

USEFUL AXIOMS

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ABSTRACT. We give a brief survey of the interplay between forcing axioms and various other non-constructive principles widely used in many fields of abstract mathematics, such as the axiom of choice and Baire’s category theorem.

First of all we outline how, using basic partial order theory, it is possible to reformulate the axiom of choice, Baire’s category theorem, and many large cardinal axioms as specific instances of forcing axioms. We then address forcing axioms with a model-theoretic perspective and outline a deep analogy existing between the standard Loś Theorem for ultraproducts of first order structures and Shoenfield’s absoluteness for Σ_2^1 -properties. Finally we address the question of whether and to what extent forcing axioms can provide “complete” semantics for set theory. We argue that to a large extent this is possible for certain initial fragments of the universe of sets: The pioneering work of Woodin on generic absoluteness show that this is the case for the Chang model $L(\text{Ord}^\omega)$ (where all of mathematics formalizable in second order number theory can be developed) in the presence of large cardinals, and recent works by the author with Asperó and with Audrito show that this can also be the case for the Chang model $L(\text{Ord}^{\omega_1})$ (where one can develop most of mathematics formalizable in third order number theory) in the presence of large cardinals and maximal strengthenings of Martin’s maximum or of the proper forcing axiom. A major question we leave completely open is whether this situation is peculiar to these Chang models or can be lifted up also to $L(\text{Ord}^\kappa)$ for cardinals $\kappa > \omega_1$.

INTRODUCTION

Since its introduction by Cohen in 1963 forcing has been the key and the most effective tool for obtaining independence results in set theory. This method has found applications in set theory and in virtually all fields of pure mathematics: in the last forty years natural problems of group theory, functional analysis, operator algebras, general topology, and many other subjects were shown to be undecidable by means of forcing. Starting from the early seventies and during the eighties it became transparent that many of these consistency results could all be derived by a short list of set theoretic principles, which are known in the literature as forcing axioms. These axioms gave set theorists and mathematicians a very powerful tool to obtain independence results: for any given mathematical problem we are most likely able to compute its (possibly different) solutions in the constructible

Key words and phrases. MSC 2010: 03E25, 03E30, 03E40, 03E57, 03C20, 03C90.

universe L and in models of strong forcing axioms. These axioms settle basic problems in cardinal arithmetic like the size of the continuum and the singular cardinal problem (see among others the works of Foreman, Magidor, Shelah [10], Veličković [28], Todorćević [25], Moore [17], Caicedo and Veličković [5], and the author [29]), as well as combinatorially complicated ones like the basis problem for uncountable linear orders (see Moore’s result [18] which extends previous work of Baumgartner [4], Shelah [23], Todorćević [24], and others). Interesting problems originating from other fields of mathematics and apparently unrelated to set theory have also been settled by appealing to forcing axioms, as is the case (to cite two of the most prominent examples) for Shelah’s results [22] on Whitehead’s problem in group theory and Farah’s result [8] on the non-existence of outer automorphisms of the Calkin algebra in operator algebra. Forcing axioms assert that for a large class of compact topological spaces X Baire’s category theorem can be strengthened to the statement that any family of \aleph_1 -many dense open subsets of X has non empty intersection. In light of the success these axioms have met in solving problems a convinced platonist may start to argue that these principles may actually give a “complete” theory of a suitable fragment of the universe of sets. However, it is not clear how one could formulate such a result. The aim of this paper is to explain in which sense we can show that forcing axioms can give such a “complete” theory and why they are so “useful”.

Section 1 starts by showing that two basic non-constructive principles which play a crucial role in the foundations of many mathematical theories, the axiom of choice and Baire’s category theorem, can both be formulated as specific instances of forcing axioms. In section 2 we also argue that many large cardinal axioms can be reformulated in the language of partial orders as specific instances of a more general kind of forcing axiom. Sections 3 and 4 show that Shoenfield’s absoluteness for Σ_2^1 -properties and Loś Theorem for ultraproducts of first order models are two sides of the same coin: recast in the language of boolean valued models, Shoenfield’s absoluteness shows that there is a more general notion of boolean ultrapower (of which the standard ultrapowers encompassed by Loś Theorem are just special cases) and that in the specific case in which one takes a boolean ultrapower of a compact, second countable space X , the natural embedding of X in its boolean ultrapower is at least Σ_2 -elementary. Section 5 embarks on a rough analysis of what is a maximal forcing axiom. We are led by two driving observations, one rooted in topological considerations and the other in model-theoretic arguments. First of all we outline how Woodin’s generic absoluteness results for $L(\text{Ord}^\omega)$ entail that in the presence of large cardinals the natural embeddings of a separable compact Hausdorff space X in its boolean ultrapowers are not only Σ_2 -elementary but fully elementary. We then present other recent results by the author, with Asperó [1] and with Audrito [2] which show that, in the presence of natural strengthenings of Martin’s maximum or of the proper forcing axiom, an exact analogue of Woodin’s generic

absoluteness result can be established also at the level of the Chang model $L(\text{Ord}^{\omega_1})$ and/or for the first order theory of H_{\aleph_2} . The main question left open is whether these generic absoluteness results are specific to the Chang models $L(\text{Ord}^{\omega_i})$ for $i = 0, 1$ or can be replicated also for other cardinals. The paper is meant to be accessible to a wide audience of mathematicians; specifically the first two sections do not require any special familiarity with logic or set theory other than some basic cardinal arithmetic. The third section requires a certain familiarity with first order logic and the basic model theoretic constructions of ultraproducts. The fourth and fifth sections, on the other hand, presume the reader has some familiarity with the forcing method.

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1. THE AXIOM OF CHOICE AND BAIRE'S CATEGORY THEOREM AS
FORCING AXIOMS

The axiom of choice AC and Baire's category theorem BCT are non-constructive principles which play a prominent role in the development of many fields of abstract mathematics. Standard formulations of the axiom of choice and of Baire's category theorem are the following:

Definition 1.1. $\text{AC} \equiv \prod_{i \in I} A_i$ is non-empty for all families of non empty sets $\{A_i : i \in I\}$, i.e. there is a choice function $f : I \rightarrow \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.

Theorem 1.2. $\text{BCT}_0 \equiv$ For all compact Hausdorff spaces (X, τ) and all countable families $\{A_n : n \in \mathbb{N}\}$ of dense open subsets of X , $\bigcap_{n \in \mathbb{N}} A_n$ is non-empty.

There are large numbers of equivalent formulations of the axiom of choice and it may come as a surprise that one of these is a natural generalization of Baire's category theorem and naturally leads to the notion of forcing axiom.

Definition 1.3. (P, \leq) is a *partial order* if \leq is a reflexive and transitive relation on P .

Notation 1.4. Given a partial order (P, \leq) ,

$$\uparrow A = \{p \in P : \exists q \in A : q \leq p\}$$

denotes the *upward closure* of A and similarly $\downarrow A$ will denote its *downward closure*.

- $A \subseteq P$ is *open* if it is a downward closed subset of P .
- The *order topology* τ_P on P is given by the downward closed subsets of P .
- D is *dense* if for all $p \in P$ there is some $q \in D$ refining p (q refines p if $q \leq p$),
- $G \subseteq P$ is a *filter* if it is upward closed and all $q, p \in G$ have a common refinement $r \in G$.
- p is *incompatible* with q ($p \perp q$) if no $r \in P$ refines both p and q .
- X is a *predense* subset of P if $\downarrow X$ is open dense in P .
- X is an *antichain* of P if it is composed of pairwise incompatible elements, and a maximal one if it is also predense.
- X is a *chain* of P if \leq is a total order on X .

The terminology for open and dense subsets of P comes from the observation that the collection τ_P of downward closed subsets of P is a topology on the space of points P (though in general not a Hausdorff one), whose dense sets are exactly those satisfying the above property. Notice also that the downward closure of a dense set is open dense in this topology.

A simple proof of the Baire Category Theorem is given by a basic enumeration argument (which however needs some amount of the axiom of choice to be carried):

Lemma 1.5. $\text{BCT}_1 \equiv$ Let (P, \leq) be a partial order and $\{D_n : n \in \mathbb{N}\}$ be a family of predense subsets of P . Then there is a filter $G \subseteq P$ meeting all the sets D_n .

Proof. Build by induction a decreasing chain $\{p_n : n \in \mathbb{N}\}$ with $p_n \in \downarrow D_n$ and $p_{n+1} \leq p_n$ for all n . Let $G = \uparrow\{p_n : n \in \mathbb{N}\}$. Then G meets all the D_n . \square

Baire's category theorem can be proved from the above Lemma (without any use of the axiom of choice) as follows:

Proof of BCT_0 from BCT_1 . Given a compact Hausdorff space (X, τ) and a family of dense open sets $\{D_n : n \in \mathbb{N}\}$ of X , consider the partial order $(\tau \setminus \{\emptyset\}, \subseteq)$ and the family $E_n = \{A \in \tau : \text{Cl}(A) \subseteq D_n\}$. Then it is easily checked that each E_n is dense open in the order topology induced by the partial order $(\tau \setminus \{\emptyset\}, \subseteq)$. By Lemma 1.5, we can find a filter $G \subseteq \tau \setminus \{\emptyset\}$ meeting all the sets E_n . This gives that for all $A_1, \dots, A_n \in G$

$$\text{Cl}(A_1) \cap \dots \cap \text{Cl}(A_n) \supseteq A_1 \cap \dots \cap A_n \supseteq B \neq \emptyset$$

for some $B \in G$ (where $\text{Cl}(A)$ is the closure of $A \subseteq X$ in the topology τ .) By the compactness of (X, τ) ,

$$\bigcap \{\text{Cl}(A) : A \in G\} \neq \emptyset.$$

Any point in this intersection belongs to the intersection of all the open sets D_n . \square

Notice the interplay between the order topology on the partial order $(\tau \setminus \{\emptyset\}, \subseteq)$ and the compact topology τ on X . Modulo the prime ideal theorem (a weak form of the axiom of choice), BCT_1 can also be proved from BCT_0 .

It is less well-known that the axiom of choice has also an equivalent formulation as the existence of filters on posets meeting sufficiently many dense sets. In order to proceed further, we need to introduce the standard notion of forcing axiom.

Definition 1.6. Let κ be a cardinal and (P, \leq) be a partial order.

$\text{FA}_\kappa(P) \equiv$ For all families $\{D_\alpha : \alpha < \kappa\}$ of predense subsets of P , there is a filter G on P meeting all these predense sets.

Given a class Γ of partial orders $\text{FA}_\kappa(\Gamma)$ holds if $\text{FA}_\kappa(P)$ holds for all $P \in \Gamma$.

Definition 1.7. Let λ be a cardinal. A partial order (P, \leq) is $< \lambda$ -closed if every decreasing chain $\{P_\alpha : \alpha < \gamma\}$ indexed by some $\gamma < \lambda$ has a lower bound in P .

Γ_λ denotes the class of $< \lambda$ -closed posets. Ω_λ denotes the class of posets P for which $\text{FA}_\lambda(P)$ holds.

It is almost immediate to check that Γ_{\aleph_0} is the class of all posets, and that BCT_1 states that $\Omega_{\aleph_0} = \Gamma_{\aleph_0}$. The following formulation of the axiom of choice in terms of forcing axioms was handed to me by Todorćević, I'm

not aware of any published reference. In what follows, let ZF denote the standard first order axiomatization of set theory in the first order language $\{\in, =\}$ (excluding the axiom of choice) and ZFC denote ZF+ the first order formalization of the axiom of choice.

Theorem 1.8. *The axiom of choice AC is equivalent (over the theory ZF) to the assertion that $\text{FA}_\kappa(\Gamma_\kappa)$ holds for all regular cardinals κ .*

We sketch a proof of Theorem 1.8, the interested reader can find a full proof in [20, Chapter 3, Section 2] (see the following hyperlink: *Tesi-Parente*). First of all, it is convenient to prove 1.8 using a different equivalent formulation of the axiom of choice.

Definition 1.9. Let κ be an infinite cardinal. The *principle of dependent choices* DC_κ states the following:

For every non-empty set X and every function $F: X^{<\kappa} \rightarrow \mathcal{P}(X) \setminus \{\emptyset\}$, there exists $g: \kappa \rightarrow X$ such that $g(\alpha) \in F(g \upharpoonright \alpha)$ for all $\alpha < \kappa$.

Lemma 1.10. *AC is equivalent to $\forall \kappa \text{DC}_\kappa$ modulo ZF.*

The reader can find a proof in [20, Theorem 3.2.3]. We prove the Theorem assuming the Lemma:

Proof of Theorem 1.8. We prove by induction on κ that DC_κ is equivalent to $\text{FA}_\kappa(\Gamma_\kappa)$ over the theory $\text{ZF} + \forall \lambda < \kappa \text{DC}_\lambda$. We sketch the ideas for the case κ -regular¹:

Assume DC_κ ; we prove (in ZF) that $\text{FA}_\kappa(\Gamma_\kappa)$ holds. Let (P, \leq) be a $<\kappa$ -closed partially ordered set, and $\{D_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(P)$ a family of predense subsets of P .

Given a sequence $\langle p_\beta : \beta < \alpha \rangle$ call $\xi_{\vec{p}}$ the least ξ such that $\langle p_\beta : \xi \leq \beta < \alpha \rangle$ is a decreasing chain if such a ξ exists, and fix $\xi_{\vec{p}} = \alpha$ otherwise. Notice that when the length α of \vec{p} is successor then $\xi_{\vec{p}} < \alpha$.

We now define a function $F: P^{<\kappa} \rightarrow \mathcal{P}(P) \setminus \{\emptyset\}$ as follows: given $\alpha < \kappa$ and a sequence $\vec{p} \in P^{<\kappa}$,

$$F(\vec{p}) = \begin{cases} \{p_0\} & \text{if } \xi_{\vec{p}} = \alpha \\ \{d \in \downarrow D_\alpha : d \leq p_\beta \text{ for all } \xi_{\vec{p}} \leq \beta < \alpha\} & \text{otherwise.} \end{cases}$$

The latter set is non-empty since (P, \leq) is $<\kappa$ -closed, $\alpha < \kappa$, and D_α is predense. By DC_κ , we find $g: \kappa \rightarrow P$ such that $g(\alpha) \in F(g \upharpoonright \alpha)$ for all $\alpha < \kappa$. An easy induction shows that for all α the sequence $g \upharpoonright \alpha$ is decreasing, so $g(\alpha) \in \downarrow D_\alpha$ for all $\alpha < \kappa$. Then

$$G = \{p \in P : \text{there exists } \alpha < \kappa \text{ such that } g(\alpha) \leq p\}$$

is a filter on P , such that $G \cap D_\beta \neq \emptyset$ for all $\beta < \kappa$.

Conversely, assume $\text{FA}_\kappa(\Gamma_\kappa)$, we prove (in ZF) that DC_κ holds.

¹In this case the assumption $\forall \lambda < \kappa \text{DC}_\lambda$ is not needed, but all the relevant ideas in the proof of the equivalence are already present.

Let X be a non-empty set and $F: X^{<\kappa} \rightarrow \mathcal{P}(X) \setminus \{\emptyset\}$. Define the partially ordered set

$$P = \{s \in X^{<\kappa} : \text{for all } \alpha \in \text{dom}(s), s(\alpha) \in F(s \upharpoonright \alpha)\},$$

with $s \leq t$ if and only if $t \subseteq s$. Let $\lambda < \kappa$ and let $s_0 \geq s_1 \geq \dots \geq s_\alpha \geq \dots$, for $\alpha < \lambda$, be a chain in P . Then $\bigcup_{\alpha < \lambda} s_\alpha$ is clearly a lower bound for the chain. Since κ is regular, we have $\bigcup_{\alpha < \lambda} s_\alpha \in P$ and so P is $<\kappa$ -closed. For every $\alpha < \kappa$, define

$$D_\alpha = \{s \in P : \alpha \in \text{dom}(s)\},$$

and note that D_α is dense in P . Using $\text{FA}_\kappa(\Gamma_\kappa)$, there exists a filter $G \subset P$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Then $g = \bigcup G$ is a function $g: \kappa \rightarrow X$ such that $g(\alpha) \in F(g \upharpoonright \alpha)$ for all $\alpha < \kappa$. \square

2. LARGE CARDINALS AS FORCING AXIOMS

From now on, we focus on boolean algebras rather than posets.

2.1. A fast briefing on boolean algebras.

Definition 2.1. A *boolean algebra* \mathbf{B} is a boolean ring i.e. a ring in which every element is idempotent. Equivalently a boolean algebra is a complemented distributive lattice $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ (see [11]).

Notation 2.2. Given a boolean algebra $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$, the poset $(\mathbf{B}^+; \leq_{\mathbf{B}})$ is given by its non-zero elements, with order relation given by $b \leq_{\mathbf{B}} q$ iff $b \wedge q = b$ iff $b \vee q = q$.

A boolean ring $(\mathbf{B}, +, \cdot, 0, 1)$ has a natural structure of complemented distributive lattice $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$, for which the sum on the boolean ring becomes the operation Δ of symmetric difference ($a \Delta b = a \vee b \wedge \neg(a \wedge b)$) on the complemented distributive lattice, and the multiplication of the ring the operation \wedge .

We refer to filters, antichains, dense sets, predense sets, open sets on \mathbf{B} , meaning that these notions are declined for the corresponding partial order $(\mathbf{B}^+; \leq_{\mathbf{B}})$.

We also recall the following:

- An *ideal* I on \mathbf{B} is a non-empty downward closed subset of \mathbf{B} with respect to $\leq_{\mathbf{B}}$ which is also closed under \vee (equivalently it is an ideal on the boolean ring $(\mathbf{B}, \Delta, \wedge, 0, 1)$). Its *dual filter* \check{I} is the set $\{\neg a : a \in I\}$. It is a filter on the poset $(\mathbf{B}^+; \leq_{\mathbf{B}})$ (equivalently I is an ideal in the boolean ring \mathbf{B}).
- An ideal I on \mathbf{B} is $< \delta$ -*complete* (δ -complete) if all the subsets of I of size less than δ (of size δ) have an upper bound in I .
- A *maximal* ideal I is an ideal properly contained in \mathbf{B} and maximal with respect to this property (equivalently it is a prime ideal on the boolean ring $(\mathbf{B}, \Delta, \wedge, 0, 1)$). Its dual filter is an *ultrafilter*. An ideal I is maximal if and only if $a \in I$ or $\neg a \in I$ for all $a \in \mathbf{B}$.

- \mathbf{B} is $< \delta$ -complete (δ -complete) if all subsets of size less than δ (of size δ) have a supremum and an infimum.
- Given an ideal I on \mathbf{B} , \mathbf{B}/I is the quotient boolean algebra given by equivalence classes $[a]_I$ obtained by $a =_I b$ iff $a \Delta b \in I$.
- \mathbf{B}/I is $< \kappa$ -complete if I and \mathbf{B} are both $< \kappa$ -complete.
- \mathbf{B} is *atomless* if there are no minimal elements in the partial order $(\mathbf{B}^+; \leq_{\mathbf{B}})$.
- \mathbf{B} is *atomic* if the set of minimal elements in the partial order $(\mathbf{B}^+; \leq_{\mathbf{B}})$ is open dense.

Usually we insist in the formulation of forcing axioms on the requirement that for certain partial orders P any family of predense subsets of P of some fixed size κ can be met in a single filter. In order to obtain a greater variety of forcing axioms, we need to consider a much richer variety of properties which characterizes the families of predense sets of P which can be met in a single filter. Using boolean algebras, by considering partial orders of the form $(\mathbf{B}^+; \leq_{\mathbf{B}})$ for some boolean algebra \mathbf{B} , we can formulate (using the algebraic structure of \mathbf{B}) a wide spectrum of properties each defining a distinct forcing axiom.

2.2. Measurable cardinals. A cardinal κ is measurable if and only if there is a uniform $< \kappa$ -complete ultrafilter on the boolean algebra $\mathcal{P}(\kappa)$. The requirement that G is uniform amounts to saying that G is disjoint from the ideal I on the boolean algebra $(\mathcal{P}(\kappa), \cap, \cup, \emptyset, \kappa)$ given by the bounded subsets of $\mathcal{P}(\kappa)$. This means that we are actually looking for an ultrafilter G on the boolean algebra $\mathcal{P}(\kappa)/I$. This is an atomless boolean algebra which is $< \kappa$ -complete. The requirement that G is $< \kappa$ -complete amounts to asking that G selects a unique member of any partition of κ in $< \kappa$ -many pieces, moreover any maximal antichain $\{[A_i]_I : i < \gamma\}$ in the boolean algebra $\mathcal{P}(\kappa)/I$ of size γ less than κ is induced by a partition of κ in γ -many pairwise disjoint pieces.

All in all, we have the following characterization of measurability:

Definition 2.3. κ is a *measurable* cardinal if and only if there is an ultrafilter G on $\mathcal{P}(\kappa)/I$ (where I is the ideal of bounded subsets of κ) which meets all the maximal antichains on $\mathcal{P}(\kappa)/I$ of size less than κ .

In particular the measurability of κ holds if and only if $(\mathcal{P}(\kappa)/I)^+$ satisfies a certain forcing axiom stating that certain collections of predense subsets of $(\mathcal{P}(\kappa)/I)^+$ can be simultaneously met in a filter.

We are led to the following definitions:

Definition 2.4. Let (P, \leq) be a partial order and \mathcal{D} be a family of non-empty subsets of P . A filter G on P is \mathcal{D} -generic if $G \cap D$ is non-empty for all $D \in \mathcal{D}$.

Let $\phi(x, y)$ be a property and (P, \leq) a partial order. $\text{FA}_{\phi}(P)$ holds if for any family \mathcal{D} of predense subsets of P such that $\phi(P, \mathcal{D})$ holds there is some \mathcal{D} -generic filter G on P .

For instance, $\text{FA}_\kappa(P)$ says that $\text{FA}_\phi(P)$ holds for $\phi(x, y)$ being the property:

“ x is a partial order and y is a family of predense subsets of x of size κ ”

The measurability of κ amounts to saying that $\text{FA}_\phi(P)$ holds with $\phi(x, y)$ being the property

“ x is the partial order $(\mathcal{P}(\kappa)/I)^+$ and y is the (unique) family of predense subsets of x consisting of maximal antichains of $(\mathcal{P}(\kappa)/I)^+$ of size less than κ ”

We do not want to expand further on this topic but many other large cardinal properties of a cardinal κ can be formulated as axioms of the form $\text{FA}_\phi(P)$ for some property ϕ (for example this is the case for supercompactness, hugeness, almost hugeness, strongness, superstrongness, etc....).

In these first two sections we have already shown that the language of partial orders can accomodate three completely distinct and apparently unrelated families of non-constructive principles which are essential tools in the development of many mathematical theories (as is the case for the axiom of choice and of Baire’s category theorem) and of crucial importance in the current developments of set theory (as is the case for large cardinal axioms).

3. BOOLEAN VALUED MODELS, ŁOŚ THEOREM, AND GENERIC ABSOLUTENESS

We address here the correlation between forcing axioms and generic absoluteness results. We show how Shoenfield’s absoluteness for Σ_2^1 -properties and Łoś Theorem are two sides of the same coin: more precisely they are distinct specific cases of a unique general theorem which follows from AC.

After recalling the basic formulation of Łoś Theorem for ultraproducts, we introduce boolean valued models, and we argue that Łoś Theorem for ultraproducts is the specific instance for complete atomic boolean algebras of a more general theorem which applies to a much larger class of boolean valued models. Then we introduce the concept of boolean ultrapower of a first order structure on a Polish space X endowed with Borel predicates R_1, \dots, R_n , and show that Shoenfield’s absoluteness for Σ_2^1 -properties amounts to saying that the boolean ultrapower of $\langle X, R_1, \dots, R_n \rangle$ by any complete boolean algebra is a Σ_2 -elementary superstructure of $\langle X, R_1, \dots, R_n \rangle$.

3.1. Łoś Theorem.

Theorem 3.1. *Let $\{\mathfrak{M}_l : l \in L\}$ be models in a given first order signature*

$$\mathcal{L} = \{R_i : i \in I, f_j : j \in J, c_k : k \in K\},$$

i.e. each $\mathfrak{M}_l = (M_l, R_i^l : i \in I, f_j^l : j \in J, c_k^l : k \in K)$. Let G be an ultrafilter on L (i.e. its dual is a prime ideal on the boolean algebra $\mathcal{P}(L)$). Let

$$[f]_G = \left\{ g \in \prod_{l \in L} M_l : \{l \in L : g(l) = f(l)\} \in G \right\}$$

for each $f \in \prod_{l \in L} M_l$, and set

$$\prod_{l \in L} M_l / G = \left\{ [f]_G : f \in \prod_{l \in L} M_l \right\}.$$

For each $i \in I$ let $\bar{R}_i([f_1]_G, \dots, [f_n]_G)$ hold on $\prod_{l \in L} M_l / G$ if and only if

$$\left\{ l \in L : \mathfrak{M}_l \models R_i^l(f_1(l), \dots, f_n(l)) \right\} \in G.$$

Similarly interpret $\bar{f}_j : \prod_{l \in L} (M_l / G)^n \rightarrow \prod_{l \in L} M_l / G$ and $\bar{c}_k \in \prod_{l \in L} M_l^n / G$ for each $j \in J$ and $k \in K$.

Then:

(1) For all formulae $\phi(x_1, \dots, x_n)$ in the signature \mathcal{L}

$$\left(\prod_{l \in L} M_l / G, \bar{R}_i : i \in I, \bar{f}_j : j \in J, \bar{c}_k : k \in K \right) \models \phi([f_1]_G, \dots, [f_n]_G)$$

if and only if

$$\left\{ l \in L : \mathfrak{M}_l \models \phi(f_1(l), \dots, f_n(l)) \right\} \in G.$$

(2) Moreover if $\mathfrak{M}_l = \mathfrak{M}$ for all $l \in L$ (i.e. $\prod_{l \in L} M_l / G$ is the ultrapower of M by G), we have that the map $m \mapsto [c_m]_G$ (where $c_m : L \rightarrow M$ is constant with value m) defines an elementary embedding.

It is a useful exercise to check that the axiom of choice is essentially used in the induction step for existential quantifiers in the proof of Łoś Theorem. Moreover Łoś Theorem is clearly a strengthening of the axiom of choice, for the very existence of an element in $\prod_{l \in L} M_l / G$ implies that $\prod_{l \in L} M_l$ is non-empty.

One peculiarity of the above formulation of Łoś theorem is that it applies just to ultrafilters on $\mathcal{P}(X)$. We aim to find a “most” general formulation of this Theorem, which makes sense also for other kind of “ultraproducts” and of ultrafilters on boolean algebras other than $\mathcal{P}(X)$. This forces us to introduce the boolean valued semantics.

3.2. A fast briefing on complete boolean algebras and Stone duality. Recall that for a given topological space (X, τ) the regular open sets are those $A \in \tau$ such that $A = \text{Reg}(A) = \text{Int}(\text{Cl}(A))$ (A coincides with the interior of its closure) and that $\text{RO}(X, \tau)$ is the complete boolean algebra whose elements are regular open sets and whose operations are given by $A \wedge B = A \cap B$, $\bigvee_{i \in I} A_i = \text{Reg}(\bigcup_{i \in I} A_i)$, $\neg A = X \setminus \text{Cl}(A)$.

For any partial order (P, \leq) the map $i : P \rightarrow \text{RO}(P, \tau_P)$ given by $p \mapsto \text{Reg}(\downarrow \{p\})$ is order and incompatibility preserving and embeds P as a dense subset of the non-empty regular open sets in $\text{RO}(P, \tau_P)$.

Recall also that the Stone space $\text{St}(\mathbf{B})$ of a boolean algebra \mathbf{B} is given by its ultrafilters G and it is endowed with a compact topology $\tau_{\mathbf{B}}$ whose clopen sets are the sets $N_b = \{G \in \text{St}(\mathbf{B}) : b \in G\}$ so that the map $b \mapsto N_b$ defines a natural isomorphism of \mathbf{B} with the boolean algebra $\text{CLOP}(\text{St}(\mathbf{B}))$ of clopen subset of $\text{St}(\mathbf{B})$. Moreover a boolean algebra \mathbf{B} is complete if and only if $\text{CLOP}(\text{St}(\mathbf{B})) = \text{RO}(\text{St}(\mathbf{B}), \tau_{\mathbf{B}})$. Spaces X satisfying the property that their regular open sets are closed are extremally (or extremely) disconnected.

We refer the reader to [11] or [33, Chapter 1] (available at the following hyperlink: *Notes on Forcing*) for a detailed account of these matters.

3.3. Boolean valued models. In a first order model, a formula can be interpreted as true or false. Given a complete boolean algebra \mathbf{B} , \mathbf{B} -boolean valued models generalize Tarski semantics associating to each formula a value in \mathbf{B} , so that propositions are not only true and false anymore (that is, only associated to $1_{\mathbf{B}}$ and $0_{\mathbf{B}}$ respectively), but take also other “intermediate values” of truth. A complete account of the theory of these boolean valued models can be found in [21]. We now recall some basic facts; an expanded version of the material of this section can be found in [26] (see also the following hyperlink: *Tesi-Vaccaro*) and in [33, Chapter 3]. In order to avoid unnecessary technicalities, we define boolean valued semantics just for relational first order languages (i.e. signatures with no function symbols).

Definition 3.2. Given a complete boolean algebra \mathbf{B} and a first order relational language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{c_j : j \in J\}$$

a \mathbf{B} -boolean valued model (or \mathbf{B} -valued model) \mathcal{M} in the language \mathcal{L} is a tuple

$$\langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, c_j^{\mathcal{M}} : j \in J \rangle$$

where:

- (1) M is a non-empty set, called *domain* of the \mathbf{B} -boolean valued model, whose elements are called \mathbf{B} -names;
- (2) $=^{\mathcal{M}}$ is the *boolean value* of the equality:

$$\begin{aligned} =^{\mathcal{M}} : M^2 &\rightarrow \mathbf{B} \\ (\tau, \sigma) &\mapsto \llbracket \tau = \sigma \rrbracket_{\mathbf{B}}^{\mathcal{M}} \end{aligned}$$

- (3) The forcing relation $R_i^{\mathcal{M}}$ is the *boolean interpretation* of the n -ary relation symbol R_i :

$$\begin{aligned} R_i^{\mathcal{M}} : M^n &\rightarrow \mathbf{B} \\ (\tau_1, \dots, \tau_n) &\mapsto \llbracket R_i(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}}^{\mathcal{M}} \end{aligned}$$

- (4) $c_j^{\mathcal{M}} \in M$ is the *boolean interpretation* of the constant symbol c_j .

We require that the following conditions hold:

- for $\tau, \sigma, \chi \in M$,
 - (1) $\llbracket \tau = \tau \rrbracket_{\mathbf{B}}^{\mathcal{M}} = 1_{\mathbf{B}}$;
 - (2) $\llbracket \tau = \sigma \rrbracket_{\mathbf{B}}^{\mathcal{M}} = \llbracket \sigma = \tau \rrbracket_{\mathbf{B}}^{\mathcal{M}}$;
 - (3) $\llbracket \tau = \sigma \rrbracket_{\mathbf{B}}^{\mathcal{M}} \wedge \llbracket \sigma = \chi \rrbracket_{\mathbf{B}}^{\mathcal{M}} \leq \llbracket \tau = \chi \rrbracket_{\mathbf{B}}^{\mathcal{M}}$;
- for $R \in \mathcal{L}$ with arity n , and $(\tau_1, \dots, \tau_n), (\sigma_1, \dots, \sigma_n) \in M^n$,
 - (1) $(\bigwedge_{h \in \{1, \dots, n\}} \llbracket \tau_h = \sigma_h \rrbracket_{\mathbf{B}}^{\mathcal{M}}) \wedge \llbracket R(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}}^{\mathcal{M}} \leq \llbracket R(\sigma_1, \dots, \sigma_n) \rrbracket_{\mathbf{B}}^{\mathcal{M}}$;

Given a \mathbf{B} -model $\langle M, =^{\mathcal{M}} \rangle$ for equality, a forcing relation R on M is a map $R : M^n \rightarrow \mathbf{B}$ satisfying the above condition for boolean predicates.

The boolean valued semantics is defined as follows:

Definition 3.3. Let

$$\langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, c_j^{\mathcal{M}} : j \in J \rangle$$

be a \mathbf{B} -valued model in a relational language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{c_j : j \in J\},$$

ϕ a \mathcal{L} -formula whose free variables are in $\{x_1, \dots, x_n\}$, and ν a valuation of the free variables in \mathcal{M} whose domain contains $\{x_1, \dots, x_n\}$. Since \mathcal{L} is a relational language, the terms of a formula are either free variables or constants, let us define $\nu(c_j) = c_j^{\mathcal{M}}$ for c_j a constant of \mathcal{L} . We denote with $\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu}$ the *boolean value* of ϕ with the assignment ν .

Given a formula ϕ , we define recursively $\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu}$ as follows:

- for atomic formulae this is done letting

$$\llbracket t = s \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} = \llbracket \nu(t) = \nu(s) \rrbracket_{\mathbf{B}}^{\mathcal{M}},$$

and

$$\llbracket R(t_1, \dots, t_n) \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} = \llbracket R(\nu(t_1), \dots, \nu(t_n)) \rrbracket_{\mathbf{B}}^{\mathcal{M}}$$

- if $\phi \equiv \neg\psi$, then

$$\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} = \neg \llbracket \psi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu};$$

- if $\phi \equiv \psi \wedge \theta$, then

$$\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} = \llbracket \psi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} \wedge \llbracket \theta \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu};$$

- if $\phi \equiv \exists y \psi(y)$, then

$$\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} = \bigvee_{\tau \in M} \llbracket \psi(y/\tau) \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu};$$

If no confusion can arise, we omit the superscripts \mathcal{M}, ν and the subscript \mathbf{B} , and we simply denote the boolean value of a formula ϕ with parameters in \mathcal{M} by $\llbracket \phi \rrbracket$.

With elementary arguments it is possible prove the Soundness Theorem for boolean valued models.

Theorem 3.4 (Soundness Theorem). *Assume \mathcal{L} is a relational language and ϕ is a \mathcal{L} -formula which is syntactically provable by a \mathcal{L} -theory T . Assume each formula in T has boolean value at least $b \in \mathbf{B}$ in a \mathbf{B} -valued model \mathcal{M} with valuation ν . Then $\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} \geq b$ as well.*

On the other hand the completeness theorem for the boolean valued semantics with respect to first order calculi is a triviality, given that 2 is complete boolean algebra.

We get a standard Tarski model from a \mathbf{B} -valued model by passing to a quotient by an ultrafilter $G \subseteq \mathbf{B}$.

Definition 3.5. Take \mathbf{B} a complete boolean algebra, \mathcal{M} a \mathbf{B} -valued model in the language \mathcal{L} , and G an ultrafilter over \mathbf{B} . Consider the following equivalence relation on M :

$$\tau \equiv_G \sigma \Leftrightarrow \llbracket \tau = \sigma \rrbracket \in G$$

The first order model $\mathcal{M}/G = \langle M/G, R_i^{\mathcal{M}/G} : i \in I, c_j^{\mathcal{M}/G} : j \in J \rangle$ is defined letting:

- For any n -ary relation symbol R in \mathcal{L}

$$R^{\mathcal{M}/G} = \{([\tau_1]_G, \dots, [\tau_n]_G) \in (M/G)^n : \llbracket R(\tau_1, \dots, \tau_n) \rrbracket \in G\}.$$

- For any constant symbol c in \mathcal{L}

$$c^{\mathcal{M}/G} = [c^{\mathcal{M}}]_G.$$

If we require \mathcal{M} to satisfy a key additional condition, we get an easy way to control the truth value of formulas in \mathcal{M}/G .

Definition 3.6. A \mathbf{B} -valued model \mathcal{M} for the language \mathcal{L} is *full* if for every \mathcal{L} -formula $\phi(x, \bar{y})$ and every $\bar{\tau} \in M^{|\bar{y}|}$ there is a $\sigma \in M$ such that

$$\llbracket \exists x \phi(x, \bar{\tau}) \rrbracket = \llbracket \phi(\sigma, \bar{\tau}) \rrbracket$$

Theorem 3.7 (Łoś's Theorem for Boolean Valued Models). *Assume \mathcal{M} is a full \mathbf{B} -valued model for the relational language \mathcal{L} . Then for every formula $\phi(x_1, \dots, x_n)$ in \mathcal{L} and $(\tau_1, \dots, \tau_n) \in M^n$:*

- (1) For all ultrafilters G over \mathbf{B}

$$\mathcal{M}/G \models \phi([\tau_1]_G, \dots, [\tau_n]_G) \text{ if and only if } \llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket \in G.$$

- (2) For all $a \in \mathbf{B}$ the following are equivalent:

- (a) $\llbracket \phi(f_1, \dots, f_n) \rrbracket \geq a$,

- (b) $\mathcal{M}/G \models \phi([\tau_1]_G, \dots, [\tau_n]_G)$ for all $G \in N_a$,

- (c) $\mathcal{M}/G \models \phi([\tau_1]_G, \dots, [\tau_n]_G)$ for densely many $G \in N_a$.

A key observation to relate standard ultraproducts to boolean valued models is the following:

Fact 3.8. *Let $(M_x : x \in X)$ be a family of Tarski-models in the first order relational language \mathcal{L} . Then $N = \prod_{x \in X} M_x$ is a full $\mathcal{P}(X)$ -model letting for each n -ary relation symbol $R \in \mathcal{L}$, $\llbracket R(f_1, \dots, f_n) \rrbracket_{\mathcal{P}(X)} = \{x \in X : M_x \models R(f_1(x), \dots, f_n(x))\}$.*

Let G be any non-principal ultrafilter on X . Then, using the notation of the previous fact, N/G is the familiar ultraproduct of the family $(M_x : x \in X)$ by G , and the usual Łoś Theorem for ultraproducts of Tarski models is the specialization to the case of the full $\mathcal{P}(X)$ -valued model N of Theorem 3.7. Notice that in this special case, if the ultraproduct is an ultrapower of a model M , the embedding $a \mapsto [c_a]_G$ (where $c_a(x) = a$ for all $x \in X$ and $a \in M$) is elementary.

3.4. Boolean ultrapowers of compact Hausdorff spaces and Shoenfield's absoluteness. Take X a set with the discrete topology, and for any $a \in X$, let $G_a \in \text{St}(\mathcal{P}(X))$ denote the principal ultrafilter given by supersets of $\{a\}$. The map $a \mapsto G_a$ embeds X as an open, dense, discrete subspace of $\text{St}(\mathcal{P}(X))$. In particular for any topological space (Y, τ) , any function $f : X \rightarrow Y$ is continuous (since X has the discrete topology); moreover in case Y is compact Hausdorff it induces a unique continuous $\bar{f} : \text{St}(\mathcal{P}(X)) \rightarrow Y$ mapping $G \in \text{St}(\mathcal{P}(X))$ to the unique point in Y which is in the intersection of $\{\text{Cl}(A) : A \in \tau, f^{-1}[A] \in G\}$ (we are in the special situation in which $\text{St}(\mathcal{P}(X))$ is also the Stone-Čech compactification of X).

This gives that for any compact Hausdorff space (Y, τ) , the space $C(X, Y) = Y^X$ of (continuous) functions from X to Y is canonically isomorphic to the space $C(\text{St}(\mathcal{P}(X)), Y)$ of continuous functions from $\text{St}(\mathcal{P}(X))$ to Y .

What if we replace $\mathcal{P}(X)$ with an arbitrary (complete) boolean algebra? In view of the above remarks, it is reasonable to regard $C(\text{St}(\mathbf{B}), Y)$ is the \mathbf{B} -ultrapower of Y for any compact Hausdorff space Y , since this is exactly what occurs for the case $\mathbf{B} = \mathcal{P}(X)$.

Let us examine closely this situation in the case $Y = 2^\omega$ with the product topology. This will unfold the relation existing between the notion of boolean ultrapowers of 2^ω and Shoenfield's absoluteness.

Let us fix \mathbf{B} arbitrary (complete) boolean algebra, and set $M = C(\text{St}(\mathbf{B}), 2^\omega)$. Fix also R a Borel relation on $(2^\omega)^n$. The continuity of an n -tuple $f_1, \dots, f_n \in M$ implies that the set

$$\{G : R(f_1(G), \dots, f_n(G))\} = (f_1 \times \dots \times f_n)^{-1}[R]$$

has the Baire property in $\text{St}(\mathbf{B})$ (i.e. it has meager symmetric difference with a unique regular open set — see [13, Lemma 11.15, Def. 32.21]), where $f_1 \times \dots \times f_n(G) = (f_1(G), \dots, f_n(G))$. So we can define

$$\begin{aligned} R^M : M^n &\rightarrow \mathbf{B} \\ (f_1, \dots, f_n) &= \text{Reg}(\{G : R(f_1(G), \dots, f_n(G))\}). \end{aligned}$$

Also, since the diagonal is closed in $(2^\omega)^2$,

$$=^M (f, g) = \text{Reg}(\{G : f(G) = g(G)\})$$

is well defined.

It is not hard to check that, for any Borel relation R on $(2^\omega)^n$, the structure $(M, =^M, R^M)$ is a full \mathbf{B} -valued extension of $(2^\omega, =, R)$, where 2^ω is

copied inside M as the set of constant functions. It is also not hard to check that whenever G is an ultrafilter on $St(\mathbf{B})$, the map $i_G : 2^\omega \rightarrow M/G$ given by $x \mapsto [c_x]_G$ (the constant function with value x) defines an injective morphism of the 2-valued structure $(2^\omega, R)$ into the 2-valued structure $(M/G, R^M/G)$. Nonetheless it is not clear whether this morphism is an elementary map or not. This is the case for $\mathbf{B} = \mathcal{P}(X)$, since in this case we are analyzing the standard embedding of the first order structure $(2^\omega, R)$ in its ultrapowers induced by ultrafilters on $\mathcal{P}(X)$. What are the properties of this map if \mathbf{B} is some other complete boolean algebra?

We can relate the degree of elementarity of the map i_G with Shoenfield's absoluteness for Σ_2^1 -properties. This can be done if one is willing to accept as a black-box the identification of the \mathbf{B} -valued model $C(St(\mathbf{B}), 2^\omega)$ with the \mathbf{B} -valued model given by the family of \mathbf{B} -names for elements of 2^ω in $V^{\mathbf{B}}$ (which is the canonical \mathbf{B} -valued model for set theory); we will expand further on this identification in the next section. Modulo this identity, Shoenfield's absoluteness can be recast as a statement about boolean valued models. We choose to name Cohen's absoluteness the following statement, which gives (as we will see) an equivalent reformulation of Shoenfield's absoluteness. Its proof (as we will see in the next section) ultimately relies on Cohen's forcing theorem, hence the name.

Theorem 3.9 (Cohen's absoluteness). *Assume \mathbf{B} is a complete boolean algebra and $R \subseteq (2^\omega)^n$ is a Borel relation. Let $M = C(St(\mathbf{B}), 2^\omega)$ and $G \in St(\mathbf{B})$. Then*

$$(2^\omega, =, R) \prec_{\Sigma_2} (M/G, =^M /G, R^M/G).$$

4. GETTING COHEN'S ABSOLUTENESS FROM BAIRE'S CATEGORY THEOREM

Let us now show how Theorem 3.9 is once again a consequence of forcing axioms. To do so, we delve deeper into set theoretic techniques and assume the reader has some acquaintance with the forcing method. We give below a brief review sufficient for our aims.

4.1. Forcing. Let V denote the standard universe of sets and ZFC the standard first order axiomatization of set theory by the Zermelo-Frankel axioms. For any complete boolean algebra $\mathbf{B} \in V$ let

$$V^{\mathbf{B}} = \left\{ f : V^{\mathbf{B}} \rightarrow \mathbf{B} : f \in V \text{ is a function} \right\}$$

be the class of \mathbf{B} -names with boolean relations $\in^{\mathbf{B}}, \subseteq^{\mathbf{B}}, =^{\mathbf{B}} : (V^{\mathbf{B}})^2 \rightarrow \mathbf{B}$ given by:

(1)

$$\in^{\mathbf{B}}(\tau, \sigma) = \llbracket \tau \in \sigma \rrbracket = \bigvee_{\tau_0 \in \text{dom}(\sigma)} (\llbracket \tau = \tau_0 \rrbracket \wedge \sigma(\tau_0)).$$

(2)

$$\subseteq^{\mathbf{B}} (\tau, \sigma) = \bigwedge_{\sigma_0 \in \text{dom}(\tau)} (\neg\tau(\sigma_0) \vee \llbracket \sigma_0 \in \sigma \rrbracket).$$

(3)

$$=^{\mathbf{B}} (\tau, \sigma) = \llbracket \tau = \sigma \rrbracket = \llbracket \tau \subseteq \sigma \rrbracket \wedge \llbracket \sigma \subseteq \tau \rrbracket.$$

Theorem 4.1 (Cohen's forcing theorem I). $(V^{\mathbf{B}}, \in^{\mathbf{B}}, =^{\mathbf{B}})$ is a full boolean valued model which assigns the boolean value $1_{\mathbf{B}}$ to all axioms $\phi \in \text{ZFC}$.

V is copied inside $V^{\mathbf{B}}$ as the family of \mathbf{B} -names $\check{a} = \{\langle \check{b}, 1_{\mathbf{B}} \rangle : b \in a\}$ and has the property that for all Σ_0 -formulae (i.e with quantifiers bounded to range over sets) $\phi(x_0, \dots, x_n)$ and $a_0, \dots, a_n \in V$

$$\llbracket \phi(\check{a}_0, \dots, \check{a}_n) \rrbracket = 1_{\mathbf{B}} \text{ if and only if } V \models \phi(a_0, \dots, a_n).$$

This procedure can be formalized in any first order model $(M, E, =)$ of ZFC for any $\mathbf{B} \in M$ such that $(M, E, =)$ models that \mathbf{B} is a complete boolean algebra.

Two ingredients are still missing to prove Cohen's absoluteness (Theorem 3.9) from Baire's category theorem: the notion of an M -generic filter and the duality between $C(\text{St}(\mathbf{B}), 2^\omega)$ and the \mathbf{B} -names in $V^{\mathbf{B}}$ for elements of 2^ω . We first deal with the duality.

4.2. $C(\text{St}(\mathbf{B}), 2^\omega)$ is the family of \mathbf{B} -names for elements of 2^ω .

Definition 4.2. Let \mathbf{B} be a complete boolean algebra. Let $\sigma \in V^{\mathbf{B}}$ be a \mathbf{B} -name such that $\llbracket \sigma : \check{\omega} \rightarrow \check{2} \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$. We define $f_\sigma : \text{St}(\mathbf{B}) \rightarrow 2^\omega$ by

$$f_\sigma(G)(n) = i \iff \llbracket \sigma(\check{n}) = \check{i} \rrbracket \in G.$$

Conversely assume $g : \text{St}(\mathbf{B}) \rightarrow 2^\omega$ is a continuous function, then define

$$\tau_g = \{ \langle \check{n}, \check{i} \rangle, \{G : g(G)(n) = i\} : n \in \omega, i < 2 \} \in V^{\mathbf{B}}.$$

Observe indeed that

$$\{G \in \text{St}(\mathbf{B}) : g(G)(n) = i\} = g^{-1}[N_{n,i}],$$

where $N_{n,i} = \{f \in 2^\omega : f(n) = i\}$. Since g is continuous, $g^{-1}[N_{n,i}]$ is clopen and so it is an element of \mathbf{B} .

We can prove the following duality:

Proposition 4.3. Assume that $\llbracket \sigma : \check{\omega} \rightarrow \check{2} \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$ and $g : \text{St}(\mathbf{B}) \rightarrow 2^\omega$ is continuous. Then

- (1) $\tau_g \in V^{\mathbf{B}}$;
- (2) $f_\sigma : \text{St}(\mathbf{B}) \rightarrow 2^\omega$ is continuous;
- (3) $\llbracket \tau_{f_\sigma} = \sigma \rrbracket_{\mathbf{B}} = 1_{\mathbf{B}}$;
- (4) $f_{\tau_g} = g$.

In particular letting

$$(2^\omega)^\mathbf{B} = \left\{ \sigma \in V^\mathbf{B} : \llbracket \sigma : \check{\omega} \rightarrow \check{2} \rrbracket_\mathbf{B} = 1_\mathbf{B} \right\},$$

the 2-valued models $((2^\omega)^\mathbf{B}/G, =^\mathbf{B}/G)$ and $(C(\text{St}(\mathbf{B}), 2^\omega)/G, =^{\text{St}(\mathbf{B})}/G)$ are isomorphic for all $G \in \text{St}(\mathbf{B})$ via the map $[g]_G \mapsto [\tau_g]_G$.

This is just part of the duality, as the duality can lift the isomorphism also to all \mathbf{B} -Baire relations on 2^ω , among which are all Borel relations. Recall that for any given topological space (X, τ) a subset Y of X is meager for τ if Y is contained in the countable union of closed nowhere dense (i.e. with complement dense open) subsets of X . Y has the Baire property if $Y \Delta A$ is meager for some unique regular open set $A \in \tau$.

Definition 4.4. $R \subseteq (2^\omega)^n$ is a \mathbf{B} -Baire subset of $(2^\omega)^n$ if for all continuous functions $f_1, \dots, f_n : \text{St}(\mathbf{B}) \rightarrow 2^\omega$ we have that

$$(f_1 \times \dots \times f_n)^{-1}[A] = \{G : f_1 \times \dots \times f_n(G) \in A\}$$

has the Baire property in $\text{St}(\mathbf{B})$.

$R \subseteq (2^\omega)^n$ is universally Baire if it is \mathbf{B} -Baire for all complete boolean algebras \mathbf{B} .

It can be shown in ZFC that Borel (and even analytic) subsets of $(2^\omega)^n$ are universally Baire (see [13, Def. 32.21]).

An important result of Feng, Magidor, and Woodin [9] can be restated as follows:

Theorem 4.5. $R \subseteq (2^\omega)^n$ is \mathbf{B} -Baire if and only if there exist $\dot{R}^\mathbf{B} \in V^\mathbf{B}$ such that

$$\llbracket \dot{R}^\mathbf{B} \subseteq (2^{\check{\omega}})^n \rrbracket = 1_\mathbf{B},$$

and for all $\tau_1, \dots, \tau_n \in (2^\omega)^\mathbf{B}$

$$\text{Reg}(\{G : R(f_{\tau_1}(G), \dots, f_{\tau_n}(G))\}) = \llbracket (\tau_1, \dots, \tau_n) \in \dot{R}^\mathbf{B} \rrbracket.$$

In particular an easy corollary is the following:

Theorem 4.6. Let $R \subseteq (2^\omega)^n$ be a \mathbf{B} -baire relation. Then the map $[f]_G \mapsto [\tau_f]_G$ implements an isomorphism between

$$\langle C(\text{St}(\mathbf{B})/G, R^{\text{St}(\mathbf{B})}/G) \cong \langle (2^\omega)^\mathbf{B}/G, \dot{R}^\mathbf{B}/G \rangle$$

for any $G \in \text{St}(\mathbf{B})$.

These results can be suitably generalized to arbitrary Polish spaces. We refer the reader to [26] and [27]. [31] gives an application of this result to tackle a problem in number theory related to Schanuel's conjecture.

4.3. M -generic filters and Cohen's absoluteness.

Definition 4.7. Let (P, \leq) be a partial order and M be a set. A subset G of P is M -generic if $G \cap D$ is non-empty for all $D \in M$ predense subset of P .

By BCT_1 every countable set M admits M -generic filters for all partial orders P .

Theorem 4.8 (Cohen's forcing theorem II). *Assume (N, \in) is a transitive model of ZFC, $\mathbf{B} \in N$ is a complete boolean algebra in N , and $G \in \text{St}(\mathbf{B})$ is an N -generic filter for \mathbf{B}^+ .*

Let

$$\begin{aligned} \text{val}_G : N^{\mathbf{B}} &\rightarrow V \\ \sigma &\mapsto \sigma_G = \{\tau_G : \exists b \in G \langle \tau, b \rangle \in \sigma\}, \end{aligned}$$

and $N[G] = \text{val}_G[N^{\mathbf{B}}]$.

Then $N[G]$ is transitive, the map $[\sigma]_G \mapsto \sigma_G$ is the Mostowski collapse of the Tarski models $\langle N^{\mathbf{B}}/G, \in^{\mathbf{B}}/G \rangle$ and induces an isomorphism of this model with the model $\langle N[G], \in \rangle$.

In particular for all formulae $\phi(x_1, \dots, x_n)$ and $\tau_1, \dots, \tau_n \in N^{\mathbf{B}}$

$$\langle N[G], \in \rangle \models \phi((\tau_1)_G, \dots, (\tau_n)_G)$$

if and only if $\llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket \in G$.

Recall that:

- For any infinite cardinal λ , H_λ is the set of all sets $a \in V$ such that $|\text{trcl}(a)| < \lambda$ (where $\text{trcl}(a)$ is the transitive closure of the set a).
- If κ is a strongly inaccessible cardinal (i.e. regular and strong limit), H_κ is a transitive model of ZFC.
- A property $R \subseteq (2^\omega)^n$ is Σ_2^1 , if it is of the form

$$R = \{(a_1, \dots, a_n) \in (2^\omega)^n : \exists y \in 2^\omega \forall x \in 2^\omega S(x, y, a_1, \dots, a_n)\}$$

with $S \subseteq (2^\omega)^{n+2}$ a Borel relation.

- If $\phi(x_0, \dots, x_n)$ is a Σ_0 -formula and $M \subseteq N$ are transitive sets or classes, then for all $a_0, \dots, a_n \in M$

$$M \models \phi(a_0, \dots, a_n) \text{ if and only if } N \models \phi(a_0, \dots, a_n).$$

Observe that for any theory $T \supseteq \text{ZFC}$ there is a recursive translation of Σ_2^1 -properties (provably Σ_2^1 over T) into Σ_1 -properties over H_{ω_1} (provably Σ_1 over the same theory T) [13, Lemma 25.25].

Lemma 4.9. *Assume $\phi(x, r)$ is a Σ_0 -formula in the parameter $\vec{r} \in (2^\omega)^n$. Then the following are equivalent:*

- (1) $H_{\omega_1} \models \exists x \phi(x, r)$.
- (2) For all complete boolean algebra \mathbf{B} $\llbracket \exists x \phi(x, r) \rrbracket = 1_{\mathbf{B}}$.
- (3) There is a complete boolean algebra \mathbf{B} such that $\llbracket \exists x \phi(x, r) \rrbracket > 0_{\mathbf{B}}$.

Summing up we get: a Σ_2^1 -statement holds in V iff the corresponding Σ_1 -statement over H_{ω_1} holds in some model of the form $V^{\mathbf{B}}/G$.

Combining the above Lemma with Proposition 4.3, we can easily infer the proof of Theorem 3.9.

Proof. We shall actually prove the following slightly stronger formulation of the non-trivial direction in the three equivalences above:

$$H_{\omega_1} \models \exists x \phi(x, r) \text{ if } \llbracket \exists x \phi(x, \vec{r}) \rrbracket > 0_{\mathbf{B}} \text{ for some complete boolean algebra } \mathbf{B} \in V.$$

To simplify the exposition we prove this statement under the further assumption that there exists an inaccessible cardinal $\kappa > \mathbf{B}$. With greater care for details the large cardinal assumption can be removed. So assume $\phi(x, \vec{y})$ is a Σ_0 -formula and $\llbracket \exists x \phi(x, \vec{r}) \rrbracket > 0_{\mathbf{B}}$ for some complete boolean algebra $\mathbf{B} \in V$ with parameters $\vec{r} \in (2^\omega)^n$. Pick a model $M \in V$ such that $M \prec (H_\kappa)^V$, M is countable in V , and $\mathbf{B}, \vec{r} \in M$. Let $\pi_M : M \rightarrow N$ be its transitive collapse (i.e. $\pi_M(a) = \pi_M[a \cap M]$ for all $a \in M$) and $\mathbf{Q} = \pi_M(\mathbf{B})$. Notice also that $\pi_M(\vec{r}) = \vec{r}$: since $\omega \in M$ is a definable ordinal *contained* in M , $\pi_M(\omega) = \pi_M[\omega] = \omega$; consequently, π_M fixes also all the elements in $2^\omega \cap M$.

Since π_M is an isomorphism of M with N ,

$$N \models \text{ZFC} \wedge (b = \llbracket \exists x \phi(x, \vec{r}) \rrbracket > 0_{\mathbf{Q}}).$$

Now let $G \in V$ be N -generic for \mathbf{Q} with $b \in G$ (G exists since N is countable); then by Cohen's theorem of forcing applied in V to N , we have that $N[G] \models \exists x \phi(x, \vec{r})$. So we can pick $a \in N[G]$ such that $N[G] \models \phi(a, \vec{r})$. Since $N, G \in (H_{\omega_1})^V$, we have that V models that $N[G] \in H_{\omega_1}^V$ and thus V models that a as well belongs to $H_{\omega_1}^V$. Since $\phi(x, \vec{y})$ is a Σ_0 -formula, V models that $\phi(a, \vec{r})$ is absolute between the transitive sets $N[G] \subset H_{\omega_1}$ to which a, \vec{r} belong. In particular a witnesses in V that $H_{\omega_1}^V \models \exists x \phi(x, \vec{r})$. \square

5. MAXIMAL FORCING AXIOMS

Guided by all the previous results we want to formulate maximal forcing axioms. We pursue two directions:

- (1) A direction shaped by topological considerations: we have seen that $\text{FA}_{\aleph_0}(P)$ holds for any partial order P , and that AC is equivalent to the satisfaction of $\text{FA}_\lambda(P)$ for all regular λ and all $< \lambda$ -closed posets P .

We want to isolate the largest possible class of partial orders Γ_λ for which $\text{FA}_\lambda(P)$ holds for all $P \in \Gamma_\lambda$. The case $\lambda = \aleph_0$ is handled by Baire's category theorem, which shows that Γ_{\aleph_0} is the class of all posets. We will outline how the case $\lambda = \aleph_1$ is settled by the work of Foreman, Magidor, and Shelah [10] and leads to Martin's maximum. On the other hand, the case $\lambda > \aleph_1$ is wide open and until recently only partial results have been obtained. New techniques to handle the case $\lambda = \aleph_2$ are being developed (notably by Neeman, and also

by Asperò, Cox, Krueger, Mota, Velickovic, see among others [14, 15, 19]), however the full import of their possible applications is not clear yet.

- (2) A direction shaped by model-theoretic considerations: Baire’s category theorem implies that the natural embedding of 2^ω into $C(\text{St}(\mathbf{B}), 2^\omega)/G$ is Σ_2 -elementary, whenever 2^ω is endowed with \mathbf{B} -baire predicates (among which are all the Borel predicates). We want to reinforce this theorem in two directions:
- (A) We want to be able to infer that (at least for Borel predicates) the natural embedding of 2^ω into $C(\text{St}(\mathbf{B}), 2^\omega)/G$ yields a full elementary embedding of 2^ω into $C(\text{St}(\mathbf{B}), 2^\omega)/G$.
 - (B) We want to be able to define boolean ultrapowers $M^{\mathbf{B}}$ also for other first order structures M than 2^ω and be able to infer that the natural embedding of M into $M^{\mathbf{B}}/G$ is elementary for these boolean ultrapowers.

Both directions (the topological and the model-theoretic) converge towards the isolation of certain natural forcing axioms. Moreover for each cardinal λ , the relevant structures for which we can define a natural notion of boolean ultrapower are either the structure H_{λ^+} , or the Chang model $L(\text{Ord}^\lambda)$.

We believe that we have now a satisfactory understanding of the maximal forcing axioms one can get following both directions for the cases $\lambda = \aleph_0, \aleph_1$. The main open question remaining how to isolate (if at all possible) the maximal forcing axioms for $\lambda > \aleph_1$.

5.1. Woodin’s generic absoluteness for H_{ω_1} and $L(\text{Ord}^\omega)$. We start with the model-theoretic direction, following Woodin’s work in Ω -logic. Observe that a set theorist works either with first order calculus to justify some proofs over ZFC, or with forcing to obtain independence results over ZFC. However, in axiom systems extending ZFC there seems to be a gap between what we can achieve by ordinary proofs and the independence results that we can obtain by means of forcing. To close this gap we would like two desirable features of a “complete” first order theory T that contains ZFC, specifically with respect to the semantics given by the class of boolean valued models of T :

- T is complete with respect to its intended semantics, i.e for all statements ϕ only one among $T + \phi$ and $T + \neg\phi$ is forceable.
- Forceability over T should correspond to a notion of derivability with respect to some proof system, for instance derivability with respect to a standard first order calculus for T .

Both statements appear to be rather bold and have to be handled with care: Consider for example the statement $|\omega| = |\omega_1|$ in a theory T extending ZFC with the statements ω is the first infinite cardinal and ω_1 is the first uncountable cardinal. Then clearly T proves $|\omega| \neq |\omega_1|$, while if one forces with $\text{Coll}(\omega, \omega_1)$ one produces a model of set theory where this equality holds

(however the formula ω_1 is the first uncountable cardinal is now false in this model).

At first glance, this suggests that as we expand the language for T , forcing starts to act randomly on the formulae of T , switching the truth value of its formulae with parameters in ways which it does not seem simple to describe. However the above difficulties arise essentially from our lack of attention to define the type of formulae for which we aim to have the completeness of T with respect to forceability. We can show that when the formulae are limited to talking only about a suitable initial segment of the set theoretic universe (i.e. H_{ω_1} or $L(\text{Ord}^\omega)$), and we consider only forcings that preserve the intended meaning of the parameters by which we enriched the language of T (i.e. parameters in H_{ω_1}), this random behaviour of forcing does not show up anymore.

We take a platonist's stance towards set theory; thus we have one canonical model V of ZFC, the truths of which we try to uncover. To do this, we may use model theoretic techniques that produce new models of the part of $\text{Th}(V)$ about which we are confident. This certainly includes ZFC, and (for most platonists) all the large cardinal axioms.

We may start our quest to uncover the truth in V by first settling the theory of $H_{\omega_1}^V$ (the hereditarily countable sets), then the theory of $H_{\omega_2}^V$ (the sets of hereditarily cardinality \aleph_1) and so on and so forth, thus covering step by step all infinite cardinals. To proceed we need some definitions:

Definition 5.1. Given a theory $T \supseteq \text{ZFC}$ and a family Γ of partial orders definable in T , we say that ϕ is Γ -consistent for T if T proves that there exists a complete boolean algebra $\mathbb{B} \in \Gamma$ such that $\llbracket \phi \rrbracket_{\mathbb{B}} > 0_{\mathbb{B}}$.

Given a model V of ZFC we say that V models that ϕ is Γ -consistent if ϕ is Γ -consistent for $\text{Th}(V)$.

Definition 5.2. Let

$$T \supseteq \text{ZFC} + \{\lambda \text{ is an infinite cardinal}\}$$

Ω_λ is the definable (in T) class of partial orders P which satisfy $\text{FA}_\lambda(P)$.

In particular Baire's category theorem amounts to saying that Ω_{\aleph_0} is the class of all partial orders (denoted by Woodin as the class Ω). The following is a careful reformulation of Lemma 4.9 which does not require any ontological commitments about V .

Lemma 5.3 (Cohen's Absoluteness Lemma). *Assume $T \supseteq \text{ZFC} + \{p \subseteq \omega\}$ and $\phi(x, p)$ is a Σ_0 -formula. Then the following are equivalent:*

- $T \vdash \exists x \phi(x, p)$,
- $\exists x \phi(x, p)$ is Ω -consistent for T .

This shows that for Σ_1 -formulae with real parameters the desired overlap between the ordinary notion of provability and the semantic notion of forceability is provable in ZFC. Now it is natural to asking if we can expand the above in at least two directions:

- (1) Increase the complexity of the formula,
- (2) Expand the language allowing parameters also for other infinite cardinals.

The second direction will be pursued in the next subsection. Concerning the first direction, the extent by which we can increase the complexity of the formula requires once again some attention to the semantical interpretation of its parameters and its quantifiers. We have already observed that the formula $|\omega| = |\omega_1|$ is inconsistent but Ω -consistent in a language with parameters for ω and ω_1 . One of Woodin's main achievements² in Ω -logic shows that if we restrict the semantic interpretation of ϕ to range over the structure $L([\text{Ord}]^{\aleph_0})$ and we assume large cardinal axioms, we can get a full correctness and completeness result³ [16, Corollary 3.1.7]:

Theorem 5.4 (Woodin). *Assume T is a theory extending*

$\text{ZFC} + \{p \subset \omega\} + \text{there are class many supercompact cardinals,}$

$\phi(x, y)$ is any formula in free variables x, y , and $A \subseteq (2^\omega)^n$ is universally Baire. Then the following are equivalent (where $\dot{A}^{\mathbb{B}}$ is the \mathbb{B} -name given by the lifting of A to $V^{\mathbb{B}}$ given by Theorem 4.5):

- $T \vdash [L([\text{Ord}]^{\aleph_0}, A) \models \phi(p, A)],$
- $T \vdash \exists \mathbb{B} \left\| L([\text{Ord}]^{\aleph_0}, \dot{A}^{\mathbb{B}}) \models \phi(p, \dot{A}^{\mathbb{B}}) \right\| > 0_{\mathbb{B}},$
- $T \vdash \forall \mathbb{B} \left\| L([\text{Ord}]^{\aleph_0}, \dot{A}^{\mathbb{B}}) \models \phi(p, \dot{A}^{\mathbb{B}}) \right\| = 1_{\mathbb{B}}.$

Notice that since $H_{\omega_1} \subseteq L([\text{Ord}]^{\aleph_0})$, via Theorem 4.5 and natural generalizations of [13, Lemma 25.25] establishing a correspondence between Σ_{n+1}^1 -properties and Σ_n -properties over H_{ω_1} , we obtain that for any complete boolean algebra \mathbb{B} and any Σ_n^1 -predicate $R \subseteq (2^\omega)^n$ the map $x \mapsto [c_x]_G$ of $(2^\omega, R)$ into $(C(\text{St}(\mathbb{B}, 2^\omega), R^{\text{St}(\mathbb{B})}))$ is an elementary embedding. In particular the above theorem provides a first fully satisfactory answer to the question of whether the natural embeddings of 2^ω into its boolean ultrapowers can be elementary: the answer is yes if we accept the existence of large cardinals!

The natural question to address now is whether we can step up this result also for uncountable λ . If so, to which form?

5.2. Topological maximality: Martin's maximum MM. Let us now address the quest for maximal forcing axioms from the topological direction. Specifically: what is the largest class of partial orders Γ for which we can posit $\text{FA}_{\aleph_1}(\Gamma)$?

Shelah proved that $\text{FA}_{\aleph_1}(P)$ fails for any P which does not preserve stationary subsets of ω_1 . Nonetheless it cannot be decided in ZFC whether this is a necessary condition for a poset P in order to have the failure of

²We follow Larson's presentation as in [16].

³The large cardinal assumptions on T of the present formulation can be significantly reduced. See [16, Corollary 3.1.7].

$\text{FA}_{\aleph_1}(P)$. For example let P be a forcing which shoots a club of ordertype ω_1 through a projectively stationary and costationary subset of $P_{\omega_1}(\omega_2)$ by selecting countable initial segments of this club: for all such P , it is provable in ZFC that P preserve stationary subsets of ω_1 . However in L , $\text{FA}_{\aleph_1}(P)$ fails for some such P while in a model of Martin's maximum MM, $\text{FA}_{\aleph_1}(P)$ holds for all such P .

The remarkable result of Foreman, Magidor, and Shelah [10] is that the above necessary condition is consistently also a sufficient condition: it can be forced that $\text{FA}_{\aleph_1}(P)$ holds if and only if P is a forcing notion preserving all stationary subsets of ω_1 . This axiom is known in the literature as Martin's maximum MM. In view of Theorem 1.8, MM realizes a maximality property for forcing axioms: it can be seen as a maximal strengthening of the axiom of choice $\text{AC} \upharpoonright_{\omega_2}$ for \aleph_1 -sized families of non-empty sets. Can we strengthen this further? If so, in which form? It turns out that stronger and stronger forms of forcing axioms can be expressed in the language of categories and provide means to extend Woodin's generic absoluteness results to third order arithmetic or more generally to larger and larger fragments of the set theoretic universe.

5.3. Category forcings and category forcing axioms. Assume Γ is a class of complete boolean algebras and \rightarrow^Θ is a family of complete homomorphisms between elements of Γ closed under composition and containing all identity maps. $(\Gamma, \rightarrow^\Theta)$ is the category whose objects are the complete boolean algebras in Γ and whose arrows are given by complete homomorphisms $i : B \rightarrow Q$ in \rightarrow^Θ . We call embeddings in \rightarrow^Θ , Θ -correct embeddings. Notice that these categories immediately give rise to natural class pre-orders associated with them, pre-orders whose elements are the complete boolean algebras in Γ and whose order relation is given by the arrows in \rightarrow^Θ (i.e. $B \leq_\Theta C$ if there exists $i : C \rightarrow B$ in \rightarrow^Θ). We denote these class partial orders by (Γ, \leq_Θ) .

Depending on the choice of Γ and \rightarrow^Θ these partial orders can be trivial (as forcing notions), for example:

Remark 5.5. Assume $\Omega = \Omega_{\aleph_0}$ is the class of all complete boolean algebras and \rightarrow^Ω is the class of all complete embeddings, then any two conditions in (Γ, \leq_Ω) are compatible, i.e. (Γ, \leq_Ω) is forcing equivalent to the trivial partial order. This is the case since for any pair of partial orders P, Q and X of size larger than $2^{|P|+|Q|}$ there are complete injective homomorphisms of $\text{RO}(P)$ and $\text{RO}(Q)$ into the boolean completion of $\text{Coll}(\omega, X)$ (see [16, Thm A.0.7] and its following remark). These embeddings witness the compatibility of $\text{RO}(P)$ with $\text{RO}(Q)$.

On the other hand these class partial orders will in general be non-trivial: let SSP be the class of stationary set preserving forcings. Then the Namba forcing shooting a cofinal ω -sequence on ω_2 and $\text{Coll}(\omega_1, \omega_2)$ are incompatible conditions in $(\text{SSP}, \leq_\Omega)$: any forcing notion absorbing both of them makes the cofinality of ω_2^V at the same time of cofinality ω_1^V (using the generic filter

for $\text{Coll}(\omega_1, \omega_2)$) and countable (using the generic filter for Namba forcing); this means that this forcing must collapse ω_1^V to become a countable ordinal, hence cannot be stationary set preserving.

Forcing axioms as density properties of category forcings. The following results are among the main reasons to analyze in more detail these types of class forcings:

Theorem 5.6 (Woodin, Thm. 2.53 [34]). *Assume there are class many supercompact cardinals. Then the following are equivalent for any complete cba \mathbb{B} and cardinal κ :*

- (1) $\text{FA}_\kappa(\mathbb{B})$;
- (2) *there is a complete homomorphism of \mathbb{B} into a presaturated tower inducing a generic ultrapower embedding with critical point κ^+ .*

Theorem 5.7 (V. Thm. 2.12 [32]). *Assume there are class many supercompact cardinals. Then the following are equivalent:*

- (1) MM^{++} ;
- (2) *the class of presaturated normal towers is dense in $(\text{SSP}, \leq_{\text{SSP}})$.*

It is not in the scope of this paper to delve into the definition and properties of presaturated tower forcings and of the axiom MM^{++} . Let us just remark the following two facts:

- MM^{++} is a natural strengthening of Martin's maximum whose consistency is proved by exactly the same methods producing a model of Martin's maximum.
- A presaturated tower \mathcal{T} inducing a generic ultrapower embedding with critical point κ^+ is such that whenever G is V -generic for \mathcal{T} we have that

$$(1) \quad H_{\kappa^+}^V \prec H_{\kappa^+}^{V[G]}.$$

In particular the above theorems show that forcing axioms can be also stated as density properties of class partial orders. Below we will describe assumptions $\text{AX}(\Gamma, \kappa)$ yielding a dense class of forcings in (Γ, \leq_Γ) whose generic extensions satisfy (1), and producing generic absoluteness results. We refer the reader to [3, 2, 30] for details.

5.4. Iterated resurrection axioms and generic absoluteness for H_{κ^+} . The results and ideas of this subsection expand on the seminal work of Hamkins and Johnstone [12] on resurrection axioms.

Definition 5.8. Let Γ be a definable class of complete Boolean algebras closed under two-step iterations. The *cardinal preservation degree* $\text{cpd}(\Gamma)$ of Γ is the largest cardinal κ such that every $\mathbb{B} \in \Gamma$ forces that every cardinal $\nu \leq \kappa$ is still a cardinal in $V^{\mathbb{B}}$. If all cardinals are preserved by Γ , we say that $\text{cpd}(\Gamma) = \infty$.

The *distributivity degree* $\text{dd}(\Gamma)$ of Γ is the largest cardinal κ such that every $\mathbb{B} \in \Gamma$ is $<\kappa$ -distributive.

We remark that the supremum of the cardinals preserved by Γ is preserved by Γ , and the same holds for the property of being $<\kappa$ distributive. Furthermore, $\text{dd}(\Gamma) \leq \text{cpd}(\Gamma)$ and $\text{dd}(\Gamma) \neq \infty$ whenever Γ is non trivial (i.e., it contains a Boolean algebra that is not forcing equivalent to the trivial Boolean algebra). Moreover $\text{dd}(\Gamma) = \text{cpd}(\Gamma)$ whenever Γ is closed under two-step iterations and contains the class of $<\text{cpd}(\Gamma)$ -closed posets.

Definition 5.9. Let Γ be a definable class of complete Boolean algebras. We let $\gamma = \gamma_\Gamma = \text{cpd}(\Gamma)$.

For example, $\gamma = \omega$ if Γ is the class of all posets, while for axiom- A , proper, SP , SSP we have that $\gamma = \omega_1$, and for $<\kappa$ -closed we have that $\gamma = \kappa$.

We aim to isolate for each cardinal γ classes of forcings Δ_γ and axioms $\text{AX}(\Delta_\gamma)$ such that:

- (1) $\gamma = \text{cpd}(\Delta_\gamma)$ and assuming certain large cardinal axioms, the family of $\mathbb{B} \in \Delta_\gamma$ which force $\text{AX}(\Delta_\gamma)$ is dense in $(\Delta_\gamma, \leq_{\Delta_\gamma})$;
- (2) $\text{AX}(\Delta_\gamma)$ gives generic absoluteness for the theory with parameters of H_{γ^+} with respect to all forcings in Δ_γ which preserve $\text{AX}(\Delta_\gamma)$;
- (3) the axioms $\text{AX}(\Delta_\gamma)$ are mutually compatible for the largest possible family of cardinals γ simultaneously;
- (4) the classes Δ_γ are the largest possible for which the axioms $\text{AX}(\Delta_\gamma)$ can possibly be consistent.

Towards this aim remark the following:

- $\text{dd}(\Gamma)$ is the least possible cardinal γ such that $\text{AX}(\Gamma)$ is a non-trivial axiom asserting generic absoluteness for the theory of H_{γ^+} with parameters. In fact, $H_{\text{dd}(\Gamma)}$ is never changed by forcings in Γ .
- $\text{cpd}(\Gamma)$ is the largest possible cardinal γ for which an axiom $\text{AX}(\Gamma)$ as above can grant generic absoluteness with respect to Γ for the theory of H_{γ^+} with parameters. To see this, let Γ be such that $\text{cpd}(\Gamma) = \gamma$ and assume towards a contradiction that there is an axiom $\text{AX}(\Gamma)$ yielding generic absoluteness with respect to Γ for the theory with parameters of H_λ with $\lambda > \gamma^+$.

Assume that $\text{AX}(\Gamma)$ holds in V . Since $\text{cpd}(\Gamma) = \gamma$, there exists a $\mathbb{B} \in \Gamma$ which collapses γ^+ . Let $\mathbb{C} \leq_\Gamma \mathbb{B}$ be obtained by property (1) above for $\Gamma = \Delta_\gamma$, so that $\text{AX}(\Gamma)$ holds in $V^{\mathbb{C}}$, and remark that γ^+ cannot be a cardinal in $V^{\mathbb{C}}$ as well. Then γ^+ is a cardinal in H_λ and not in $H_\lambda^{\mathbb{C}}$, witnessing failure of generic absoluteness and contradicting property (2) for $\text{AX}(\Gamma)$.

We argue that there are axioms $\text{RA}_\omega(\Gamma)$ satisfying the first two of the above requirements, and which are consistent for a variety of forcing classes Γ . These axioms also provide natural examples for the last two requirements. We will come back later on with philosophical considerations outlining why the last two requirements are also natural. We can prove the consistency

of $\text{RA}_\omega(\Gamma)$ for forcing classes which are definable in Gödel-Bernays set theory with classes NBG , closed under two-step iterations, weakly iterable (a technical definition asserting that most set sized descending sequences in \leq_Γ have lower bounds in Γ , see [2] or [3] for details), and contain all the $< \text{cpd}(\Gamma)$ -closed forcings.

The axioms $\text{RA}_\alpha(\Gamma)$ for α an ordinal can be formulated in the Morse Kelley axiomatization of set theory MK as follows:

Definition 5.10. Given an ordinal α and a definable⁴ class of forcings Γ closed under two-steps iterations, the axiom $\text{RA}_\alpha(\Gamma)$ holds if for all $\beta < \alpha$ the class

$$\left\{ \mathbf{B} \in \Gamma : H_{\gamma^+} \prec H_{\gamma^+}^{\mathbf{B}} \wedge V^{\mathbf{B}} \models \text{RA}_\beta(\Gamma) \right\}$$

is dense in (Γ, \leq_Γ) (where $\gamma = \gamma_\Gamma$).

$\text{RA}_{\text{Ord}}(\Gamma)$ holds if $\text{RA}_\alpha(\Gamma)$ holds for all α .

Remark 5.11. The above definition can be properly formalized in MK (but most likely not in ZFC if α is infinite). The problem is the following: the axioms $\text{RA}_\alpha(\Gamma)$ can be formulated only by means of a transfinite recursion over a well-founded relation which is not set-like. It is a delicate matter to argue that this transfinite recursion can be carried out. [2] shows that this is the case if the base theory is MK .

The axiom $\text{RA}_\omega(\Gamma)$ yields generic absoluteness by the following elementary argument:

Theorem 5.12. *Suppose $n \in \omega$, Γ is well behaved, $\text{RA}_n(\Gamma)$ holds, and $\mathbf{B} \in \Gamma$ forces $\text{RA}_n(\Gamma)$. Then $H_{\gamma^+} \prec_n H_{\gamma^+}^{\mathbf{B}}$ (where $\gamma = \gamma_\Gamma$).*

Proof. We proceed by induction on n . Since $\gamma^+ \leq (\gamma^+)^{V^{\mathbf{B}}}$, $H_{\gamma^+} \subseteq H_{\gamma^+}^{\mathbf{B}}$ and the thesis holds for $n = 0$ by the fact that for all transitive structures M, N , if $M \subset N$ then $M \prec_0 N$. Suppose now that $n > 0$, and fix G V -generic for \mathbf{B} . By $\text{RA}_n(\Gamma)$, let $\mathbf{C} \in V[G]$ be such that whenever H is $V[G]$ -generic for \mathbf{C} , $V[G * H] \models \text{RA}_{n-1}(\Gamma)$ and $H_{\gamma^+}^V \prec H_{\gamma^+}^{V[G * H]}$. Hence we have the following diagram:

$$\begin{array}{ccc} H_{\gamma^+}^V & \xrightarrow{\Sigma_\omega} & H_{\gamma^+}^{V[G * H]} \\ & \searrow \Sigma_{n-1} & \nearrow \Sigma_{n-1} \\ & & H_{\gamma^+}^{V[G]} \end{array}$$

obtained by inductive hypothesis applied both on V , $V[G]$ and on $V[G]$, $V[G * H]$ since in all those classes $\text{RA}_{n-1}(\Gamma)$ holds.

⁴ Γ must be definable by a formula with no class quantifier and no class parameter to be on the safe side with respect to the definability issues regarding the iterated resurrection axioms raised by the remark right after this definition. All usual classes of forcings such as proper, semiproper, stationary set preserving, $< \kappa$ -closed, etc.... are definable by formulae satisfying these restrictions.

Let $\phi \equiv \exists x\psi(x)$ be any Σ_n formula with parameters in $H_{\gamma^+}^V$. First suppose that ϕ holds in V , and fix $\bar{x} \in V$ such that $\psi(\bar{x})$ holds. Since $H_{\gamma^+}^V \prec_{n-1} H_{\gamma^+}^{V[G]}$ and ψ is Π_{n-1} , it follows that $\psi(\bar{x})$ holds in $V[G]$ hence so does ϕ . Now suppose that ϕ holds in $V[G]$ as witnessed by $\bar{x} \in V[G]$. Since $H_{\gamma^+}^{V[G]} \prec_{n-1} H_{\gamma^+}^{V[G*H]}$ it follows that $\psi(\bar{x})$ holds in $V[G*H]$, hence so does ϕ . Since $H_{\gamma^+}^V \prec H_{\gamma^+}^{V[G*H]}$, the formula ϕ holds also in V concluding the proof. \square

Corollary 5.13. *Assume Γ is closed under two-steps iterations and contains the $< \text{cpd}(\Gamma)$ -closed forcings. If $\text{RA}_\omega(\Gamma)$ holds, and $\mathbb{B} \in \Gamma$ forces $\text{RA}_\omega(\Gamma)$, then $H_{\gamma^+} \prec H_{\gamma^+}^{\mathbb{B}}$ (where $\gamma = \gamma_\Gamma$).*

Regarding the consistency of the axioms $\text{RA}_\omega(\Gamma)$ we have the following:

Proposition 5.14. *Assume there are class-many Woodin cardinals. Then $\text{RA}_{\text{Ord}}(\Omega)$ holds.*

Theorem 5.15. *$\text{RA}_1(\Gamma)$ implies $H_{\gamma^+} \prec_1 V^{\mathbb{B}}$ for all $\mathbb{B} \in \Gamma$, hence it is a strengthening of the bounded forcing axiom⁵ $\text{BFA}_\gamma(\Gamma)$ (where $\gamma = \gamma_\Gamma$).*

Theorem 5.16 ([2]). *Assume there is a super huge cardinal.⁶*

Then $\text{RA}_{\text{Ord}}(\text{SSP}) + \text{MM}^{++}$ and $\text{RA}_{\text{Ord}}(\text{proper}) + \text{PFA}^{++}$ are consistent.

For the consistency of $\text{RA}_{\text{Ord}}(\text{proper})$ a Mahlo cardinal suffices.

Moreover it is also consistent relative to a Mahlo cardinal that $\text{RA}_{\text{Ord}}(\Gamma_\kappa)$ holds simultaneously for all cardinals κ (where Γ_κ is the class of $< \kappa$ -closed forcings)⁷.

In this regard the axioms $\text{RA}_\alpha(\Gamma)$ for $\Gamma \supseteq \Gamma_\kappa$ (Γ_κ being the class of $< \kappa$ -closed forcings) appear to be natural companions of the axiom of choice, while the axioms $\text{RA}_{\text{Ord}}(\Omega)$ and $\text{RA}_{\text{Ord}}(\text{SSP}) + \text{MM}$ are natural maximal strengthenings of the axiom of choice at the levels ω and ω_1 . Hence it is in our opinion natural to try to isolate classes of forcings Δ_κ as κ ranges among the cardinals such that:

- (1) $\kappa = \text{cpd}(\Delta_\kappa)$ for all κ .
- (2) $\Delta_\kappa \supseteq \Gamma_\kappa$ for all κ .
- (3) $\text{FA}_\kappa(\Delta_\kappa)$ and $\text{RA}_\omega(\Delta_\kappa)$ are simultaneously consistent for all κ .
- (4) For all cardinals κ , Δ_κ is the largest possible Γ with $\text{cpd}(\Gamma) = \kappa$ for which $\text{FA}_\kappa(\Delta_\kappa)$ and $\text{RA}_\omega(\Delta_\kappa)$ are simultaneously consistent (and if possible for all κ simultaneously).

⁵The bounded forcing axiom $\text{BFA}_\gamma(\Gamma)$ asserts that $H_{\gamma^+} \prec_1 V^{\mathbb{B}}$ for all $\mathbb{B} \in \Gamma$.

⁶A cardinal κ is *super huge* iff for every ordinal α there exists an elementary embedding $j : V \rightarrow M \subseteq V$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \alpha$ and $j^{(\kappa)}M \subseteq M$.

⁷It is also consistent the following:

$$\text{RA}_{\text{Ord}}(\Omega_{\aleph_0}) + \text{RA}_{\text{Ord}}(\text{SSP}) + \forall \kappa > \omega_1 \text{RA}_{\text{Ord}}(\Gamma_\kappa)$$

Compare the above requests with requirements (3) and (4) in the discussion motivating the introduction of the iterated resurrection axioms on page 26. In this regard it appears that we have now a completely satisfactory answer about what Δ_ω and Δ_{ω_1} are: i.e., respectively the class of *all* forcing notions and the class of all SSP-forcing notions.

5.5. Boosting Woodin’s absoluteness to $L(\text{Ord}^\kappa)$: the axioms $\text{CFA}(\Gamma)$.

We gave detailed arguments leading us to axioms which can be stated as density properties of certain category forcings and yielding generic absoluteness results for the theory of H_{κ^+} for various cardinals κ . Exploring Woodin’s proof for the generic absoluteness of the Chang model $L(\text{Ord}^\omega)$ one can get an even stronger type of category forcing axiom yielding generic absoluteness results for the Chang models $L(\text{Ord}^\kappa)$. The best result we can currently produce is the following (we refer the interested reader to [1, 3, 30] for details):

Theorem 5.17. *Let Γ be a κ -suitable class of forcings⁸.*

Let MK^ stand for⁹*

$\text{MK} +$ there are stationarily many inaccessible cardinals.

There is an axiom¹⁰ $\text{CFA}(\Gamma)$ which implies $\text{FA}_\kappa(\Gamma)$ as well as $\text{RA}_{\text{Ord}}(\Gamma)$ and is such that for any T^ extending*

$$\text{MK}^* + \text{CFA}(\Gamma) + \kappa \text{ is a regular cardinal} + S \subset \kappa,$$

and for any formula $\phi(S)$, the following are equivalent:

- (1) $T^* \vdash [L(\text{Ord}^\kappa) \models \phi(S)]$,
- (2) T^* proves that for some forcing $\mathbb{B} \in \Gamma$

$$\llbracket \text{CFA}(\Gamma) \rrbracket_{\mathbb{B}} = \llbracket L(\text{Ord}^\kappa) \models \phi(S) \rrbracket_{\mathbb{B}} = 1_{\mathbb{B}}.$$

We also have that

Theorem 5.18 ([1, 3]). *Assume Γ is κ -suitable. Then $\text{CFA}(\Gamma)$ is consistent relative to the existence of a 2-superhuge cardinal¹¹.*

⁸This is a lengthy and technical definition; roughly it requires that:

- Γ is closed under two-step iterations, and contains all the $< \kappa$ -closed posets (where $\kappa = \text{cpd}(\Gamma)$),
- there is an iteration theorem granting that all set sized iterations of posets in Γ has a limit in Γ ,
- Γ is defined by a syntactically simple formula (i.e. Σ_2 in the Levy hierarchy of formulae),
- Γ has a dense set of Γ -rigid elements (i.e. the $\mathbb{B} \in \Gamma$ admitting at most one $i : \mathbb{B} \rightarrow \mathbb{C}$ witnessing that $\mathbb{C} \leq_{\Gamma} \mathbb{B}$ for all $\mathbb{C} \in \Gamma$ form a dense subclass of Γ).

⁹In MK one can define the club filter on the class Ord , hence the notion of stationarity for classes of ordinals makes sense.

¹⁰ $\text{CFA}(\Gamma)$ can be formulated as a density property of the class forcing (Γ, \leq_{Γ}) .

¹¹A cardinal κ is 2-superhuge if it is supercompact and this can be witnessed by 2-huge embeddings.

While the definition of κ -suitable Γ is rather delicate, it can be shown that many interesting classes are ω_1 -suitable, among others: proper, semiproper, ω^ω -bounding and (semi)proper, preserving a suslin tree and (semi)proper. [1] contains a detailed list of classes which are ω_1 -suitable. It is not known whether there can be κ -suitable classes Γ for some $\kappa > \omega_1$.

6. SOME OPEN QUESTIONS

Here is a list of questions for which we do not have many clues.....

- (1) What are the Γ which are κ -suitable for a given cardinal $\kappa > \aleph_1$ (i.e. such that $\text{CFA}(\Gamma)$ is consistent)?
- (2) Do they even exist for $\kappa > \aleph_1$?
- (3) In case they do exist for some $\kappa > \aleph_1$, do we always have a unique maximal Γ such that $\text{cpd}(\Gamma) = \kappa$ as is the case for $\kappa = \aleph_0$ or $\kappa = \aleph_1$?

Any interesting iteration theorem for a class $\Gamma \supseteq \Gamma_{\omega_2}$ closed under two-step iterations can be used to prove that $\text{RA}_{\text{Ord}}(\Gamma)$ is consistent relative to suitable large cardinal assumptions and freezes the theory of H_{ω_3} with respect to forcings in Γ preserving $\text{RA}_\omega(\Gamma)$ (see [2]). It is nonetheless still a mystery which classes $\Gamma \supseteq \Gamma_{\omega_2}$ can give us a nice iteration theorem, even if the recent works by Neeman, Asperò, Krueger, Mota, Velickovic and others are starting to shed some light on this problem (see among others [14, 15, 19]).

We can dare to be more ambitious and replicate the above type of issue at a much higher level of the set theoretic hierarchy. There is a growing set of results regarding the first-order theory of $L(V_{\lambda+1})$ assuming λ is a very large cardinal (i.e., for example admitting an elementary $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ with critical point smaller than λ , see for example [6, 7, 35]). It appears that large fragments of this theory are generically invariant with respect to a great variety of forcings.

Assume $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ is elementary with critical point smaller than λ . Can any of the results presented in this paper be of any use in the study of which type of generic absoluteness results may hold at the level of $L(V_{\lambda+1})$?

The reader is referred to [1, 3, 2, 30, 32] for further examinations of these topics.

Acknowledgements. This paper owes much of its clarity to the suggestions of Raphael Carroy, and takes advantage of several several fruitful discussions we shared on the material presented here. I wish to thank Jeffrey Bergflak for his many useful comments.

The author acknowledges support from: Kurt Gödel Research Prize Fellowship 2010, PRIN grant 2012: Logica, modelli e insiemi, San Paolo Junior PI grant 2012.

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