Forcing axioms and cardinal arithmetic

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CARDINAL ARITHMETIC

Since the introduction of the notion of cardinal and ordinal numbers by Cantor, Cardinal Arithmetic has been a central subject of research in Set Theory.

Question 1 *What is the value of κ^λ for cardinals κ and λ?*

In particular the continuum problem is the instance of this general question to the case:

$$\kappa = 2, \;\; \lambda = \aleph_0.$$

The continuum hypothesis CH asserts that:

$$2^{\aleph_0} = \aleph_1.$$

In the usual ZFC framework there is no definite answer to almost all the instances of this question.

Up to the seventies, König's inequality was one of the sharpest result on cardinal arithmetic. However this inequality dates back to the beginning of the XX-th century:

Lemma 1 *(König 190?)* $\texttt{cof}(2^\kappa) > \kappa$ *for any regular* κ.

Soon after the discovery of forcing (mids of the sixties), Easton showed that this inequality was the sharpest possible result when we restrict our attention on the class of regular cardinals.

The situation for singular cardinals is much different.

When singular cardinals are concerned upper bounds for the exponential function can be computed in ZFC, in particular:

Theorem 2 *(Shelah)* $\aleph_\omega^{\aleph_0} < (2^{\aleph_0})^+ + \aleph_{\omega_4}$

The singular cardinal hypothesis SCH asserts that $\lambda^{\mathtt{cof}\lambda} = \lambda^+$ for all singular cardinal $\lambda \geq 2^{\mathtt{cof}\lambda}$.

Silver has shown the following:

Theorem 3 *(Silver) Let λ be the least singular cardinal $\kappa \geq 2^{\mathtt{cof}\kappa}$ such that $\lambda^{\mathtt{cof}\lambda} > \lambda^+$, then $\mathtt{cof}\lambda = \omega$.*

These results are almost optimal for the theory ZFC. Assuming large cardinals one can build models of ZFC in which SCH first fails at $\aleph_\omega$.

The failure of SCH has large cardinal consequences. In particular if SCH fails, there is an inner model with measurable cardinals.

There are models of ZFC in which $2^{\aleph_0} < \aleph_\omega$ and $\aleph_\omega^{\aleph_0} > \aleph_\alpha$ for any countable ordinal α.

It remains nonentheless open whether $(2^{\aleph_0})^+ + \aleph_{\omega_4}$ is the best possible bound for $\aleph_\omega^{\aleph_0}$ if no other extra axioms are assumed (cfr Magidor, Gitik).

It remains also open what are the possible configurations of the function $\kappa^{\mathtt{cof}\,\kappa}$ for κ ranging over all singular cardinals (cfr. Cummings, Gitik).

FORCING AXIOMS AND LARGE CARDINALS

Starting from the sixties on two kind of axioms have received considerable attention by the community of set theorists: large cardinals axioms and forcing axioms.

These axioms arose naturally in the course of investigation of Set Theory because forcing and large cardinals are in fact the only known means to generate models of ZFC with new sets.

LARGE CARDINALS

The large cardinals axioms asserts certain combinatorial properties of infinite cardinals which try to capture Cantor's intuition that the sequence of infinite cardinals cannot be described by a definite set of rules. For example:

- The existence of an inaccessible cardinal asserts that the standard ZFC operations cannot generate all the infinite cardinal just by iterating the process of set formation starting from $\aleph_0$.
- The existence of a measurable cardinals implies that $V \neq L$, i.e. that the universe of sets cannot be obtained just as the closure of the class of ordinals by Gödel operations.

FORCING

Forcing is the unique known method to produce from a model M a new model N of set theory with more sets but the same ordinals.

Forcing can be seen as an algebra on sets which with input P a partial poset and $\mathcal{F}$ a family of dense sets of P provides as output a filter for P which has non-empty intersection with all the elements of $\mathcal{F}$.

By an algebra on sets I mean a function which takes a family of sets as an input and produce a set as an output.

Taking this approach, forcing axioms can be presented as generalizations of the Baire's category theorem:

Theorem 4 *(Baire) Let $\mathcal{F}$ be a countable family of dense open subsets of $\mathbb{R}$ then $\bigcap \mathcal{F}$ is non-empty*

Forcing axioms are obtained enlarging the class of topological spaces and the size of the family of dense sets.

> $FA(\mathcal{A}, \kappa)$ *holds if whenever $\mathcal{A}$ is a class of topological spaces and for some $X \in \mathcal{A}$, $\mathcal{F}$ is a family of less than κ dense open sets of X, then $\bigcap \mathcal{F}$ is non-empty.*

They are natural combinatorial statements which decide many of the questions left open by the usual axioms ZFC of set theory, in particular we will concentrate on their effects on cardinal arithmetic.

SOME HISTORY ON FORCING AXIOMS

Late 1960s. Solovay and Tennenbaum introduce iterated forcing. Martin and Solovay formulate Martin's axiom MA as an abstraction of Solovay and Tennenbaum's approach to solving Suslin's problem. $\mathsf{MA} + \neg\mathsf{CH}$ provides a rich structure theory for the reals.

Early 1980s. Shelah develops the theory of countable support iterations of proper forcings. Baumgartner and Shelah formulate the proper forcing axiom PFA. Very successful in resolving questions left open by MA. The proof of the consistency of PFA uses a supercompact cardinal.

Mid to late 1980s. Shelah develops revised countable support iteration of semi proper forcings. Foreman, Magidor, and Shelah formulate Martin's Maximum MM - the provably maximal forcing axiom. Resolves questions left open by PFA such as the saturation of the nonstationary ideal NS_{ω_1}, Chang's Conjecture, etc.

Key questions in cardinal arithmetic left open by ZFC and resolved by forcing axioms are the value of the continuum $\mathfrak{c}$ and the SCH which is also a consequence of large cardinals.

In the early seventies Solovay showed that if λ is strongly compact then $\kappa^{\theta} = \kappa$ for all regular $\kappa \geq \lambda$ and for all $\theta < \lambda$.

Combining this result with Silver's theorem one obtains that $\kappa^{\mathtt{cof}(\kappa)} = \kappa^{+}$ for all singular $\kappa \geq \lambda$, i.e. the SCH above λ.

Foreman, Magidor and Shelah later showed that MM implies that $\mathfrak{c} = \aleph_2$ and the SCH.

Their proof reposes on the fact that in a model of MM the properties of regular cardinals κ greater or equal than $\aleph_2$ resembles in many respects to the properties of regular cardinals $\kappa \geq \lambda$ for some λ supercompact.

They can show that a variation of Solovay arguments yield that MM implies that $\kappa^{\omega_1} = \kappa$ for all regular $\kappa \geq \omega_2$.

A further simple argument is needed to obtain that $2^{\omega} > \aleph_1$. The SCH is once again obtained combining their theorem with Silver's theorem.

THE EFFECTS OF THE PROPER FORCING AXIOM ON CARDINAL ARITHMETIC.

To obtain the same conclusions from PFA has demanded different ideas. In fact the models of PFA loose a great amount of the properties of the supercompact λ from which they are so far obtained.

For this reason there are no straigthforward means to modify Solovay's argument in order to fit also with this situation.

However Veličković and Todorčević in the late eighties were the first to obtain a proof of $2^{\aleph_0} = \aleph_2$ from PFA.

Another proof of this result led to isolate a very interesting combinatorial principle which follows from PFA, the open coloring axiom OCA.

Later on another argument yielding a similar conclusion led to the formulation of the P-ideal dichotomy PID. Another very successful combinatorial principle which is a consequence of PFA.

These principles deserves a particular interest because they made apparent an unexpected resemblance between the combinatorics of uncountable cardinals and certain analytic properties of the reals.

OCA has an exact counterpart in the context of analytic sets of reals. It is one of the few examples of a property of pure set theory which led to new theorems of descriptive set theory (cfr. Todorčević).

The PID generalizes to the level of uncountable cardinals phenomena which were already studied in the context of descriptive set theory (cfr. Solecki).

The P-Ideal Dichotomy

Let Z be any uncountable set. $\mathcal{I} \subseteq [Z]^\omega$ is a P-ideal if for every family $\{X_n : n \in \omega\} \subseteq \mathcal{I}$, there is $X \in \mathcal{I}$ which contains every X_n modulo finite.

Let $\mathcal{I}$ be a P-ideal on $[Z]^\omega$ then one of the following holds:

(i) there is an uncountable $Y \subseteq Z$ such that $[Y]^\omega \subseteq \mathcal{I}$,

(ii) there is a family $\{A_n : n \in \omega\}$ such that for every n, $[A_n]^\omega \cap \mathcal{I} = \emptyset$ and $\bigcup_n A_n = Z$.

Some facts about the PID (all the results are by Todorčević):

Theorem 5 PFA *implies the* PID.

Theorem 6 PID *implies that* $\square(\kappa)$ *fails for all regular* $\kappa \geq \aleph_2$.

Theorem 7 PID *implies* $\mathbf{b} \leq \aleph_2$.

$\mathbf{b}$ is the bounding number, i.e. the least cardinal κ for which there is a family $\mathcal{A}$ of subsets of ω of size κ such that for every $X \subseteq \omega$ there is $Y \in \mathcal{A}$ with $Y \setminus X$ infinite.

Theorem 8 PID *implies that there are no Souslin tree on* $\aleph_1$.

Theorem 9 PID *is compatible with* GCH.

This is the result I want to present:

Theorem 10 PID *implies* $\kappa^\omega = \kappa$ *for all regular* $\kappa \geq 2^{\aleph_0}$.

Once again a combination of this result with Silver's theorem yields the SCH.

PROOF:

The spirit of the proof is the same of the original argument of Todorčević and Abraham that PID implies that every (ω_1, ω_1) gap on $(P(\omega), \subseteq^*)$ is an Haussdorff gap.

For any cardinal κ of countable cofinality,

$$\mathcal{C} = (K(n, \beta) : n < \omega \,, \beta \in \kappa^+)$$

is a covering matrix for κ^+ if:

(i) for all n and α, $|K(n, \alpha)| < \kappa$,

(ii) for all $\alpha \in \kappa^+$, $K(n, \alpha) \subseteq K(m, \alpha)$ for $n < m$,

(iii) for all $\alpha \in \kappa^+$, $\alpha + 1 = \bigcup_n K(n, \alpha)$,

(iv) for all $\alpha < \beta \in \kappa^+$, if $\alpha \in K(n, \beta)$, then $K(n, \alpha) \subseteq K(n, \beta)$.

(v) for all $X \in [\kappa]^\omega$ there is $\gamma_X < \kappa^+$ such that for all β, there is n such that $K(m, \beta) \cap X = K(m, \gamma_X)$ for all $m \geq n$.

Fact 1 *For any κ singular cardinal of countable cofinality, there is a covering matrix $\mathcal{C}$ on κ^+.*

Proof: We show it just in the case $\kappa > \mathfrak{c}$, however appealing to the approachability ideal $\mathcal{I}[\kappa^+]$ we can drop this assumption. Let for all β, $\phi_\beta : \kappa \to \beta$ be a surjection and $(\kappa_n)_n$ be a strictly increasing sequence of regular cardinals converging to κ.

Define $K(n, \beta)$ by induction on β as follows:

$$\phi_\beta[\kappa_n] \cup \{K(n, \gamma) : \gamma \in \phi_\beta[\kappa_n]\}$$

It is easy to check *(i)*,$\cdots$,*(iv)*.

To see *(v)* let $X \in [\kappa^+]^\omega$ be arbitrary.

Since $\mathfrak{c} < \kappa^+$ and there are at most $\mathfrak{c}$ many subsets of X, there is a stationary subset S of κ^+ and a fixed decomposition of X as the increasing union of sets X_n such that $X \cap K(n, \alpha) = X_n$ for all α in S and for all n.

Now property *(iv)* of the matrix guarantees that this property of S is enough to get *(v)* for X with $\gamma_X = \min(S)$. $\square$

Definition 11 *(*$\mathsf{CP}(\mathcal{D})$*):)* *Let $\mathcal{D}$ be a covering matrix for κ^+. $\mathcal{D}$ has the "Covering Property" if there is an unbounded subset A_0 of κ^+ such that $[A_0]^\omega$ is covered by $\mathcal{D}$.*

CP is the statement: $\mathsf{CP}(\mathcal{D})$ *holds for all covering matrices $\mathcal{D}$ on any $\kappa^+ > 2^{\aleph_0}$ successor of a singular cardinal of countable cofinality.*

Fact 2 *Assume* CP. *Then* $\lambda^{\aleph_0} = \lambda$, *for every* $\lambda \geq 2^{\aleph_0}$ *of uncountable cofinality.*

Proof: We will prove this fact by induction.

The base case is trivial.

If $\lambda = \kappa^+$ with $\mathtt{cof}(\kappa) > \omega$, then:

$$\lambda^{\aleph_0} = \lambda \cdot \kappa^{\aleph_0} = \lambda \cdot \kappa = \lambda,$$

by the inductive hypothesis on κ and using the Haussdorff formula $(\kappa^+)^\lambda = \kappa^\lambda + \kappa^+$.

If λ is a limit cardinal and $\mathtt{cof}(\lambda) > \omega$, then:

$$\lambda^{\aleph_0} = \sup\{\mu^{\aleph_0} : \mu < \lambda \,\&\, \mu \, regular\},$$

so the result also follows by the inductive hypothesis.

The only interesting case is when $\lambda = \kappa^+$, with κ singular of countable cofinality for we cannot apply the inductive hypothesis to κ^ω, since κ has countable cofinality.

In this case let $\mathcal{D}$ be a covering matrix for κ^+.

Remark that by our inductive assumptions, since every $K(n, \beta) \in \mathcal{D}$ has size less than κ, $|[K(n, \beta)]^\omega|$ has size less than κ.

So $\bigcup\{[K(n, \beta)]^\omega : n < \omega \,\&\, \beta \in \kappa^+\}$ has size κ^+. Use $\mathsf{CP}(\mathcal{D})$ to find $A_0 \subseteq \kappa^+$ unbounded in κ^+, such that $[A_0]^\omega$ is covered by $\mathcal{D}$.

Then:

$$[A_0]^\omega \subseteq \bigcup\{[K(n, \beta)]^\omega : n < \omega \,\&\, \beta \in \kappa^+\},$$

from which the conclusion follows. $\square$

Here comes the P-ideal dichotomy:

Lemma 12 PID *implies* $\mathsf{CP}(\mathcal{D})$ *for every covering matrix* $\mathcal{D}$.

Let κ be a cardinal of countable cofinality and $\mathcal{D}$ be a covering matrix on κ^+ and set

$$\mathcal{I} = \{X \in [\kappa^+]^\omega : \text{ for all } n,\ \alpha \\ X \cap K(n, \alpha) \text{ is finite } \}$$

Claim 13 $\mathcal{I}$ *is a P-ideal.*

Proof of the claim:

Let $\{Y_n : n \in \omega\} \subseteq \mathcal{I}$. We need to find $X \in \mathcal{I}$ containing modulo finite every Y_n.

Let $Y = \bigcup_n Y_n$. Now appealing to property *(v)* of $\mathcal{D}$ find γ_Y such that for all $\beta \geq \gamma_Y$ there is n such that $K(m, \beta) \cap Y = K(m, \gamma_Y) \cap Y$ for all $m \geq n$.

Now it is easy to build $X \subseteq Y$ which has finite intersection with $K(n, \gamma_Y)$ for all n and which contains modulo finite every X_n.

Using properties *(iv)* for the $\alpha < \gamma_Y$ and *(v)* for the others α it is immediate to check that X has in fact a finite intersection with all $K(n, \alpha)$.

To build this X, notice that for every n, m, $Y_n \cap K(m, \alpha)$ is finite, set $X(n, m)$ to be the finite set

$$Y_n \cap K(m, \gamma_Y) \setminus K(m-1, \gamma_Y)$$

and let:

$$X = \bigcup_n \bigcup_{j \geq n} X(n, j).$$

Notice that $X_n = \bigcup_j X(n, j)$ and $\bigcup_{j \geq n} X(n, j) \subseteq X$, so we have that $X_n \subseteq^* X$.

Moreover $X \cap K(n, \alpha) = \bigcup_{j \leq i \leq n} X(j, i)$, so it is finite. $\square$

Now remark that if $Z \subseteq \kappa$ is any set of ordinals of size $\aleph_1$ and $\alpha \in \kappa^+$ is larger than $\sup(Z)$, there must be an n such that $Z \cap K(n, \alpha)$ is uncountable. This means that $\mathcal{I} \not\subseteq [Z]^\omega$, since any countable subset of $Z \cap K(n, \alpha)$ is not in $\mathcal{I}$.

This forbids $\mathcal{I}$ to satisfy the first alternative of the P-ideal dichotomy.

So the second possibility must be the case, i.e. we can split κ^+ in countably many sets A_n such that $\kappa = \bigcup_n A_n$ and for each n, $[A_n]^\omega \cap \mathcal{I} = \emptyset$.

Claim 14 *For every n, $[A_n]^\omega$ is covered by $\mathcal{D}$.*

Proof of the claim: Suppose not and find n and $X \in [A_n]^\omega$ such that for all α and m, $X \not\subseteq K(m, \alpha)$.

This means that for γ_X we can find $X_0 \subseteq X$ infinite such that for all m, $X_0 \cap K(m, \gamma_X)$ is finite.

However by property *(iv)* and *(v)* of $\mathcal{D}$, X_0 has in fact finite intersection with all $K(n, \alpha)$, i.e. $X_0 \in \mathcal{I}$ a contradiction. $\square$

Now any A_n which is unbounded in κ^+ witnesses $\mathsf{CP}(\mathcal{D})$. This concludes the proof of everything. $\blacksquare$