\langle p, q\rangle : \alpha \in \mathtt{dom}(p), p \Vdash p(\alpha) \in \dot{C}, \exists n\, q \Vdash p(\alpha) \in \dot{H}^\xi_n\}$$

are dense. If we choose a filter $G * H$ $\{D^\xi_\alpha : \alpha, \xi \in \omega_1\}$-generic, then letting $\delta = \{p(\xi) : \exists q \in H\, \langle p, q\rangle \in G * H,\, \xi \in \omega_1\}$, $C = \{p(\xi) : \exists q \in H\, \langle p, q\rangle \in G * H,\, \xi \in \omega_1\}$, $H^\xi_n = \{p(\xi) : \exists q \in H\, \langle p, q\rangle \in G * H,\, \langle p, q\rangle \vDash p(\xi) \in \dot{H}^\xi_n,\, \xi \in \omega_1\}$ we have that $\delta < \kappa$ and has cofinality ω_1, C is a club in δ such that $\forall \gamma \in C$ and $\forall \xi \in \omega_1$ there is an n such that $\exists \langle p, q\rangle \in G * H$ such that $\langle p, q\rangle \vDash \gamma \in \dot{H}^\xi_n$, hence $\gamma \in H^\xi_n$.

Note that $|\{\xi \in \omega_1 : \chi_A(\xi) = 1\}| = |A| = \aleph_1$ and

$$\xi \in A_\alpha \iff \exists C \subseteq \alpha, C \text{ club and } K^\xi_{\chi_A(\xi)}\text{-countable and } \chi_A(\xi) = 1 \iff \xi \in A.$$

□

4.3.2 The oscillation map

We now present a sequence of ω_1 colorings that have the property expressed by 4.1 and that will give us an upper bound to the cardinality of the continuum. Without loss of generality we can work on the space of all the increasing function from ω to ω that we will indicate as (ω^ω). From now on we fix an unbounded family $\mathcal{F}$ in (ω^ω).

Definition 4.3.6. *Let $\mathcal{F} \subseteq (\omega^\omega)$ be a family totally ordered by eventual domination, in order type $\mathfrak{b}$. For $k \in \omega$, we say that $\mathcal{X} \subseteq [\mathcal{F}]^k$ is unbounded if $\forall f \in \mathcal{F}\, \exists A \in \mathcal{X}$ such that $f <^* g$ for every $g \in A$.*

The following theorem is crucial and we state it without proof. It is essentially due to Todorčević; see [21].

Theorem 4.3.7. ***(Todorčević)*** *There is a function $\sigma : (\omega^\omega)^2 \to \omega$ such that for any unbounded family $\mathcal{X} \subseteq [(\omega^\omega)]^k$ and any function $u : k \times k \to \omega$, there are $a, b \in \mathcal{X}$ such that*

$$\forall i, j < k\ \sigma(a_i, b_j) = u(i, j)$$

We can now define a sequence of coloring for which the Property 4.1 will hold.

Definition 4.3.8. *Fix an increasing enumeration of $\{f_\alpha : \alpha \in \mathfrak{b}\}$ of $\mathcal{F}$; then for every $\xi \in \omega_1$ we define the following coloring.*

For $\alpha < \beta < \mathfrak{b}$

$$\{\alpha, \beta\} \in K^\xi_0 \iff \sigma(f_{\omega_1 \cdot \alpha + \xi}, f_{\omega_1 \cdot \beta + \xi}) \text{ is even.}$$

Not that $\mathcal{F}_\xi = \{f_{\omega_1 \cdot \alpha + \xi} : \alpha \in \mathfrak{b}\}$ is unbounded in $\mathcal{F}$ and $\mathcal{F}_\xi$ form a disjoint family; i.e. $\mathcal{F}_\xi \cap \mathcal{F}_\gamma = \varnothing$, for $\xi, \gamma \in \omega_1$.

Lemma 4.3.9. *Given a sequence of colorings as in Definition 4.3.8, we have the following property.*

$$\forall s \in Fn(\omega_1, 2) \prod_{\xi \in dom(s)} Q^\xi_{s(\xi)} \text{ is } \mathfrak{b}\text{-c.c.}$$

Proof. We show that the product of any two elements is $\mathfrak{b}$-c.c (i.e. if $|\mathtt{dom}(s)| = 2$). This proof can be easily generalised to the case of any finite product. Thus fix a $Q^\xi_{s(\xi)}$ and $Q^\gamma_{s(\gamma)}$. By definition $Q^\xi_{s(\xi)} = \prod_{n\in\omega} \bar{Q}^\xi_{s(\xi)}$ and $Q^\gamma_{s(\gamma)} = \prod_{n\in\omega} \bar{Q}^\gamma_{s(\gamma)}$, so if we show that $\bar{Q}^\xi_{s(\xi)} \times \bar{Q}^\gamma_{s(\gamma)}$ is $\mathfrak{b}$-c.c. then, modulo the generalisation to the case of any finite product, the Lemma will follow.

Let $\mathcal{A} = \{p_\alpha \in \bar{Q}^\xi_{s(\xi)} \times \bar{Q}^\gamma_{s(\gamma)} : \alpha \in \mathfrak{b}\}$ be a family of incompatible conditions. Note that $p_\alpha \in \bar{Q}^\xi_{s(\xi)} \times \bar{Q}^\gamma_{s(\gamma)}$ iff $p_\alpha = (X_\alpha, Y_\alpha)$, where $X_\alpha, Y_\alpha \in [\kappa]^{<\omega}$, X_α is homogeneous for $K^\xi_{s(\xi)}$ and Y_α is homogeneous for $K^\gamma_{s(\gamma)}$. Assume $K^\xi_{s(\xi)} = K^\xi_0$ and $K^\gamma_{s(\gamma)} = K^\gamma_1$. We claim that is possible to find two compatible conditions in $\mathcal{A}$.

Since $\mathfrak{b}$ is uncountable, we can assume that, for all $\alpha \in \mathfrak{b}$, $|X_\alpha| = n$ and $|Y_\alpha| = m$, for fixed $n, m \in \omega$. Moreover, by the Δ-system Lemma there are roots r_0, r_1 such that $X_\alpha \cap X_\beta = r_0$ and $Y_\alpha \cap Y_\beta = r_1$, for $\alpha, \beta \in \mathfrak{b}$. Moreover, by the Δ-system Lemma we can refine A so that $max(X_\alpha \cup Y_\alpha) < min(X_\beta \cup Y_\beta)$, for $\alpha < \beta$.

Define

$$\mathcal{B} = \{X_\alpha \setminus r_0 : \alpha \in \mathfrak{b}\}$$

and

$$\mathcal{C} = \{Y_\alpha \setminus r_1 : \alpha \in \mathfrak{b}\}.$$

We now work with $\mathcal{B} \times \mathcal{C}$, because (r_0, r_1) is not going to raise problems in seeking two compatible conditions. Let $|X_\alpha \setminus r_0| = k$ and $|Y_\alpha \setminus r_1| = l$, for every α, then $X_\alpha \setminus r_0 = \{\delta_0, \dots, \delta_{k-1}\}$ and $Y_\alpha \setminus r_1 = \{\beta_0, \dots, \beta_{l-1}\}$.

We can now define $\mathcal{X} \subseteq [\mathcal{F}]^{k+l}$ as follows:

$Z = \{f_{\omega_1 \cdot \delta_0 + \xi}, \dots, f_{\omega_1 \cdot \delta_{k-1} + \xi}, f_{\omega_1 \cdot \beta_0 + \gamma}, \dots, f_{\omega_1 \cdot \beta_{l-1} + \gamma}\} \in \mathcal{X} \iff \exists \alpha \ \ X_\alpha \setminus r_0 = \{\delta_0, \dots, \delta_{k-1}\} \in \mathcal{B}$ and $Y_\alpha \setminus r_1 = \{\beta_0, \dots, \beta_{l-1}\} \in \mathcal{C}$.

Define a label function $g : |X_\alpha \setminus r_0| + |Y_\alpha \setminus r_1| = k + l \to \kappa \times 2$

$$g(n) = \begin{cases} (\eta_i, 0), & \text{if } \exists j \ \eta_i = \delta_j; \\ (\eta_i, 1), & \text{if } \exists j \ \eta_i = \beta_j. \end{cases}$$

where $\{\eta_0, \dots, \eta_{l+k-2}\} = X_\alpha \setminus r_0 \cup Y_\alpha \setminus r_1$.

Let now $u : (k+l) \times (k+l) \to \omega$ be the following function:

$$u(i,j) = \begin{cases} 0, & \text{if } g(i) = (x,0) \text{ and } g(j) = (x,0); \\ 1, & \text{if } g(i) = (x,1) \text{ and } g(j) = (x,1); \\ 2, & \text{if } g(i) = (x,1) \text{ and } g(j) = (x,0); \\ 3, & \text{if } g(i) = (x,0) \text{ and } g(j) = (x,1). \end{cases}$$

By Theorem 4.3.7 there are $Z = \{f_{\omega_1\cdot\mu_0+\xi}, \dots, f_{\omega_1\cdot\mu_{k+l-2}+\gamma}\} \in \mathcal{X}$ and $W = \{f_{\omega_1\cdot\nu_0+\xi}, \dots, f_{\omega_1\cdot\nu_{k+l-2}+\gamma}\} \in \mathcal{X}$ such that, for $\theta \in \{\xi, \gamma\}$,

$$\sigma(f_{\omega_1\cdot\mu_i+\theta}, f_{\omega_1\cdot\nu_j+\theta}) = u(i,j)$$

Hence there are $\alpha, \beta \in \mathfrak{b}$ such that $X_\alpha \setminus r_0 \cup Y_\alpha \setminus r_1 = Z$ and $X_\beta \setminus r_0 \cup Y_\beta \setminus r_1 = W$. Note that, by definition of $\mathcal{X}$, $\{\mu_i : i < k+l-1\} \in \mathcal{B}$ and $\{\nu_i : i < k+l-1\} \in \mathcal{C}$. So, by the definition of K_0^ξ and of K_1^γ, we can conclude that

$$\forall i,j < k \quad \{\mu_i, \nu_j\} \in K_0^\xi$$

and

$$\forall k \le i,j < k+l-1 \quad \{\mu_i, \nu_j\} \in K_1^\gamma$$

Thus $(X_\alpha \setminus r_0, Y_\alpha \setminus r_1) \cup (X_\beta \setminus r_0, Y_\beta \setminus r_1)^2$ is K_0^ξ homogeneous on the first k components and K_1^γ homogeneous on the last ones. So the same happens for $(X_\alpha, Y_\alpha) \cup (X_\beta, Y_\beta) = p_\alpha \cup p_\beta$. $\square$

Lemma 4.3.9 is true in V, hence when forcing with a σ-closed poset P that collapse $\mathfrak{b}$ to ω_1, in the generic extension we will have

$$\Vdash_P \forall s \in Fn(\omega_1, 2) \prod_{\xi \in \mathtt{dom}(s)} Q^\xi_{s(\xi)} \text{ is c.c.c.}$$

that is the property 4.1.

As a simple corollary we obtain the upper bound we promised.

Corollary 4.3.10. *Assuming FA(σ∗c,$\aleph_1$), we have that* $2^{\aleph_1} = \aleph_2$.

Proof. We managed to construct a sequence of ω_1 coloring on $\mathfrak{b}$, such that 4.1 holds, so that Theorem 4.3.5 applies. Hence we have an injective map from $\mathcal{P}(\omega_1)$ in $\mathfrak{b}$: the one that to each $A \subseteq \omega_1$ associates $min\{\alpha : A_\alpha = A\}$. Moreover FA($\sigma$∗c,$\aleph_1$) implies OCA which implies $\mathfrak{b} = \aleph_2$. Hence $2^{\aleph_1} = \aleph_2$. $\square$

[2] Where the union is to be taken component by component.

We thus have the following important corollary.

Corollary 4.3.11. *FA($\sigma{*}c$,$\aleph_1$) implies* $2^{\aleph_0} = \aleph_2$.

Proof. Under FA($\sigma{*}$c,$\aleph_1$) we have that

$$2^{\aleph_1} \geq 2^{\aleph_0} = |\omega^\omega| \geq \mathfrak{b} = \aleph_2 = 2^{\aleph_1}$$

hence $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. □

Note that in all this chapter we used poset of size $\leq 2^{\aleph_0}$, so we actually showed something stronger: FA($\Gamma, \aleph_1$) implies that $2^{\aleph_0} = \aleph_2$, where $\Gamma = \{P : P \text{ is a } \sigma{*}c \text{ poset and } |P| \leq 2^{\aleph_0}\}$. This weaker form of FA($\sigma{*}$c,$\aleph_1$) is equiconsistent with a weakly compact cardinal; this is a large cardinal assumption much weaker than the one used for PFA, that for example is consistent with $V = L$.

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