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APPLICATIONS OF THE PROPER FORCING AXIOM TO CARDINAL ARITHMETIC

Matteo Viale

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DIRECTEUR:	Boban Veličković
CO-DIRECTEUR:	Alessandro Andretta
JURY :	Mirna Džamonja
	Ilijas Farah
	Justin Moore (rapporteur)
AUTRES RAPPORTEURS :	Menachem Magidor
	Hugh Woodin

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Introduction

Dans cette thèse j'analyse quelques conséquences de l'axiome du forcing propre PFA sur les propriétés combinatoires des cardinaux infinis. Le principal résultat obtenu est le suivant :

Théorème 1 *L'axiome du forcing propre implique l'hypothèse des cardinaux singuliers*[1].

La preuve de ce théorème mérite l'attention pour plusieurs raisons: Premièrement on montre que SCH est une conséquence de cet axiome de forcing et pour cette raison on résout dans le sens attendu un problème classique dans ce domaine. La preuve repose sur une propriété des cardinaux non-dénombrables qui appartient aux cardinaux réguliers au dessus d'un cardinal fortement compact et qui est une conséquence d' au moins deux principes combinatoires qui découlent de PFA. Le premier, la dichotomie des P-idéaux, a été isolé par Todorčevic et Abraham. Le deuxième, introduit par Moore, est le principe de réflexion MRP. La preuve demande presque toute la force de consistance connue de PFA ou d' un cardinal fortement compact parce que on peut aisément la prolonger en une preuve de la négation du principe du carré.

La thèse est organisée en cinq chapitres et deux appendices:

- Dans le premier chapitre je donne une brève présentation des axiomes du forcing et des grands cardinaux en prenant inspiration du programme de Gödel. Je les présente comme une solution plausible d'un grand nombre de problèmes en théorie des ensembles qui ont été posés au cours du siècle dernier.
- Dans le deuxième chapitre, je concentre mon intérêt sur la dichotomie des P-idéaux introduite par Todorčević en [41] en généralisant les travaux de lui-même et Abraham parus dans [1]. Ce principe est une propriété combinatoire qui découle de PFA mais qui en même temps est compatible avec l'hypothèse du continu généralisé GCH. Il y a plein de conséquences de PFA qui peuvent être déduites a partir de PID. En particulier on remarque que PID implique la non-existence d'arbres de Souslin, la négation du principe

[1] SCH dans la suite.

du carré et une représentation très simple de la structure $(P(\omega)/FIN, \subseteq^*)$. J'analyserai en détail la preuve des deux premiers théorèmes parce que ces démonstrations suivent un schéma général qui peut être utilisé pour déduire SCH à partir de PID.

- Dans le troisième chapitre, j'introduis le principe de réflexion MRP présenté par Moore dans [32] et j'analyse quelques unes de ses conséquences. Ce principe est l'unique principe de réflexion qui découle de PFA. On verra qu'il peut être utilisé comme tous les autres principes de réflexion pour démontrer des résultats d'arithmétique cardinale: l'égalité $\mathfrak{c} = \omega_2$ et SCH sont des conséquences de ce principe. En plus ce principe a été un élément essentiel pour établir la validité de l'existence d'une base finie pour les ordres linéaires non-dénombrables dans les modèles de PFA. Même si les preuves de SCH a partir de MRP et de PID sont similaires, ces deux principes sont indépendants parce que MRP est compatible avec l'existence d'un arbre de Souslin et refuse GCH.

- Dans le quatrième chapitre je présente le résultat principal de cette thèse, c'est-à-dire la preuve que PFA implique SCH. On introduit une propriété de recouvrement qu'on appelle CP et qui permet de démontrer SCH d'une façon similaire à la preuve de ce principe à partir du lemme de recouvrement de Jensen. Mais CP est une hypothèse forte: un argument simple permet de nier le principe du carré en acceptant CP. Le résultat principal du chapitre est la preuve que CP est une conséquence à la fois de l'existence d'un cardinal fortement compact, de PID et de MRP. Quelques autres conséquences de ce principe de recouvrement sont montrées. En particulier on montre qu'une forme faible de réflexion pour les ensembles stationnaires découle de PFA.

- Dans le cinquième chapitre on étudie la rigidité des modèles de CP. Récemmment Veličković et Caicedo [6] ont montré que deux modèles $M \subseteq V$ de PFA avec le même ω_2 ont les mêmes réels. Dans le but de généraliser ce résultat à des cardinaux plus grands on montre que si V est un modèle de CP et M est un modèle interne avec les mêmes cardinaux et les mêmes réels, alors le plus petit κ tel que $\kappa^\omega \setminus M$ est non-vide n'est pas régulier en M. En plus je montrerai que si $\aleph_\omega$ est le plus petit κ alors $\aleph_\omega$ est presque un cardinal de Jónsson en M. D'autres restrictions sur κ sont aussi démontrées.

- Dans le premier appendice je démontre que PFA implique PID.

- Dans le deuxième appendice je prouve que PFA implique MRP et j'esquisse une démonstration du fait que MRP et PID sont deux principes mutuellement indépendants.

Les axiomes de forcing et l'arithmétique des cardinaux

L'arithmétique des cardinaux a été un des principaux domaines de recherche à partir des travaux de Cantor [7] dans la dernière partie du 19-siècle. Au cours de la définition des propriétés de base des cardinaux infinis, Cantor introduisait la notion d' exponentielle des cardinaux. Il s' apercevait rapidement qu' il n' était pas capable de calculer la valeur de la fonction exponentielle $\kappa \mapsto 2^\kappa$ même pour le plus petit cardinal infini $\aleph_0$. Il a posé donc le problème du continu dans lequel il demandait la valeur du cardinal $2^{\aleph_0}$ et a formulé la célèbre hypothèse du continu CH qui affirme que $2^{\aleph_0} = \aleph_1$.

Plus de quatre-vingt ans de recherches ont été nécessaires pour montrer que CH est indépendante de ZFC, l'habituelle axiomatisation de la théorie des ensembles. En 1939 Gödel [15] a introduit le modèle minimal de ZFC, le modèle L des ensembles constructibles et a montré que CH est valide dans ce modèle. Mais il n' était pas satisfait par son résultat, et ses spéculations successives sur ce problème aboutiront dans son article de 1947 [16] et [17][2] où il a présenté son point de vue personnel sur le problème du continu et plus en général sur le statut ontologique de la théorie des ensembles et des problèmes qui apparaissaient dans ce domaine. Sur ce sujet il a pris une position platonicienne et a déclaré sur CH:

> ...if the meaning of the primitive terms of set theory... are accepted as sound, it follows that the set-theorethical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality; ([17], section 3, p. 476)

En effet il a conjecturé que ZFC était une théorie trop faible pour régler le problème du continu et a supposé que CH était indépendant des axiomes de ZFC. Il a proposé aussi quelques arguments heuristiques qui permettaient de distinguer entre les cardinaux $\aleph_1$ et $2^{\aleph_0}$ ([17], section 4, pp. 478 – 479). Il a présenté plusieurs exemples qui marquaient une différence entre les deux cardinaux. Nous en soulignons deux:

1. Le fait que par un résultat de Sierpinski il est possible de déduire à partir de ZFC l'existence d' un ensemble de réels de taille $\aleph_1$ avec intersection maigre avec tous les sous-ensemble parfaits de $\mathbb{R}$ ([22], p.269), mais aucun moyen de produire un tel ensemble de taille $2^{\aleph_0}$ était connu.

2. Le fait que par un résultat de Sierpinski et Luzin [26] CH implique l'existence d' un ensemble non-dénombrable de mesure fortement nulle, mais aucun moyen de produire un tel ensemble à partir de ZFC n'était connu [3].

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[2] Il y a deux éditions de cet article, celle [16] et celui de 1964 [17]. J' utiliserai dans la suite celui de 1964, qui exprime quelque réflexions supplémentaires que Gödel a fait sur ce sujet après la publication de [16].

[3] Laver a montré en utilisant le forcing qu'il y a des modèles de ZFC sans ensemble non-dénombrable de mesure fortement nulle.

Il en a tiré la conclusion qu'il était nécessaire de rechercher des nouveaux axiomes qui pouvaient donner une solution satisfaisante au problème du continu et aux autres problèmes sans réponse qui étaient apparu en ce domaine.
Gödel a donné plusieurs critères que ces nouveaux axiomes devraient satisfaire pour être acceptés comme vrais. En particulier, il a souligné que ces axiomes devraient satisfaire une propriété de "maximum" qu' il n'a pas énoncé clairement mais qu' il a opposé génériquement aux conditions de minimalité satisfaites par l'univers L des ensembles constructibles ([17], section 4, p.478, note 19). Il s' attendait au fait que les progrès dans l'étude du problème du continu auraient permis d'avoir une intuition plus nette sur la nature de cette propriété de maximum. Il a pu aussi proposer une classe d' axiomes qui satisfaisaient ces critères, plus précisément les axiomes forts de l'infini ou dans la terminologie courante, les axiomes des grands cardinaux ([17], section 4, pp. $476-477$ et note 16). Approximativement on peut dire que ces axiomes établissent certaines propriétés combinatoires des cardinaux infinis qui cherchent à donner une signification mathématique précise à l'intuition de Cantor et de Gödel qu' on ne peut pas avoir un ensemble précis de règles qui puisse décrire le processus de génération de la séquence des cardinaux infinis. La propriété du maximum est piégée par ces axiomes dans le sens que l'univers des ensembles est clos par un quelconque moyen consistant à générer des nouveaux cardinaux infinis.

Une étape cruciale en théorie des ensembles fut la découverte de la méthode de forcing en 1963 par Cohen [8]. Cette méthode a permit à Cohen de construire un modèle de ZFC dans lequel CH est faux. Le forcing est une technique générale qui permet de construire à partir d'un modèle donné M de ZFC, un modèle N qui contient strictement M mais qui a les mêmes ordinaux que M. En effet le forcing a rapidement permis de construire plusieurs modèles qui ont montré l'indépendance de ZFC de presque toutes les conjectures encore ouvertes dans le domaine jusqu'à présent.

A partir des années soixante-dix, la recherche des effets des grands cardinaux d' une part et la maîtrise de plus en plus affinée de la méthode du forcing d'autre part ont donné de nouvelles idées sur la façon d' approcher le programme de Gödel.

En poussant vers les limites l'esprit du forcing, une autre interprétation claire de la propriété du maximum a pu être proposée. Pour cela on a besoin du concept d' axiome du forcing, qui dit à peu près que si un ensemble satisfaisant certaines conditions existe dans un modèle produit en utilisant la méthode du forcing, alors cet ensemble existe vraiment. Cela peut être énoncé comme un principe de saturation de l'univers par rapport au forcing. Cette formulation de la propriété du maximum est complémentaire aux axiomes des grands cardinaux qui de leur côté postulent l'existence de cardinaux et ordinaux de plus en plus grands[4]. Ces axiomes se sont révélés suffisamment puissants pour pouvoir résoudre un grand nombre de problèmes classiques de la théorie des ensembles. Par exemple il y a plusieurs démonstrations à partir des certaines axiomes de forcing que $2^{\aleph_0} = \aleph_2$ (voir en particulier [11], [31], [32], [40], [42], [44]).

[4]En effet on montrera dans la suite que ces deux types d' axiomes sont strictement liés.

Il y a une présentation topologique simple de ce type d' axiomes. Le théorème des catégories de Baire affirme que pour une grande variété d' espaces topologiques entre lesquels il y a en particulier les nombres réels, l'intersection d' un ensemble dénombrable d' ouverts denses est non-vide. Les axiomes du forcing peuvent être présentés comme un renforcement de cet théorème.

Définition 1 $\mathsf{FA}(\mathcal{A}, \kappa)$ *est vrai si chaque fois que* $\mathcal{A}$ *est une classe d' espaces topologiques et pour n'importe quel* $X \in \mathcal{A}$, $\mathcal{F}$ *est une famille de moins de* κ *ouverts denses de* X, *alors* $\bigcap \mathcal{F}$ *est non vide.*

Le théorème des catégories de Baire peut être formulé comme $\mathsf{FA}(\{\mathbb{R}\}, \aleph_1)$. Au début des années soixante-dix, Solovay et Martin ont introduit le premier axiome de forcing non-trivial, l'axiome de Martin (MA). Celui-ci est $FA(K, 2^{\aleph_0})$, où K est la classe des espaces topologiques compacts qui satisfont aussi la propriété des antichaînes dénombrables, une classe d' espaces qui inclut les nombres réels et concerne beaucoup des propriétés topologiques de ces nombres. Au début des années quatre-vingt, Baumgartner et Shelah [4] ont introduit l'axiome du forcing propre (PFA), et à la fin des années quatre-vingt, Foreman, Magidor et Shelah [11] ont présenté Martin' s maximum (MM), l'axiome de forcing démonstrativement plus fort[5] au moins pour $\kappa = \aleph_2$. Entre autre ils ont montré dans cet article que MM implique $\mathfrak{c} = \aleph_2$. Quelques années après Veličković et Todorčević ont obtenu la même conclusion en supposant le plus faible axiome PFA. D'autres preuves que PFA implique $\mathfrak{c} = \aleph_2$ ont été présentées par la suite aussi par Todorčević [40], Moore [32], [31] et Caicedo et Veličković [6]. Ces preuves sont aussi intéressantes pour d' autres raisons: elles ont permis d' isoler des principes qui ont conduit à des nouveaux théorèmes dans la théorie combinatoire et descriptive des ensembles[6].

Pendant les années quatre-vingt-dix, l'attention des chercheurs s'est tournée vers la version bornée des axiomes de forcing en particulier vers le bounded Martin' s maximum BMM et le bounded proper forcing axiom BPFA. Ces derniers semblent être une intéressante solution du problème du continu et en partie l'accomplissement du programme de Gödel. Ceci est du en partie à la caractérisation simple de ces axiomes donnée par Bagaria qui a montré que la version bornée des axiomes de forcing peut être présentée comme une généralisation au niveau de $H(\aleph_2)$ du théorème d' absoluté de Schoenfield. Ce théorème affirme

[5] Cette assertion a besoin de quelques précisions: PFA, MA et tous les axiomes de forcing qui peuvent être formulés en accord avec la définition 1 sont une conséquence de MM. En effet MM est $\mathsf{FA}(SSP, \aleph_2)$, où SSP dénote la classe des algèbres de Boole complètes qui préservent les sous-ensembles stationnaires de ω_1. Il a été montrée dans [11] que si X est une algèbre de Boole complète qui n' est pas SSP alors il y a un ouvert Y de X et une famille $\mathcal{F}$ de $\aleph_1$ sous-ensembles ouverts de Y tels que $\bigcap \mathcal{F}$ est vide. Pour cette raison on peut affirmer que MM est l'axiome démonstrativement plus fort. Mais par exemple l'axiome $(*)$ de Woodin peut être qualifié aussi axiome de forcing même s'il est indépendant de MM. De toute façon je ne sais pas si dans un quelconque modèle de ZFC, $(*)$ peut être présenté en accord avec la définition 1.

[6] Je me réfère en particulier à l'axiome OCA introduit par Todorčević comme conséquence de PFA et qui a conduit à la découverte des nouvelles dichotomies en théorie descriptive des ensembles et à l'axiome MRP introduit par Moore [32] qui a permis de démontrer la consistance de l'existence d' une base finie pour les ordres linéaires non-dénombrables (voir [33] et [20]).

que $L_{\aleph_1} \prec_1 V$. Les axiomes bornés de forcing peuvent être presentés comme un renforcement de ce théorème de la forme $H(\aleph_2) \prec_1 V^P$, où P varie dans la classe appropriée de forcing. D' autre part Moore a montré que le faible BPFA implique que $\mathfrak{c} = \aleph_2$ [32].

Finalement les efforts de plusieurs logiciens, y compris Woodin, ont permis de combiner le grand nombre des résultats partiels obtenus à partir des axiomes de forcing avec ceux qui lient la théorie de $L(\mathbb{R})$ et l'axiome de détermination avec les hypothèses des grands cardinaux. Premièrement Woodin a montré que l'existence d' une classe propre de cardinaux de Woodin implique que la théorie de $L(\mathbb{R})$ ne peut pas être modifiée par le forcing et de cette façon a pu renforcer le théorème d' absoluté de Schoenfield aux plus grandes généralités possible en supposant que les grands cardinaux et le forcing sont les uniques moyens de produire un témoin d' une propriété projective des nombre réels. En poussant ces idées au niveau de $H(\aleph_2)$, il a introduit la Ω-logique. C'est la logique qui a comme modèles les univers V_α^B où B est une algèbre de Boole complète. Woodin a montré que la notion de validité pour cette logique est invariante par forcing (i.e. $V \models$ "$T \models_\Omega \phi$" si et seulement si $V^B \models$ "$T \models_\Omega \phi$" pour tous les algèbres de Boole complètes B). Il a aussi introduit une notion syntactique de preuve $\vdash_\Omega$ qui, d'après les mots de Woodin,"is a natural transfinite generalization of the classical notion of proof for first order logic" [47]. La Ω-conjecture demande si le théorème de complétude de Gödel peut être démontré aussi pour la Ω-logique. Si cette question a une réponse positive, Woodin a montré qu'il y a une forme forte de l'axiome[7] BMM qui décide en Ω-logique toute la théorie de $H(\aleph_2)$ (voir [48], [49] or [3]). Il a ensuite montré que chaque axiome ϕ avec cette propriété décidera en Ω-logique que CH est faux. Par rapport à cela, la Ω-logique se présente comme un renforcement du concept de démontrabilité qui donne une plausibilité à tous les résultats obtenus jusqu'à maintenant dans le domaine des axiomes de forcing qui tendent à montrer que $\mathfrak{c} = \aleph_2$. En reprenant encore une fois les mots de Gödel sur les effets de son théorème d' incomplétude et sur les remèdes possibles:

> It is well known that in whichever way you make [the concept of demonstrability] precise by means of a formalism, the contemplation of this very formalism gives rise to new axioms which are exactly as evident as those with which you started, and that this process can be iterated into the transfinite. So there cannot exist any formalism which would embrace all these steps; but this does not exclude that all these steps.... could be described and collected together in some non constructive way. ([14] p. 151)

Par rapport à cela, la Ω-Logic semble être une candidate plausible pour le système de preuve non-constructive que Gödel proposait de rechercher.

Après la découverte du forcing, une étude poussée des comportements possibles de la fonction exponentielle sur les cardinaux non dénombrables a été poursuivie. Des travaux sur cet argument avaient déjà été réalisés. L'hypothèse

[7] C' est en effet son axiome $(*)$

généralisée du continu GCH affirme que $2^\kappa = \kappa^+$ pour tous les cardinaux infinis κ. Par exemple Gödel [15] avait déjà montré que GCH est vrai dans L. Mais dans une des premières applications du forcing Easton [10] a généralisé le résultat de Cohen en montrant qu'il y a très peu de restrictions sur le comportement de la fonction exponentielle $\kappa \mapsto 2^\kappa$ sur la classe des cardinaux réguliers. La situation pour les cardinaux singulier est beaucoup plus subtile. On rappelle que SCH affirme que $\kappa^{\mathtt{cof}\kappa} = \kappa^+ + 2^{\mathtt{cof}\kappa}$ pour tous les cardinaux singuliers κ. Premièrement Solovay a montré que SCH est vrai au dessus d' un cardinal fortement compact (voir aussi le chapitre 4 pour une preuve différente de ce théorème). Peu après Silver [37] a montré que SCH ne peut pas faillir en premier sur un cardinal singulier de co-finalité non-dénombrable. Finalement un des résultats majeurs de Shelah [36] est la preuve que $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4} + \mathfrak{c}^+$ est vrai en ZFC. Donc contrairement à la situation pour les cardinaux réguliers, il y a des bornes sur le comportement de la fonction exponentielle sur les cardinaux singuliers qui peuvent être calculés en ZFC. De toute façon, on sait que SCH peut faillir[8]. Le ròle des grands cardinaux en ce contexte est bivalent. D' une part ils sont nécessaires pour la construction de modèles où SCH est faux, parce que chacun de ces modèles a un modèle interne avec des cardinaux mesurables (voir Gitik [12]). D'autre part, comme on l'a déjà souligné, SCH est vrai au dessus d' un cardinal fortement compact.

MM implique des principes de réflexions similaires à ceux qui ont été utilisés dans la preuve de Solovay. En effet une variation de l'argument de Solovay montre que SCH est aussi une conséquence de cet axiome de forcing (voir [11]). Ce résultat a été amélioré par Velickovic [44] peu d' années après, qui montrait que SCH découle aussi de PFA$^+$ un axiome de forcing un peu plus fort que PFA mais plus faible que MM. Ce qui a été montré en [44] est que si $\theta > \aleph_1$ est régulier et les sous ensembles stationnaires de $[\theta]^\omega$ sont réflexifs, alors $\theta^\omega = \theta$. Ceci combiné avec le résultat de Silver suffit pour montrer SCH. Le problème de déterminer si SCH est une conséquence de PFA restait ouvert.

Peu de progrès ont été réalisés pendant plus de quinze années sur ce problème parce que aucun principe de réflexion découlant de PFA et ressemblant à ceux qui avaient été utilisés dans les autres preuves de SCH n'était connu. Les difficultés ressemblaient à celles qui avaient rendu complexe une preuve de $\mathfrak{c} = \aleph_2$ à partir de PFA. Avant l'argument de Veličković et Todorčević, les preuves connues de $\mathfrak{c} = \aleph_2$ à partir de MM utilisaient des arguments de réflexion qui n' étaient pas une conséquence de PFA.

Un travail supplémentaire a conduit à isoler des principes découlant de PFA qui paraissent être utiles dans la recherche d' une preuve de SCH à partir de PFA. En 2001 Todorčević [41] en travaillant sur des idées précédents de Shelah, Abraham et lui-même, a introduit une dichotomie pour les P-idéaux (dans la suite PID) qui d' une part donne de nouveaux arguments pour prouver que PFA implique $\mathfrak{b} \leq \omega_2$, et d'autre part permette d'obtenir une nouvelle preuve de la négation du principe du carré.

[8]Ce résultat est du a Magidor, voir [27] et [28], [13] est une monographie des forcings du type Prikry et des applications possibles a l'étude de la combinatoire des cardinaux singuliers.

En 2003, Moore a introduit un nouveau principe de réflexion, le mapping reflection principle MRP et l'a déduit à partir de PFA. Il a montré que MRP implique que le continuum est $\aleph_2$ et la négation de $\square(\kappa)$ pour tous les $\kappa > \aleph_1$ et réguliers. MRP partage beaucoup de propriétés avec les principes de réflexion qui découlent de MM. Donc il était plausible de s' attendre que MRP pouvait influencer les comportements de la fonction exponentielle sur les autres cardinaux aussi. En effet, Moore a montré dans [34] que si MRP est vrai et $\kappa > \aleph_1$ est un cardinal régulier contenant un sous-ensemble stationnaire de points de co-finalité ω qui ne réflet pas alors $\kappa^{\aleph_1} = \kappa$. Ceci combiné avec le résultat précédent de Veličković et le fait que tous les modèles connus de PFA sont aussi des modèles de SCH semblait suggérer que PFA implique SCH.

Dans la suite on montrera que MRP et PID peuvent être utilisés pour obtenir une preuve de SCH à partir de PFA en utilisant une variation de l'argument de Solovay pour montrer la négation du principe du carré au dessus d' un cardinal fortement compact.

A brief introduction for the expert reader

In[9] this thesis I analyze some of the effects of the proper forcing axiom PFA on the combinatorial properties of infinite cardinals. The main result that is achieved is the following:

Theorem 1 *The proper forcing axiom implies the singular cardinal hypothesis*[10].

The proof of this theorem is interesting in many respects: first of all it shows that SCH is a consequence also of this forcing axiom and thus solves positively a folklore problem in the area. The core of the proof relies on the isolation of a property of uncountable cardinals which holds above a strongly compact cardinal and which is a consequence of at least two very simple combinatorial principles which follow from PFA. The first has been isolated by Todorčević and Abraham is the P-ideal dichotomy PID. The second isolated by Moore is the mapping reflection principle MRP. Moreover the proof seems to require almost all of the known large cardinal strength of PFA or of a strongly compact cardinal since it can be extended without many difficulties to a proof of the failure of square.

The thesis is organized in five chapters and two appendixes:

- In the first chapter I give a brief presentation of forcing axioms and large cardinals and taking inspiration from Gödel's program I present them as a plausible solution to many of the mathematical problems arising in set theory in the past century.

- In the second chapter I concentrate my attention on the P-ideal dichotomy introduced by Todorčević in [41] developing ideas from him and Abraham in [1]. This is a very simple combinatorial principle which follows from PFA but is compatible with GCH and which is already strong enough to

[9]The nonexpert reader may skip directly to the next paragraph or even to the first chapter which is a more general introduction to the subject of this thesis.

[10]SCH in what follows.

capture many of the relevant consequences of this forcing axiom. The most interesting consequences of this dichotomy are the non-existence of Souslin trees, the failure of square and a very simple picture of the gaps in $(P(\omega)/FIN, \subseteq^*)$. In particular I will analyze in some details the proofs of the first two results. The reason is that these proofs follow a general pattern which I will follow in order to show that PID implies SCH.

- In the third chapter I turn my attention to the mapping reflection principle MRP recently introduced by Moore in [32] and I analyze some of its effects. This is the only known reflection principle which is a consequence of PFA. We will see that it can be used as all other reflection principles to prove interesting bounds in cardinal arithmetic. Moreover it has been an essential tool in establishing the validity of the existence of a five element basis for uncountable linear orders in models of PFA (see [33]). However I will be interested in this principle mainly for its applications to cardinal arithmetic. Even if it allows for a proof of SCH in the same fashion as PID does, it is mutually independent with PID since it is compatible with the existence of a Souslin tree and it entails that $\mathfrak{c} = \aleph_2$. However its proof requires more out of PFA than the proof of PID in order to go trough.

- In the fourth chapter I will present the main result, i.e. that PFA implies SCH. This will be done isolating a very simple covering property which I will call CP and which allows for a proof of SCH in the same fashion that Jensen covering lemma does. However this covering property is a strong hypothesis. A simple argument will yield that CP implies the failure of square. The main result of the chapter is that CP is a consequence of the P-ideal dichotomy, of the mapping reflection principle and of the existence of a strongly compact cardinal. Some other simple consequences of CP are outlined. In particular we prove that a weak form of reflection for stationary sets holds under PFA.

- In the fifth chapter I will investigate the rigidity of models of CP. Recently Veličković and Caicedo [6] have shown that any two models $M \subseteq V$ of PFA with the same ω_2 have the same reals. In an attempt to generalize this result to larger cardinals I will show that if V is a model of CP and M is an inner model with the same reals and the same cardinals, then the least κ such that $\kappa^\omega \setminus M$ is nonempty is not regular in V. Moreover I will show that if $\aleph_\omega$ is the least such κ, then it is close to be a Jónsson cardinal. Other interesting restrictions on κ as well as a weakening of the hypothesis are shown.

- In appendix one I prove that PFA implies PID.

- In appendix two I prove that PFA implies MRP and sketch a proof that MRP and the P-ideal dichotomy are mutually indipendent principles.

Contents

Chapter 1

Forcing axioms and cardinal arithmetic

Cardinal arithmetic has been one of the main fields of research in set theory since the foundational works by Cantor in the last quarter of the 19-th century [7]. While giving shape to the theory of cardinal arithmetic, Cantor defined the notion of exponentiation of infinite cardinals. He soon realized that he was not able to compute the value of the exponential function $\kappa \mapsto 2^\kappa$ even for the least infinite cardinal $\aleph_0$. This led him to formulate the celebrated continuum problem in which he asked for the specific value of $2^{\aleph_0}$. This later became the first of the twenty-three Hilbert problems. Cantor's conjecture, known as the continuum hypothesis (CH), states that $2^{\aleph_0} = \aleph_1$.

1.1 Gödel's program and the continuum problem

Over eighty years of work have been necessary to show that CH is undecidable in ZFC, the current axiomatic framework of set theory. In 1939 Gödel [15] isolated the constructible universe L, the minimal model of ZFC, and showed that CH holds in this model. However he was not satisfied by his result and his subsequent speculations on this problem culminated in his 1947 papers [16] and [17][1] where he presented his own views on the continuum problem and more generally on the onthological status of set theory and the issues arising in this field. In this respect, he assumed a platonistic point of view and asserted on CH:

> ...if the meaning of the primitive terms of set theory... are accepted as sound, it follows that the set-theorethical concepts and theorems describe some well-determined reality, in which Cantor's conjecture

[1] There are two editions of this paper, the 1947 [16] and the 1964 [17]. I will refer in what follows to the 1964 revised version, which expresses some of the further speculations Gödel had on this subject after the publication of [16].

> must be either true or false and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality; ([17], section 3, p. 476)

Subsequently he conjectured that ZFC was too weak to settle the continuum problem and expected CH to be independent of ZFC. He also gave some heuristic arguments in which he suggested the possibility of distinguishing between $2^{\aleph_0}$ and $\aleph_1$ (see [17], section 4, pp. 478 – 479). He presented many examples outlining a difference between the two cardinals, among which we mention two:

1. The fact that by a result of Sierpiński it was possible to deduce from ZFC the existence of a set of reals of size $\aleph_1$ with meager intersection with all perfect subsets of $\mathbb{R}$ (see [22], p.269), however no means of obtaining from ZFC a set with this property of power continuum was known.

2. The fact that by a result of Sierpiński and Luzin [26] CH entailed the existence of strong measure zero sets of power continuum, however no means of obtaining an uncountable strong measure zero set of reals from ZFC alone were known[2].

He concluded that it was necessary to seek new natural axioms that could give a satisfactory solution to the continuum problem as well as to all other natural problems arising in the field. This became later known as Gödel's program.
Gödel gave several criteria that these new axioms should satisfy in order to be accepted as true. In particular, he stressed that these axioms should satisfy a maximum property which he could not state clearly but which he opposed generically to the minimality conditions satisfied by L (see [17], section 4, p.478, note 19). He expected that advancements towards a solution of the continuum problem would give better insights into the nature of this maximum property. He could also propose a class of axioms that could fit with his requirements, namely, the strong axioms of infinity or, in the current terminology, the large cardinals axioms ([17], section 4, pp. 476 – 477 and note 16). Roughly speaking, these axioms assert certain combinatorial properties of infinite cardinals which try to capture Cantor's and Gödel's intuition that there cannot be a definite set of rules from which the sequence of cardinals can be generated. Thus, the maximum property as suggested by these axioms asserts the closure of the universe of sets with respect to any consistent means of generating new infinite cardinals.

A crucial step forward in set theory came in 1963 [8] when Paul Cohen discovered forcing and used it to produce a model of ZFC where CH fails. This confirmed Gödel's original intuition on the undecidability of CH in ZFC. Forcing is a general method by which one starts with a model of set theory and produces a larger model with the same ordinals and containing more sets. In fact, the forcing techniques turned out to be a means of producing a rich variety of models of set theory and to show the undecidability in ZFC of almost all problems in the area which were still open at that time.

[2]Laver has later shown [25] by means of forcing that there are models of ZFC with no uncountable strong measure zero sets.

1.2 Forcing axioms

Starting in the seventies, the combined investigations of the effects of large cardinals and of the forcing techniques gave new insights into possible ways to attack Gödel's program, once again confirming Gödel's idea that any advance towards the solution of the continuum problem would ultimately result in progress also in Gödel's program. Pushing the limits in the philosophy of forcing, another clear meaning of the maximum property can be proposed. For this, one needs the concept of a forcing axiom, which roughly says that if a set matching a certain simple description can be shown to exist in an ideal universe obtained by forcing, then the set actually exists. Thus, in this context, the maximum property can be formulated as a principle of saturation of the universe of sets with respect to forcing[3]. This formulation of the maximum property complements the large cardinals axioms which on their side assert the existence of ever larger ordinals and cardinals[4]. These axioms turned out to be strong enough to settle many of the central problems in set theory. The most striking effect being that of giving a wide spectrum of arguments to decide that $2^{\aleph_0} = \aleph_2$ (see for example [11], [31], [32], [40], [42], [44]).

There is a nice topological characterization of forcing axioms. Baire's Category theorem states that in many topological spaces, among which are all complete second countable spaces (i.e.: the reals), the intersection of countably many open dense sets is non empty. It turned out that forcing axioms could be presented as a strengthening of Baire's Category:

Definition 1.1 $\mathsf{FA}(\mathcal{A}, \kappa)$ *holds if whenever* $\mathcal{A}$ *is a class of topological spaces and for some* $X \in \mathcal{A}$, $\mathcal{F}$ *is a family of less than* κ *dense open sets of* X, *then* $\bigcap \mathcal{F}$ *is non-empty.*

Thus, the usual Baire's Category theorem is $\mathsf{FA}(\{\mathbb{R}\}, \aleph_1)$. A number of forcing axioms have been introduced together with proofs of their consistency relative to large cardinals outlining a duality between these two types of axioms. This duality establishes a correspondence between the consistency strength of a forcing axiom and the consistency strength of a large cardinal. There is a natural notion of reduction between logical theories. We say that a logical theory T is weaker than a theory T' if from a model of T' one can construct a model of T. We say that the two theories T and T' have the same consistency strength, or are equiconsistent if T is weaker than T' and conversely. The correspondence between large cardinals and forcing axioms is done using this notion of reduction. There are many methods which build a model of a large cardinal starting from a model of a forcing axiom and conversely. Moreover, this correspondence establishes a nice linear hierarchy between theories in the sense that given any two theories T and T' of a forcing axiom or of a large cardinal axiom, it is

[3]This is not accurate since not all kind of forcing can be admitted. However it gives an intuition of what kind of saturation principles this type of axioms try to capture.

[4]However as we will see below forcing axioms and large cardinals turn out to be strictly intertwined.

almost always the case[5] that one can show that $T \leq T'$ or $T' \leq T$.

In the early 1970s, Solovay and Martin presented the first non-trivial forcing axiom, Martin's axiom (MA). This is $FA(K, 2^{\aleph_0})$, where K is the class of compact Haussdorff spaces satisfying the countable chain condition, a class of spaces which shares many of the essential properties of the reals. In the beginning of the 1980s, Baumgartner and Shelah [4] introduced the proper forcing axiom (PFA), while in the late 1980s, Shelah, Magidor and Foreman [11] presented Martin's maximum (MM), the provably strongest forcing axiom[6] and showed that MM implies $2^{\aleph_0} = \aleph_2$. Later on Todorčević and Veličković [44] reached the same conclusion assuming the weaker PFA. As outlined before, other proofs that PFA implies $2^{\aleph_0} = \aleph_2$ have been presented by Todorčević [40], Moore [32], [31] and Caicedo and Veličković [6]. Many of these proofs are interesting also in another respect: they were the means to isolate principles which led to new theorems in combinatorial and descriptive set theory. For example the open coloring axiom OCA introduced by Todorčević as a consequence of PFA has an exact counterpart in descriptive set theory which has a number of interesting applications for analytic sets (see [39]). On another side the mapping reflection principle MRP has been a key tool in proving that it is consistent that the uncountable linear orders have a five element basis and that this type of basis exists in models of the proper forcing axiom. This latter result is due to Moore [33]. Speculating on this König, Moore, Larson, and Veličković were able to obtain the consistency of the existence of such a basis from much milder large cardinals assumptions [20].

In the 1990s, attention turned to the bounded forcing axioms and, in particular, to the bounded Martin's maximum (BMM) and to the weaker bounded proper forcing axiom (BPFA). The latters are nowadays a promising approach towards a satisfactory solution of the continuum problem and to a partial completion of Gödel's program. This is due in part to the nice logical characterization provided by Bagaria [2], who showed that bounded forcing axioms can be presented as a generalization at the level of $H(\aleph_2)$ of Schoenfield's absoluteness theorem. This theorem asserts that $L_{\aleph_1} \prec_1 V$. Bounded forcing axioms can be presented as a strengthening of this theorem of the form $H(\aleph_2) \prec_1 V^P$ whenever P ranges over the appropriate class of forcing notions. On the other hand, Moore has shown that already the weak BPFA decides that $2^{\aleph_0} = \aleph_2$ [32].

Finally the efforts of many logicians, including Woodin, have been able to combine the large number of partial results obtained in the area of forcing

[5]Paul Larson has shown that Woodin's axiom $(*)$ doesn't follow from MM [23]. Even if it is natural to consider $(*)$ as a forcing axiom, it is not known whether it is compatible with MM. In fact I don't know either whether in some model of ZFC, $(*)$ can be formulated according to definition 1.1 (see also the next footnotes for further discussion on this point).

[6]This assertion needs some precisions. PFA, MA and all forcing axioms whose formulation can be given according to definition 1.1 are a consequence of MM. In fact MM is $\mathsf{FA}(SSP, \aleph_2)$, where SSP denotes the class of stationary set preserving complete boolean algebras. It has been shown in [11] that if X is a complete boolean algebra which is nowhere stationary set preserving, then there is a family of size $\aleph_1$ of dense subsets of X whose intersection is empty. In this respect MM is the provably strongest forcing axiom. However as pointed out in the previous footnote MM and Woodin's axiom $(*)$ may be independent principles.

axioms with those linking the theory of $L(\mathbb{R})$ and the Axiom of Determinacy to large cardinals hypothesis. First of all Woodin has shown that the existence of a proper class of Woodin Cardinals (a certain large cardinal hypothesis) implies that the theory of $L(\mathbb{R})$ cannot be modified by forcing and thus has strengthened Schoenfield's absoluteness theorem to the largest possible extent at least under the assumption that forcing and large cardinals are the unique means to produce a witness of a projective property of the reals. Pushing these ideas to the the level of $H(\aleph_2)$ he has come to the analysis of Ω-Logic. This is the logic whose models range over the boolean valued sets V_α^B where B is any complete boolean algebra. Woodin has shown that the notion of validity $\models_\Omega$ for this logic is invariant under forcing (i.e. $V \models$ "$T \models_\Omega \phi$" if and only if $V^B \models$ "$T \models_\Omega \phi$" for any complete boolean algebra B). He has also introduced a sound syntactical notion of "proof" $\vdash_\Omega$ which, according to Woodin,"is a natural transfinite generalization of the classical notion of proof for first order logic" [47]. The Ω-Conjecture asks whether Gödel's completeness theorem can be proved also for Ω-Logic. Assuming a positive answer to the Ω-Conjecture, Woodin has then shown that there is a strong form of BMM which[7] combined with ZFC decides in Ω-Logic the whole theory of $H(\aleph_2)$ (see [48], [49] or [3]). Moreover he has shown, always assuming the Ω-Conjecture, that any axiom ϕ with this property will decide in Ω-Logic that CH fails. Thus Ω-Logic appears to be a suitable strengthening of the concept of demonstrability which gives further plausibility to all the results obtained so far in the forcing axiom area pointing to $2^{\aleph_0} = \aleph_2$. Quoting Gödel again on the effects of his incompleteness theorem and the possible remedies to it:

> It is well known that in whichever way you make [the concept of demonstrability] precise by means of a formalism, the contemplation of this very formalism gives rise to new axioms which are exactly as evident as those with which you started, and that this process can be iterated into the transfinite. So there cannot exist any formalism which would embrace all these steps; but this does not exclude that all these steps.... could be described and collected together in some non constructive way. ([14] p. 151)

In this respect Ω-Logic appears to be a plausible candidate for the non constructive proof system that Gödel suggested.

1.3 Forcing axioms, large cardinals and the singular cardinal problem

Soon after the discovery of forcing attention has been devoted to the study of the behavior of the exponential function on uncountable cardinals. Some works in this direction had been already done. The generalized continuum hypothesis GCH states that $2^\kappa = \kappa^+$ for all infinite cardinals κ. For example Gödel [15]

[7] This is in fact his axiom $(*)$.

had already shown that GCH holds in L, while it is a folklore result that if κ is measurable and GCH holds below κ, then $2^\kappa = \kappa^+$ ([19] is a reference text for the historic development of the theory of large cardinals). However in one of the first applications of the forcing techniques, Easton [10] generalized Cohen result and showed that the exponential function $\kappa \mapsto 2^\kappa$ on regular cardinals can be arbitrary modulo some mild restrictions. Nonetheless the situation for singular cardinals turned out to be much more subtle.

Recall that SCH states that $\kappa^{\mathrm{cof}\,\kappa} = \kappa^+$ for all singular cardinals $\kappa > 2^{\mathrm{cof}\,\kappa}$. First of all Solovay proved [38] that SCH holds above a strongly compact cardinal (see also chapter 4 for a different proof of this theorem). Soon after Silver [37] showed that the singular cardinal hypothesis SCH cannot fail first at a singular cardinal of uncountable cofinality. Finally one of the main achievement of Shelah is that $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4} + \mathfrak{c}^+$ holds in ZFC [36]. Thus, contrary to the situation for regular cardinals, there are interesting bounds on the behavior of the exponential function on singular cardinals which can be computed in ZFC. Moreover, by Solovay's theorem, under suitable large cardinals assumptions, the value of 2^κ for a large enough singular cardinal κ is completely determined by the behavior of the exponential function on smaller cardinals.

However, it is known that SCH can fail first even at $\aleph_\omega$ (Magidor, see [27] and [28]). The role of large cardinals in this context is twofold. On one hand they are necessary for the construction of models of the negation of SCH since any such model has an inner model with measurable cardinals (see [13] for a survey of Prikry type forcings and applications to SCH). On the other hand as we mentioned above SCH holds above a strongly compact cardinal.

MM implies reflection principles similar to the one used in Solovay's proof and in fact a variation of Solovay's argument shows that SCH is a consequence of this forcing axiom [11]. This was later improved by Veličković [44] who also showed that SCH follows from PFA^+ a forcing axiom sligthly stronger than PFA and weaker than MM. In fact, what is shown in [44] is that if $\theta > \aleph_1$ is regular and stationary subsets of $[\theta]^\omega$ reflect to an internally closed and unbounded set, then $\theta^\omega = \theta$. This, combined with Silver's theorem, implies SCH. At this point, it was left open whether SCH is a consequence of PFA.

Very little progress was made on this problem for over fifteen years since no reflection principle resembling those used in the above proofs was known to follow from PFA. The situation appeared close to the one that made it difficult to prove that PFA implies $\mathfrak{c} = \aleph_2$. Before Veličković and Todorčević's argument, the known proofs of $\mathfrak{c} = \aleph_2$ from MM used reflection's arguments which could not be derived from PFA.

Further work led to the isolation of some consequences of PFA which could be of interest in the search for a solution of this problem. In 2001 Todorčević [41] elaborating on previous works by Shelah [35], Abraham and himself [1] introduced a dichotomy for P-ideals (in the sequel PID) that on one side gave a new argument for a proof that PFA implies $\mathfrak{b} = \aleph_2$, on the other side allowed for another proof of the failure of square under PFA which is reminiscent of Solovay's argument that square fails above a strongly compact cardinal (which also appeared in [38]).

In 2003, Moore [32] introduced a new reflection principle, the mapping reflection principle MRP and deduced it from PFA. He showed that MRP implies the continuum is equal to $\aleph_2$ and the failure of $\square(\kappa)$, for all $\kappa > \aleph_1$. MRP has many features in common with the reflection principles which follow from MM, so it should be expected that MRP could affect the behaviour of the exponential function also on higher cardinals. In fact, Moore showed in [34] that if MRP holds and $\kappa > \omega_1$ is a regular cardinal with a non-reflecting stationary set consisting of points of countable cofinality, then $\kappa^{\omega_1} = \kappa$. This, combined with the above result of Veličković and the fact that all the known models of PFA are models of SCH, strongly suggested that PFA implies SCH. In the sequel we will show that MRP and PID can be used to obtain a proof of SCH from PFA using a variation of Solovay's proof that square fails above a strongly compact cardinal.

1.4 Notation and definitions

In this section we introduce the technical definitions and the main notational conventions which will be relevant in what follows. The reader may not be acquainted with forcing and still be able to understand the relevant parts of this thesis and the proofs of the main results. For this reason we feel free to omit any introduction to forcing and we refer the reader to [18] for a presentation of this subject and as a source for the standard notational conventions. For a regular cardinal θ, we use $H(\theta)$ to denote the structure $\langle H(\theta), \in, <\rangle$ whose domain is the collection of sets whose transitive closure is of size less than θ and where $<$ is a predicate for a fixed well ordering of $H(\theta)$. For cardinals $\kappa \geq \lambda$ we let $[\kappa]^\lambda$ be the family of subsets of κ of size λ. In a similar fashion we define $[\kappa]^{<\lambda}$, $[\kappa]^{\leq\lambda}$, $[X]^\lambda$, where X is an arbitrary set. If X is an uncountable set, $\mathcal{E} \subseteq [X]^\omega$ is unbounded if for every $Z \in [X]^\omega$, there is $Y \in \mathcal{E}$ containing Z. $\mathcal{E}$ is bounded otherwise. $\mathcal{E}$ is closed if whenever $X = \bigcup_n X_n$ and $X_n \subseteq X_{n+1}$ are in $\mathcal{E}$ for all n, then also $X \in \mathcal{E}$. It is a well known fact that $\mathcal{C} \subseteq [X]^\omega$ is closed and unbounded (club) iff there is $f : [X]^{<\omega} \to X$ such that $\mathcal{C}$ contains the set of all $Y \in [X]^\omega$ such that $f[Y]^{<\omega} \subseteq Y$. $S \subseteq [X]^\omega$ is stationary if it intersects all club subsets of $[X]^\omega$. The f-closure of X is the smallest Y containing X such that $f[Y]^{<\omega} \subseteq Y$. Given f as above $\mathcal{E}_f$ is the club of $Z \in [X]^\omega$ such that Z is f-closed. If X is a set of ordinals then $\overline{X}$ denotes the topological closure of X in the order topology. For regular cardinals $\lambda < \kappa$, $S^{\leq\lambda}_\kappa$ denotes the subset of κ of points of cofinality $\leq \lambda$, in similar fashion we define S^λ_κ and $S^{<\lambda}_\kappa$. We say that a family $\mathcal{D}$ is covered by a family $\mathcal{E}$ if for every $X \in \mathcal{D}$ there is a $Y \in \mathcal{E}$ such that $X \subseteq Y$.

Definition 1.2 *(Shelah [35]) P is a proper forcing notion if every $\mathcal{S}$ stationary subset of $[X]^\omega$ for some uncountable $X \in V$ is stationary in $V[G]$, where G is a P-generic filter.*

Let P be a forcing notion and $M \prec H(\theta)$ be a countable model such that $P, p \in M$. $q \leq p$ is an M-generic condition below p if for all $D \in M$ dense subset of P and for all $r \leq q$ there is $s \in D \cap M$ such that s and r are compatible conditions.

Theorem 2 *(Shelah [35]) The following are equivalent:*

- *P is proper*
- *There is $\mathcal{C}$ club in $[H((2^{|P|})^+)]^\omega$ such that for all $M \in \mathcal{C}$ and for all $p \in M$ there is $q \leq p$, M-generic condition below p.*

For a proof see theorem 31.7 of [18]. ■

Definition 1.3 *(Baumgartner [4])* PFA*: The proper forcing axiom holds if whenever $\{D_\alpha : \alpha < \omega_1\}$ is a family of dense open subsets of a proper poset P, there is $G \subseteq P$ filter on P with non-empty intersection with all the D_α.*

Definition 1.4 *λ is a strongly compact cardinal if for every $\kappa \geq \lambda$ there is $\mathcal{U}$ λ-complete ultrafilter on $[\kappa]^{<\lambda}$ such that $\{Y \in [\kappa]^{<\lambda} : X \subseteq Y\} \in \mathcal{U}$ for all $X \in [\kappa]^{<\lambda}$.*

Definition 1.5 SCH*: The Singular cardinal Hypothesis holds if $\kappa^{\mathtt{cof}(\kappa)} = \kappa^+ + 2^{\mathtt{cof}(\kappa)}$ for all infinite cardinals κ.*

Theorem 3 *(Silver [37]) Assume κ has uncountable cofinality and $\lambda^{\mathtt{cof}(\lambda)} = \lambda^+ + 2^{\mathtt{cof}(\lambda)}$ for all $\lambda < \kappa$. Then $\kappa^{\mathtt{cof}(\kappa)} = \kappa^+ + 2^{\mathtt{cof}(\kappa)}$.*

Definition 1.6 *(Shelah [36]) Let κ be an uncountable cardinal. $\mathcal{I}[\kappa]$ is the family of $S \subseteq \kappa$ such that there is $\{a_\alpha : \alpha < \kappa\}$ such that for all $\alpha \in S$, $\mathtt{otp}(a_\alpha) = \mathtt{cof}(\alpha)$ and for all $\gamma < \alpha$, $a_\alpha \cap \gamma \in \{a_\xi : \xi < \alpha\}$.*

Fact 1.7 *(Shelah [36]) $\mathcal{I}[\kappa]$ is a normal ideal.* □

Theorem 4 *(Shelah [36]) Assume κ and λ are regular uncountable cardinals with $\lambda^+ < \kappa$. Then there is S stationary subset of S^λ_κ in $\mathcal{I}[\kappa]$.* ■

Chapter 2

The P-ideal dichotomy

The purpose of this chapter is to introduce the reader to an interesting combinatorial principle which captures many of the essential features of the proper forcing axiom but nonetheless is still consistent with CH.

Let Z be an uncountable set. $\mathcal{I} \subseteq [Z]^{\leq\omega}$ is a P-ideal if it is an ideal and for every countable family $\{X_n\}_n \subseteq \mathcal{I}$ there is an $X \in \mathcal{I}$ such that for all n, $X_n \subseteq^* X$ (where $\subseteq^*$ is inclusion modulo finite).

Definition 2.1 *(Todorčević, [41])*
*The P-ideal dichotomy (*PID*) asserts that for every P-ideal $\mathcal{I}$ on $[Z]^{\leq\omega}$ for some fixed uncountable Z, one of the following holds:*

(i) *There is Y uncountable subset of Z such that $[Y]^{\leq\omega} \subseteq \mathcal{I}$.*

(ii) *$Z = \bigcup_n A_n$ with the property that A_n is orthogonal to $\mathcal{I}$ (i.e. $X \cap Y$ is finite for all $X \in [A_n]^{\omega}$ and $Y \in \mathcal{I}$) for all n.*

As we will see below PID is a principle strong enough to rule out many of the standard consequences of $V = L$, like the existence of a Souslin tree or the existence of a square sequence. Due to this latter fact the consistency strength of this principle is considerable. Nonetheless this principle is consistent with CH. This shows that the standard pattern to obtain a model of PID can be undertaken taking into account only a fragment of the class of proper forcing notions. The current strategy to obtain a model of PID is to organize a countable support iteration of proper forcings P_α such that at each successor stage $\alpha + 1$, for some P-ideal $\mathcal{I}$ on $[Z]^{\leq\omega}$ for which *(ii)* fails, $P_{\alpha+1}$ adds an uncountable $Y \subseteq Z$ such that $[Y]^{\omega} \subseteq \mathcal{I}$. Todorčević has shown that these forcing notions are not only proper, but have the property that any countable support iteration of such kind of forcings does not add reals[1]. Some of the nice properties of these forcing notions will be analyzed in more details in the appendix where we will give a proof that PID follows from PFA.

[1] This generalizes a result of Abraham and Todorčević in [1].

Recall that $(T, <_T)$ is a tree if $<_T$ is a well founded partial order on T such that for all t_0, t_1, $t_2 \in T$, if $t_0 \leq_T t_2$ and $t_1 \leq_T t_2$, then $t_0 \leq_T t_1$ or $t_1 \leq_T t_0$. For any $t \in T$, we let $\mathtt{prc}(t)$ be the set $\{s <_T t : s \in T\}$ and $\mathtt{ht}_T(t)$ be the ordinal type of $(\mathtt{prc}(t), <_T)$. For any $\alpha < |T|^+$ we let $T_\alpha = \{t \in T : ht_T(t) = \alpha\}$ and $\mathtt{ht}(T) = \sup_{t \in T} \mathtt{ht}_T(t)$.

T is normal if the set of t such $s \leq t$ has cardinality $|T|$ for every $s \in T$.

$Z \subseteq T$ is an antichain if for every $s \neq t \in Z$ neither $s <_T t$, nor $t <_T s$. $Z \subseteq T$ is a branch if $(Z, <_T)$ is a total order.

Definition 2.2 *A Souslin tree is a normal tree of size $\aleph_1$ with no uncountable antichains and no uncountable branches.*

Theorem 5 *(Abraham, Todorčević [1])* PID *implies that there are no Souslin trees.*

Let X be an infinite set of ordinals and $\mathcal{A}$ and $\mathcal{B}$ two orthogonal families in $[X]^\omega$. They form a gap if there is no $c \in [X]^\omega$ such that $a \setminus c$ is finite for all $a \in \mathcal{A}$ and $c \cap b$ is finite for all $b \in \mathcal{B}$. A (κ, λ)-gap on $[X]^\omega$ is a gap where $\mathcal{A}$ has size κ and $\mathcal{B}$ has size λ.

If $\mathcal{A}$ and $\mathcal{B}$ are orthogonal families of size ω_1 which form a gap and $\mathcal{A}$ is countably directed (i.e. for all $X \in [\mathcal{A}]^\omega$ there is $a \in \mathcal{A}$ such that $d \setminus a$ is finite for all $d \in X$) the gap is Haussdorff if for all n and for all $b \in \mathcal{B}$ the set of $a \in \mathcal{A}$ such that $a \cap b \subseteq n$ is finite.

Classical results are that there are Haussdorff gaps and $(\mathfrak{b}, \omega)$-gaps on $[\omega]^\omega$, where $\mathfrak{b}$ is the smallest size of an unbounded family in $[\omega]^\omega$ i.e. a family $\mathcal{A}$ such that for all $b \in [\omega]^\omega$ there is an $a \in \mathcal{A}$ such that $a \setminus b$ is infinite.

Theorem 6 *(Todorčević [41])* PID *implies that there are only Haussdorff gaps and (κ, ω)-gaps on $[\omega]^\omega$.*

In particular from this result it is possible to show[2] that $\mathfrak{b} \leq \omega_2$

Definition 2.3 *Let κ be an infinite cardinal. $\square_\kappa$ asserts the existence of a sequence $(C_\alpha : \alpha < \kappa^+)$ with the following properties:*

(i) for every limit α, C_α is a closed unbounded subset of α of order type at most κ,

(ii) if α is a limit point of C_β, $C_\beta \cap \alpha = C_\alpha$,

(iii) $C_{\beta+1} = \{\beta\}$.

Definition 2.4 *Let κ be an infinite regular cardinal $\square(\kappa)$ asserts the existence of a sequence $(C_\alpha : \alpha < \kappa)$ with the following properties:*

[2]See section 29 of [18] pp. $574 - 578$ and in particular the proof of theorem 29.8 for this latter result and some informations on gaps. The part on Haussdorff gaps already appeared in [1].

(i) for every limit α, C_α is a closed unbounded subset of α,

(ii) if α is a limit point of C_β, $C_\alpha = C_\beta \cap \alpha$,

(iii) there is no club C in κ such that for all α there is $\beta \geq \alpha$ such that $C \cap \alpha \subseteq C_\beta$,

(iv) $C_{\beta+1} = \{\beta\}$.

Sequences satisfying definitions 2.3 and 2.4 are called square sequences. Clearly $\square_\kappa$ implies $\square(\kappa^+)$ and $\square_\omega$ is true as witnessed by any sequence $(C_\alpha : \alpha < \omega_1)$ satisfying *(i)* and *(iii)* of def. 2.3. Moreover $V = L$ implies $\square(\kappa)$ for all regular cardinals κ, while the existence of a strongly compact cardinal λ entails the failure of $\square(\kappa)$ for all regular $\kappa \geq \lambda$. In fact the existence of square sequences is compatible with considerably strong large cardinals assumptions (i.e. hypothesis which contradicts $V = L$), for example it is possible to have a model of ZFC in which $\square_\kappa$ holds for a measurable cardinal κ. Thus asking whether the failure of $\square_\kappa$ follows from a certain combinatorial principle is a very useful question to test the consistency strength of this principle. There are plenty of interesting constructions that can be performed in ZFC assuming that $\square_\kappa$ holds, in particular when κ is a singular cardinal. For example $\square_{\aleph_\omega}$ entails the existence of an Aronszjain tree and of a Kurepa tree on $\aleph_{\omega+1}$. We refer the interested reader to [43] for a survey of many of these results.

Theorem 7 *(Todorčević) PID implies $\square(\kappa)$ fails for every regular $\kappa > \omega_1$.*

Other interesting applications of PID by Abraham, Todorčević and Veličković are outlined in [45], [41] and [1]. I will give below in many details a proof of theorems 5 and 7. This will give a general idea on how to apply the dichotomy.

2.1 PID implies that there are no Souslin trees

In this section I prove theorem 5. Assume PID and let $(T, <_T)$ be a Souslin tree. We will reach a contradiction. Since T is Souslin we have that T_α is at most countable for all $\alpha < \omega_1$, since for any α, T_α is an antichain. Thus we can enumerate T_α as $\{t^\alpha_n : n \in \omega\}$ for all $\alpha < \omega_1$. Now we can set for $n \in \omega$ and $\alpha \in \omega_1$, $K(n, \alpha) = \bigcup_{m \leq n} \{t : t \leq_T t^\alpha_m\}$. In this way we define a matrix $(K(n, \beta) : n \in \omega, \beta \in \omega_1)$ with the following crucial coherence property:

Claim 2.5 *For every n, $\alpha < \beta$, there is m such that $K(n, \alpha) \subseteq K(m, \beta)$ and $\{t \in K(n, \beta) : ht_T(t) \leq \alpha\} \subseteq K(m, \alpha)$.*

Proof: To see this, fix n, $\alpha < \beta$ and let $\{s_1, \cdots, s_n\} \subseteq T_\alpha$ be such that for every $i \leq n$, $s_i <_T t^\beta_i$. Now it is enough to take m large enough in order that $\{s_1, \cdots, s_n\} \subseteq \{t^\alpha_j : j \leq m\}$ and that for every $i \leq n$, there is a $j \leq m$ such that $t^\alpha_i <_T t^\beta_j$. $\square$

We will use this coherence property to show that the ideal $\mathcal{I} \subseteq [T]^{\leq\omega}$ of X such that $X \cap K(n,\beta)$ is finite for all n and β is a P-ideal. Then we will apply PID to obtain with some further work that option *(i)* of the dichotomy gives an uncountable antichain, while option *(ii)* can be used to obtain an uncountable branch trough T. In any case the tree T cannot be Souslin, a contradiction with our assumption.

Claim 2.6 *$\mathcal{I}$ is a P-ideal.*

Proof: Let $\{X_n : n \in \omega\} \subseteq \mathcal{I}$, we need to find $X \in \mathcal{I}$ containing each X_n modulo finite. Let α be large enough in order that $\bigcup_n X_n \subseteq \bigcup_\alpha T_\xi$. Let $X_n^m = X_n \cap K(m,\alpha) \setminus K(m-1,\alpha)$. Set

$$X = \bigcup_n \bigcup_{m \geq n} X_n^m.$$

Then it is immediate to check that $X_n \subseteq^* X$ for all n. Moreover

$$X \cap K(n,\alpha) = \bigcup_{j \leq m \leq n} X_j^m$$

and thus is finite for all n. Now we can use the coherence properties of the matrix to conclude that $X \cap K(n,\beta)$ is finite for all n and for all β. To see this notice that if $\beta \leq \alpha$ we have that for each n there is m such that $X \cap K(n,\beta) \subseteq X \cap K(m,\alpha)$ which is finite for every m. Now if $\beta > \alpha$, we have that $X \subseteq \bigcup_{\gamma \leq \alpha} T_\gamma$. Now for every n there is m such that $\bigcup_{\gamma \leq \alpha} T_\gamma \cap K(n,\beta) \subseteq K(m,\beta)$. Thus we can conclude that for every n there is an m such that $X \cap K(n,\beta) \subseteq X \cap K(m,\alpha)$ which is finite. This shows that $X \in \mathcal{I}$ and concludes the proof of the claim. $\Box$

Now assume that *(i)* of PID applies to this $\mathcal{I}$ and find an uncountable $Y \subseteq T$ such that $[Y]^{\leq\omega} \subseteq \mathcal{I}$. Consider the tree $(Y, <_T)$.

Claim 2.7 $\mathtt{ht}(Y) \leq \omega$

Proof: Notice that if $t \in Y$, the set $A_t = \{s \in Y : s <_T t\}$ is finite. If not, A_t would be a countable subset of Y which is not in $\mathcal{I}$, since if $\alpha = \mathtt{ht}_T(t)$ and $t = t_n^\alpha$ in the enumeration of T_α, we have that $A_t \subseteq K(n,\alpha)$ and thus A_t cannot be in $\mathcal{I}$ since it has countable intersection with $K(n,\alpha)$. $\Box$

This means that for some $n < \omega$, $\{t \in Y : \mathtt{ht}_Y(t) = n\}$ is uncountable. But this is an uncountable antichain in Y and in fact it is still an antichain also in T. This is not possible since we supposed T to be Souslin.
To complete our proof we will show that option *(ii)* for $\mathcal{I}$ is also not possible. For suppose that $T = \bigcup_l Z_l$ and for each l, $[Z_l]^\omega \cap \mathcal{I} = \emptyset$. Then for some l Z_l is uncountable, since T is uncountable. Fix such an l.

Claim 2.8 *For each α, there is m_α such that $Z_l \cap \bigcup_{\gamma \leq \alpha} T_\gamma \subseteq K(m_\alpha,\alpha)$.*

Proof: Suppose not, then we can find $X \subseteq Z_l$ countable and such that for some fixed α, $X \cap K(m, \alpha)$ is finite for all m. Due to the coherence properties of the matrix exactly as in the proof that $\mathcal{I}$ is a P-ideal we can see that $X \in \mathcal{I}$ contradicting the definition of Z_l. □

Now let for any α, m_α be such that $Z_l \subseteq K(m_\alpha, \alpha)$ and find an uncountable $W \subseteq \omega_1$ such that $m_\alpha = m$ for all $m \in W$. By induction, refine W to an uncountable W_0 in order that for each $\beta < \alpha \in W_0$, there is $s^\alpha \in Z_l$ such that for some $j \leq m$, $s_\alpha \leq_T t_j^\alpha$ and $\mathtt{ht}(s_\alpha) > \mathtt{ht}(s_\beta)$. Let $\mathcal{U}$ be a uniform ultrafilter on ω_1 (i.e. an ultrafilter with no countable elements) which concentrates on W_0. For each $\alpha \in W_0$ and $i \leq m$, let $W(\alpha, i) = \{\beta \in W_0 : s_\alpha <_T t_i^\beta\}$. Let for each α, i_α be such that $W(\alpha, i_\alpha) \in \mathcal{U}$. Finally let $W^* \in \mathcal{U}$ be such that for all $\alpha \in W^*$, $i_\alpha = i$. Now if $\alpha < \beta \in W^*$, $W(\alpha, i) \cap W(\beta, i) \in \mathcal{U}$, so there is $\gamma > \beta, \alpha$ and in $W(\alpha, i) \cap W(\beta, i)$. Thus $s_\alpha, s_\beta <_T t_i^\gamma$. This means that $s_\alpha <_T s_\beta$. Thus we can conclude that $\{s_\alpha : \alpha \in W^*\}$ is an uncountable branch through T contradicting the fact that T is Souslin. This concludes the proof of theorem 5. ■

2.2 PID implies the failure of square

In this section we will prove theorem 7. We will follow exactly the same pattern of the previous proof, i.e. we will assume PID and that for some regular $\kappa > \aleph_1$, $\square(\kappa)$ holds. We will then reach a contradiction. To this aim we will use the square sequence $\square(\kappa)$ to define a matrix $(K(n, \beta) : n \in \omega, \beta < \kappa)$ of subsets of κ with coherence properties similar to the ones of the matrix considered in the previous theorem. We will use these coherence properties to show that the ideal $\mathcal{I}$ of $X \in [\kappa]^{\leq\omega}$ which have finite intersection with all $K(n, \beta)$ is a P-ideal. Finally we will show that both the alternatives provided by PID lead to a contradiction. To achieve this we are going to analyze some of the ρ-functions introduced by Todorčević in order to describe the combinatorics of uncountable cardinals. The reference text for what follows is [43].

2.2.1 Some properties of the ρ-functions provided by a square sequence.

Let $(C_\alpha : \alpha < \kappa)$ be a square sequence (see def. 2.4) Define for $\alpha \leq \beta < \kappa$, the trace function $\mathtt{tr}(\alpha, \beta)$ and the code of the walk function $\rho_0(\alpha, \beta)$ by induction as follows:

(i) $\mathtt{tr}(\alpha, \alpha) = \emptyset$,

(ii) $\mathtt{tr}(\alpha, \beta + 1) = \mathtt{tr}(\alpha, \beta) \cup \{\beta + 1\}$,

(iii) if β is limit, $\mathtt{tr}(\alpha, \beta) = \{\beta\} \cup \mathtt{tr}(\alpha, \min(C_\beta \setminus \alpha))$.

(i) $\rho_0(\alpha, \alpha) = \emptyset$,

(ii) $\rho_0(\alpha, \beta + 1) = \langle 0 \rangle^\frown \rho_0(\alpha, \beta)$,

(iii) if β is limit, $\rho_0(\alpha,\beta) = \langle \mathtt{otp}(C_\beta \cap \alpha)\rangle ^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha))$.

Now set the number of steps function $\rho_2(\alpha,\beta)$ to be equal to $|\rho_0(\alpha,\beta)|$. We refer the reader to sections 6 and 8 of [43] which is our source in what follows. We first remark the following crucial relation between ρ_0 and $\mathtt{tr}$:

Claim 2.9 *Assume that $\alpha \le \beta \le \gamma$. Then $\rho_0(\alpha,\gamma) = \rho_0(\beta,\gamma)^\frown\rho_0(\alpha,\beta)$ iff $\beta \in \mathtt{tr}(\alpha,\gamma)$.*

Proof: The right to left direction is not difficult to show. For the other implication assume that $\rho_0(\alpha,\gamma) = \rho_0(\beta,\gamma)^\frown\rho_0(\alpha,\beta)$ and let $\xi = \min(\mathtt{tr}(\alpha,\gamma)\setminus\beta)$. If $\xi > \beta$, then $\mathtt{otp}(C_\xi \cap \beta) = \mathtt{otp}(C_\xi \cap \alpha)$ since $\rho_0(\beta,\gamma)$ is an initial segment of $\rho_0(\alpha,\gamma)$ and $\xi \in \mathtt{tr}(\beta,\gamma)$. Thus $\min(C_\xi \setminus \beta) = \min(C_\xi \setminus \alpha)$. But this contradicts the minimality of ξ. □

We also outline the following simple property of $\mathtt{tr}$:

Claim 2.10 *Assume $\alpha \le \beta \le \gamma$ and $\beta \in \mathtt{tr}(\alpha,\gamma)$. Then $\mathtt{tr}(\alpha,\gamma) = \mathtt{tr}(\alpha,\beta) \cup \mathtt{tr}(\beta,\gamma)$.* □

Let for $\alpha < \beta$, $\Lambda(\alpha,\beta)$ be the largest limit point of $C_\beta \cap \alpha + 1$. Now set by induction on $\alpha \le \beta$:

(i) $F(\alpha,\alpha) = \{\alpha\}$,

(ii) $F(\beta,\alpha) = F(\alpha,\beta) = F(\alpha, \min(C_\beta \setminus \alpha)) \cup \bigcup\{F(\xi,\alpha) : \xi \in [\Lambda(\alpha,\beta),\alpha) \cap C_\beta\}$

Lemma 2.11 *For all $\alpha \le \beta \le \gamma$:*

(a) $\rho_0(\alpha,\beta) = \rho_0(\min(F(\beta,\gamma)\setminus\alpha)),\beta)^\frown\rho_0(\alpha,\min(F(\beta,\gamma)\setminus\alpha))$

(b) $\rho_0(\alpha,\gamma) = \rho_0(\min(F(\beta,\gamma)\setminus\alpha)),\gamma)^\frown\rho_0(\alpha,\min(F(\beta,\gamma)\setminus\alpha))$

Proof: The proof is by a simultaneous induction and is interesting just when γ is limit. We will prove *(a)* for the triple $\alpha \le \beta \le \gamma$ assuming *(a)* and *(b)* for all smaller triples. Then we will prove *(b)* for the triple $\alpha \le \beta \le \gamma$ assuming *(b)* for all smaller triples and *(a)* for all triples including the triple α,β,γ.

Let $\lambda = \Lambda(\beta,\gamma)$, $\gamma_1 = \min(C_\gamma \setminus \alpha)$, $\alpha_1 = \min(F(\beta,\gamma)\setminus\alpha)$. Pick $\xi \in \{\min(C_\gamma \setminus \beta)\} \cup (C_\gamma \cap [\lambda,\beta))$ such that $\alpha_1 \in F(\xi,\beta)$ (or $\alpha_1 \in F(\beta,\xi)$ if $\xi = \min(C_\gamma\setminus\beta)$). Then by the inductive hypothesis *(b)* on the triple $\alpha \le \xi \le \beta$ or *(a)* on the triple $\alpha \le \beta \le \xi$:

$$\begin{aligned}&\rho_0(\alpha,\beta) = \rho_0(\min(F(\xi,\beta)\setminus\alpha,\beta))^\frown\rho_0(\alpha,\min(F(\xi,\beta)\setminus\alpha)) = \\ &= \rho_0(\alpha_1,\beta)^\frown\rho_0(\alpha,\alpha_1)\end{aligned}$$

In any case:

$$\rho_0(\alpha,\beta) = \rho_0(\alpha_1,\beta)^\frown\rho_0(\alpha,\alpha_1) \tag{2.1}$$

This proves *(a)* for the triple $\alpha \le \beta \le \gamma$ assuming *(a)* and *(b)* for all smaller triples.

We now prove *(b)* for the triple $\alpha \leq \beta \leq \gamma$ assuming *(b)* for all smaller triples and *(a)* for all triples including the triple α, β, γ. First assume $\lambda \geq \alpha$. Now $F(\lambda, \beta) \subseteq F(\beta, \gamma)$. So $\alpha_2 = \min(F(\lambda, \beta) \setminus \alpha) \geq \alpha_1$. By our inductive assumptions on *(a)* and *(b)*:

$$\rho_0(\alpha, \beta) = \rho_0(\min(F(\lambda, \beta) \setminus \alpha), \beta)\hat{\ }\rho_0(\alpha, \min(F(\lambda, \beta) \setminus \alpha)) = \rho_0(\alpha_2, \beta)\hat{\ }\rho_0(\alpha, \alpha_2) \tag{2.2}$$

$$\rho_0(\alpha, \lambda) = \rho_0(\min(F(\lambda, \beta) \setminus \alpha), \lambda)\hat{\ }\rho_0(\alpha, \min(F(\lambda, \beta) \setminus \alpha)) = \rho_0(\alpha_2, \lambda)\hat{\ }\rho_0(\alpha, \alpha_2) \tag{2.3}$$

By equations (2.1) and (2.2) and appealing to claim 2.9, we get that $\alpha_1 \in \mathtt{tr}(\alpha, \beta)$ and $\alpha_2 \in \mathtt{tr}(\alpha, \beta)$. In particular $\alpha_1 \in \mathtt{tr}(\alpha, \alpha_2)$, since $\alpha_2 \geq \alpha_1$. We can conclude that:

$$\rho_0(\alpha, \alpha_2) = \rho_0(\alpha_1, \alpha_2)\hat{\ }\rho_0(\alpha, \alpha_1) \tag{2.4}$$

Thus equation (2.3) can be written as:

$$\rho_0(\alpha, \lambda) = \rho_0(\alpha_2, \lambda)\hat{\ }\rho_0(\alpha_1, \alpha_2)\hat{\ }\rho_0(\alpha, \alpha_1) = \rho_0(\alpha_1, \lambda)\hat{\ }\rho_0(\alpha, \alpha_1) \tag{2.5}$$

Now $C_\lambda = C_\gamma \cap \lambda$, thus $\rho_0(\alpha, \lambda) = \rho_0(\alpha, \gamma)$ and $\rho_0(\alpha_1, \gamma) = \rho_0(\alpha_1, \lambda)$, so we can conclude that $\rho_0(\alpha, \gamma) = \rho_0(\alpha_1, \gamma)\hat{\ }\rho_0(\alpha, \alpha_1)$. This proves *(b)* in the case that $\lambda \geq \alpha$.

Now assume $\lambda < \alpha$, then $\Lambda(\alpha, \gamma) = \Lambda(\beta, \gamma) = \lambda$. Now assume $\gamma_1 = \min(C_\gamma \setminus \beta)$ or $\gamma_1 \in [\lambda, \beta) \cap C_\gamma$. In any case $F(\gamma_1, \beta) \subseteq F(\beta, \gamma)$ and thus $\gamma_2 = \min F(\gamma_1, \beta) \geq \alpha_1$. Then by the inductive hypothesis *(b)* on the triple $\alpha \leq \gamma_1 \leq \beta$ or *(a)* on the triple $\alpha \leq \beta \leq \gamma_1$:

$$\begin{aligned} &\rho_0(\alpha, \gamma_1) = \rho_0(\min(F(\gamma_1, \beta) \setminus \alpha), \gamma_1)\hat{\ }\rho_0(\alpha, \min(F(\gamma_1, \beta) \setminus \alpha)) = \\ &\rho_0(\gamma_2, \gamma_1)\hat{\ }\rho_0(\alpha, \gamma_2) \end{aligned}$$

In any case we get that:

$$\rho_0(\alpha, \gamma_1) = \rho_0(\gamma_2, \gamma_1)\hat{\ }\rho_0(\alpha, \gamma_2) \tag{2.6}$$

Since *(a)* holds for the triple α, β, γ we can appeal to equation (2.1) and claim 2.9 to infer that $\alpha_1 \in \mathtt{tr}(\alpha, \beta)$. Now applying *(b)* on the triple $\alpha \leq \gamma_1 \leq \beta$ or *(a)* on the triple $\alpha \leq \beta \leq \gamma_1$:

$$\begin{aligned} &\rho_0(\alpha, \beta) = \rho_0(\min(F(\beta, \gamma) \setminus \alpha), \beta)\hat{\ }\rho_0(\alpha, \min(F(\beta, \gamma) \setminus \alpha)) = \\ &= \rho_0(\gamma_2, \beta)\hat{\ }\rho_0(\alpha, \gamma_2) \end{aligned}$$

Appealing again to claim 2.9, we can infer that $\gamma_2 \in \mathtt{tr}(\alpha, \beta)$. So since $\alpha_1 \leq \gamma_2$ and also $\alpha_1 \in \mathtt{tr}(\alpha, \beta)$ by our previous remark, we get that $\alpha_1 \in \mathtt{tr}(\alpha, \gamma_2)$, and so we can conclude that:

$$\rho_0(\alpha, \gamma_2) = \rho_0(\alpha_1, \gamma_2)\hat{\ }\rho_0(\alpha, \alpha_1) \tag{2.7}$$

Finally, by the inductive hypothesis *(b)* on the triple $\alpha_1 \leq \gamma_1 \leq \beta$ or *(a)* on the triple $\alpha_1 \leq \beta \leq \gamma_1$:

$$\begin{aligned} &\rho_0(\alpha_1, \gamma_1) = \rho_0(\min(F(\gamma_1, \beta) \setminus \alpha_1), \gamma_1)\hat{\ }\rho_0(\alpha_1, \min(F(\gamma_1, \beta) \setminus \alpha_1)) = \\ &= \rho_0(\gamma_2, \gamma_1)\hat{\ }\rho_0(\alpha_1, \gamma_2) \end{aligned}$$

So we obtain that:

$$\rho_0(\alpha_1,\gamma_1)=\rho_0(\gamma_2,\gamma_1)^\frown\rho_0(\alpha_1,\gamma_2) \tag{2.8}$$

Now combining together equations (2.6), (2.7), (2.8), we get that:

$$\begin{aligned}&\rho_0(\alpha,\gamma)=\langle\mathtt{otp}(C_\gamma\cap\alpha)\rangle^\frown\rho_0(\alpha,\gamma_1)=\\&=\langle\mathtt{otp}(C_\gamma\cap\alpha)\rangle^\frown\rho_0(\gamma_2,\gamma_1)^\frown\rho_0(\alpha,\gamma_2)=\langle\mathtt{otp}(C_\gamma\cap\alpha)\rangle^\frown\rho_0(\gamma_2,\gamma_1)^\frown\rho_0(\alpha_1,\gamma_2)^\frown\rho_0(\alpha,\alpha_1)=\\&=\langle\mathtt{otp}(C_\gamma\cap\alpha)\rangle^\frown\rho_0(\alpha_1,\gamma_1)^\frown\rho_0(\alpha,\alpha_1)=\rho_0(\min(F(\beta,\gamma)\setminus\alpha),\gamma)^\frown\rho_0(\alpha,\min(F(\beta,\gamma)\setminus\alpha))\end{aligned}$$

This concludes the proof of lemma 2.11. ∎

Lemma 2.12 *For all* $\alpha<\beta<\kappa^+$, $\sup_{\xi<\alpha}|\rho_2(\xi,\alpha)-\rho_2(\xi,\beta)|<\omega$.

Proof: $\rho_2(\alpha,\beta)=|\rho_0(\alpha,\beta)|$. Using the previous claim we can conclude that for all $\xi<\alpha$, $|\rho_2(\xi,\alpha)-\rho_2(\xi,\beta)|\leq\max\{|\rho_2(\eta,\beta)-\rho_2(\eta,\alpha)|:\eta\in F(\alpha,\beta)\}$. This is an integer which depends only on α and β. ∎

Here is another crucial property of the function ρ_2:

Theorem 8 *Let* $(C_\alpha:\alpha<\kappa)$ *be a square sequence and* A, B *be unbounded subset of* κ *on a regular* κ*. Then for every* n *there are* $\alpha\in A$ *and* $\beta\in B$ *such that* $\rho_2(\alpha,\beta)>n$.

Proof: Let $\overrightarrow{\mathcal{C}}=(C_\alpha:\alpha<\kappa)$ witness $\square(\kappa)$. We prove the following stronger statement:

> *For every* A *and* B *unbounded subsets of* κ *and every* n *there are a tail subset* A' *of* A *and an unbounded subset* B' *of* B *such that* $\rho_2(\alpha,\beta)>n$ *for all* $\alpha\in A'$ *and* $\beta\in B'$.

We proceed by induction over n. So we assume the above staement holds for $n-1$ and we prove it for n. To this aim, fix A, B arbitrary unbounded subsets of κ and a continuous $\in$-sequence $(M_\xi:\xi<\kappa)$ of submodels of $H(\kappa^+)$ containing all relevant informations and such that $M_\xi\cap\kappa\in\kappa$ for all ξ. Now consider the club C of ξ such that $M_\xi\cap\kappa=\xi$ and using *(iii)* of def. 2.4 find ξ such that $C\cap\xi\not\subseteq C_\eta$ for all $\eta\geq\xi$. Pick $\beta\in B\setminus\xi$ and $\eta\in C\cap\xi$ such that $\eta\notin C_\beta$. Then $\alpha_0=\sup(C_\beta\cap\eta)<\eta$. Since $\beta\in B\setminus M_\eta$, M_η models that for all $\alpha\in A\setminus\alpha_0$ there is $\beta(\alpha)\in B\setminus(\alpha+1)$ such that $\sup(C_{\beta(\alpha)}\cap\alpha)=\alpha_0$. Thus by elementarity the latter is true in the universe. Now let $A_0=A\setminus\alpha_0$ and

$$D=\{\min(C_{\beta(\alpha)}\setminus\alpha):\alpha\in A_0\}.$$

They are both unbounded subsets of κ so by our inductive assumptions there is a tail subset A' of A_0 and an unbounded subset D' of D such that $\rho_2(\alpha,\gamma)>n-1$ for all $\alpha\in A'$ and $\gamma\in D'$. Now let

$$B'=\{\beta(\alpha):\min(C_{\beta(\alpha)}\setminus\alpha)\in D'\}.$$

Then for all $\beta \in B'$ and $\alpha \in A'$:

$$\rho_0(\alpha,\beta) = \langle \mathtt{otp}(C_\beta \cap \alpha)\rangle\hat{\ }\rho_0(\alpha, \min(C_{\beta(\alpha)} \setminus \alpha)).$$

So:

$$\rho_2(\alpha,\beta) = \rho_2(\alpha, \min(C_{\beta(\alpha)} \setminus \alpha)) + 1 > n - 1 + 1 = n.$$

This concludes the proof. ■

2.2.2 Proof of theorem 7

We are now ready to prove theorem 7. We will use ρ_2 to define the matrix. Set for all $n \in \omega$ and $\beta < \kappa^+$, $K(n,\beta) = \{\alpha \leq \beta : \rho_2(\alpha,\beta) \leq n\}$.

Claim 2.13 *For every $\alpha < \beta$ and for every n, there is m such that $K(n,\alpha) \subseteq K(m,\beta)$ and $K(n,\beta) \cap \alpha \subseteq K(m,\alpha)$.*

Proof: This follows from lemma 2.12. □

By the same argument of the proof of theorem 5 (in particular see the proof of claim 2.6), we can conclude that the ideal $\mathcal{I} \subseteq [\kappa^+]^{\leq\omega}$ of X which have finite intersection with all $K(n,\beta)$ is a P-ideal.

Alternative *(i)* of PID cannot hold for $\mathcal{I}$ since, if $Z \subseteq \kappa^+$ is of size $\aleph_1$ and $\alpha > \sup Z$, then there is n such that $Z \cap K(n,\alpha)$ is uncountable. Thus any countable $X \subseteq Z \cap K(n,\alpha)$ is not in $\mathcal{I}$.

We show that alternative *(ii)* of PID is also not possible. Assume that $\kappa = \bigcup_n A_n$ with each A_n orthogonal to $\mathcal{I}$. Then there is some n such that A_n is unbounded in κ.

Claim 2.14 *For all β there is m_β such that $A_n \cap \beta \subseteq K(m_\beta,\beta)$.*

Proof: Suppose not and find α such that for all m, $A_n \not\subseteq K(m,\alpha)$. In particular take Y to be a countable subset of α with finite intersection with all $K(m,\alpha)$. Then $Y \in \mathcal{I}$ by the same argument of claims 2.6 and 2.8. Thus $[A_n]^\omega \cap \mathcal{I} \neq \emptyset$ which is a contradiction. □

Pick A unbounded subset of A_n such that $m_\beta = m$ for all $\beta \in A$. This means that $\rho_2(\alpha,\beta) \leq m$ for all $\alpha < \beta \in A$. However this contradicts theorem 8. The proof of theorem 7 is now completed. ■

Chapter 3

The mapping reflection principle

Almost all known applications of MM which do not follow from PFA are a consequence of some form of reflection for stationary sets. These types of reflection principles are the fundamental source in order to obtain proofs of all cardinal arithmetic result that follows from MM. In particular SCH and the fact that $\mathfrak{c} \leq \omega_2$ are a consequence of many of the known reflection principles which holds under MM. However up to a very recent time there was no such kind of principle which could be derived from PFA alone. This has been the main difficulty in the search for a proof that PFA implies $\mathfrak{c} = \omega_2$, a result which has been obtained by Todorčević and Veličković appealing to combinatorial arguments which are not dissimilar from the P-ideal dichotomy. Later on this has also been the crucial obstacle in the search for a proof of SCH from PFA.
In 2003 Moore [32] has found an interesting form of reflection which can be derived from PFA, the mapping reflection principle MRP. He has then used this principle to show that BPFA implies that $\mathfrak{c} = \aleph_2$ and also that this principle is strong enough to entail the non-existence of square sequences. He has also shown in [34] that MRP could be a useful tool in the search of a proof of SCH from PFA. I first obtained my proof of this latter theorem elaborating from [34]. Many other interesting consequences of this reflection principle have been found by Moore and others. A complete presentation of this subject will be found in [5]

Definition 3.1 *Let θ be a regular cardinal, let X be uncountable, and let $M \prec H(\theta)$ be countable such that $[X]^\omega \in M$. A subset Σ of $[X]^\omega$ is M-stationary if for all $\mathcal{E} \in M$ such that $\mathcal{E} \subseteq [X]^\omega$ is club, $\Sigma \cap \mathcal{E} \cap M \neq \emptyset$.*

Recall that the Ellentuck topology on $[X]^\omega$ is obtained by declaring a set open if it is the union of sets of the form

$$[x, N] = \{Y \in [X]^\omega : x \subseteq Y \subseteq N\}$$

where $N \in [X]^\omega$ and $x \subseteq N$ is finite.

Definition 3.2 *Σ is an open stationary set mapping if there is an uncountable set X and a regular cardinal θ such that $[X]^\omega \in H(\theta)$, the domain of Σ is a club in $[H(\theta)]^\omega$ of countable elementary submodels M such that $X \in M$ and for all M, $\Sigma(M) \subseteq [X]^\omega$ is open in the Ellentuck topology on $[X]^\omega$ and M-stationary.*

The mapping reflection principle (MRP) asserts that:

> If Σ is an open stationary set mapping, there is a continuous $\in$-chain $\vec{N} = (N_\xi : \xi < \omega_1)$ of elements in the domain of Σ such that for all limit ordinals $0 < \xi < \omega_1$ there is $\nu < \xi$ such that $N_\eta \cap X \in \Sigma(N_\xi)$ for all η such that $\nu < \eta < \xi$.

If $(N_\xi : \xi < \omega_1)$ satisfies the conclusion of MRP for Σ then it is said to be a reflecting sequence for Σ.

We will analyze in some detail the following two theorems to give an idea of how to apply[1] MRP.

Theorem 9 *(Moore [32]) Assume* MRP*. Then there is a well ordering of $P(\omega_1)/NS_{\omega_1}$ in type $\aleph_2$ definable in $H(\omega_2)$ by a Δ_2-formula with parameter a subset of ω_1. In particular* MRP *implies that $2^{\aleph_1} = \aleph_2$.*

A simple argument will then yield that BPFA $\models \mathfrak{c} = 2^{\omega_1} = \omega_2$, while a detailed analysis of the proof[2] of this theorem will show that also MRP implies that $\mathfrak{c} = \omega_2$.

Theorem 10 *(Moore [32]) Assume* MRP*. Then $\square(\kappa)$ fails for all regular $\kappa \geq \aleph_2$.*

Other interesting consequences of MRP are the following:

Theorem 11 *(Caicedo, Veličković [6]) Assume* MRP*. Then there is a well-ordering of the reals which is Δ_1-definable in $H(\aleph_2)$ using as parameter a subset of $\aleph_1$.*

Elaborating some more on their argument, they can obtain as a corollary the following:

Theorem 12 *(Caicedo, Veličković [6]) Assume $M \subseteq V$ are models of* BPFA *with the same ω_2 then $P(\omega_1) \subseteq M$.*

A folklore problem in combinatorial set theory for the last twenty years has been the consistency of the existence of a five element basis for the uncountable linear orders, i.e. the statement that there are five uncountable linear orders such that at least one of them embeds in any other uncountable linear order.

[1] $P(\omega_1)/NS_{\omega_1}$ denotes the quotient of $P(\omega_1)$ by the ideal of non-stationary subsets of ω_1.
[2] The interested reader is referred to [32].

Theorem 13 *(Moore [33]) Assume* BPFA *and* MRP*. Then there is a five element basis for the uncountable linear orders.*

Definition 3.3 *A Kurepa tree is a tree of height $\aleph_1$ with countable levels and at least $\aleph_2$-many branches.*

Theorem 14 *(König, Larson, Moore, Veličković [20])* MRP *implies that there are no Kurepa trees.*

We now turn to the proofs of theorems 9 and 10.

3.1 MRP implies that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$

We will just prove that assuming MRP there is a first order definable injection of $P(\omega_1)/NS_{\omega_1}$ into ω_2. We will also explain why this is enough to obtain that $2^{\aleph_1} = \aleph_2$. A proof of the other parts of the theorem can be found in [32].

Fact 3.4 $|P(\omega_1)/NS_{\omega_1}| = |P(\omega_1)|$

Proof: Clearly $|P(\omega_1)/NS_{\omega_1}| \leq |P(\omega_1)|$. To see the other direction fix $\{A_\xi : \xi < \omega_1\}$ partition of ω_1 in disjoint stationary sets. Now let

$$\phi : P(\omega_1) \to P(\omega_1)/NS_{\omega_1}$$

be defined as follows:

$$\phi(A) = [\bigcup\{A_\xi : \xi \in A\}]$$

It is immediate to check that ϕ is injective. So $|P(\omega_1)/NS_{\omega_1}| \geq |P(\omega_1)|$ □

We now turn to the interesting part of the theorem. Fix $\{C_\eta : \eta < \omega_1\}$ such that for all limit η, C_η is a cofinal sequence in η of type ω. We show that MRP allows to define an injection of $P(\omega_1)/NS_{\omega_1}$ into ω_2 definable in $H(\omega_2)$ by the parameter $\{C_\eta : \eta < \omega_1\}$.

Let $Y \subseteq X$ be countable sets and π_X be the transitive collapse of X. Set $\alpha_X = \sup(X \cap \omega_1)$ and $\beta_X = \sup(X \cap \omega_2)$, $n_1(Y, X) = |\alpha_Y \cap C_{\alpha_X}|$, $n_2(Y, X) = |\mathtt{otp}(\beta_Y \cap X) \cap C_{\mathtt{otp}(\beta_X \cap X)}|$.

Now let A be a subset of ω_1 and $[A]$ denote its equivalence class in $P(\omega_1)/NS_{\omega_1}$. We say that δ codes A if $\delta = \bigcup_{\omega_1} X_\xi$ with the property that:

(i) X_ξ is countable for all ξ,

(ii) if $\xi < \eta$, $X_\xi \subseteq X_\eta$

(iii) $X_\eta = \bigcup_{\xi<\eta} X_\xi$ for all limit η,

(iv) for all η such that $X_\eta \cap \omega_1 = \eta$, $\eta \in A$ if there is $\gamma < \eta$ such that $n_1(X_\xi, X_\eta) \leq n_2(X_\xi, X_\eta)$ for all $\xi \in (\gamma, \eta)$ and $\eta \notin A$ if there is $\gamma < \eta$ such that $n_1(X_\xi, X_\eta) > n_2(X_\xi, X_\eta)$ for all $\xi \in (\gamma, \eta)$.

Claim 3.5 *If δ codes A and B, then $[A] = [B]$.*

Proof: Let $\{X_\xi : \xi < \omega_1\}$ witness that δ codes A and $\{Y_\xi : \xi < \omega_1\}$ witness that δ codes B. Find C club in ω_1 such that for all $\xi \in C$ $X_\xi = Y_\xi$ and $X_\xi \cap \omega_1 = \xi$. Then for all $\eta \in C$, $\eta \in A$, if for some $\gamma < \eta$ and for all $\xi \in (\gamma, \eta)$, $n_1(X_\xi, X_\eta) \leq n_2(X_\xi, X_\eta)$, if $\eta \in B$. $\eta \notin A$, if for some $\gamma < \eta$ and for all $\xi \in (\gamma, \eta)$, $n_1(X_\xi, X_\eta) > n_2(X_\xi, X_\eta)$, if $\eta \notin B$. So $A \cap C = B \cap C$, i.e. $[A] = [B]$. □

Thus $\phi : P(\omega_1)/NS_{\omega_1} \to \omega_2$ which sends $[A]$ to the least δ such that δ codes A is injective on its domain and is definable by a first order formula[3] in $H(\omega_2)$ with parameter the sequence $\{C_\eta : \eta < \omega_1\}$. We will complete the proof of the theorem once we show the following:

Lemma 3.6 *Assume* MRP. *Then for all $A \subseteq \omega_1$ there is δ which codes A.*

Proof: Let $M \prec H(\theta)$ be a countable model containing all relevant information. Let $\Sigma_0(M)$ be the set of $Y \subseteq M \cap \omega_2$ such that $n_1(Y, M) \leq n_2(Y, M) < \omega$ and $\Sigma_1(M)$ be the set of $Y \subseteq M \cap \omega_2$ such that $\omega > n_1(Y, M) > n_2(Y, M)$.

Let $\Sigma_A(M) = \Sigma_0(M)$ if $\alpha_M \in A$ and $\Sigma_A(M) = \Sigma_1(M)$ if $\alpha_M \notin A$.

We will show that $\Sigma_A(M)$ is open and M-stationary for all the relevant M. Assume that this is the case and let $\{M_\eta : \eta < \omega_1\}$ be a reflecting sequence for Σ_A and let $\delta = \bigcup_\eta M_\eta \cap \omega_2$. It is immediate to check that $\{M_\eta \cap \omega_2 : \eta < \omega_1\}$ witnesses that δ codes A. So the proof of the lemma will be completed once we show the following claim:

Claim 3.7 *$\Sigma_i(M)$ is open and M-stationary for $i = 0, 1$.*

Proof: First we show that both $\Sigma_i(M)$ are open subsets of $[\omega_2]^\omega$. So assume that $X \in \Sigma_i(M)$. Let $\gamma = \max(C_{\alpha_M} \cap \alpha_X)$ and $\eta = \max(\mathtt{otp}(\beta_X \cap M) \cap C_{\mathtt{otp}(\beta_M \cap M)})$. Then $\gamma < \alpha_X$ and $\eta < \mathtt{otp}(\beta_X \cap M)$. So there are $\alpha_0 \in X \cap (\gamma, \alpha_X]$ and $\nu \in (\eta, \mathtt{otp}(\beta_X \cap M)] \cap \pi_M[X]$. Let $\beta_0 = \pi_M^{-1}(\nu)$. It is immediate to check that $\alpha_0 \leq \alpha_Y \leq \alpha_X$ and $\beta_0 \leq \beta_Y \leq \beta_X$ for all $Y \in [\{\alpha_0, \beta_0\}, X]$. So $n_1(Y, M) = n_1(X, M)$ and $n_2(Y, M) = n_2(X, M)$ for all $Y \in [\{\alpha_0, \beta_0\}, X]$. Thus $Y \in \Sigma_i(M)$ iff $X \in \Sigma_i(M)$.

We turn to the proof that both $\Sigma_i(M)$ are M-stationary sets. We first prove that $\Sigma_0(M)$ is M-stationary. So let $f : [\omega_2]^{<\omega} \to \omega_2$ be in M. Let for all $\alpha \in \omega_1$

$$E_\alpha = \{\sup(X) : X \text{ is countable and } f\text{-closed } \& \, \alpha_X = \alpha\}$$

Then there is α such that E_α is unbounded in $\aleph_2$. Otherwise if $\delta = \sup_{\omega_1}(E_\alpha) < \omega_2$, there is X f-closed and countable such that $\delta + 1 \in X$. Then $\sup(X) > \delta$ which is a contradiction. Moreover by elementarity there is $\alpha \in M$ such that E_α is unbounded. Now let $n_1(\alpha, \alpha_M) = m$ and find $\beta \in M$ such that $|\mathtt{otp}(M \cap \beta) \cap C_{\mathtt{otp}(M \cap \beta)}| = l > m$. Find $X \in M$ countable and such that

[3] A closer inspection of the formula will show that it is a Δ_2-formula.

$\alpha_X = \alpha$ and $\beta_X > \beta$. Then $n_1(X, M) = m < l \leq n_2(X, M)$ so X is f-closed and $X \in \Sigma_0(M)$.

We now prove that $\Sigma_1(M)$ is M-stationary. So let $f : [\omega_2]^{<\omega} \to \omega_2$ be in M. Let for all $\beta \in \omega_2$:

$$F_\beta = \{\alpha_X : X \text{ is countable and } f\text{-closed} \,\&\, \beta_X = \beta\}.$$

We prove that there is $\beta < \omega_2$ such that F_β is unbounded in $\aleph_1$. Otherwise there is a stationary set B of β such that $\sup(F_\beta) = \alpha$ for all $\beta \in B$. Now let $C = \{\delta : f[[\delta]^{<\omega}] \subseteq \delta\}$. Find $\delta \in C \cap B$ and let X be the f-closure of $\{\alpha + 1\} \cup Z$, where Z is a cofinal countable subset of δ. Then $\sup(X) = \beta$ and $\alpha_X > \alpha$. This contradicts the fact that $\alpha = \sup(F_\delta)$. By elementarity there is $\beta \in M$ such that F_β is unbounded in ω_1. Now let $n_2(\beta \cap M, M) = m$ and find $\alpha \in M$ such that $|\alpha \cap C_{\alpha_M}| = l > m$. Find $X \in M$ countable and such that $\beta_X = \beta$ and $\alpha_X > \alpha$. Then $n_1(X, M) \geq n_1(\alpha, M) = l > m = n_2(X, M)$. So $X \in \Sigma_1(M)$. This concludes the proof of the claim. □

The lemma and the part of the theorem relevant for our purposes are now proved. ■

3.2 MRP implies that square fails

Assume MRP and towards a contradiction let $\{C_\alpha : \alpha < \kappa\}$ witness $\square(\kappa)$ for a regular $\kappa > \omega_1$ (see definition 2.4). Let $M \prec H(\theta)$ be a countable model containing all relevant information and for all countable X, let $\delta_X = \sup(X \cap \kappa)$. Set $\Sigma(M)$ to be the family of all $X \subseteq M$ such that δ_X is not a limit point of C_{δ_M}. We will show that Σ_M is open and M-stationary. Assume this is the case and apply MRP to obtain a reflecting sequence $\{M_\xi : \xi < \omega_1\}$ for Σ. Let $\delta = \sup_{\omega_1} \delta_{M_\xi}$. Find S stationary in ω_1 such that for some γ and for all $\eta \in S$, $M_\xi \cap \kappa \in \Sigma(M_\eta)$ for all $\xi \in (\gamma, \eta)$. Finally find $\eta < \beta \in S$ such that δ_{M_η} and δ_{M_β} are limit point of C_δ. Then $C_\delta \cap \delta_{M_\eta} = C_{\delta_{M_\eta}}$ and $C_\delta \cap \delta_{M_\beta} = C_{\delta_{M_\beta}}$. So δ_{M_η} is a limit point of $C_{\delta_{M_\beta}}$. However $M_\eta \cap \kappa \in \Sigma(M_\beta)$ since $\eta \in (\gamma, \beta)$, i.e. δ_{M_η} is not a limit point of $C_{\delta_{M_\beta}}$, a contradiction.

We are left with the proof that $\Sigma(M)$ is open and M stationary. To see that $\Sigma(M)$ is open let $X \in \Sigma(M)$. Since δ_X is not a limit point of C_{δ_M} and C_{δ_M} is closed in δ_M, $\max(C_{\delta_M} \cap \delta_X) = \gamma < \delta_X$. Now pick $\xi \in X \cap (\gamma, \delta_X]$. Then $[\{\xi\}, X] \subseteq \Sigma(M)$ since any $Y \in [\{\xi\}, X]$ is such that $\gamma < \delta_Y \leq \delta_X$ so δ_Y is not a limit point of C_{δ_M}.

To see that $\Sigma(M)$ is M-stationary, suppose that this is not the case and let $f : [\kappa]^{<\omega} \to \kappa$ be a function in M such that for all X which are f-closed $X \notin \Sigma(M)$. Let $C = \{\delta : f[[\delta]^{<\omega}] \subseteq \delta\} \in M$. Now for all $\alpha \in C \cap M$ of countable cofinality pick $X \in M$, countable, f-closed and such that $\delta_X = \alpha$. Since $X \notin \Sigma(M)$, $\alpha = \delta_X$ is a limit point of C_{δ_M}. So $C_{\delta_M} \cap \alpha = C_\alpha$ for all $\alpha \in C \cap M$ of countable cofinality. In particular we can conclude that for all $\alpha < \beta \in C \cap M$ and of countable cofinality $C_\alpha = C_\beta \cap \alpha$. So M models that for all $\alpha < \beta \in C$ of countable cofinality, $C_\alpha = C_\beta \cap \alpha$. By elementarity of M this is true in $H(\theta)$. Set $D = \bigcup\{C \cap C_\alpha : \alpha \in C \cap S^\omega_\kappa\}$. Then D is a club in κ since if $\{\xi_\eta : \eta < \delta\} \subseteq D$ with $\delta < \kappa$ and $\{\alpha_\eta : \eta < \delta\} \subseteq C \cap S^\omega_\kappa$ witnesses this, we can pick $\alpha > \sup\{\alpha_\eta : \eta < \delta\}$ in $C \cap S^\omega_\kappa$. Then since $C_\alpha \cap \alpha_\eta = C_{\alpha_\eta}$ for all η we get $\{\xi_\eta : \eta < \delta\} \subseteq C \cap C_\alpha$ which is a closed set and thus $\sup\{\xi_\eta : \eta < \delta\} \in C \cap C_\alpha \subseteq D$. Now D is a club which contradicts property *(iii)* of def. 2.4. This is the desired contradiction. Theorem 10 is now proved. ■

Chapter 4

A covering property

In this section we introduce the main original concept of this thesis. In fact a careful analysis of Solovay's proof that $\square(\kappa)$ fails for all regular $\kappa \geq \lambda$, where λ is a strongly compact cardinal, lead to the isolation of a covering property CP which one side is strong enough to entail both SCH and the failure of square, on the other side is weak enough to be a consequence either of the existence of a strongly compact cardinal, or of PID or of MRP.

Definition 4.1 *For any cardinal κ, $\mathcal{D} = (K(n,\beta) : n < \omega,\ \beta \in \kappa)$ is a covering matrix for κ if:*

(i) For all α, $\alpha + 1 = \bigcup_n K(n,\alpha)$,

(ii) for all α and n, $|K(n,\alpha)| < \kappa$.

(iii) for all α, $K(n,\alpha) \subseteq K(m,\alpha)$ for $n < m$,

(iv) for all $\alpha < \beta$ and for all m there is n such that $K(m,\alpha) \subseteq K(n,\beta)$.

(v) for all $X \in [\kappa]^\omega$ there is $\gamma_X < \kappa$ such that for all β and n there is m such that $K(n,\beta) \cap X \subseteq K(m,\gamma_X)$

$\beta_{\mathcal{D}} \leq \kappa$ is the least such that for all n and β, $\mathtt{otp}(K(n,\beta)) < \beta_{\mathcal{D}}$

The matrices $\mathcal{D}$ that we considered in the other applications of the PID (see theorems 7 and 5) satisfied *(i)*, *(ii)*, *(iii)*, *(iv)* and a much stronger form of *(v)*. In fact they satisfied the requirement that for all $\alpha < \beta$ and for all n, there is m such that $K(n,\beta) \cap \alpha \subseteq K(m,\alpha)$. However we will see below that *(v)* is a requirement on $\mathcal{D}$ which still allows for a proof that the family of $X \in [\kappa]^{\leq\omega}$ with finite intersections with all $K(n,\beta) \in \mathcal{D}$ is a P-ideal. The proof that PFA implies SCH will be a fruitful application of this fact.

Lemma 4.2 *Assume that κ is a singular cardinal of countable cofinality. Then there is a covering matrix $\mathcal{C}$ for κ^+ with $\beta_{\mathcal{C}} = \kappa$.*

Proof: The lemma holds without any cardinal arithmetic assumption and in fact the matrices we are going to define satisfy *(i),⋯,(iii)* and a stronger coherence property than what is required by *(iv)* and *(v)* of the above definition. They will satisfy the following properties *(iv*)* and *(v*)* from which *(iv)* and *(v)* immediately follow[1]:

(iv)* For all $\alpha < \beta$ there is n such that $K(m, \alpha) \subseteq K(m, \beta)$ for all $m \geq n$.

(v)* For all $X \in [\kappa^+]^\omega$ there is $\gamma_X < \kappa$ such that for all $\beta \geq \gamma_X$, there is n such that $K(m, \beta) \cap X = K(m, \gamma_X) \cap X$ for all $m \geq n$.

We first fix some notation. Let $\phi_\eta : \kappa \to \eta$ be a surjection for all $\eta < \kappa^+$. Fix also $\{\kappa_n : n < \omega\}$ increasing sequence of regular cardinals cofinal in κ with $\kappa_0 \geq \aleph_1$. We prove the lemma first assuming that $\kappa > \mathfrak{c}$ since in this case the proof is simpler[2]. In this case let

$$K(n, \beta) = \bigcup\{K(n, \gamma) : \gamma \in \phi_\beta[\kappa_n]\}.$$

It is immediate to check that $\mathcal{D} = (K(n, \beta) : n \in \omega, \beta < \kappa^+)$ satisfies *(i)*, *(ii)*, *(iii)* of definition 4.1, property *(iv*)* and that $\beta_{\mathcal{D}} = \kappa$. To prove *(v*)*, let $X \in [\kappa^+]^\omega$ be arbitrary. Now since $\mathfrak{c} < \kappa^+$ and there are at most $\mathfrak{c}$ many subsets of X, there is a stationary subset S of κ^+ and a fixed decomposition of X as the increasing union of sets X_n such that $X \cap K(n, \alpha) = X_n$ for all α in S and for all n. Now properties *(i)⋯(iv)* of the matrix guarantees that this property of S is enough to get *(v*)* for X with $\gamma_X = \min(S)$.

We now prove the lemma without assuming $\mathfrak{c} < \kappa$. Thus let S be a stationary set of points of cofinality ω_1 in $\mathcal{I}[\kappa^+]$ and[3] $\{a_\alpha : \alpha < \kappa^+\} \subseteq [\kappa^+]^{<\kappa}$ witness that $S \in \mathcal{I}[\kappa^+]$, i.e. such that for all $\alpha \in S$, $\mathtt{otp}(a_\alpha) = \mathtt{cof}(\alpha) = \omega_1$ and for all $\gamma < \alpha$, $a_\alpha \cap \gamma \in \{a_\xi : \xi < \alpha\}$. Assume that $(K(n, \alpha) : n \in \omega, \alpha < \beta)$ have been defined for all n and $\alpha < \beta$. If $\beta \notin S$, let $\phi_\beta : \kappa \to \beta$ be a surjection and set:

$$\begin{aligned} K(n, \beta) = &\{\beta\} \cup \phi_\beta[\kappa_n] \cup \bigcup\{K(n, \gamma) : \gamma \in \phi_\beta[\kappa_n]\} \\ &\cup \bigcup\{a_\eta : \eta \in \phi_\beta[\kappa_n] \,\&\, |a_\eta| \leq \kappa_n\} \\ &\cup \bigcup\{K(n, \gamma) : \gamma \in a_\eta \,\&\, \eta \in \phi_\beta[\kappa_n] \,\&\, |a_\eta| \leq \kappa_n\}. \end{aligned}$$

[1]We decided to use the weaker *(iv)* and *(v)* in the definition of a covering matrix in order to include among these matrices the one produced by a square sequence using the ρ_2-function (see section 2.2), since this matrix satisfies *(iv)* and *(v)*, but neither *(iv*)* nor *(v*)*. This is made in order to have a simple criterion to evaluate the consistency strength of the covering property CP that we are going to introduce below. It remains open whether the restriction of the covering property to matrices satisfying *(i),…,(iii)* and *(iv*), (v*)* has any large cardinal strength at all. We felt free to adopt this weakening, since properties *(i),…,(v)* on a matrix are the minimal requirements we isolated up to now in order to run all the applications of the PID that we know of.

[2]The only relevant application of the existence of such matrices $\mathcal{D}$ that we have found up to now is the proof of SCH from PFA. In this situation $\mathfrak{c} = \aleph_2$. So the assumption $\mathfrak{c} < \kappa$ is trivially satisfied.

[3]All the properties of this ideal that will be used can be found in section 1.4. The reference text is [36].

If $\beta \in S$ set:

$$K(n,\beta) = \{\beta\} \cup \bigcup\{K(n,\alpha) : \alpha \in a_\beta\}.$$

We show that $\mathcal{D} = \{K(n,\beta : n \in \omega,\ \beta \in \kappa^+\}$ is a covering matrix with $\beta = \kappa$. First of all, it is easy to show by induction on α, that $|K(n,\alpha)| \leq \kappa_n$ for all n. Now we prove by induction on $\alpha < \beta$ that:

(a) If $\alpha \in K(n,\beta)$, then $K(n,\alpha) \subseteq K(n,\beta)$.

First suppose $\beta \notin S$. If $\alpha \in K(n,\beta)$, then:

- $\alpha \in \phi_\beta[\kappa_n]$. In this case $K(n,\alpha) \subseteq K(n,\beta)$.
- For some $\gamma \in \phi_\beta[\kappa_n]$, $\alpha \in K(n,\gamma)$. In this case we can apply the inductive hypothesis on $\alpha < \gamma$ to get that $K(n,\alpha) \subseteq K(n,\gamma) \subseteq K(n,\beta)$.
- There is $\eta \in \phi_\beta[\kappa_n]$ such that $|a_\eta| \leq \kappa_n$ and $\alpha \in a_\eta$. By definition of $K(n,\beta)$, $K(n,\alpha) \subseteq K(n,\beta)$.
- There is $\eta \in \phi_\beta[\kappa_n]$ and $\gamma \in a_\eta$, such that $|a_\eta| \leq \kappa_n$ and $\alpha \in K(n,\gamma)$. In this case we can apply the inductive assumption to get that $K(n,\alpha) \subseteq K(n,\gamma) \subseteq \{K(n,\xi) : \xi \in a_\eta\} \subseteq K(n,\beta)$.

Now suppose $\beta \in S$. In this case if $\alpha \in K(n,\beta)$, either $\alpha \in a_\beta$ and in this case $K(n,\alpha) \subseteq K(n,\beta)$ by definition of $K(n,\beta)$ or there is $\gamma \in a_\beta$ such that $\alpha \in K(n,\gamma)$. Now we can apply the inductive hypothesis on $\alpha < \gamma$ to get that $K(n,\alpha) \subseteq K(n,\gamma) \subseteq K(n,\beta)$, so we are done once again. Thus $\mathcal{D}$ satisfies *(i)*, *(ii)*, *(iii)*, *(iv*)* and $\beta_\mathcal{D} = \kappa$.

We now show that $\mathcal{D}$ satisfies also *(v*)*. So fix $X \in [\kappa^+]^\omega$. For any $\beta \in S \setminus \sup(X)$, we have that

$$K(n,\beta) = \bigcup\{K(n,\eta) : \eta \in a_\beta\}.$$

Since $\beta \in S$ has uncountable cofinality, there is $\xi < \beta$ such that

$$K(n,\beta) \cap X \subseteq \bigcup\{K(n,\eta) : \eta \in a_\beta \cap \xi\}.$$

Since $\beta \in S$ there is $\eta < \beta$ such that $a_\beta \cap \xi = a_\eta$. Now pick $\gamma_\beta \in (\eta,\beta)$ and not in S. Then $a_\eta \subseteq K(m,\gamma_\beta)$ for eventually all m. Now for all such m and for all $\xi \in a_\eta$, $K(m,\xi) \subseteq K(m,\gamma_\beta)$, from which it follows that for eventually all m:

$$K(m,\beta) \cap X \subseteq \bigcup\{K(m,\xi) : \xi \in a_\eta\} \subseteq K(m,\gamma_\beta).$$

Find A stationary subset of S such that for all $\beta \in A$, $\gamma_\beta = \gamma_X$. Now let ξ be any ordinal below κ^+ and let $\beta \geq \xi$ be in A. If $\xi \in K(n,\beta)$ then by *(a)* above for all $m \geq n$, $K(m,\xi) \subseteq K(m,\beta)$. Now for eventually all m, $K(m,\beta) \cap X \subseteq K(m,\gamma_X)$. We can conclude that for eventually all m, $K(m,\xi) \cap X \subseteq K(m,\beta) \cap X \subseteq K(m,\gamma_X) \cap X$. However if $\gamma_X \in K(l,\beta)$ the inclusion $K(m,\gamma_X) \cap X \subseteq K(m,\xi) \cap X$ for all $m \geq l$. Thus for eventually all m, $K(m,\xi) \cap X = K(m,\gamma_X) \cap X$. Since X is arbitrary, this shows that $\mathcal{D}$ satisfies also *(v*)* and concludes the proof of the lemma. ■

Definition 4.3 $\mathsf{CP}(\kappa)$*: κ has the covering property if for every $\mathcal{D}$ covering matrix for κ there is an unbounded subset A of κ such that $[A]^\omega$ is covered by $\mathcal{D}$. CP is the statement that $\mathsf{CP}(\kappa)$ holds for all regular $\kappa > \mathfrak{c}$.*

Fact 4.4 *Assume $\mathsf{CP}(\kappa^+)$ for all singular κ of countable cofinality. Then $\lambda^{\aleph_0} = \lambda$, for every $\lambda \geq 2^{\aleph_0}$ of uncountable cofinality.*

Proof: By induction. The base case is trivial. If $\lambda = \kappa^+$ with $\mathtt{cof}(\kappa) > \omega$, then $\lambda^{\aleph_0} = \lambda \cdot \kappa^{\aleph_0} = \lambda \cdot \kappa = \lambda$, by the inductive hypothesis on κ. If λ is a limit cardinal and $\mathtt{cof}(\lambda) > \omega$ then $\lambda^{\aleph_0} = \sup\{\mu^{\aleph_0} : \mu < \lambda\}$, so the result also follows by the inductive hypothesis. Thus, the only interesting case is when $\lambda = \kappa^+$, with κ singular of countable cofinality. In this case we will show, using CP, that $(\kappa^+)^{\aleph_0} = \kappa^+$. To this aim let $\mathcal{D}$ be a covering matrix for κ^+ with $\beta_{\mathcal{D}} = \kappa$. Remark that by our inductive assumptions, since every $K(n,\beta)$ has order type less than κ, $|[K(n,\beta)]^\omega|$ has size less than κ. So $\bigcup\{[K(n,\beta)]^\omega : n < \omega \,\&\, \beta \in \kappa^+\}$ has size κ^+. Use CP to find $A \subseteq \kappa^+$ unbounded in κ^+, such that $[A]^\omega$ is covered by $\mathcal{D}$. Then $[A]^\omega \subseteq \bigcup\{[K(n,\beta)]^\omega : n < \omega \,\&\, \beta \in \kappa^+\}$, from which the conclusion follows. ■

The following theorems motivate our attention for this property:

Theorem 15 *Assume λ is strongly compact. Then $\mathsf{CP}(\kappa)$ holds for all regular $\kappa \geq \lambda$.*

Theorem 16 *Assume* PID. *Then* CP *holds.*

On the other hand MRP allows us to infer a slightly weaker conclusion then the one of the previous theorems[4].

Theorem 17 *Assume MRP and let $\mathcal{D}$ be a covering matrix for κ such that $K(n,\beta)$ is a closed set of ordinals for all $K(n,\beta)$. Then there is A unbounded in κ such that $[A]^\omega$ is covered by $\mathcal{D}$.*

In particular we obtain:

Corollary 1 PFA *implies* SCH.

Proof: PFA implies PID and PID implies CP. In particular PFA implies that $\kappa^\omega = \kappa$ for all regular $\kappa \geq \mathfrak{c}$. By Silver's theorem the least singular $\kappa > 2^{\mathtt{cof}\kappa}$ such that $\kappa^{\mathtt{cof}\kappa} > \kappa^+$ has countable cofinality. Now assume PFA and let κ have countable cofinality. By fact 4.4, $\kappa^{\mathtt{cof}(\kappa)} \leq (\kappa^+)^\omega = \kappa^+$. Thus assuming PFA there cannot be a singular cardinal of countable cofinality which violates SCH. Combining this fact with Silver's result we get that SCH holds under PFA. ■

Before proving all the above theorems we analyze in more details the effects of CP.

[4]Moore has first noticed that a similar covering property followed from MRP reading a draft of [46].

4.1 Some simple consequences of $\mathsf{CP}(\kappa)$

Fact 4.5 *Let $\mathcal{D} = \{K(n,\beta) : n \in \omega, \beta < \kappa\}$ be a covering matrix on a regular $\kappa > \mathfrak{c}$, $\lambda < \kappa$ have uncountable cofinality and A be an unbounded subset of κ. The following are equivalent:*

(i) $[A]^\omega$ *is covered by* $\mathcal{D}$.

(ii) $[A]^\lambda$ *is covered by* $\mathcal{D}$.

Proof: *(ii)* implies *(i)* is evident. To prove the other direction, assume *(i)* and let $Z \subseteq A$ have size λ. We need to find n, β such that $Z \subseteq K(n,\beta)$. For $X \in [Z]^\omega$ let by *(i)* n_X, β_X be such that $X \subseteq K(n_X, \beta_X)$. By fact 4.4, $\lambda^\omega = \lambda + \mathfrak{c} < \kappa$. For this reason $\beta = \sup_{X \in [Z]^\omega} \beta_X < \kappa$. Now by property *(iii)* of $\mathcal{D}$, we have that for all $X \in [Z]^\omega$, $X \subseteq K(m_X, \beta)$ for some m_X. Let $\mathcal{C}_m$ be the set of X such that $m_X = m$. Now notice that for at least one m, $\mathcal{C}_m$ must be unbounded, otherwise $[Z]^\omega$ would be a countable union of bounded subsets which is not possible since Z is uncountable. Then $Z \subseteq K(m,\beta)$, since every $\alpha \in Z$ is in some $X \in \mathcal{C}_m$ because $\mathcal{C}_m$ is unbounded. This completes the proof of the fact. □

If we restrict ourselves to $\aleph_\omega$ we do not have to make any assumption on the size of $\mathfrak{c}$ to obtain a similar conclusion.

Fact 4.6 *Let $\mathcal{D} = \{K(n,\beta) : n \in \omega, \beta < \kappa\}$ be a covering matrix on a regular $\kappa \leq \aleph_{\omega+1}$, and A be an unbounded subset of κ. The following are equivalent:*

(i) $[A]^\omega$ *is covered by* $\mathcal{D}$.

(ii) $[A]^\lambda$ *is covered by* $\mathcal{D}$ *for all regular* $\lambda < \kappa$.

Proof: We just need to show that *(i)* implies *(ii)*. So let $\kappa = \aleph_j$ with $j < \omega$ or $j = \omega + 1$. We proceed by induction on $m < j - 1$ to show that $[A]^{\aleph_{m+1}}$ is covered by $\mathcal{D}$. If X is any subset of A of size $\aleph_{m+1}$, let $X = \bigcup\{X_\eta : \eta < \aleph_{m+1}\}$ with each X_η of size $\aleph_m$ and for all $\eta < \xi$, $X_\eta \subseteq X_\xi$. Then by our inductive assumptions for each η there are l and $\gamma_\eta < \aleph_n$ such that $X_\eta \subseteq K(l, \gamma_\eta)$. Let $\gamma = \sup_{\aleph_{m+1}} \gamma_\eta$. Then $\gamma < \aleph_j$. Find l such that for an unbounded $S \subseteq \aleph_{m+1}$, $X_\eta \subseteq K(l,\gamma)$ for all $\eta \in S$. Then $X \subseteq K(l,\gamma)$. □

A covering matrix $\mathcal{D}$ for κ^+ with $\beta_\mathcal{D} = \kappa$ is an object simple to define when κ has countable cofinality. If $\kappa \geq \omega_1$ does not have countable cofinality the existence of a covering matrix $\mathcal{D}$ for κ^+ with $\beta_\mathcal{D} < \kappa^+$ is not compatible with PFA. This is a simple consequence of the above facts:

Fact 4.7 *Assume κ has uncountable cofinality,* $\mathsf{CP}(\kappa^+)$ *and that either $\kappa \geq \mathfrak{c}$ or $\kappa < \aleph_\omega$. Then there is no covering matrix $\mathcal{D}$ on κ^+ with $\beta_\mathcal{D} < \kappa^+$.*

Proof: Assume not and let $\mathcal{D}$ be a covering matrix for κ^+ with $\beta_\mathcal{D} < \kappa^+$. By $\mathsf{CP}(\kappa^+)$ there should be an A unbounded in κ^+ such that $[A]^\omega$ is covered by $\mathcal{D}$.

Appealing to fact 4.5 if $\kappa \geq \mathfrak{c}$, or to fact 4.6 if $\kappa < \aleph_\omega$ we can conclude in any case that $[A]^\kappa$ is covered by $\mathcal{D}$. Take β large enough in order that $\mathtt{otp}(A \cap \beta) > \beta_{\mathcal{D}}$. Since $A \cap \beta$ has size at most κ there are n, γ such that $A \cap \beta \subseteq K(n, \gamma)$. Thus $\beta_{\mathcal{D}} < \mathtt{otp}(A \cap \beta) \leq \mathtt{otp} K(n, \beta) < \beta_{\mathcal{D}}$ a contradiction. ■

Fact 4.8 *Assume κ is regular and either $\kappa > \mathfrak{c}$ or $\kappa < \aleph_\omega$. Then $\mathsf{CP}(\kappa)$ implies that $\square(\kappa)$ fails.*

Proof: This is so because assuming $\square(\kappa)$, the matrix $\mathcal{D}$, whose entries $K(n, \beta) = \{\alpha < \beta : \rho_2(\alpha, \beta) \leq n\}$ are defined using the square sequence, is a covering matrix for κ, however it has a much stronger coherence property: for every $\alpha < \beta$ and n there is m such that $K(n, \alpha) \subseteq K(m, \beta)$ and $K(n, \beta) \cap \alpha \subseteq K(m, \alpha)$. Using $\mathsf{CP}(\mathcal{D})$ find A unbounded in κ and such that $[A]^\omega$ is covered by $\mathcal{D}$. Using this coherence property of $\mathcal{D}$ one gets that for all $\beta \in A$, $A \cap \beta \subseteq K(m_\beta, \beta)$ for some m_β. Thus one can refine A to an unbounded B such that for a fixed m, $B \cap \beta \subseteq K(m, \beta)$ for all $\beta \in B$. This contradicts theorem 8. □

The main difficulty towards a proof that PFA implies SCH has been the fact that all standard principles of reflection for stationary sets do not hold for PFA. In particular PFA is compatible with the existence on $\aleph_2$ of a never reflecting stationary subset of $S^\omega_{\aleph_2}$. However the following form of reflection holds:

Fact 4.9 *Assume CP and let $\kappa > \mathfrak{c}$ or $\aleph_1 < \kappa < \aleph_\omega$ be regular and $\mathcal{D}$ be a covering matrix for κ with all $K(n, \beta)$ closed. Let $\lambda < \kappa$ be a regular cardinal and let $(S_\eta : \eta < \lambda)$ be an arbitrary family of stationary subsets of $S^{\leq\lambda}_\kappa$. Then there exist n and β such that $S_\eta \cap K(n, \beta)$ is non-empty for all $\eta < \lambda$.*

Proof: By $\mathsf{CP}(\kappa)$ and facts 4.5 and 4.6, there is X unbounded in κ such that $[X]^\lambda$ is covered by $\mathcal{D}$. Since $K(n, \beta)$ is closed for all n and β, we have that $[\overline{X} \cap S^{\leq\lambda}_\kappa]^\lambda$ is covered by $\mathcal{D}$. To see this, let Z be in this latter set and find $Y \subseteq X$ of size λ such that $Z \subseteq \overline{Y}$. Now find n and β such that $Y \subseteq K(n, \beta)$. Since $K(n, \beta)$ is closed, $Z \subseteq \overline{Y} \subseteq K(n, \beta)$.
Now pick $M \prec H(\Theta)$ with Θ large enough such that $|M| = \lambda \subseteq M$ and $\lambda, X, (S_\eta : \eta < \lambda) \in M$. Then $S_\eta \cap \overline{X} \cap S^{\leq\lambda}_\kappa$ is non-empty for all η. By elementarity, M sees this and so $M \cap S_\eta \cap \overline{X} \cap S^{\leq\lambda}_\kappa$ is non-empty for all η. However $M \cap \overline{X} \cap S^{\leq\lambda}_\kappa$ has size λ so there are n and β such that $M \cap \overline{X} \cap S^{\leq\lambda}_\kappa \subseteq K(n, \beta)$. So $S_\eta \cap K(n, \beta)$ is non-empty for all η. □

4.2 CP holds above a strongly compact cardinal

We will need the following trivial consequence of the existence of a strongly compact cardinal:

Lemma 4.10 *Assume λ is strongly compact. Then for every regular $\kappa \geq \lambda$, there is $\mathcal{U}$, λ-complete ultrafilter on κ which concentrates on $S^{<\lambda}_\kappa$.*

Proof: Assume λ is strongly compact and $\kappa \geq \lambda$ is regular. By definition of λ there is a λ-complete ultrafilter $\mathcal{W}$ on $[\kappa]^{<\lambda}$ such that for all $X \in [\kappa]^{<\lambda}$, $\{Y \in [\kappa]^{<\lambda} : X \subseteq Y\} \in \mathcal{W}$. Set $\mathcal{U}$ to be the family of $A \subseteq \kappa$ such that $\{X \in [\kappa]^{<\lambda} : \sup(X \cap \alpha) = \alpha\} \in \mathcal{W}$ for all $\alpha \in A$. It is immediate to check that $\mathcal{U}$ is a λ-complete ultrafilter which concentrates on $S^{<\lambda}_{\kappa}$. □

Now let $\mathcal{D} = (K(n, \beta) : n \in \omega, \beta \in \kappa)$ be a covering matrix for κ. Let $A^{\gamma}_n = \{\beta > \gamma : \gamma \in K(n, \beta)\}$ and $A_n = \{\gamma \in S^{<\lambda}_{\kappa} : A^{\gamma}_n \in \mathcal{U}\}$. By the λ-completeness of $\mathcal{U}$, for every $\gamma \in S^{<\lambda}_{\kappa}$, there is a least n such that $A^{\gamma}_n \in \mathcal{U}$. Thus $\bigcup_n A_n = S^{<\lambda}_{\kappa}$. So there is n such that $A_n \in \mathcal{U}$. In particular A_n is unbounded. Now let X be a countable subset of A_n. Then $A^{\gamma}_n \in \mathcal{U}$ for all $\gamma \in X$. Since $|X| = \aleph_0 < \lambda$, $\bigcap_X A^{\gamma}_n \in \mathcal{U}$ and thus is non-empty. Pick β in this latter set. Then $X \subseteq K(n, \beta)$. Since X is an arbitrary countable subset of A_n, we conclude that $[A_n]^{\omega}$ is covered by $\mathcal{D}$. This concludes the proof of theorem 15. ■

4.3 PID implies SCH

We turn to the proof of theorem 16. As we will see below a model of the PID retains enough properties of the supercompact cardinals from which it is obtained in order that a variation of the above argument can be run also in this context. We break the proof of theorem 16 in two parts. Assume κ is regular and let $\mathcal{D} = (K(n, \alpha) : n \in \omega,\ \beta < \kappa)$ be a covering matrix on κ. Let $\mathcal{I}$ be the family of $X \in [A]^{\omega}$ such that for all $\alpha \in A$ and for all n, $X \cap K(n, \alpha)$ is finite.

Claim 4.11 *$\mathcal{I}$ is a P-ideal.*

Proof: Let $\{X_n : n \in \omega\} \subseteq \mathcal{I}$. Let $Y = \bigcup_n X_n$. Let γ_Y witness *(v)* for $\mathcal{D}$ relative to Y. Now since for every n, m, $X_n \cap K(m, \gamma_Y)$ is finite, let $X(n, m)$ be the finite set

$$X_n \cap K(m, \gamma_Y) \setminus K(m-1, \gamma_Y)$$

and let:

$$X = \bigcup_n \bigcup_{j \geq n} X(n, j).$$

Notice that $X_n = \bigcup_j X(n, j)$ and $\bigcup_{j \geq n} X(n, j) \subseteq X$, so we have that $X_n \subseteq^* X$. Moreover $X \cap K(n, \gamma_Y) = \bigcup_{j \leq i \leq n} X(j, i)$, so it is finite. We claim that $X \in \mathcal{I}$. If not there would be some β and some l such that $X \cap K(l, \beta)$ is infinite. Now $X \cap K(l, \beta) \subseteq Y \cap K(l, \beta) \subseteq K(m, \gamma_Y)$ for some m. Thus we would get that $X \cap K(m, \gamma_Y)$ is infinite for some m contradicting the very definition of X. □

Now remark that if $Z \subseteq \kappa$ is any set of ordinals of size $\aleph_1$ and $\alpha = \sup(Z)$, there must be an n such that $Z \cap K(n, \alpha)$ is uncountable. This means that $\mathcal{I} \not\subseteq [Z]^{\omega}$, since any countable subset of $Z \cap K(n, \alpha)$ is not in $\mathcal{I}$. This forbids $\mathcal{I}$ to satisfy the first alternative of the P-ideal dichotomy. So the second possibility must be the case, i.e. we can split κ in countably many sets A_n such that $\kappa = \bigcup_n A_n$ and for each n, $[A_n]^{\omega} \cap \mathcal{I} = \emptyset$.

Claim 4.12 *For every n, $[A_n]^\omega$ is covered by $\mathcal{D}$.*

Proof: Assume that this is not the case and let $X \in [A_n]^\omega$ be such that $X \setminus X(l,\beta)$ is non-empty for all l,β. Now let X_0 be a subset of X such that $X_0 \cap K(l,\gamma_X)$ is finite for all l. Then exactly as in the proof of claim 4.11 we can see that $X_0 \in [A]^\omega \cap \mathcal{I}$. This contradicts the definition of A. □

This concludes the proof of theorem 16. ■

4.4 MRP implies SCH

We prove theorem 17. Thus assume MRP and let $\mathcal{D}$ be a covering matrix on κ such that $K(n,\beta)$ is a closed set of ordinal for all n and β. Assume that for all X unbounded in κ, $[X]^\omega$ is not covered by $\mathcal{D}$. We will reach a contradiction. For each $\delta < \kappa$ of countable cofinality, fix C_δ cofinal in δ of order type ω. Let M be a countable elementary submodel of $H(\Theta)$ for some large enough regular Θ. Let $\delta_M = \sup(M \cap \kappa)$ and β_M be the ordinal $\gamma_{M\cap\kappa}$ provided by property *(v)* of $\mathcal{D}$ applied to $M \cap \kappa$. Set $\Sigma(M)$ to be the set of all countable $X \subseteq M \cap \kappa$ such that

$$\sup(X) \notin K(|C_{\delta_M} \cap \sup(X)|, \beta_M).$$

We will show that $\Sigma(M)$ is open and M-stationary. Assume this is the case and let $\{M_\eta : \eta < \omega_1\}$ be a reflecting sequence for Σ. Let $\delta_{M_\xi} = \delta_\xi$ and $\delta = \sup_{\omega_1} \delta_\xi$. Find $C \subseteq \omega_1$ club such that $\{\delta_\xi : \xi \in C\} \subseteq K(n,\delta)$ for some n (which is possible since the $K(n,\delta)$ are closed subsets of κ). Let α be a limit point of C. Let $M = M_\alpha$ and notice that by our choice of β_M for all m, there is l such that $K(m,\delta) \cap M \subseteq K(l,\beta_M)$. This means that for all $\eta \in C \cap \alpha$, $\delta_\eta \in K(n,\delta) \cap M \subseteq K(l,\beta_M)$ for some fixed l. Since α is a limit point of C there is $\eta \in \alpha \cap C$ such that $|C_{\delta_M} \cap \delta_\eta| > l$ and $M_\eta \cap \kappa \in \Sigma(M)$. But this is impossible, since $M_\eta \in \Sigma(M)$ means that $\delta_\eta \notin K(|C_{\delta_M} \cap \delta_\eta|, \beta_M)$, i.e. $\delta_\eta \notin K(l,\beta_M)$.

We now show that Σ_M is open and M-stationary:

Claim 4.13 $\Sigma(M)$ *is open.*

Proof: Assume $X \in \Sigma(M)$, we will find $\gamma \in X$ such that $[\{\gamma\}, X] \subseteq \Sigma(M)$. To this aim notice that $C_{\delta_M} \cap \sup(X)$ is a finite subset of X. Let $n_0 = |C_{\delta_M} \cap \sup(X)|$ and $\gamma_0 = \max(C_{\delta_M} \cap \sup(X)) + 1$. Since $X \in \Sigma(M)$, $\sup(X) \notin K(n_0,\beta_M)$ and so, since $K(n_0,\beta_M)$ is closed, $\gamma_1 = \max(K(n_0,\beta_M) \cap \sup(X)) < \sup(X)$. Thus, let $\gamma \in X$ be greater or equal than $\max\{\gamma_1 + 1, \gamma_0\}$. If $Y \in [\{\gamma\}, X]$, then $\gamma_0 \leq \sup(Y) \leq \sup(X)$, so $|C_{\delta_M} \cap \sup(Y)| = |C_{\delta_M} \cap \sup(X)| = n_0$ and

$$\gamma_1 = \max(K(n_0,\beta_M) \cap \sup(X)) < \sup(Y) \leq \sup(X) < \min(K(n_0,\beta_M) \setminus \sup(X)).$$

Thus $Y \notin K(|C_{\delta_M} \cap \sup(Y)|, \beta_M)$, i.e. $Y \in \Sigma(M)$. □

Claim 4.14 $\Sigma(M)$ *is M-stationary.*

Proof: Let $f : [\kappa]^{<\omega} \to \kappa$ in M. We need to find $X \in \Sigma(M)$ such that $f[[X]^{<\omega}] = X$. Let $N \prec H(\kappa^+)$ be a countable submodel in M such that $f \in N$ and let $C = \{\delta < \kappa : f[[\delta]^{<\omega}] = \delta\}$. Let also $n_0 = |C_{\delta_M} \cap \sup(N \cap \kappa)|$ and $\gamma_0 \in N$ be larger than $\max(C_{\delta_M} \cap \sup(N \cap \kappa))$. Then $(C \setminus \gamma_0) \in N$. We assumed that no A unbounded in κ is such that $[A]^\omega$ is covered by $\mathcal{D}$. So in particular by elementarity of N:

$$N \models [(C \setminus \gamma_0) \cap S^\omega_\kappa]^\omega \textit{is not covered by } \mathcal{D}$$

Thus there exists $X \in N$ countable subset of $(C \setminus \gamma_0) \cap S^\omega_\kappa$ such that for all n and β, $X \setminus K(n, \beta)$ is non-empty. Let $\gamma \in X \setminus K(n_0, \beta_M)$. Now find $Z \in N$ countable and cofinal in γ and let Y be the f-closure of Z. Then $Y \in N \subseteq M$. Now $\gamma \in C$ so $\sup(Y) = \sup(Z) = \gamma \notin K(n_0, \beta_M)$. Moreover $\gamma = \sup(Y) \in (C \setminus \gamma_0) \cap N$, so $\gamma_0 < \sup(Y) < \sup(N \cap \kappa)$, i.e. $|C_{\delta_M} \cap \sup(Y)| = |C_{\delta_M} \cap \sup(N \cap \kappa)| = n_0$. Thus:

$$\sup(Y) \notin K(|C_{\delta_M} \cap \sup(Y)|, \beta_M).$$

I.e. $Y \in \Sigma(M)$. □

This concludes the proof of theorem 17 ■

Chapter 5

Inner models of CP

Since forcing axioms have been able to settle many of the classical problems of set theory, we can expect that the models of a forcing axiom are in some sense categorical. There are many ways in which one can give a precise formulation to this concept. For example, one can study what kind of forcings can preserve PFA, or else if a model V of a forcing axiom can have an interesting inner model M of the same forcing axiom. There are many results in this area, some of them very recent. For instance, König and Yoshinobu [21, Theorem 6.1] showed that PFA is preserved by ω_2-closed forcing. The same holds for BPFA. In fact, BPFA is preserved by any proper forcing that does not add subsets of ω_1. In the other direction, in [44] Veličković showed that if MM holds and M is an inner model such that $\omega_2^M = \omega_2$, then $\mathcal{P}(\omega_1) \subseteq M$ and in a very recent paper Caicedo and Veličković [6] showed, using the mapping reflection principle MRP introduced by Moore in [34], that if $M \subseteq V$ are models of BPFA and $\omega_2^M = \omega_2$ then $\mathcal{P}(\omega_1) \subseteq M$. We first show, using the CP, that almost all known cardinal preserving forcing notions to add ω-sequences like Prikry forcing or diagonal Prikry forcing destroy PFA. On the other hand we show that if it is possible to have two models of PFA, $W \subseteq V$ with the same cardinals, the same bounded subsets of $\aleph_\omega$ and such that $[\aleph_\omega]^\omega \not\subseteq W$, then $\aleph_\omega$ is Jónsson in W. We thus relate this problem to a very large cardinal issue.

5.1 PFA fails in Prikry type forcing extensions

Theorem 18 *Let V be a model of set theory and W be an inner model such that there is κ regular W-cardinal satisfying:*

(i) $\kappa > \aleph_0$ is an ordinal of countable cofinality,

(ii) for all $\lambda < \kappa$, $[\lambda]^\omega \subseteq W$,

(iii) $\kappa^+ = (\kappa^+)^W$.

Then V does not model CP.

In particular this gives another proof that Prikry forcing destroys PFA.

Theorem 19 *Assume that V is a model of* CP *and that W is an inner model with the same reals and the same $\aleph_2$. Assume that for the least W cardinal κ such that $\kappa^\omega \setminus W$ is nonempty, $\kappa^+ = (\kappa^+)^W$. Then for every $\lambda < \kappa$ regular cardinal of V there is S, W-stationary subset of $(S^\lambda_{\kappa^+})^W$, which is not anymore stationary in V.*

In particular any forcing P satisfying the κ^+-chain condition and such that κ is the least ordinal to which P adds a new ω-sequence destroys PFA. Thus also diagonal Prikry forcing kills PFA.

Proof of theorem 18: Assume otherwise, and let κ satisfy the hypotheses of the theorem. Let $g \in \kappa^\omega \setminus W$ be a strictly increasing cofinal sequence. We will reach a contradiction. Fix a surjection $\phi_\delta : \kappa \to \delta$ for all $\delta < \kappa^+$. Define in W, $\mathcal{D}$ to be the matrix indexed by $\kappa \times \kappa^+$, whose entries are the sets:

$$K(\alpha, \beta) = \bigcup \{K(\alpha, \gamma) : \gamma \in \phi_\beta[\alpha]\}$$

Remark that by the minimality of κ and the fact that $K(\alpha, \beta) \in W$ has size less than κ we get that $K(\alpha, \beta)^\omega \subseteq W$ for all α, β. Again in W, fix $(C_\delta : \delta < \kappa^+) \in W$ such that for all δ limit, C_δ is a club in δ of minimal W-order-type and for all δ, $C_{\delta+1} = \{\delta\}$. So for each δ limit, C_δ has order type at most κ. Define by recursion on $\alpha < \beta < \kappa^+$, $\rho_1(\alpha, \alpha) = 0$ and:

$$\rho_1(\alpha, \beta) = \max\{\rho_1(\alpha, \min C_\beta \setminus \alpha), \mathtt{otp}(C_\beta \cap \alpha)\}.$$

Then $\rho_1 \in W$ and ρ_1 has this crucial property:

Fact 5.1 *For all $\nu < \kappa$ and for all $\kappa < \alpha < \kappa^+$: $\{\eta < \alpha : \rho_1(\eta, \alpha) \leq \nu\}$ is a closed subset of α of size at most $|\nu| + \aleph_0$.*

For a proof see [43], Lemma 5.1, page 58. □

Define for all $\alpha < \kappa$, for all $\beta < \kappa^+$, $c(\alpha, \beta)$ to be the supremum of:

$$\{\rho_1(\gamma, \eta) : \gamma, \eta \in K(\alpha, \beta)\}.$$

Then, since $c \in W$ and κ is regular in W, $c(\alpha, \beta) < \kappa$, for all $\alpha < \kappa$ and $\beta < \kappa^+$. From now on work in V, and let $\mathcal{D}^* \in V$ be the matrix produced by the sets $K(g(n), \beta)$. It is easy to check that this matrix is a covering matrix for κ^+. Now in V, $\kappa^+ > \mathfrak{c}$ since

$$W \models \kappa \geq \mathfrak{c}$$

and W and V have the same reals. Find A unbounded in κ^+ such that $[A]^\omega$ is covered by $\mathcal{D}^*$ and let $M \in V$ be a countable elementary submodel of some $H(\theta)$ containing all relevant information. Then $M \cap A \subseteq K(g(n), \beta)$ for some n, β, so $M \cap A \in W$. Let m be such that $g(m) > c(g(n), \beta)$, and let $\gamma \in A$ be such that $\mathtt{otp}(A \cap \gamma) = \kappa$, then by elementarity $\gamma \in M \cap A$. Using again the elementarity of M and fact 5.1 find $\eta < \gamma \in A$ such that $\rho_1(\eta, \gamma) > g(m)$. Now we have a contradiction since:

$$g(m) > c(g(n), \beta) \geq \rho_1(\eta, \gamma) > g(m)$$

This concludes the proof of Theorem 18. ■

Proof of theorem 19: By the previous theorem κ cannot be regular in W, otherwise V cannot model CP. Since κ is the least such that $\kappa^\omega \setminus W$ is non empty and is not regular in W, we can conclude that κ is in W a cardinal of countable cofinality. So there is a covering matrix $\mathcal{D} \in W$ for $(\kappa^+)^W$ with $\beta_\mathcal{D} = \kappa$ and such that $K(n, \beta)$ is closed for all n and β. Since $\kappa^+ = (\kappa^+)^W$ $\mathcal{D}$ is still a covering matrix in V for κ^+ with $\beta_\mathcal{D} = \kappa$. Assume towards a contradiction that there is some $\lambda < \kappa$ regular cardinal of V such that every W-stationary subset of $S^\lambda_{\kappa^+}$ is still stationary in V. So fix in W some $\{A_\alpha : \alpha < \kappa\} \in W$ partition of $(S^\lambda_{\kappa^+})^W$ in κ-many W-stationary sets. Now by our assumption, this is still a family of disjoint stationary subsets of V. Let for every n, β, $D(n, \beta)$ be the set of $\alpha < \kappa$ such that $A_\alpha \cap K(n, \beta)$ is nonempty. Now $D(n, \beta) \in W$ and $|D(n, \beta)| \leq |K(n, \beta)|$ has size less than κ. So, by minimality of κ, we have that $D(n, \beta)^\omega \subseteq W$, else there would be a new ω-sequence in $|D(n, \beta)|$. Apply in V the CP and find X unbounded in κ^+ such that $[X]^\omega$ is covered by $\mathcal{D}$. Exactly as in the proof of fact 4.9, we can see that

$$[\overline{X} \cap S^{\leq\lambda}_{\kappa^+}]^\lambda$$

is covered by $\mathcal{D}$. Now pick M countable elementary submodel containing all relevant information. Then $A_{g(n)} \cap \overline{X} \cap M \cap S^\lambda_{\kappa^+}$ is nonempty for all n, by elementarity of M. Now $M \cap \overline{X} \cap S^\lambda_{\kappa^+} \subseteq K(n, \beta)$ for some n, β. This means that $g \in D(n, \beta)^\omega \subseteq W$ and we are done. ■

5.2 CP and $\aleph_\omega$

Woodin has shown that assuming enough large cardinals, it is possible to define a class generic extension $V[\mathcal{G}]$ of V with the same bounded subsets of $\aleph_\omega$ but such that $([\aleph_\omega]^\omega)^{V[\mathcal{G}]} \setminus V \neq \emptyset$. However $\aleph_\omega^+$ is collapsed in $V[\mathcal{G}]$ to an ordinal of countable cofinality[1]. Can this be done without collapsing $\aleph_\omega^+$? We show that if this is possible then $\aleph_\omega$ is a Jónsson cardinal in the smaller model.

Say that κ is $\mathcal{E}$-Jónsson for a family $\mathcal{E}$ of structures $\mathbb{A} = \langle X, \kappa, (R_i : i < \omega)\rangle$ if every $\mathbb{A} \in \mathcal{E}$ has a submodel Y of size κ such that $Y \cap \kappa$ is a proper subset of κ. κ is Jónsson if $\mathcal{E}$ is everything.

Theorem 20 *Let $W \subseteq V$ be models of* ZFC *with the same cardinals and reals. Assume that $V \models$* CP, *$2^\omega < \aleph_\omega$ and that $[\aleph_\omega]^\omega \not\subseteq W$. Then $\aleph_\omega$ is W-Jónsson.*

[1] A detailed exposition of the method of generic ultrapowers used to produce this kind of models can be found in chapter 2 of [24].

Theorem 21 *Let $W \subseteq V$ be models of* ZFC *with the same cardinals. Assume that $V \models$* CP*, $2^\omega < \aleph_\omega$, that for all $\delta < \aleph_\omega$, $[\delta]^{\leq\delta} \subseteq W$ and that $[\aleph_\omega]^\omega \not\subseteq W$. Then $\aleph_\omega$ is Jónsson in W.*

The rest of this section is devoted to the proof of the above theorems.
We will prove both theorems in a sequence of two lemmas. We first set some notation. We fix $W \subseteq V$ models of ZFC with the same cardinals and reals. We assume that $V \models$ CP and that $\mathfrak{c} < \aleph_\omega$. By what we have shown before, this implies that $W \models (\aleph_\omega)^\omega = \aleph_{\omega+1}$. Fix in $\mathcal{W}$ the following objects:

- A scale $\mathcal{F} = \{h_\alpha : \alpha < \kappa^+\}$ on $(\prod_n \aleph_n, <)$, cofinal sequence under the partial order $<$ of full dominance and strictly increasing in the partial order $<^*$ of eventual dominance.
- $\mathcal{D} \in W$ is a covering matrix for $\aleph_{\omega+1}$ with $\beta_\mathcal{D} = \kappa$.
- $\mathcal{C} = (C_\alpha : \alpha < \aleph_{\omega+1})$ a sequence such that for all α limit, C_α is a club in α of minimal order type and for all α, $C_{\alpha+1} = \{\alpha\}$.
- For any structure $\mathbb{A} \in W$, $\mathbb{A} = \langle H(\theta)^W, \mathcal{D}, \mathcal{F}, <, \cdots \rangle$, $\phi_\mathbb{A} \in W$ a Skolem function.

It is not hard to check by a simple enumeration argument that a scale $\mathcal{F} \in W$ as required above exists.
Finally fix $A \in V$ is an unbounded set witnessing CP relative to $\mathcal{D}$.
We will reach the desired conclusions once we prove the following two lemmas[2].

Lemma 5.2 *There is an n_0 such that for all $m \geq n_0$, $\sup_A h_\alpha(m) < \aleph_m$.*

Let $g(m) = \sup_A h_\alpha(m)$ for all m.

Lemma 5.3 *Let $\mathbb{A}$ be any structure in W, and let $M_\mathbb{A}$ be the $\phi_\mathbb{A}$-closure of A. Then there is $n_\mathbb{A} \geq n_0$ such that for all $\delta \in \aleph_{\omega+1} \cap M_\mathbb{A}$ such that $\texttt{cof}(\delta) = \aleph_m \geq \aleph_{n_\mathbb{A}}$ we have that $\sup M_\mathbb{A} \cap \delta \leq C_\delta(g(m))$.*

Lemmas 5.2 and 5.3 suffices to show that $\aleph_\omega$ is W-Jónsson, since for any $\mathbb{A} \in W$, $M_\mathbb{A} \cap \aleph_\omega$ is a proper subset of $\aleph_\omega$ of size $\aleph_\omega$. However we can still get more out of them, in particular theorem 21. For every structure $\mathbb{A}$ and real $r \in \omega^\omega$ consider the tree $T^r_\mathbb{A}$ of finite sequences $s = \langle M_i : i < |s| \rangle$ of models $M_i \prec \mathbb{A}$ such that $|M_j \cap \aleph_i| = \aleph_{r(i)}$, $M_j = \phi_\mathbb{A}[M_j \cap \aleph_j]$, and $M_i \subseteq M_j$ for all $i < j < |s|$, ordered by end extension. $T^r_\mathbb{A} \in W$ for all r and $\mathbb{A} \in W$. Moreover since $[\delta]^\delta \subseteq W$ for all $\delta < \aleph_\omega$, $T^r_\mathbb{A}$ gets the same interpretation in V and in W for all r and $\mathbb{A} \in W$. Now for a fixed $\mathbb{A} \in W$, let $r_0(n) = |M_\mathbb{A} \cap \aleph_n|$ for all n. Then in V, $T^{r_0}_\mathbb{A} \in W$ is ill-founded: if M_n is the $\mathbb{A}$-closure of $M_\mathbb{A} \cap \aleph_n$, $\langle M_n : n \in \omega \rangle$ is an infinite

[2] We remark that Cummings, Foreman and Magidor proved in ZFC a similar generalized version of lemma 5.2 (see theorem 7.3 of [9]). Applied to the specific situation we are considering, their theorem would say that if there is $\mathcal{F} \in W$ which is a scale in V on $\prod_n \aleph_n$, then W covers $[\aleph_\omega]^\omega$. I will reach this conclusion assuming CP and the slightly weaker hypothesis that there is an infinite $A \subseteq \omega$ and $\mathcal{F} \in W$ scale in V on $\prod_A \aleph_n$.

branch through $T^{r_0}_{\mathbb{A}}$. By absoluteness $T^{r_0}_{\mathbb{A}}$ is ill-founded in W. It is evident that an infinite branch through $T^{r_0}_{\mathbb{A}}$ gives raise to a substructure M of $\mathbb{A}$ such that $M \cap \aleph_\omega$ is a proper subset of $\aleph_\omega$ of size $\aleph_\omega$. Thus W models that $\aleph_\omega$ is Jónsson. ■

We are left with the proof of the two lemmas.

Proof of lemma 5.2: Let A witness the covering property for $\mathcal{D}$. Assume towards a contradiction that there are infinitely many n such that

$$\sup_A h_\alpha(n) = \aleph_n.$$

We will reach a contradiction showing that $(\aleph_\omega)^\omega \subseteq W$ contrary to our hypothesis. The idea is to approximate any $f \in (\aleph_\omega)^\omega$ by a sequence $(f_n : n \in \omega) \in W$ such that:

(i) for all n, m, $f_n(m) \geq f_{n+1}(m) \geq f(m)$,

(ii) for all n, if $f_n(m) > f(m)$ then $f_n(m) > f_{n+1}(m)$.

Suppose this can be achieved and let G be the set of n such that for all j, $f_j(n) > f(n)$. We claim that G is empty, otherwise if $n \in G$, by *(ii)*, we would get that $(f_j(n) : j \in \omega)$ is a strictly decreasing sequence of ordinals. So, since G is empty, we get that for all n, $f(n) = \min_{j\in\omega} f_j(n)$. So we get that $f \in W$ since $(f_n : n \in \omega) \in W$.
We now proceed from any given f to build a sequence which approximate f. Let $r = \{n : \sup_A h_\alpha(n) = \aleph_n\}$ and $X_n = \{h_\alpha(n) : \alpha \in A\}$.
Recall this other crucial property of the function ρ_1 which follows from fact 5.1:

Fact 5.4 *Let X be any unbounded subset of $\aleph_n$, then $\rho_1[[X]^2]$ is unbounded in $\aleph_{n-1}$.*

Using this property and a simple inductive argument it is easy to show the following:

Fact 5.5 *Let $\rho_1^1 = \rho_1$ and $\rho_1^{n+1} : [\aleph_{\omega+1}]^{2^{n+1}} \to \aleph_\omega$ be defined by:*

$$\rho_1^{n+1}(\alpha_1, \cdots, \alpha_{2^{n+1}}) = \rho_1^n(\rho(\alpha_1, \alpha_2), \cdots, \rho_1(\alpha_{2^{n+1}-1}\alpha_{2^{n+1}})).$$

Then if X is unbounded in $\aleph_{n+m}$, $\rho_1^m[[X]^{2^m}]$ is unbounded in $\aleph_n$.

Now consider the following operations:

- for any $\delta < \aleph_\omega^+$, $c(n, \delta) = h_\delta(n)$,
- for any $\gamma \leq \aleph_\omega$ limit and for any $\beta < \texttt{cof}\gamma$, $d(\gamma, \beta) = C_\gamma(\beta)$,
- for any $\gamma < \aleph_\omega$, $p(\gamma + 1) = \gamma$,
- for any given $D \in [(\aleph_\omega)^\omega]^\omega$, $h_D(n) = \min\{f(n) : f \in D\}$.

First remark that by the above facts on the function ρ_1 and the fact that there are infinitely many n for which $\sup_X h_\alpha(n) = \aleph_n$, we can get for every $m < \omega$ a set X_m unbounded in $\aleph_m$ and such that any element η of X_m is obtained by a finite set of ordinals in A by a finite application of the operations c, ρ_1.
Let for any countable set of ordinals $X \subseteq \aleph_{\omega+1}$, $G(X)$ be the least set closed under the operations c, d, p, ρ_1. It is clear that if $X \in W$ then $G(X) \in W$. Moreover since X is countable $G(X) \in W$ is countable and so $G(X)^\omega \subseteq W$. Now let $E(X)$ be the set of f such that there is $Y \in [G(X)^\omega]^{\leq\omega}$ such that $f = h_Y$.

Claim 5.6 *If $X \in W$ is countable, then $E(X) = (E(X))^W$.*

Proof: Since $G(X)^\omega = (G(X)^\omega)^W \subseteq W$, if $\varphi : G(X)^\omega \to \mathfrak{c}$ is a bijection in W of $G(X)^\omega$ with its size, φ is still a bijection in V of $G(X)^\omega$ with $\mathfrak{c}$. Now $[\mathfrak{c}]^\omega \subseteq W$. So if $Y \in [G(X)^\omega]^\omega \setminus W$, we have that $\varphi[Y] \in [\mathfrak{c}]^\omega \setminus W$ and this is impossible. Now:

$$E(X) = \{f : \exists Y \in [G(X)^\omega]^{\leq\omega} \text{ such that } f(n) = \min_{h \in Y} h(n)\}$$

So $E(X) \subseteq W$. □

Given any X subset of A of size at most ω, by the covering property and by our hypothesis on W we get that $X \in [K(n,\beta)]^\omega \subseteq W$. So we get that $E(X \cup \{\aleph_\omega\} \cup (\omega)^W) \subseteq W$. Build by induction a decreasing sequence of functions $\{f_n : n < \omega)\}$ and of sets $Y_n \subseteq [A]^{\leq\omega}$ in order that:

(i) $\{f_j : j \leq n\}$ is a subset of $G(Y_n \cup \{\aleph_\omega\} \cup \omega)$,

(ii) for all n and j, $f_n(j) \geq f_{n+1}(j) \geq f(j)$

(iii) for all n, if $f_n(j) > f(j)$, then $f_n(j) > f_{n+1}(j)$.

To define such a sequence first of all set $f_0(j) = \aleph_\omega$ for all j and $Y_0 = \emptyset$. Now if $\{f_j : j \leq k\}$ and Y_k have been defined, let B_k be the set of m such that $f_k(m) > f(m)$. Let P be the set of $m \in B_k$ such that $f_k(m)$ is limit, Q be the set of $m \in B_k$ such that $f_k(m)$ is a successor ordinal. For any $j \in Q$ let $f_{k+1}(j) = f_k(j) - 1$. For any $j \in P$ let $\aleph_l = \mathtt{cof} f_k(j)$. Now pick $\gamma \in X_l$ such that $\eta_j = C_{f_k(j)}(\gamma) > f(j)$. Notice that such a γ can be found because X_l is unbounded in $\aleph_l$. Let $\{\eta_0^j, \cdots, \eta_{n_j}^j\}$ be a finite subset of A such that γ is obtained by this set applying finitely many times the operations c, ρ_1. Let for all $j \in P$, $f_{k+1}(j) = \eta_j$. If $j \notin B_k$, let $f_{k+1}(j) = f_k(j) = f(j)$. Let $Y_{k+1} = Y_k \bigcup \{\{\eta_0^j, \cdots, \eta_{n_j}^j\} : j \in P\}$. It is easy to check that:

- $\{f_j : j \leq k+1\} \subseteq G(Y_{k+1} \cup \{\aleph_\omega\} \cup \omega)^\omega$,
- for all j such that $f_k(j) > f(j)$, $f_k(j) > f_{k+1}(j) \geq f(j)$,
- $Y_{k+1} \in [A]^\omega \subseteq W$.

Now if we continue for ω-many steps and we set $Y = \bigcup_n Y_n$, we get that $Y \in [A]^\omega \subseteq W$ by the CP and that $\{f_n : n \in \omega\} \subseteq G(Y \cup \{\aleph_\omega\} \cup \omega)^\omega$. Since $Y \cup \{\aleph_\omega\} \cup \omega \in W$, $f = h_{\{f_n : n \in \omega\}} \in E(Y \cup \{\aleph_\omega\} \cup \omega) \subseteq W$, so $f \in W$. $\square$

Proof of lemma 5.3: By lemma 5.2 we get that whenever f is greater or equal than g on an infinite set, $f \notin W$ else there is $\alpha \in A$ such that $f <^* h_\alpha$ but this is impossible since $h_\alpha <^* g$. Now suppose that for some $\mathbb{A}$, lemma 5.3 is false. This means that there is an infinite set $\{\delta_n : n \in Y\} \subseteq M_{\mathbb{A}}$ such that for all $n \in Y$, $\mathtt{cof}\delta_n = \aleph_n$ and $\sup C_{\delta_n} \cap M_{\mathbb{A}} > C_{\delta_n}(g(n))$. Let $(\eta_n : n \in Y)$ be a sequence in $M_{\mathbb{A}}$ such that for all $n \in Y$, $C_{\delta_n}(g(n)) \leq \eta_n < \delta_n$.
Notice now that for each n there are $\{\xi_0^n, \cdots, \xi_{m_n}^n\} \subseteq C \cap S^\omega_{\aleph_{\omega+1}}$ such that $\{\delta_n, \eta_n\} = \phi_{\mathbb{A}}(\xi_0^n, \cdots, \xi_{m_n}^n)$. So let $X = \bigcup_n \{\xi_0^n, \cdots, \xi_{m_n}^n\}$, then X is a countable subset of A so it is in W, so the Skolem hull under $\phi_{\mathbb{A}}$ of X is countable and in W. So by arguments now standard the function $f \in W$ defined by $f(j) = 0$ if $j \notin Y$, $f(j) = \mathtt{otp}(C_{\delta_j} \cap \eta_j)$ if $j \in Y$ is greater than g on an infinite set, which is not possible. $\square$.

This completes the proof of both theorems. $\blacksquare$

Chapter 6

PFA implies PID

Theorem 22 *(Todorčevicć [41]) The proper forcing axiom implies the P-ideal dichotomy.*

Proof: We prove PID by induction on θ. So we assume the for any $\gamma < \theta$ and for any $\mathcal{E}$, P-ideal on $[\gamma]^{\leq\omega}$ either *(i)* or *(ii)* of PID holds. We let $\mathcal{I}$ be any P-ideal on $[\theta]^{\leq\omega}$ such that *(ii)* fails for $\mathcal{I}$ and we force an uncountable $X \subseteq \theta$ such that $[X]^\omega \subseteq \mathcal{I}$ by a proper forcing. First assume θ is a cardinal of countable cofinality. Let θ_n be an increasing sequence of regular cardinals converging to θ and let $\mathcal{I}_n = \mathcal{I} \cap [\theta_n]^{\leq\omega}$. Then, since the θ_n are regular cardinals, it is immediate to check that $\mathcal{I}_n$ is a P-ideal for all n. Now, by our inductive assumption, either for some n there is $X \subseteq \theta_n$ uncountable such that $[X]^\omega \subseteq \mathcal{I}_n \subseteq \mathcal{I}$ and this shows that *(i)* holds for $\mathcal{I}$, or, for all n there is a splitting $\{A^n_i : i \in \omega\}$ of θ_n in countably many sets orthogonal to $\mathcal{I}$. Then $\{A^n_i : i \in \omega, n \in \omega\}$ is a splitting of θ in countably many sets orthognal to $\mathcal{I}$.
Now assume that θ has uncountable cofinality and let $\mathcal{I}$ be a P-ideal on $[\theta]^{\leq\omega}$ such that *(ii)* fails for $\mathcal{I}$ but holds for $\mathcal{I} \cap [\gamma]^{\leq\omega}$ for all $\gamma < \theta$.
Let P be the poset of $p = \langle x_p, \mathcal{Z}_p \rangle$ such that $x_p \in \mathcal{I}$ and $\mathcal{Z}_p$ is a countable family of stationary subsets of $[\mathcal{I}]^\omega$. For any $A \in [\mathcal{I}]^\omega$ let $a_A \in \mathcal{I}$ be a subset of $\bigcup A$ such that $a \setminus a_A$ is finite for all $a \in A$. Now let $p \leq q$ if x_p end extends x_q and $X(\mathcal{F}, p, q) = \{A \in \mathcal{F} : x_p \setminus x_q \subseteq a_A\} \in \mathcal{Z}_p$ for all $\mathcal{F} \in \mathcal{Z}_q$.

Lemma 6.1 *P is proper in the following stronger form: for every relevant countable elementary substructure M and for every $p \in M \cap P$, there is $q \leq p$ in all dense open subsets of P which are in M.*

Lemma 6.2 *For all $\gamma < \omega_1$, $D_\gamma = \{p : \mathtt{otp}(x_p) \geq \gamma\}$ is dense.*

If we can prove both lemmas it is clear that if G is $\{D_\alpha : \alpha < \omega_1\}$-generic, then $[\bigcup\{x_p : p \in G\}]^\omega \subseteq \mathcal{I}$. We begin proving the following:

Claim 6.3 *For any $p \in P$ and any $\gamma \in (\max(x_p), \theta)$, there is $q \leq p$ such that $x_q \setminus \gamma$ is non-empty.*

Proof: Assume not and let p, γ contradict the claim. Then for every $\beta \in (\gamma, \theta)$, there is $\mathcal{F}_\beta \in \mathcal{Z}_p$ such that $\mathcal{F}(\beta) = \{A \in \mathcal{F}_\beta : \beta \in a_A\}$ is not stationary in $[\mathcal{I}]^\omega$ else $q = \langle x_p \cup \{\beta\}, \mathcal{Z}_p \cup \{\mathcal{F}(\beta) : \mathcal{F} \in \mathcal{Z}_p\}\rangle$ is a condition below p such that $x_q \setminus \gamma$ is non-empty. Now let $A_\mathcal{F} = \{\beta : \mathcal{F}_\beta = \mathcal{F}\}$. We assert that each $A_\mathcal{F}$ is orthogonal to $\mathcal{I}$. If not let $a \in \mathcal{I} \cap [A_\mathcal{F}]^\omega$. Let $\mathcal{F}' = \{A \in \mathcal{F} : a \in A\}$. Then $\mathcal{F}'$ is stationary in $[\mathcal{I}]^\omega$. Now for every $A \in \mathcal{F}'$, there is a finite $F_A \subseteq a$ such that $a \setminus F_A \subseteq a_A$. Use the pressing down lemma to find $\mathcal{F}_0$ stationary subset of $\mathcal{F}'$ such that F_A is the same for all $A \in \mathcal{F}_0$. Pick $\beta \in a \setminus F_A$ then $\mathcal{F}_0 \subseteq \mathcal{F}(\beta)$ contradicting the assumption that $\mathcal{F}(\beta)$ is not stationary.
Now by our assumption on $\mathcal{I}$ there is a family $\{A_n : n \in \omega\}$ such that $\bigcup A_n = \gamma + 1$ and each A_n is orthogonal to $\mathcal{I}$. Then $\{A_n : n \in \omega\} \cup \{A_\mathcal{F} : \mathcal{F} \in \mathcal{Z}_p\}$ is a countable family $\mathcal{Z}$ of sets orthogonal orthogonal to $\mathcal{I}$ and such that $\bigcup \mathcal{Z} = \theta$ contradicting our assumption on $\mathcal{I}$. □

Assume that lemma 6.1 has been proved. We can now prove lemma 6.2 by induction on $\alpha < \omega_1$. D_0 is clearly a dense set. If $\alpha = \beta + 1$ and D_β is dense, we can appeal to the above claim to obtain that $D_{\beta+1}$ is also dense. If α is limit and for all $\beta < \alpha$, D_β is dense, given any $p \in P$, take $M \prec H(\kappa)$ such that $p, \alpha \in M$ and let q be a condition below p and in all dense open subsets of P which are in M. Now for each $\beta < \alpha$, $q \in D_\beta \in M$. So $\mathtt{otp}(q) \geq \alpha$.

We are left with the proof of lemma 6.1. So let M be a countable elementary submodel of some $H(\kappa)$ with κ regular and large enough and such that all the relevant objects are in M. Let p be an arbitrary condition of P in M. We need to find an M-generic condition below p. Let also $\{D_n : n \in \omega\}$ be an enumeration of the dense sets of P which are in M and $b \subseteq M \cap \theta$ be any element of $\mathcal{I}$ such that $a \subseteq^* b$ for all $a \in \mathcal{I} \cap M$. We will build a decreasing sequence of conditions $\{p_n : n \in \omega\} \subseteq M$ such that $p_0 = p$, each $p_{n+1} \in D_n \cap M$ and such that if $x_\omega = \cup_n x_{p_n}$ then:

(i) $x_\omega \setminus x_{p_n} \subseteq b$,

(ii) $\mathcal{E}(\mathcal{F}, n) = \{A \in [\mathcal{I}]^\omega : x_\omega \setminus x_{p_n} \subseteq a_A\}$ is stationary in $[\mathcal{I}]^\omega$ for all $\mathcal{F} \in \mathcal{Z}_{p_n}$.

If this can be done

$$q = \langle x_\omega, \bigcup\{\mathcal{Z}_{p_n} : n \in \omega\} \cup \{\mathcal{E}(\mathcal{F}, n) : \exists n\, \mathcal{F} \in Z_{p_n}\}\rangle$$

is an M-generic condition below each p_n.
Assume that $p_0, \cdots, p_n$ have been defined. Let $\{\mathcal{F}^i_m : m \in \omega\}$ be an enumeration of the stationary subsets of $[\mathcal{I}]^\omega$ which are in p_i. Let $\mathcal{F}(i, m, n) = \{A \in \mathcal{F}^i_m : x_n \setminus x_i \subseteq a_A\}$. Then this set is stationary and in M, since p_n is a condition below p_i. By the pressing down lemma, find $\mathcal{G}(i, m, n)$ stationary subset of $\mathcal{F}(i, m, n)$ (but not in M) and $F(i, m, n)$ finite subset of $M \cap \theta$ such that for all $A \in \mathcal{G}(i, m, n)$, $b \subseteq a_A \cup F(i, m, n)$. We will extend p_n with a $p_{n+1} \in D_n \cap M$ such that $(x_{p_{n+1}} \setminus x_{p_n}) \subseteq b$ and $\min(x_{p_{n+1}} \setminus x_{p_n}) > \max\{\max(F(i, m, n)) : i, m \leq n\}$. Then for all $i, m \leq n < l$ and for all $A \in \mathcal{G}(i, m, n)$,

$$x_{p_l} \setminus x_{p_i} = (x_{p_l} \setminus x_{p_n}) \cup (x_{p_n} \setminus x_{p_i}) \subseteq b \setminus \max(F(i, m, n)) \cup a_A \subseteq a_A$$

Thus if we can proceed this way for all n, then for all i, m once that $n \geq \max\{m, i\}$ we get that for all $A \in \mathcal{G}(i, m, n)$,

$$x_\omega \setminus x_{p_i} = \bigcup_{l>n} (x_{p_l} \setminus x_{p_n}) \cup (x_{p_n} \setminus x_{p_i}) \subseteq a_A$$

Thus $\mathcal{G}(i, m, n) \subseteq \mathcal{E}(\mathcal{F}^i_m, i)$, i.e. $\mathcal{E}(\mathcal{F}^i_m, i)$ is stationary for all m and i, so

$$q = \langle x_\omega, \bigcup \{\mathcal{Z}_{p_n} : n \in \omega\} \cup \{\mathcal{E}(\mathcal{F}, n) : \exists n \mathcal{F} \in Z_{p_n}\}$$

is an M-generic condition below p.

Now assume that $p_0, \cdots, p_n$ have been defined as well as $F(i, m, n)$ and $\mathcal{G}(i, m, n)$ for all $i, m \leq n$. Let $c = b \setminus \bigcup \{F(i, m, n) : i, m \leq n\}$. Suppose towards a contradiction that p_{n+1} as above cannot be found. Then for all $r \in D_n \cap M$ below p_n, $x_r \setminus x_{p_n} \not\subseteq c$. Let for all $a \in M \cap \mathcal{I}$, $F_a = a \setminus c$. Let Y_0 be the collection of $A \in [\mathcal{I}]^\omega$ such that for some finite $F_A \subseteq a_A$, there is no $r \in D_n$ such that $x_r \setminus x_{p_n} \subseteq a_A \setminus F_A$. Then $Y_0 \in M$ since it is defined by a first-order formula with parameters in M. Moreover every $A \in M \cap [\mathcal{I}]^\omega$ is in Y_0. To see this, let $F_A = a_A \setminus c$ and suppose that for some $r \in D_n \cap M$, $x_r \setminus x_{p_n} \subseteq a_A \setminus F_A$. Then $x_r \setminus x_{p_n} \subseteq c$ contradicting our assumption. Then

$$M \models Y_0 = [\mathcal{I}]^\omega$$

So by the pressing down lemma applied in M there is $\mathcal{F} \in M$ stationary subset of $[\mathcal{I}]^\omega$ and a finite F such that $F_A = F$ for all $A \in \mathcal{F}$. Set $s = \langle x_{p_n}, \mathcal{Z}_{p_n} \cup \{\mathcal{F}\} \rangle$. Then $s \in M$. By the previous claim 6.3 find $t \leq s$ such that $\min(x_t \setminus x_s) > \max(F)$. Now find $r \leq t$ and in $D_n \cap M$. Then $\mathcal{G} = \{A \in \mathcal{F} : x_r \setminus x_{p_n} \subseteq a_A\}$ is stationary since $\mathcal{F} \in \mathcal{Z}_s$ and $r \leq s$. Moreover $\min(x_r \setminus x_{p_n}) = \min(x_t \setminus x_{p_n}) > \max(F)$. Thus if $A \in \mathcal{G}$, $x_r \setminus x_{p_n} \subseteq a_A \setminus F_A$ so $A \notin Y_0$. This is impossible since we have shown that $[\mathcal{I}]^\omega = Y_0$. Thus a p_{m+1} as required can be found. This concludes the proof of lemma 6.1. ■

We conclude this part remarking that the forcing P is more than proper. In fact P satisfies a condition ($(*)$ in what follows) isolated by Shelah (see [35] chapter V) with the property that any generic extension produced by a countable support iteration of forcing which satisfies $(*)$ preserves $\aleph_1$ and has the same reals of the ground model. Using this fact one can produce a model of $\mathsf{PID} + \mathsf{GCH}$ starting from a ground model of GCH with a supercompact cardinal λ and use the standard countable support iteration of length λ to force PID in the generic extension. Since all elements of the iteration are forcings which satisfy $(*)$, CH holds in the extension. Moreover since the iteration satisfies the λ-chain condition, the cardinal arithmetic above λ is the same in the ground model and in the extension. It is also not difficult to see that in the extension $2^{\aleph_1} = \lambda = \aleph_2$. So GCH holds in the extension. The interested reader is referred to section 5 of [41].

Chapter 7

PFA implies MRP

Theorem 23 *(Moore [32]) Assume* PFA*. Then* MRP *holds.*

Proof: Let X be uncountable and assume that for a club $\mathcal{C}$ of countable models M of $H(\kappa)$, where κ is a large enough regular cardinal, $\Sigma(M)$ is an open M-stationary subset of $[X]^\omega$. Let P be the poset of $p = \langle M_\eta : \eta \leq \alpha_p \rangle$ such that for all limit $\eta \leq \alpha_p$ there is a $\gamma < \eta$ such that for all $\xi \in (\gamma, \eta)$, $M_\xi \cap X \in \Sigma(M)$. $p \leq q$, if p is an end extension of q. Once again we will show the following:

Lemma 7.1 *P is proper in the following stronger form: for every relevant countable elementary substructure M and for every $p \in M \cap P$, there is $q \leq p$ in all dense open subsets of P which are in M.*

Lemma 7.2 *For all $\alpha < \omega_1$, $D_\alpha = \{p : \alpha_p \geq \alpha\}$ is dense in P.*

Once this is proved, if we let G be a $\{D_\alpha : \alpha < \omega_1\}$-generic filter, it is clear that the sequence $\bigcup G$ is a reflecting sequence for Σ.

Now assume lemma 7.1 holds. We can prove lemma 7.2 by induction on $\alpha < \omega_1$. D_0 is clearly a dense set. If $\alpha = \beta + 1$ and D_β is dense, $D_{\beta+1}$ is also dense. If α is limit and for all $\beta < \alpha$, D_β is dense, given any $p \in P$, take M such that $p, \alpha \in M$ and let q be an condition below p belonging to all dense open subsets of P which are in M.. Now for each $\beta < \alpha$, $q \in D_\beta$ since q is M-generic and $D_\beta \in M$. So $\alpha_q \geq \alpha$.

We are left with the proof of lemma 7.1. So let $M \prec H(\lambda)$ be countable with λ regular and large enough and such that P, Σ and all the relevant objects are in M. Let $p \in P \cap M$ and let $\{D_n : n \in \omega\}$ enumerate the dense open sets of P in M, we will build a decreasing sequence of conditions p_n such that $p_0 = p$, $p_{n+1} \in D_n$ and for all $\xi \in (\alpha_{p_0}, \alpha_{p_n}]$, $M_\xi \in \Sigma(M \cap H(\kappa))$. Set $\alpha_M = \bigcup_n \alpha_{p_n}$. Then $q = \bigcup_n p_n \cup \{\langle \alpha_M, M \cap H(\kappa) \rangle\}$ is the desired M-generic condition below p. First of all for all $\xi \in (\alpha_{p_0}, \alpha_M)$, $M_\xi \in \Sigma(M \cap H(\kappa))$. While for all $\eta < \alpha_M$ and limit, since $\eta < \alpha_{p_n}$ for some n and p_n is a condition, there is $\gamma < \eta$ such that $M_\xi \in \Sigma(M_\eta)$ for all $\xi \in (\gamma, \eta)$. Thus q is a condition. Then it is clear that it extends each p_n, so it is in all dense sets of M.

Now assume that p_n has been defined according to our constraint and let

$$\mathcal{E}_n = \{N \prec H(\lambda^+) : p_n,\, D_n \in N\}$$

Then $\mathcal{E}_n \in M$ is a club in $[H(\lambda^+)]^\omega$ so $\{N \cap X : N \in \mathcal{E}_n\}$ is a club in $[X]^\omega$ and is in M. Since $\Sigma(M)$ is M-stationary there is $N \in \mathcal{E}_n$ such that $N \cap X \in \Sigma(M)$. Since $\Sigma(M)$ is open there is a finite subset s of $X \cap N$ such that $[s, N \cap X] \subseteq \Sigma(M)$. Let $M_0 \in \mathcal{C} \cap N$ be such that $p_n, s \in M_0$. Set $q_0 = p_n \cup \{\langle \alpha_{p_n} + 1, M_0 \rangle\}$. Then $q_0 \in N$, so we can find $p_{n+1} = \langle M_\xi : \xi \leq \alpha_{p_{n+1}} \rangle \in N \cap D_n$ below q_0. Then for all $\xi \in (\alpha_{p_n}, \alpha_{p_{n+1}}]$, we have that $s \subseteq M_\xi \cap X \subseteq N \cap X$. So $M_\xi \in \Sigma(M \cap H(\kappa))$ for all $\xi \in (\alpha_{p_n}, \alpha_{p_{n+1}}]$. This concludes the proof of the lemma. □

The proof of the theorem is now completed. ■

7.1 MRP and PID are mutually independent

We sketch a proof of the following result from Miyamoto [30]:

Theorem 24 *(Miyamoto [30]) Assume that there is a supercompact cardinal. Then there is a model of* MRP *in which there is a Souslin tree.*

The theorem is a consequence of the following two lemmas:

Lemma 7.3 *(Miyamoto [29]) Assume that T is Souslin in V and that P is a countable support iteration such that at each successor stage $\xi + 1$,*

$$\Vdash_{P_{\xi+1}} T \text{ is Souslin.}$$

Let G be a P-generic filter. Then T is Souslin in $V[G]$.

Proof: For a proof see [29]. □

Lemma 7.4 *Let X be an uncountable set and $\Sigma(M)$ be an open M-stationary subset of $[X]^\omega$ for a club of countable $M \prec H(\kappa)$, with κ regular and large enough. Let P_Σ be the forcing notion that produces a reflecting sequence for Σ. Assume that T is a Souslin tree. Then*

$$\Vdash_{P_\Sigma} T \text{ is Souslin.}$$

Now assuming both lemmas, let V be a model of set theory with a Souslin tree T and a supercompact cardinal λ. Let P be the standard countable support iteration of length λ to force MRP. Then in $V[G]$ MRP holds and T is Souslin.

We sketch a proof of lemma 7.4. First of all we remark the following:

Fact 7.5 *Let $(T, <_T)$ be a Souslin tree, P be a forcing and $\dot{A}$ be a P-name for a maximal antichain in T. Then for all $p \in P$ the following set is open dense in T:*

$$D_p = \{t \in T : \exists\, s <_T t \text{ and } q \leq p \text{ such that } q \Vdash_P s \in \dot{A}\}$$

Now we can prove the following:

Claim 7.6 *Let N be a countable elementary submodel of $H(\theta)$ for a large enough regular θ and containing all relevant information. Let $\alpha_N = N \cap \omega_1$, $t \in T_{\alpha_N}$ and $p \in P \cap N$. Then there is $q \in P \cap N$ below p and $s <_T t$ such that $q \Vdash_P s \in \dot{A}$.*

Proof: Notice that t is an N-generic condition for T, since T is a c.c.c. partial order. Now $D_p \in N$ so there is $t_0 \in D_p \cap N$ compatible with t. This means that there are $s <_T t_0$ and $q \leq p$ such that $q \Vdash_P s \in \dot{A}$. Then $s \in T \cap N$ since $ht_T(s) < ht_T(t)$. So by elementarity of N there is $q \in N$ and below p such that $q \Vdash_P s \in \dot{A}$. This proves the claim. □

Now let X be an uncountable set and $\Sigma(N)$ be an open and N-stationary subset of $[X]^\omega$ for a club of $N \prec H(\theta)$. Let also T be a Souslin tree, $\dot{A}$ be a P_Σ-name for a maximal antichain of T and $p \in P_\Sigma$. Let M be a countable model containing all relevant information and $\alpha_M = M \cap \omega_1$. We will show that there is $q \leq p$, M-generic condition for P_Σ and such that $q \Vdash_{P_\Sigma} \dot{A} \subseteq \bigcup_{\alpha_M} T_\xi$. This shows that the set of such q is dense, so we can conclude that:

$$\Vdash_{P_\Sigma} \dot{A} \text{ is a countable antichain.}$$

Since $\dot{A}$ is an arbitrary name for a maximal antichain, we can conclude that:

$$\Vdash_{P_\Sigma} T \text{ is Souslin.}$$

To obtain such a q it is enough to modify the construction of the generic condition performed in lemma 7.1 as follows. First of all let $\{D_n : n \in \omega\}$ enumerate the dense open sets of $P_\Sigma \in M$ and $\{t_n : n \in \omega\} = T_{\alpha_M}$. At odd stages $2n-1$, assuming that p_{2n-1} has been defined, proceed exactly as in the proof of lemma 7.1 to define $p_{2n} \in D_n$. At even stages $2n$, assuming that p_{2n} has been defined, let $\mathcal{E}_{2n}$ be the club of countable $N \prec H(|P_\Sigma|^+)$ such that $p_{2n} \in N$. Now Pick $N \in \mathcal{E}_n$ such that $N \cap X \in \Sigma(M \cap H(\theta))$. As in the previous lemma let s be a finite subset of $X \cap N$ such that $Y \in \Sigma(M \cap H(\theta))$ for all $Y \in [s, N \cap X]$ and $N_0 \in N$ be a countable elementary submodel of $H(\theta)$ such that $p_{2n}, s \in N$. Let $q_0 = p_{2n} \cup \{\langle \alpha_{p_{2n}} + 1, N_0 \rangle\}$. Now let $t \prec t_n$ have $ht_T(t) = N \cap \omega_1$. By claim 7.6 we can find $p_{2n+1} \in N$ below q_0 and $s_n <_T t$ such that

$$p_{2n+1} \Vdash_{P_\Sigma} s_n \in \dot{A}.$$

Exactly as in the proof of lemma 7.1 we can see that $q = \bigcup_n p_n \cup \{\langle \alpha_M, M \cap H(\theta) \rangle\}$ is an M-generic condition below p. Moreover for all n there is $s_n <_T t_n$ such that $q \Vdash s_n \in \dot{A}$. Then

$$q \Vdash \dot{A} = \{s_n : n \in \omega\}.$$

This is so since any $t \in T$ is compatible with some t_n so it is also compatible with some s_n. So $\{s_n : n \in \omega\}$ is a maximal antichain. This concludes the proof of the lemma. ■

We can now conclude that MRP and PID are mutually independent. On on side MRP implies $\mathfrak{c} = \aleph_2$, while PID is compatible with CH. So MRP does not imply PID. On the other side PID implies that there are no Souslin tree, while MRP is compatible with the existence of such trees, so PID does not imply MRP.

Bibliography

[1] U. Abraham and S. Todorčević. Partition Properties of ω_1 Compatible with CH. *Fundamenta Mathematicae*, 152:165–180, 1997.

[2] J. Bagaria. Bounded forcing axioms as principles of generic absoluteness. *Arch. Math. Logic*, 39(6):393–401, 2000.

[3] J. Bagaria. Natural axioms on set theory and the continuum problem. *CRM Preprint*, 591:19, 2004.

[4] J. Baumgartner. Applications of the proper forcing axiom. In Amsterdam North-Holland, editor, *Handbook of set-theoretic topology*, pages 913–959. 1984.

[5] A. Caicedo and B. Veličković. The Mapping Reflection Principle. in preparation.

[6] A. Caicedo and B. Veličković. Bounded proper forcing axiom and well orderings of the reals. To appear in Mathematical Research Letters, 18 pages, 2005.

[7] G. Cantor. Ein beitrag zur mannifaltigkeitslehre. *J.f. Math.*

[8] J. P. Cohen. The Independence of the Continuum Hypothesis. volume 50(6), pages 1143–1148, 1963.

[9] J. Cummings, M. Foreman, and M. Magidor. Canonical structure in the universe of set theory. I. *Annals of Pure and Applied Logic*, 129(1-3):211–243, 2004.

[10] W. B. Easton. Powers of regular cardinals. *Ann. Math. Logic*, 1:139–178, 1970.

[11] M. Foreman, M. Magidor, and S. Shelah. Martin's Maximum, saturated ideals and nonregular ultrafilters. *Ann.of Math. (2)*, 127(1):1–47, 1988.

[12] M. Gitik. The strength of the failure of SCH. *Annals of Pure and Applied Logics*, 51(3):215–240, 1991.

[13] M. Gitik. Prikry Type Forcings. In M. Foreman, A. Kanamori, and M. Magidor, editors, *Handbook of Set Theory.* North Holland, to appear.

[14] K. Gödel. Remarks before the princeton bicentennial conference on problems in mathematics, 1946. In *Kurt Gödel COLLECTED WORKS, vol. II*, page 407.

[15] K. Gödel. *The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis with the Axioms of Set Theory.* Princeton University Press, 1940.

[16] K. Gödel. What is Cantor's continuum problem? *American Mathematical Monthly*, 54:515–525, 1947.

[17] K. Gödel. What is Cantor's continuum problem? In P. Benacerraf and H. Putnam, editors, *Philosophy of mathematics selected readings*, pages 470–485. Cambridge University press, 1983.

[18] T. Jech. *Set theory, The Third Millennium Edition, Revised and Expanded.* Springer, 2002.

[19] A. Kanamori. *The Higher Infinite. Large Cardinals in Set Theory from their Beginnigs.* Springer-verlag, 1994.

[20] B. König, P. Larson, J. T. Moore, and B. Veličković. Bounding the consistency strength of a five element linear basis. preprint.

[21] B. König and Y. Yoshinobu. Fragments of Martin's Maximum in generic extensions. *Mathematical Logic Quarterly*, 50(3):297–302, 2004.

[22] K. Kuratowski. Topologie I. In *Monografie Matematyczne*, volume 3. Garasínski.

[23] P. B. Larson. *Martin's Maximum and the Pmax axiom* $(*)$., volume 106(1-3). 2000.

[24] P. B. Larson. *The Stationary Tower: Notes on a Course by W. Hugh Woodin.* AMS, 2004.

[25] R. Laver. On the consistency of Borel's conjecture. *Acta Math.*, 137(3-4):151–169, 1976.

[26] N. Luzin and W. Sierpinski. Sur quelques proprietés des ensembles (A). *Bulletin international de l'Académie des sciences de Cracovie, classe des sciences mathématiques et naturelles*, serie A:35–48.

[27] M. Magidor. On the singular cardinal problem I. *Israel journal of mathematics*, 28:1–31, 1977.

[28] M. Magidor. On the singular cardinal problem II. *Annals of mathematics*, 106:517–549, 1977.

[29] T. Miyamoto. ω_1-souslin trees under countable support iterations. *Fundamenta Mathematicae*, 142(3):257–261, 1993.

[30] T. Miyamoto. a memo. unpublished, 2 pages, 2006.

[31] J. T. Moore. Open colorings, the continuum, and the second uncountable cardinal. *Proceedings of the American Mathematical Society*, 130(9):2753–2759, 2002.

[32] J. T. Moore. Set mapping reflection. *Journal of Mathematical Logic*, 5(1):87–97, 2005.

[33] J. T. Moore. A five element basis for the uncountable linear orders. *Annals of Mathematics*, 163(2):669–688, 2006.

[34] J. T. Moore. The Proper Forcing Axiom, Prikry forcing, and the Singular Cardinals Hypothesis. *Annals of Pure and Applied Logic*, 140(1-3):128–132, 2006.

[35] S. Shelah. *Proper Forcing.* Springer, 1982.

[36] S. Shelah. *Cardinal Arithmetic.* Oxford University Press, 1994.

[37] J. H. Silver. On the singular cardinal problem. volume 1, pages 265–268, 1975.

[38] R. M. Solovay. Strongly compact cardinals and the GCH. pages 365–372, 1971.

[39] S. Todorčević. *Partition Problems in Topology.* AMS, 1989.

[40] S. Todorčević. Comparing the continuum with the first two uncountable cardinals. In *Logic and scientific methods (Firenze, 1995)*, volume 259 of *Synthese Lib.*, pages 145–155. Kluwer Acad. Publ., 1997.

[41] S. Todorčević. A dichotomy for P-ideals of countable sets. *Fundamenta Mathematicae*, 166(3):251–267, 2000.

[42] S. Todorčević. Generic absoluteness and the continuum. *Math. Res. Lett.*, 9(4):465–471, 2002.

[43] S. Todorčević. Coherent sequences. In M. Foreman, A. Kanamori, and M. Magidor, editors, *Handbook of Set Theory.* North Holland, to appear.

[44] B. Veličković. Forcing axioms and stationary sets. *Advances in Mathematics*, 94(2):256–284, 1992.

[45] B. Veličković. CCC Forcing and Splitting Reals. *Israel Journal of Mathematics*, 147:209–220, 2005.

[46] M. Viale. The Proper Forcing Axiom and the Singular Cardinal Hypothesis. *Journal of Symbolic Logic*, 71(2):473–479, 2006.

[47] W. H. Woodin. The Ω-conjecture. unpublished notes of the conference held in Midrasha Mathematicae, Cardinal Arithmetic at Work, Jerusalem, 23-29 March 2004.

[48] W. H. Woodin. The Continuum Hypothesis. Part I. *Notices of the AMS*, 48(6):567–576, 2001.

[49] W. H. Woodin. The Continuum Hypothesis. Part II. *Notices of the AMS*, 48(7):681–690, 2001.