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C^*-algebras and B-names for Complex Numbers

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Contents

Introduction

In this dissertation we will present some connections between the theory of commutative unital C^*-algebras, a specific domain of functional analysis, and the theory of Boolean valued models, which pertains to logic and set theory. More specifically, the main purpose will be to show that a commutative unital C^*-algebra $\mathcal{A}$, whose spectrum is extremely disconnected, can be identified with the B-names for complex numbers in the boolean valued model for set theory V^{B}, where B is the complete boolean algebra given by clopen sets on the spectrum of $\mathcal{A}$.

This study is divided in four chapters. Chapters 1, 2, 3 provide the tools needed in Chapter 4.

Chapter 1 is devoted to functional analysis, here the basics of the theory of C^*-algebras are outlined. We are interested in commutative unital C^*-algebras. These, by Gelfand-Naimark's Theorem, are Banach spaces of the form $\mathcal{C}(X, \mathbb{C})$, with X a compact Hausdorff topological space.

Chapter 2 introduces the logical tools we will need in chapters 3 and 4. Sections 2.1 and 2.2 are an overview of elementary model theory and of the basic properties of boolean algebras, respectively. In the last section of Chapter 2 we introduce the notion of boolean valued model for an arbitrary first order language $\mathcal{L}$, and we study the theory of these structures from a general point of view. Given B a complete boolean algebra, the concept of B-valued model $\mathcal{M}$ comes from pure set theory, and generalizes the usual two-valued Tarski semantics, associating to each formula φ a value $[\![\varphi]\!]^{\mathcal{M}} \in \mathsf{B}$. Boolean valued models are generally used to obtain independence proofs, and they are strictly related to the forcing method introduced by Cohen to prove the independence of the continuum hypothesis.

In chapter 3 we present some classical results from pure set theory and we briefly introduce forcing. The forcing method, developed by Cohen in 1963, is the most powerful tool (as of now) used in mathematics in order to prove independence results. However in section 3.5 we reverse the common perception of forcing, and we show how to use forcing as a tool to derive theorems within ZFC by means of the concept of generic absoluteness.

Finally, in Chapter 4, we build a bridge between the theories exposed earlier. In section 4.1 we introduce and study the space of functions $\mathcal{C}(St(\mathsf{B}), X)$ where B is a complete boolean algebra, $St(\mathsf{B})$ is the space of ultrafilters–or equivalently maximal ideals–on B, and X a topological space with some specific properties. In particular we show that this type of spaces can be interpreted as B-valued models which contain X as a proper submodel. In section 4.3 we analyze $\mathbb{C}^{\mathsf{B}}$, the set of B-names for complex numbers in the boolean model V^{B}. This family of objects is shown to be isomorphic (in the sense of B-valued models) to the space of function $\mathcal{C}^+(St(\mathsf{B}), \mathbb{C})$, which is the set of all continuous functions from the Stone space of B with image in $\mathbb{C} \cup \{\infty\}$ (seen as the one point compactification of $\mathbb{C}$) such that the preimage of $\{\infty\}$ is meager.

The boolean isomorphism between $\mathbb{C}^{\mathsf{B}}$ and $\mathcal{C}^+(St(\mathsf{B}), \mathbb{C})$ might be an interesting tool to translate ideas and results arising in set theory to ideas and results arising in the study of commutative C^*-algebras, and conversely. This could be the case since, by the results of section 4.2, commutative C^*-algebras can be studied in this context appealing to Gelfand Transform:

given a commutative unital C^*-algebra $\mathcal{A}$ with extremely disconnected spectrum, there is an isomorphism (which can be defined using the Gelfand Transform) of the C^*-algebras $\mathcal{A}$ and $\mathcal{C}(St(\mathsf{B}), \mathbb{C})$, where B is the boolean algebra given by clopen sets in the weak* topology on the spectrum of $\mathcal{A}$. By means of this isomorphism $\mathcal{A}$ can be therefore embedded in $\mathbb{C}^{\mathsf{B}}$, and the spectrum of $\mathcal{A}$ is mapped homeomorphically on the Stone space given by the ultrafilters (or dually by the maximal ideals) of the boolean algebra B.

We conclude the fourth chapter with an application of the absoluteness properties proved in Chapter 3. In particular, we will prove that Σ_2-formulae are absolute between the first order structures $\mathbb{C}$ (even endowed with arbitrary Borel predicates) and $\mathbb{C}^{\mathsf{B}}/G \cong \mathcal{C}^+(St(\mathsf{B}), \mathbb{C})/G$ (where the latter is also endowed with natural liftings of the Borel predicates), which means that the truth value of these formulae is the same in $\mathbb{C}$ and $\mathcal{C}^+(St(\mathsf{B}), \mathbb{C})/G$. We remark that in this context $\mathcal{C}^+(St(\mathsf{B}), \mathbb{C})/G$ is the ring of germs of continuos functions $f : St(\mathsf{B}) \to \mathbb{C}$ in the point $G \in St(\mathsf{B})$.

A concrete example is given letting $\mathsf{B} = \mathsf{MALG}$, the complete boolean algebra of measurable sets modulo null measure sets in $\mathbb{C}$: in this case we obtain by Gelfand's transform that $\mathcal{C}(St(\mathsf{MALG})) \cong L^\infty(\mathbb{C})$. We can now consider $L^{\infty+}(\mathbb{C})$ to be the family of Lebesgue measurable functions with range in $\mathbb{C} \cup \{\infty\}$ which takes value ∞ on a null set, and lift the above isomorphism to an isomorphism of $\mathcal{C}^+(St(\mathsf{MALG})) \cong L^{\infty+}(\mathbb{C})$. We conclude that whenever G is an ultrafilter on MALG, $L^{\infty+}(\mathbb{C})/G$ is an algebraically closed field extension of $\mathbb{C}$ which preserves the truth value of Σ_2-formulae of $\mathbb{C}$.

More generally, we obtain a way to carry properties from the theory of C^*-algebras to the first order theory of complex numbers in boolean valued models of ZFC, and conversely.

Chapter 1

Functional Analysis

The aim of this chapter is to briefly present the tools we need from functional analysis. The final intention is to define C^*-algebras and enunciate the Gelfand-Naimark Theorem. The reference texts for this part, where to find details and proofs, are [2] and [11] for the general part of functional analysis, and the first chapter of [4] for the part regarding C^*-algebras.

1.1 Banach Spaces and Weak Topologies

Since our aim is to present the theory of C^*-algebras, we will only work with vector spaces on the complex field $\mathbb{C}$.

Definition 1.1.1. Given a vector space $\mathcal{V}$, a **norm** is a function:

$$\|.\| : \mathcal{V} \to \mathbb{R}$$

such that for $x, y \in \mathcal{V}$, $\lambda \in \mathbb{C}$:

- $\|x\| \geq 0$,
- $\|x\| = 0 \Leftrightarrow x = 0$,
- $\|\lambda x\| = |\lambda| \|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$.

A **normed vector space** is a pair $(\mathcal{V}, \|.\|_{\mathcal{V}})$ (the norm could be omitted if no confusion can arise).

A distance can be defined on such spaces through the norm, as follows:

$$d(x, y) = \|y - x\|$$

We can consider therefore the topology induced by this distance.

Continuous linear functions between normed vector spaces have a strong characterization, given by the following proposition:

Proposition 1.1.2. *Assume T is a linear function between two normed spaces $(\mathcal{V}, \|.\|_{\mathcal{V}})$ and $(\mathcal{U}, \|.\|_{\mathcal{U}})$. Then the following are equivalent:*

1. *T is continuous;*

2. *T is uniformly continuous;*

3. *T is continuous at 0;*

4. *T is bounded, i. e. there is $C > 0$ such that for all $x \in \mathcal{V}$ $\|T(x)\|_{\mathcal{U}} \leq C\|x\|_{\mathcal{V}}$ holds.*

Proof. A proof can be found in [11, Lemma 4.1]. □

Definition 1.1.3. Given a normed vector space $\mathcal{V}$, its **dual** space is the set:

$$\mathcal{V}^* = \{f : \mathcal{V} \to \mathbb{C} \mid f \text{ is linear and continuous}\}$$

Remark 1.1.4. The previous proposition tells us that, if $f \in \mathcal{V}^*$, the value $\sup_{\|x\|\leq 1}|f(x)|$ is bounded, hence we can define the following function:

$$\begin{aligned} \|.\| : \mathcal{V}^* &\to \mathbb{R} \\ f &\mapsto \sup_{\|x\|\leq 1} |f(x)| \end{aligned}$$

and this happens to be a norm (see [11, Lemma 4.15]). This means that the dual space of a normed vector space is a normed vector space itself, and it is always complete in the topology induced by the norm ([11, Theorem 4.27]). This is an example of a *Banach space*.

Definition 1.1.5. A **Banach space** is a normed vector space $(\mathcal{E}, \|.\|)$ which is complete for the metric induced by the norm.

Besides the topology induced by the norm, in functional analysis there are two other topologies which are often defined on Banach spaces (actually the second can be defined only on the dual of a Banach space).

Definition 1.1.6. Given a set X, a family of functions $\mathcal{F} = \{f_i\}_{i\in I}$, and a family of topological spaces $\{Y_i\}_{i\in I}$ such that $f_i : X \to Y_i$, then the **coarsest topology** for X associated to $\mathcal{F}$ is the smallest topology on X which renders each element in $\mathcal{F}$ continuous.

The proof of the existence of such a topology can be found in the first section of Chapter 3 in [2], where it is also explained how to find a basis for this topology:

Proposition 1.1.7. *A basis for the coarsest topology of X associated to $\mathcal{F}$ is given by the family of finite intersections of sets in $\{U_\lambda\}_{\lambda\in\Lambda}$, where $U_\lambda = f_i^{-1}(V)$ for some $f_i \in \mathcal{F}$ and V open set in Y_i*

Definition 1.1.8. Given a Banach space $\mathcal{E}$, the **weak topology** (usually denoted with $\sigma(\mathcal{E}, \mathcal{E}^*)$) on $\mathcal{E}$ is the coarsest topology on $\mathcal{E}$ associated to $\mathcal{E}^*$.

Remark 1.1.9. Since the functions in $\mathcal{E}^*$ are continuous in the topology induced by the norm, the norm topology on $\mathcal{E}$ contains the weak topology $\sigma(\mathcal{E}, \mathcal{E}^*)$.

Proposition 1.1.10. *The weak topology on $\mathcal{E}$ is generated by the following basis of neighborhoods, as $f_1, \ldots, f_k \in \mathcal{E}^*$, $\epsilon \in \mathbb{R}^+$ and $x_0 \in \mathcal{E}$ vary:*

$$V_{x_0}(f_1, \ldots, f_k; \epsilon) = \{x \in \mathcal{E} : |f_i(x - x_0)| < \epsilon \ \forall i = 1, \ldots, k\}.$$

A proof of this can be found section 3.2 of [2] (Proposition 3.4), with a deeper explanation of this topology, and all the details which have been omitted here.

We are more interested on a third topology which can be defined on the dual of a Banach space.

Definition 1.1.11. The **bidual space** $\mathcal{E}^{**}$ of a Banach space is the dual space of $\mathcal{E}^*$.

The following map is an embedding of $\mathcal{E}$ into $\mathcal{E}^{**}$:

$$J : \mathcal{E} \to \mathcal{E}^{**}$$
$$x \mapsto \hat{x}$$

where, given $f \in \mathcal{E}^*$, $\hat{x}(f) = f(x)$. The linearity of $\hat{x}$ follows from the linearity of f, and the continuity holds because of the following:

$$|\hat{x}(f)| = |f(x)| \leq \|f\|\|x\|.$$

Definition 1.1.12. Given a Banach space $\mathcal{E}$, the **weak* topology** on $\mathcal{E}^*$ (denoted by $\sigma(\mathcal{E}^*, \mathcal{E})$) is the coarsest topology of $\mathcal{E}^*$ associated to $J(\mathcal{E})$.

Remark 1.1.13. If $\mathcal{E}$ is such that J is surjective, then the weak topology and the weak* topology on $\mathcal{E}^*$ overlap.

For our purposes, only few properties of this topology will be enunciated. All the facts we have already stated and the properties we will present from now on about the weak* topology can be found in Chapter 3, Section 4, in [2]. In particular, the reason why this topology is considered in analysis is briefly explained in *Remark* 8 of that Chapter.

Proposition 1.1.14. *The weak* topology on $\mathcal{E}^*$ is generated by the following base of neighborhoods, as $x_1, \dots, x_k \in \mathcal{E}$, $f_0 \in \mathcal{E}^*$ and $\epsilon \in \mathbb{R}^+$ vary:*

$$V_{f_0}(x_1, \dots, x_k; \epsilon) = \{f \in \mathcal{E} : |\hat{x}_i(f - f_0)| < \epsilon \ \forall i = 1, \dots, k\}.$$

Proposition 1.1.15. *The space $\mathcal{E}^*$ equipped with the weak* topology is an Hausdorff space.*

An interesting aspect of this topology is that the convergence of a sequence to an element is the pointwise convergence:

Proposition 1.1.16. *A sequence (f_n) in $\mathcal{E}^*$ converges to f in the weak* topology if and only if $f_n(x)$ converges to $f(x)$ for all $x \in \mathcal{E}$.*

In defining $\sigma(\mathcal{E}^*, \mathcal{E})$ many open set from the topology defined by the norm on $\mathcal{E}^*$ were removed (the weak* topology is contained in the one induced by the norm). This operation was done in order to have more compact sets and in fact we have the following result:

Theorem 1.1.17 (Banach-Alaoglu-Bourbaki)**.** *In the space $\mathcal{E}^*$ with weak* topology, the closed ball*

$$B_0(1) = \{f \in \mathcal{V}^* : \|f\| \leq 1\}$$

is compact.

Proof. See [2, Theorem 3.16]. □

Remark 1.1.18. The theorem is in general false for the topology induced by the norm, since we are working with vector spaces which may have infinite dimension (see [11, Theorem 2.26]).

1.2 C^*-algebras and Gelfand Transform

Definition 1.2.1. A $\mathbb{C}$-algebra $\mathcal{A}$ with a norm (so that $\mathcal{A}$ is a normed vector space) is a **Banach algebra** if it is a Banach space and for x, $z \in \mathcal{A}$:

$$\|xz\| \leq \|x\|\|z\|$$

A Banach algebra is **unital** if there is a neutral element for the product.

Definition 1.2.2. We call **involution** on a $\mathbb{C}$-algebra $\mathcal{A}$ an operator $* : \mathcal{A} \to \mathcal{A}$ such that for all x, $y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$:

- $(x + y)^* = x^* + y^*$,
- $(xy)^* = y^* x^*$,
- $(\lambda x)^* = \overline{\lambda} x^*$,
- $x^{**} = x$.

A $\mathbb{C}$-algebra $\mathcal{A}$ equipped with an involution is called a **$*$-algebra**.
A **C^*-algebra** is a Banach $*$-algebra which satisfies the property

$$\|x^* x\| = \|x\|^2$$

Remark 1.2.3. As we can see, all these definitions tend to generalize operations which are well-known in the field of complex numbers: $\mathbb{C}$ equipped with its usual product and with conjugation as involution is the most elementary example of C^*-algebra.

Definition 1.2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{C}$-algebras. A **homomorphism** from $\mathcal{A}$ to $\mathcal{B}$ is a map $\varphi : \mathcal{A} \to \mathcal{B}$ such that:

- φ is linear;
- φ bounded (hence continuous);
- $\varphi(xy) = \varphi(x)\varphi(y)$ for each x, $y \in \mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras, φ is said to be a **$*$-homomorphism** if it satisfies $\varphi(x^*) = \varphi(x)^*$ as well. An **isomorphism** of $\mathbb{C}$-algebras is a bijective homomorphism whose inverse is an homomorphism. A **$*$-isomorphism** is a bijective $*$-homomorphism whose inverse is a $*$-homomorphism

There are many interesting examples of C^*-algebras which motivate the study of these objects, but we are essentially interested in the following one:

Example 1.2.5. Let X be an Hausdorff and compact space. Consider

$$\mathcal{C}(X) = \{f : X \to \mathbb{C} \ \mid \ f \text{ is continuous}\}$$

with pointwise sum and product and the uniform norm. If we also define $f^* = \overline{f}$, that is $f^*(x) = \overline{f(x)}$, it is easy to check that we obtain a C^*-algebra.

From now on, we will focus on commutative unital C^*-algebras. Gelfand theory starts from a specific object which can be defined for a Banach algebra $\mathcal{A}$: its *spectrum*.

Definition 1.2.6. Given a Banach algebra $\mathcal{A}$ the set of non-zero homomorphisms from $\mathcal{A}$ to $\mathbb{C}$ is the **spectrum** of $\mathcal{A}$, and we denote it with $\sigma(\mathcal{A})$.

Proposition 1.2.7. *Given a commutative unital Banach algebra $\mathcal{A}$ and $h \in \sigma(\mathcal{A})$, the following holds:*

- $h(e) = 1$
- $|h(x)| \leq \|x\|$

Proof. See [4, Proposition 1.10]. □

Proposition 1.2.8. *The spectrum $\sigma(\mathcal{A})$ of a commutative unital Banach algebra is an Hausdorff compact subspace of $\mathcal{A}^*$ in the weak* topology.*

Proof. Since $|h(x)| \leq \|x\|$, $\sigma(\mathcal{A})$ is a subset of the ball of radius 1 centered in 0 in $\mathcal{A}^*$. This means that the spectrum can be defined equivalently as

$$\sigma(\mathcal{A}) = \{h \in B_1(0) : h(e) = 1 \wedge h(xy) = h(x)h(y) \ \forall x, y \in \mathcal{A}\}.$$

Since the conditions $h(e) = 1$ and $h(xy) = h(x)h(y)$ are preserved under pointwise limits, $\sigma(\mathcal{A})$ is closed in $B_1(0)$, hence is an Hausdorff compact subspace of $\mathcal{A}^*$ in the weak* topology. □

Definition 1.2.9. The **Gelfand transform** of a Banach algebra $\mathcal{A}$ is the operator

$$\begin{aligned} \Gamma_{\mathcal{A}} : \mathcal{A} &\to \mathcal{C}(\sigma(\mathcal{A})) \\ x &\mapsto \hat{x} \end{aligned}$$

where $\hat{x}(h) = h(x)$. We will often denote $\Gamma_{\mathcal{A}}$ as Γ.

The Gelfand-Naimark Theorem shows a strong relation between a commutative unital C^* and its spectrum.

Theorem 1.2.10 (Gelfand-Naimark)**.** *Assume $\mathcal{A}$ is a commutative and unital C^*-algebra, then Γ is an isometric $*$-isomorphism from $\mathcal{A}$ to $\mathcal{C}(\sigma(\mathcal{A}))$.*

Proof. See [4, Section 1.2, Theorem 1.20] for the details of the proof. □

Remark 1.2.11. The Gelfand-Naimark Theorem tells us that, in the study of commutative unital C^*-algebras, it is enough to focus on the function algebras of the form $\mathcal{C}(X)$ with X compact Hausdorff space, since any commutative unital C^*-algebra has an isomorphic copy among these type of C^*-algebras.

Chapter 2

First order logic and Boolean Valued Models

This chapter will be dedicated to mathematical logic, with a focus on boolean valued models and their semantics. Boolean valued models generalize the structure of first order models, combining first order model theory with boolean algebras. We will first give a presentation of the basic tools and theorems of model theory. Our exposition will be short, we refer to [9, Chapters 1,2] for further details. References for the part regarding the basic properties of boolean algebras are [5] and [7].

2.1 First order logic

Definition 2.1.1. A **language** is a set:

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

where:

- every R_i is called **relation symbol**;
- every f_j is called **function symbol**;
- every c_k is called **constant symbol**;

with a function

$$\Theta_{\mathcal{L}} : \{R_i : i \in I\} \cup \{f_j : j \in J\} \to \mathbb{N}$$

which associates to each relation (function) symbol a number n which is called **arity** of the relation (function).

We will usually refer to a language omitting the function $\Theta_{\mathcal{L}}$ (if no confusion can arise).

Definition 2.1.2. Given a language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

a **$\mathcal{L}$-structure** $\mathcal{M}$ is a tuple $\langle M, R_i^{\mathcal{M}} : i \in I, f_j^{\mathcal{M}} : j \in J, c_k^{\mathcal{M}} : k \in K\rangle$ where:

- M is a non-empty set, called **domain** of the structure;

- $R_i^{\mathcal{M}}$ is a subset of M^n (where n is the arity of R_i) called **interpretation** of R_i in $\mathcal{M}$;
- $f_j^{\mathcal{M}}$ is a function from M^n to M (where n is the arity of f_j) called **interpretation** of f_j in $\mathcal{M}$;
- $c_k^{\mathcal{M}}$ is a element of M called **interpretation** of c_k in M.

The relevant maps between $\mathcal{L}$-structures are those which preserve the interpretations of symbols in $\mathcal{L}$.

Definition 2.1.3. Given two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, an **$\mathcal{L}$-embedding** is an injective map $\varphi : M \to N$ such that:

- given a relation symbol $R \in \mathcal{L}$ whose arity is n,

$$(a_1, \dots, a_n) \in R^{\mathcal{M}} \Leftrightarrow (\varphi(a_1), \dots, \varphi(a_n)) \in R^{\mathcal{N}}$$

- given a function symbol $f \in \mathcal{L}$ whose arity is n,

$$\varphi(f^{\mathcal{M}}(a_1, \dots, a_n)) = f^{\mathcal{N}}(\varphi(a_1), \dots, \varphi(a_n))$$

- given a constant symbol $c \in \mathcal{L}$,

$$\varphi(c^{\mathcal{M}}) = c^{\mathcal{N}}$$

An **isomorphism** is a bijective $\mathcal{L}$-embedding.

Definition 2.1.4. If $M \subseteq N$ and $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{L}$-structures such that the immersion of M in N is an $\mathcal{L}$-embedding, we say that $\mathcal{M}$ is a **substructure** of $\mathcal{N}$, or that $\mathcal{N}$ is an **extension** of $\mathcal{M}$.

Remark 2.1.5. At this stage of model theory, the aim is to formalize the idea that a formula is *true* in a certain structure. In order to do this, we need to formalize the notion of formula in a language $\mathcal{L}$. Among the symbols we want to appear in a formula there are variables, hence, from now on, when we will consider a certain language $\mathcal{L}$, we will also assume to have a countable set $\mathcal{V} = \{x_1, x_2, \dots\}$ of variables.

Definition 2.1.6. Given a language $\mathcal{L}$, the set of **$\mathcal{L}$-terms** is the smallest set $\mathcal{T}_{\mathcal{L}}$ such that:

- if c is a constant symbol in $\mathcal{L}$, then $c \in \mathcal{T}_{\mathcal{L}}$;
- if $x \in \mathcal{V}$, then $x \in \mathcal{T}_{\mathcal{L}}$;
- if $t_1, \dots, t_n \in \mathcal{T}_{\mathcal{L}}$ and f is a function symbol in $\mathcal{L}$ with arity n, then $f(t_1, \dots, t_n) \in \mathcal{T}_{\mathcal{L}}$.

A **closed term** is a term in which no variables occur.

Definition 2.1.7. Given a language $\mathcal{L}$, we say that φ is an **atomic $\mathcal{L}$-formula** (or atomic formula) if it is one of the following:

- $t_1 = t_2$, where t_1 and t_2 are $\mathcal{L}$-terms;
- $R(t_1, \dots, t_n)$ where R is relation symbol in $\mathcal{L}$ of arity n and t_i are terms.

The set of all **$\mathcal{L}$-formulae** (or formulae) is the smallest set $\mathcal{F}_{\mathcal{L}}$ such that it contains the atomic $\mathcal{L}$-formulae and such that:

- if $\varphi \in \mathcal{F}$ then $\neg\varphi \in \mathcal{F}$;
- if $\varphi, \psi \in \mathcal{F}$ then $\varphi \wedge \psi \in \mathcal{F}$;
- if $\varphi \in \mathcal{F}$ then $\exists x\varphi \in \mathcal{F}$.

We will also use the following abbreviations:

- $\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$;
- $\varphi \to \psi \equiv \neg\varphi \vee \psi$;
- $\varphi \leftrightarrow \psi \equiv (\varphi \to \psi) \wedge (\psi \to \varphi)$;
- $\forall x\varphi \equiv \neg\exists x\neg\varphi$.

Remark 2.1.8. Given a formula φ, an occurrence of a variable x is **bound** if in such occurrence x is immediately after a quantification symbol $\exists$ or $\forall$. An occurrence of x is **free** if is not bound. The set of **free variables** of a formula φ is the set of those variables which have at least one free occurrence in φ. If the set of free variables of φ is contained in $\{x_1, \dots, x_n\}$ we will write $\varphi(x_1, \dots, x_n)$ instead of φ. If a $\mathcal{L}$-formula has no free variables, we will call it a **$\mathcal{L}$-statement** (or a statement).

Definition 2.1.9. Let $\{x_1, \dots, x_n\}$ be a set of variables. A **valuation** in an $\mathcal{L}$-structure $\mathcal{M}$ of these variables is a function

$$\nu : \{x_1, \dots, x_n\} \to M$$

If $\nu(x_i) = a_i$, we will also write $\nu = (x_1/a_1, \dots, x_n/a_n)$.

Given a formula $\varphi(x_1, \dots, x_n)$, with $\varphi(\nu)$ or $\varphi(a_1, \dots, a_n)$ we will denote the formula φ in which every occurrence of a variable x_i is substituted with $\nu(x_i) = a_i$. We obtain this way a **formula with parameters** in $\mathcal{M}$.

Definition 2.1.10. Given an $\mathcal{L}$-term t whose variables are included in the domain of a valuation ν in $\mathcal{M}$, we inductively define $t(\nu) \in M$ as follows:

- if t is a constant c, then $t(\nu) = c$;
- if t is a variable x, then $t(\nu) = \nu(x)$;
- if $t = f(t_1, \dots, t_n)$, where f is a function symbol in $\mathcal{L}$ of arity n and t_i are $\mathcal{L}$-terms, then $t(\nu) = f(t_1(\nu), \dots, t_n(\nu))$.

Definition 2.1.11 (Tarski's Semantic)**.** Given an $\mathcal{L}$-structure $\mathcal{M}$, let φ be a formula in the language $\mathcal{L}$ whose free variables are in $\{x_1, \dots, x_n\}$, and ν a valuation in $\mathcal{M}$ whose domain contains $\{x_1, \dots, x_n\}$. We say that $\varphi(\nu)$ is **true** in $\mathcal{M}$, or $\mathcal{M} \models \varphi(\nu)$, in the following cases (by recursion):

- if $\varphi \equiv t_1 = t_2$, then $\mathcal{M} \models \varphi(\nu)$ iff $t_1(\nu) = t_2(\nu)$;
- if $\varphi \equiv R(t_1, \dots, t_n)$, then $\mathcal{M} \models \varphi(\nu)$ iff $(t_1(\nu), \dots, t_n(\nu)) \in R^{\mathcal{M}}$;
- if $\varphi \equiv \neg\psi$, then $\mathcal{M} \models \varphi(\nu)$ iff $\mathcal{M} \not\models \psi(\nu)$
- if $\varphi \equiv \psi \wedge \theta$, then $\mathcal{M} \models \varphi(\nu)$ iff $\mathcal{M} \models \psi(\nu)$ and $\mathcal{M} \models \theta(\nu)$;
- if $\varphi \equiv \exists y\psi(y)$, then $\mathcal{M} \models \varphi(\nu)$ iff there is $b \in M$ such that $\mathcal{M} \models \psi(y/b, \nu)$.

Remark 2.1.12. If φ is a statement, the previous definition does not require any valuation in order to decide if $\mathcal{M} \models \varphi$.

Definition 2.1.13. Given a language $\mathcal{L}$, a **$\mathcal{L}$-theory** (or just theory) is a set T of $\mathcal{L}$-statements.

We say that a theory is **consistent** if there is a $\mathcal{L}$-structure $\mathcal{M}$ such that for each $\varphi \in T$, $\mathcal{M} \models \varphi$ holds (we might also write $\mathcal{M} \models T$).

We say that a $\mathcal{L}$-statement φ is a **logical consequence** of a $\mathcal{L}$-theory T if

$$\mathcal{M} \models T \Rightarrow \mathcal{M} \models \varphi$$

Remark 2.1.14. We can give another notion of logical consequence, assuming some rules which tell us how to obtain a formula from other formulae, as we do in a proof. A collection of such rules is called a **proof system**, and we will refer to the one defined in [12] (Chapter 2, Section 6).

Logical axioms:

- $\varphi \vee \neg\varphi$;
- $x = x$;
- $\varphi(a) \to \exists\varphi(x)$;
- $(\overline{x} = \overline{y}) \to (f(\overline{x}) = f(\overline{y}))$
- $(\overline{x} = \overline{y}) \to (\varphi(\overline{x}) \to \varphi(\overline{y}))$.

Rules of inference:

- $\varphi \vdash \varphi \vee \psi$;
- $\varphi \vee \varphi \vdash \varphi$;
- $(\varphi \vee (\psi \vee \chi)) \vdash ((\varphi \vee \psi) \vee \chi)$;
- $(\varphi \vee \psi) \wedge (\neg\varphi \vee \chi) \vdash \psi \vee \chi$;
- if x is not free in ψ, $\varphi(a) \to \psi \vdash \exists x(\varphi(x) \to \psi)$;

We will say that a statement ψ is **syntactically provable** from a theory T, and denote it with $T \vdash \psi$, if there is a finite sequence of formulae $\varphi_1, \dots, \varphi_k$ such that $\varphi_k = \psi$ and for each $1 \leq i \leq k$ one of the following holds:

- $\varphi_i \in T$;
- φ_i is a *logical axiom*;
- φ_i can be obtained from $\varphi_1, \dots, \varphi_{i-1}$ through a *rule of inference*.

Theorem 2.1.15 (Soundness Theorem)**.** *Given an $\mathcal{L}$-theory T and an $\mathcal{L}$-statement φ, if $T \vdash \varphi$ then $T \models \varphi$.*

The vice versa holds as well.

Theorem 2.1.16 (Completeness Theorem)**.** *Given an $\mathcal{L}$-theory T and an $\mathcal{L}$-statement φ, if $T \models \varphi$ then $T \vdash \varphi$.*

Proof. A proof of both theorems can be found in [12] (Chapter 2 and 4 respectively). □

We conclude this part presenting the most important theorems of elementary model theory. The omitted proofs can be found in [9, Chapter 2, Sections 1 and 3].

Theorem 2.1.17 (Compactness Theorem). *A theory T is consistent if and only if every finite subset of T is consistent.*

Definition 2.1.18. Given two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, we say that they are **elementarily equivalent**, and denote it with $\mathcal{M} \equiv \mathcal{N}$, if for every $\mathcal{L}$-statement φ

$$\mathcal{M} \models \varphi \Leftrightarrow \mathcal{N} \models \varphi$$

We say that $\mathcal{M}$ is an **elementary substructure** of $\mathcal{N}$, denoting it with $\mathcal{M} \prec \mathcal{N}$, if $M \subseteq N$ and given a $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ with $(a_1, \ldots, a_n) \in M^n$, then:

$$\mathcal{M} \models \varphi(a_1, \ldots, a_n) \Leftrightarrow \mathcal{N} \models \varphi(a_1, \ldots, a_n)$$

Let T be a set of $\mathcal{L}$-formulae. We will write

$$\mathcal{M} \prec_T \mathcal{N}$$

if the property above holds only for $\varphi \in T$.

Theorem 2.1.19 (Upward Löwenheim−Skolem Theorem). *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and κ a cardinal such that $\kappa \geq |\mathcal{L}| + |M|$. Then there is an $\mathcal{L}$-structure $\mathcal{N}$ such that $|N| = \kappa$ and $\mathcal{M} \prec \mathcal{N}$.*

Theorem 2.1.20 (Downward Löwenheim−Skolem Theorem). *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. There is $\mathcal{N}$ such that $A \subseteq N$, $|N| \leq |A| + |\mathcal{L}| + \aleph_0$ and $\mathcal{N} \prec \mathcal{M}$.*

2.2 Boolean Algebras

As mentioned earlier, good references for the material of this part are [5] and [7].

Definition 2.2.1. A partial order $(\mathsf{B}, <)$ is called a **boolean algebra** if:

- every two elements $a, b \in \mathsf{B}$ admit a **unique least upper bound** $a \vee b$;
- every two elements $a, b \in \mathsf{B}$ admit a **unique greatest lower bound** $a \wedge b$;
- there are two elements $0_\mathsf{B}, 1_\mathsf{B} \in \mathsf{B}$ such that for every $a \in \mathsf{B}$ it holds $0_\mathsf{B} \leq a \leq 1_\mathsf{B}$;
- is **distributive**, which means

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

  for every $a, b, c \in \mathsf{B}$;
- is **complemented**, that is for every $a \in \mathsf{B}$ there is a unique $\neg a$ such that $a \wedge \neg a = 0_\mathsf{B}$ and $a \vee \neg a = 1_\mathsf{B}$.

A boolean algebra is **complete** if any $\{a_i : i \in I\}$ subset of B admits a supremum $\bigvee_{i \in I} a_i$ and an infimum $\bigwedge_{i \in I} a_i$.

Remark 2.2.2. Given $a, b \in \mathsf{B}$ a boolean algebra, we have that:

$$a \leq b \Leftrightarrow a \wedge b = a$$

$$a \leq b \Leftrightarrow a \vee b = b$$

Example 2.2.3. Given a topological space X, $\mathsf{CL}(X)$ is the set of all **clopen subsets** of X. $\mathsf{CL}(X)$ is a boolean algebra with the following operations:

$$A \leq B \Leftrightarrow A \subseteq B$$

$$A \wedge B = A \cap B$$

$$A \vee B = A \cup B$$

$$\neg A = X \setminus A$$

Another example is given by the set $\mathsf{RO}(X)$, the collection of all regular open subsets of a topological space X. An open subset A of X is **regular** if $A = \mathring{\overline{A}}$. With the following operations, $\mathsf{RO}(X)$ is a complete boolean algebra:

$$A \leq B \Leftrightarrow A \subseteq B$$

$$A \wedge B = A \cap B$$

$$A \vee B = \mathring{\overline{A \cup B}}$$

$$\neg A = (X \mathring{\setminus} A)$$

$$\bigvee_{i \in I} A_i = \mathring{\overline{\bigcup A_i}}$$

$$\bigwedge_{i \in I} A_i = \mathring{\overline{\bigcap A_i}}$$

A proof of this can be found in [5, Chapter 10].

Definition 2.2.4. A **morphism** between two boolean algebras B and C is a map $\varphi : \mathsf{B} \to \mathsf{C}$ such that, given $a, b \in \mathsf{B}$:

- $\varphi(0_{\mathsf{B}}) = 0_{\mathsf{C}}$ and $\varphi(1_{\mathsf{B}}) = 1_{\mathsf{C}}$;
- if $a \leq b$ then $\varphi(a) \leq \varphi(b)$;
- $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$;
- $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$;
- $\varphi(\neg a) = \neg\varphi(a)$.

An **isomorphism** of boolean algebras is a bijective morphism. A morphism is **complete** if preserves all suprema and infima.

Lemma 2.2.5. *Every isomorphism is complete.*

Proof. Let $i : \mathsf{B} \to \mathsf{C}$ an isomorphism and consider $\{a_j : j \in J\} \subseteq \mathsf{B}$. We want to show

$$i\left(\bigvee_{j \in J} a_j\right) = \bigvee_{j \in J} i(a_j)$$

On the one hand $\bigvee_{j\in J} a_j \geq a_j$, hence $i(\bigvee_{j\in J} a_j) \geq \bigvee_{j\in J} i(a_j)$. On the other hand, if $b \geq i(a_j)$ for every $j \in J$, let $a \in \mathsf{B}$ such that $b = i(a)$. It follows $a \geq a_j$ for each $j \in J$, hence $a \geq \bigvee_{j\in J} a_j$, and in conclusion

$$b = i(a) \geq i\left(\bigvee_{j\in J} a_j\right)$$

The proof for infima is similar. □

We now turn to the topological counterparts of a boolean algebra, its *Stone space*.

Definition 2.2.6. Given B a boolean algebra, a **filter** is a subset F of B such that:

- $1_{\mathsf{B}} \in F$ and $0_{\mathsf{B}} \notin F$;
- if $a \in F$ and $b \in F$ then $a \wedge b \in F$;
- if $a \in F$ and $a \leq b$ then $b \in F$.

A **ultrafilter** G is a filter such that one of the following equivalent properties holds:

- if $a \notin G$ then $\neg a \in G$;
- if $a_1 \vee \cdots \vee a_k \in G$ then there is a a_i such that $a_i \in G$.

The **Stone space** of B is the set

$$St(\mathsf{B}) = \{G \subset \mathsf{B} : G \text{ is a ultrafilter}\}$$

Remark 2.2.7. It follows from the definitions that for a given ultrafilter G on the boolean algebra B, $a \in G \Leftrightarrow \neg a \notin G$.

Remark 2.2.8. Let $i : \mathsf{B} \to \mathsf{C}$ be an injective morphism of boolean algebras, and let F be a filter in B. We can define the set:

$$i^*(F) = \{a \in \mathsf{C} : \exists b \in F \text{ such that } a \geq i(b)\}$$

Since the only $b \in \mathsf{B}$ such that its image is 0_{C} is 0_{B}, $i^*(F)$ can be easily verified to be a filter.

Theorem 2.2.9 (Maximal Ideal Theorem)**.** *If F is a filter in a boolean algebra B, then F can be extended to a ultrafilter in B.*

Proof. A proof of this theorem can be found in [7, Chapter 2, Proposition 2.16]. □

Definition 2.2.10. Let B be a boolean algebra and $F \subseteq \mathsf{B}$ a filter. The following is an equivalence relation on B:

$$a \sim_F b \Leftrightarrow \exists f \in F : a \wedge f = b \wedge f$$

The set of all equivalence classes B/F is called **quotient algebra**, and it is a boolean algebra with the following order relation and operations:

- $[a]_F \leq [b]_F \Leftrightarrow a < b$;
- $[a \wedge b]_F = [a]_F \wedge [b]_F$;
- $[a \vee b]_F = [a]_F \vee [b]_F$;
- $\neg[a]_F = [\neg a]_F$.

Moreover $1_{\mathsf{B}/F} = [1_{\mathsf{B}}]_F$ and $0_{\mathsf{B}/F} = [0_{\mathsf{B}}]_F$ (a proof of all these facts can be found in [7, Lemma and Definition 5.22]).

Proposition 2.2.11. *Let* B *and* C *be two boolean algebras and* $i : \mathsf{B} \to \mathsf{C}$ *an injective morphism. Assume* F *is a filter in* B*, then the map*

$$\begin{aligned} i_F : \mathsf{B}/F &\to \mathsf{C}/i^*(F) \\ [a]_F &\mapsto [i(a)]_{i^*(F)} \end{aligned}$$

is a well-defined injective morphism of boolean algebras. Moreover, if i *is an isomorphism,* i_F *is an isomorphism as well.*

Proof. First, we show that i_F is well-defined. Let $a, a' \in \mathsf{B}$ such that $a \sim_F a'$, hence there is $f \in F$ such that $a \wedge f = a' \wedge f$. This implies

$$i(a) \wedge i(f) = i(a \wedge f) = i(a' \wedge f) = i(a') \wedge i(f)$$

so that $i(a) \sim_{i^*(F)} i(a')$ and i_F is a well-defined map.

The map i_F is a morphism, in fact:

- we have that

$$i_F(0_{\mathsf{B}/F}) = i_F([0_{\mathsf{B}}]_F) = [i(0_{\mathsf{B}})]_{i^*(F)} = [0_{\mathsf{C}}]_{i^*(F)} = 0_{\mathsf{C}/i^*(F)}$$

  and

$$i_F(1_{\mathsf{B}/F}) = i_F([1_{\mathsf{B}}]_F) = [i(1_{\mathsf{B}})]_{i^*(F)} = [1_{\mathsf{C}}]_{i^*(F)} = 1_{\mathsf{C}/i^*(F)}$$

- if $[a]_F \leq [b]_F \in \mathsf{B}/F$:

$$i_F([a]_F) = [i(a)]_{i^*(F)} \leq [i(b)]_{i^*(F)} = i_F([b]_F)$$

  since $a \leq b$ and i is a morphism itself;

- if $[a]_F, [b]_F \in \mathsf{B}/F$:

$$\begin{aligned} i_F([a]_F \wedge [b]_F) = i_F([a \wedge b]_F) = [i(a \wedge b)]_{i^*(F)} = [i(a) \wedge i(b)]_{i^*(F)} = \\ [i(a)]_{i^*(F)} \wedge [i(b)]_{i^*(F)} = i_F([a]_F) \wedge i_F([b]_F) \end{aligned}$$

- if $[a]_F, [b]_F \in \mathsf{B}/F$:

$$\begin{aligned} i_F([a]_F \vee [b]_F) = i_F([a \vee b]_F) = [i(a \vee b)]_{i^*(F)} = [i(a) \vee i(b)]_{i^*(F)} = \\ [i(a)]_{i^*(F)} \vee [i(b)]_{i^*(F)} = i_F([a]_F) \vee i_F([b]_F) \end{aligned}$$

- if $[a]_F \in \mathsf{B}/F$:

$$i_F(\neg[a]_F) = i_F([\neg a]_F) = [i(\neg a)]_{i^*(F)} = [\neg i(a)]_{i^*(F)} = \neg[i(a)]_{i^*(F)} = \neg i_F([a]_F)$$

The morphism i_F is injective since assuming $i(a) \sim_{i^*(F)} i(b)$, it can be found $f \in i^*(F)$ such that

$$i(a) \wedge f = i(b) \wedge f$$

This means that we can find $c \in F$ such that $f \geq i(c)$, so that

$$i(a) \wedge i(c) = i(a) \wedge (i(c) \wedge f) = i(b) \wedge (i(c) \wedge f) = i(b) \wedge i(c)$$

Since i is injective this implies $a \wedge c = b \wedge c$, thus $a \sim_F b$.

Assume i is surjective. Then i_F is surjective as well, since given $b \in N$ it can be found a $a \in M$ such that $i(a) = b$, hence $[b]_{i^*(F)} = i_F([a]_F)$. □

Remark 2.2.12. If G is a ultrafilter in B, then $\mathsf{B}/G = \{[0_B]_G, [1_B]_G\}$. In fact, there are only two classes: the one containing all the elements in G, and the one containing their complements (i. e. all the elements not in G).

If $a \in G$, then $a \wedge a = 1_{\mathsf{B}} \wedge a$, which leads to $a \sim_G 1_{\mathsf{B}}$ (this is true for any filter F). On the other side, if $a \notin G$, then $\neg a \in G$, and since $a \wedge \neg a = 0_{\mathsf{B}} \wedge \neg a$, $a \sim_G 0_{\mathsf{B}}$ holds.

On $St(\mathsf{B})$ we define the topology generated by the family of sets $\{\mathcal{O}_a\}_{a \in \mathsf{B}}$, where:

$$\mathcal{O}_a = \{G \in St(\mathsf{B}) : a \in G\}$$

Proposition 2.2.13. *Given a boolean algebra* B*,* $St(\mathsf{B})$ *is an Hausdorff compact space, and the family* $\{\mathcal{O}_a\}_{a \in \mathsf{B}}$ *is a base of clopen sets (i.e.* $St(\mathsf{B})$ *is* 0*-dimensional).*

Proof. A proof of this proposition can be found in [7, Theorem 7.8]. □

Theorem 2.2.14 (Stone's Representation Theorem)**.** *Every boolean algebra* B *is isomorphic to the boolean algebra* $\mathsf{CL}(St(\mathsf{B}))$*.*

Proof. The map

$$\begin{aligned} \Phi : \mathsf{B} &\to \mathsf{CL}(St(\mathsf{B})) \\ a &\mapsto \mathcal{O}_a \end{aligned}$$

is an isomorphism of boolean algebras. The details of the proof can be found in [7, Theorems 2.1 and 7.8]. □

Remark 2.2.15. With the following proposition we will characterize the completeness of a boolean algebra with the topology of its Stone space. In particular, a topological space X is **extremely disconnected** if $\mathsf{RO}(X) = \mathsf{CL}(X)$.

Proposition 2.2.16. *A boolean algebra* B *is complete* $\Leftrightarrow$ $St(\mathsf{B})$ *is extremely disconnected.*

Proof. A proof of this proposition can be found in [7, Proposition 7.21]. □

The following proposition gives a characterization of 0-dimensional Hausdorff compact spaces in terms of the Stone space of a specific boolean algebra. We will give an explicit proof of it since we will need the map defined in it later on.

Proposition 2.2.17. *Let* X *be a* 0*-dimensional compact Hausdorff topological space, then* X *is homeomorphic to* $St(\mathsf{CL}(X))$*.*

Proof. We shall define the map:

$$\begin{aligned} \varphi : X &\to St(\mathsf{CL}(X)) \\ x &\mapsto G_x \end{aligned}$$

where $G_x = \{A \in \mathsf{CL}(X) : x \in A\}$ can be easily checked to be a ultrafilter.

<u>Injective:</u> If $x \neq y$, let A be a clopen set such that $x \in A$, $y \in A^c$ (we can find such set since X is 0-dimensional and Hausdorff). $A \in G_x$ and $A^c \in G_y$, thus these ultrafilters have to be different.

Surjective: Let $G \in St(\mathsf{CL}(X))$; it is enough to show that $C = \bigcap G$ is a singleton. In fact, $C = \{z\}$ implies that if $A \in G$ then $z \in A$. On the other hand, if B is clopen and $a \notin B$, it cannot be $B \in G$ (otherwise $a \notin C$), hence $z \in B^c \in G$. This means that $A \in G \Leftrightarrow z \in A \Leftrightarrow A \in G_z$, so that $G = G_z$. We have that C is non-empty, because G has the finite intersection property and X is compact. Let $x \neq y$ be in C, we can therefore find a clopen set A such that $x \in A$ and $y \notin A$. It follows that

$$A \in G \Leftrightarrow A^c \notin G$$

which means

$$x \in C \Leftrightarrow y \notin C$$

and this is absurd.

Continuous and open: Let A be a clopen set in X. $x \in A$ if and only if $A \in G_x$, which means $\varphi(A) = \mathcal{O}_A$. It follows that the map φ is open and continuous.

□

Proposition 2.2.18. *Let* B *and* C *be two boolean algebras, and* f *be a continuous map:*

$$f : St(\mathsf{C}) \to St(\mathsf{B})$$

We can define the following morphism of boolean algebras:

$$\begin{aligned} f^* : \mathsf{B} &\to \mathsf{C} \\ a &\mapsto f^{-1}[a] \end{aligned}$$

where we are identifying the elements of a boolean algebra with the clopen sets of their associated Stone spaces as in Theorem 2.2.14.

Given another boolean algebra D*, and a continuous map*

$$g : St(\mathsf{B}) \to St(\mathsf{D})$$

the following properties hold:

1. $(g \circ f)^* = f^* \circ g^*$;
2. *if* $\mathsf{B} = \mathsf{C}$ *and* $f = id$*, then* $id^* = id_{\mathsf{B}}$;
3. *if* f *is an homeomorphism, then* (f^*) *is an isomorphism of boolean algebras and* $(f^*)^{-1} = (f^{-1})^*$.

Proof. A proof of this can be found in [7, Theorem 8.2]. □

Remark 2.2.19. The last proposition tells us that if two boolean algebras have homeomorphic Stone spaces, then they are isomorphic. Since the converse is trivial, we have the following:

Corollary 2.2.20. *The boolean algebras* B *and* C *are isomorphic if and only if* $St(\mathsf{B})$ *and* $St(\mathsf{C})$ *are homeomorphic.*

2.3 Boolean Valued Models

In a first order model a formula can be interpreted as true or false. Given a complete boolean algebra B, B-boolean valued models generalize Tarski semantics associating to each formula a value in B, so that there are no more only true and false propositions (those associated to 1_B and 0_B respectively), but also other "intermediate values" (even not comparable values, since B is not necessarily a linear order) of truth. This section presents and extends some of the contents of the notes [14] regarding boolean valued models.

Definition 2.3.1. Given a complete boolean algebra B, and a first order language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\},$$

a B**-boolean valued model** (or B-valued model) $\mathcal{M}$ in the language $\mathcal{L}$ is a tuple

$$\langle M, =^\mathcal{M}, R_i^\mathcal{M} : i \in I, f_j^\mathcal{M} : j \in J, c_k^\mathcal{M} : k \in K\rangle$$

where:

1. M is a non-empty set, called **domain** of the B-boolean valued model, whose elements are called B**-names**;
2. $=^\mathcal{M}$ is the **boolean value** of the equality:
$$\begin{aligned} =^\mathcal{M}\colon M^2 &\to \mathsf{B} \\ (\tau, \sigma) &\mapsto [\![\tau = \sigma]\!]_\mathsf{B}^\mathcal{M} \end{aligned}$$
3. $R_i^\mathcal{M}$ is the **boolean interpretation** of the n-ary relation symbol R_i:
$$\begin{aligned} R_i^\mathcal{M} : M^n &\to \mathsf{B} \\ (\tau_1, \dots, \tau_n) &\mapsto [\![R_i(\tau_1, \dots, \tau_n)]\!]_\mathsf{B}^\mathcal{M} \end{aligned}$$
4. $f_j^\mathcal{M}$ is the **boolean interpretation** of the n-ary function symbol f_j:
$$\begin{aligned} f_j^\mathcal{M} : M^{n+1} &\to \mathsf{B} \\ (\tau_1, \dots, \tau_n, \sigma) &\mapsto [\![f_j(\tau_1, \dots, \tau_n) = \sigma]\!]_\mathsf{B}^\mathcal{M} \end{aligned}$$
5. $c_k^\mathcal{M}$ is the **boolean interpretation** of the constant symbol c_k, and it is an element in M.

We require that the following conditions hold:

for $\tau, \sigma, \chi \in M$,

i) $[\![\tau = \tau]\!]_\mathsf{B}^\mathcal{M} = 1_\mathsf{B}$;

ii) $[\![\tau = \sigma]\!]_\mathsf{B}^\mathcal{M} = [\![\sigma = \tau]\!]_\mathsf{B}^\mathcal{M}$;

iii) $[\![\tau = \sigma]\!]_\mathsf{B}^\mathcal{M} \wedge [\![\sigma = \chi]\!]_\mathsf{B}^\mathcal{M} \leq [\![\tau = \chi]\!]_\mathsf{B}^\mathcal{M}$;

for $R_i \in \mathcal{L}$ with arity n, and $(\tau_1, \dots, \tau_n), (\sigma_1, \dots, \sigma_n) \in M^n$,

iv) $(\bigwedge_{h \in \{1,\dots,n\}} [\![\tau_h = \sigma_h]\!]_\mathsf{B}^\mathcal{M}) \wedge [\![R_i(\tau_1, \dots, \tau_n)]\!]_\mathsf{B}^\mathcal{M} \leq [\![R_i(\sigma_1, \dots, \sigma_n)]\!]_\mathsf{B}^\mathcal{M}$;

for $f_j \in \mathcal{L}$ with arity n, and $(\tau_1, \ldots, \tau_n), (\sigma_1, \ldots, \sigma_n) \in M^n$ and $\mu, \nu \in M$,

v) $(\bigwedge_{h\in\{1,\ldots,n\}} [\![\tau_h = \sigma_h]\!]^{\mathcal{M}}_{\mathsf{B}}) \wedge [\![f_j(\tau_1, \ldots, \tau_n) = \mu]\!]^{\mathcal{M}}_{\mathsf{B}} \leq [\![f_j(\sigma_1, \ldots, \sigma_n) = \mu]\!]^{\mathcal{M}}_{\mathsf{B}}$;

vi) $\bigvee_{\mu\in M} [\![f_j(\tau_1, \ldots, \tau_n) = \mu]\!]^{\mathcal{M}}_{\mathsf{B}} = 1_{\mathsf{B}}$;

vii) $[\![f_j(\tau_1, \ldots, \tau_n) = \mu]\!]^{\mathcal{M}}_{\mathsf{B}} \wedge [\![f_j(\tau_1, \ldots, \tau_n) = \nu]\!]^{\mathcal{M}}_{\mathsf{B}} \leq [\![\mu = \nu]\!]^{\mathcal{M}}_{\mathsf{B}}$.

If no confusion can arise, we will omit the pedix B and we will confuse a function or predicate symbol with its interpretation.

Remark 2.3.2. Every first order model naturally defines a B-valued model with $\mathsf{B} = \{0, 1\}$.

Remark 2.3.3. Given $\overline{\tau} = (\tau_1, \ldots, \tau_n)$ and $\overline{\sigma} = (\sigma_1, \ldots, \sigma_n)$ in M^n, we will often use the following abbreviation

$$[\![\overline{\tau} = \overline{\sigma}]\!]^{\mathcal{M}} \equiv \bigwedge_{h\in\{1,\ldots,n\}} [\![\tau_h = \sigma_h]\!]^{\mathcal{M}}$$

Definition 2.3.4. Let $\mathcal{M}$ be a B-valued model and $\mathcal{N}$ a C-valued model in the same language $\mathcal{L}$. Let

$$i : \mathsf{B} \to \mathsf{C}$$

be a morphism of boolean algebras and $\Phi \subseteq M \times N$ a relation. The couple $\langle i, \Phi\rangle$ is a **morphism** of boolean valued models if:

1. $\mathrm{dom}\Phi = M$;

2. given $(\tau_1, \sigma_1), (\tau_2, \sigma_2) \in \Phi$:

$$i([\![\tau_1 = \tau_2]\!]^{\mathcal{M}}_{\mathsf{B}}) \leq [\![\sigma_1 = \sigma_2]\!]^{\mathcal{N}}_{\mathsf{C}},$$

3. given R an n-ary relation symbol and $(\tau_1, \sigma_1), \ldots, (\tau_n, \sigma_n) \in \Phi$:

$$i([\![R(\tau_1, \ldots, \tau_n)]\!]^{\mathcal{M}}_{\mathsf{B}}) \leq [\![R(\sigma_1, \ldots, \sigma_n)]\!]^{\mathcal{N}}_{\mathsf{C}},$$

4. given f an n-ary function symbol and $(\tau_1, \sigma_1), \ldots, (\tau_n, \sigma_n), (\mu, \nu) \in \Phi$:

$$i([\![f(\tau_1, \ldots, \tau_n) = \mu]\!]^{\mathcal{M}}_{\mathsf{B}}) \leq [\![f(\sigma_1, \ldots, \sigma_n) = \nu]\!]^{\mathcal{N}}_{\mathsf{C}},$$

5. given a constant symbol c and $(\tau, \sigma) \in \Phi$:

$$i([\![\tau = c]\!]^{\mathcal{M}}_{\mathsf{B}}) \leq [\![\sigma = c]\!]^{\mathcal{N}}_{\mathsf{C}}.$$

An **injective morphism** is a morphism such that in 2 equality holds.

An **embedding** of boolean valued models is an injective morphism such that in 3-5 equality holds.

An embedding $\langle i, \Phi\rangle$ from $\mathcal{M}$ to $\mathcal{N}$ is called **isomorphism** of boolean valued models if i is an isomorphism of boolean algebras and for every $b \in N$ there is a $a \in M$ such that $(a, b) \in \Phi$.

Remark 2.3.5. When $\mathsf{B} = \mathsf{C}$ we will consider $i = id_{\mathsf{B}}$ unless otherwise stated.

Definition 2.3.6. Suppose $\mathcal{M}$ is a B-valued model and $\mathcal{N}$ a C-valued model (both in the same language $\mathcal{L}$) such that B is a complete subalgebra of C, $M \subseteq N$, and

$$\llbracket R(\tau_1, \ldots, \tau_n) \rrbracket_{\mathsf{B}}^{\mathcal{M}} = \llbracket R(\tau_1, \ldots, \tau_n) \rrbracket_{\mathsf{C}}^{\mathcal{N}}$$

$$\llbracket f(\tau_1, \ldots, \tau_n) = \sigma \rrbracket_{\mathsf{B}}^{\mathcal{M}} = \llbracket f(\tau_1, \ldots, \tau_n) = \sigma \rrbracket_{\mathsf{C}}^{\mathcal{N}}$$

$$\llbracket \tau = c \rrbracket_{\mathsf{B}}^{\mathcal{M}} = \llbracket \tau = c \rrbracket_{\mathsf{C}}^{\mathcal{N}}$$

for all relation symbols R, all function symbols f, and all constant symbols c in $\mathcal{L}$. Let J be the immersion of M into N. Then $\langle id_{\mathsf{B}}, J \rangle$ is an embedding of boolean valued models and $\mathcal{N}$ is said to be a **boolean extension** of $\mathcal{M}$.

Remark 2.3.7. The definition of **valuation** of variables in a first order model can be easily generalized to B-valued models, as the notion of formula with parameters. We will therefore use the notations defined in Definition 2.1.9 for B-valued models as well.

Definition 2.3.8. Given a B-valued model $\mathcal{M}$ in a language $\mathcal{L}$, let φ be a $\mathcal{L}$-formula whose free variables are in $\{x_1, \ldots, x_n\}$, and let ν be a valuation in $\mathcal{M}$ whose domain contains $\{x_1, \ldots, x_n\}$. We define now $\llbracket \varphi(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}$, the **boolean value** of $\varphi(\nu)$.

First, let t be an $\mathcal{L}$-term and $\tau \in M$; we define recursively $\llbracket (t = \tau)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} \in \mathsf{B}$ as follows:

- if t is a constant c, then
$$\llbracket (c = \tau)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} = \llbracket c^{\mathcal{M}} = \tau \rrbracket_{\mathsf{B}}^{\mathcal{M}}$$
- if t is a variable x, then
$$\llbracket (x = \tau)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} = \llbracket \nu(x) = \tau \rrbracket_{\mathsf{B}}^{\mathcal{M}}$$
- if $t = f(t_1, \ldots, t_n)$ where t_i are terms and f is an n-ary function symbol, then
$$\llbracket (f(t_1, \ldots, t_n) = \tau)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} = \bigvee_{\sigma_1, \ldots, \sigma_n \in M} \left(\bigwedge_{1 \leq i \leq n} \llbracket (t_i = \sigma_i)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} \right) \wedge \llbracket f(\sigma_1, \ldots, \sigma_n) = \tau \rrbracket_{\mathsf{B}}^{\mathcal{M}}$$

Given a formula φ, we define recursively $\llbracket \varphi(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}$ as follows:

- if $\varphi \equiv t_1 = t_2$, then
$$\llbracket (t_1 = t_2)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} = \bigvee_{\tau \in M} \llbracket (t_1 = \tau)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} \wedge \llbracket (t_2 = \tau)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}$$
- if $\varphi \equiv R(t_1, \ldots, t_n)$, then
$$\llbracket (R(t_1, \ldots, t_n))(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} = \bigvee_{\tau_1, \ldots, \tau_n \in M} \left(\bigwedge_{1 \leq i \leq n} \llbracket (t_i = \tau_i)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} \right) \wedge \llbracket R(\tau_1, \ldots, \tau_n) \rrbracket_{\mathsf{B}}^{\mathcal{M}}$$
- if $\varphi \equiv \neg\psi$, then
$$\llbracket \varphi(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} = \neg \llbracket \psi(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}$$
- if $\varphi \equiv \psi \wedge \theta$, then
$$\llbracket \varphi(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} = \llbracket \psi(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}} \wedge \llbracket \theta(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}$$

- if $\varphi \equiv \exists y \psi(y)$, then
$$[\![\varphi(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}} = \bigvee_{\tau \in M} [\![\psi(y/\tau, \nu)]\!]^{\mathcal{M}}_{\mathsf{B}}$$

If no confusion can arise, we omit the index $\mathcal{M}$ and the pedix B, and we simply denote the boolean value of a formula φ with parameters in $\mathcal{M}$ by $[\![\varphi]\!]$.

Remark 2.3.9. In the previous definition we considered the sup of several subsets of B, and this motivates our request to work with a complete boolean algebra. If we assume that the language does not contain function symbols, the definition of the boolean value of atomic formulae is simpler:
$$[\![(t_1 = t_2)(\nu)]\!] = [\![t_1(\nu) = t_2(\nu)]\!]$$
$$[\![(R(t_1, \ldots, t_n))(\nu)]\!] = [\![R(t_1(\nu), \ldots, t_n(\nu))]\!]$$
where for a term t which is a constant or a variable, $t(\nu)$ can be defined verbatim for B-valued models as for first order models. In this case we do not require B to be complete for the definition of the boolean value of atomic formulae, but a certain degree of completeness of B is necessary to give a semantic interpretation of the existential quantifier.

Proposition 2.3.10. *Let $\mathcal{M}$ be a B-valued model and $\mathcal{N}$ a C-valued model in the same language $\mathcal{L}$. Assume $\langle i, \Phi \rangle$ is an isomorphism of boolean valued models.*

Then for any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, and for every $(\tau_1, \sigma_1), \ldots, (\tau_n, \sigma_n) \in \Phi$ we have that:
$$i([\![\varphi(\tau_1, \ldots, \tau_n)]\!]^{\mathcal{M}}_{\mathsf{B}}) = [\![\varphi(\sigma_1, \ldots, \sigma_n)]\!]^{\mathcal{N}}_{\mathsf{C}}$$

Proof. The proof proceeds by induction on the complexity of the formula. We will write $\nu = (x_1/\tau_1, \ldots, x_n/\tau_n)$ and $\nu' = (x_1/\sigma_1, \ldots, x_n/\sigma_n)$ with $(\tau_i, \sigma_i) \in \Phi$.

<u>Atomic formulae:</u> First, consider $(\eta, \zeta) \in \Phi$ and t an $\mathcal{L}$-term. We will show
$$i([\![(t = \eta)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}}) = [\![(t = \zeta)(\nu')]\!]^{\mathcal{N}}_{\mathsf{C}}$$

If t is a constant or a variable, this follows because i is an injective morphism. If $t = f(t_1, \ldots, t_n)$ we use the fact that i is complete (see Lemma 2.2.5) and that $\langle i, \Phi \rangle$ is an embedding and proceed by induction on the terms t_i:
$$i([\![(f(t_1, \ldots, t_n) = \eta)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}}) =$$
$$\bigvee_{\overline{\chi} \in M^n} \left(\bigwedge_{1 \le i \le n} i([\![(t_i = \chi_i)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}}) \right) \wedge i([\![f(\chi_1, \ldots, \chi_n) = \eta]\!]^{\mathcal{M}}_{\mathsf{B}})$$

Since $\mathrm{Im}(\Phi) = N$ we have:
$$\bigvee_{\overline{\chi} \in M^n} \left(\bigwedge_{1 \le i \le n} i([\![(t_i = \chi_i)(\nu)]\!]^{\mathcal{M}}_{\mathsf{B}}) \right) \wedge i([\![f(\chi_1, \ldots, \chi_n) = \eta]\!]^{\mathcal{M}}_{\mathsf{B}}) =$$
$$\bigvee_{\overline{\omega} \in N^n} \left(\bigwedge_{1 \le i \le n} [\![(t_i = \omega_i)(\nu')]\!]^{\mathcal{N}}_{\mathsf{C}} \right) \wedge [\![f(\omega_1, \ldots, \omega_n) = \zeta]\!]^{\mathcal{N}}_{\mathsf{C}} =$$
$$[\![(f(t_1, \ldots, t_n) = \zeta)(\nu')]\!]^{\mathcal{N}}_{\mathsf{C}}$$

Let now $\varphi \equiv t_1 = t_2$. By induction we have:

$$i(\llbracket (t_1 = t_2)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) = \bigvee_{\chi \in M} i(\llbracket (t_1 = \chi)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) \wedge i(\llbracket (t_2 = \chi)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}})$$

From $\operatorname{Im}(\Phi) = N$ follows that:

$$\bigvee_{\chi \in M} i(\llbracket (t_1 = \chi)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) \wedge i(\llbracket (t_2 = \chi)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) =$$

$$\bigvee_{\omega \in N} \llbracket (t_1 = \omega)(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}} \wedge \llbracket (t_2 = \omega)(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}} = \llbracket (t_1 = t_2)(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}}$$

If $\varphi \equiv R(t_1, \dots, t_n)$ we have:

$$i(\llbracket R(t_1, \dots, t_n)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) =$$

$$\bigvee_{\overline{\chi} \in M^n} \left(\bigwedge_{1 \le i \le n} i(\llbracket (t_i = \chi_i)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) \right) \wedge i(\llbracket R(\chi_1, \dots, \chi_n)(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) =$$

$$\bigvee_{\overline{\omega} \in N^n} \left(\bigwedge_{1 \le i \le n} \llbracket (t_i = \omega_i)(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}} \right) \wedge \llbracket R(\omega_1, \dots, \omega_n)(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}} = \llbracket R(t_1, \dots, t_n)(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}}$$

Negation: Consider $\varphi \equiv \neg\psi$. We have that:

$$i(\llbracket \varphi(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) = \neg i(\llbracket \psi(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) = \neg \llbracket \psi(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}} = \llbracket \varphi(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}}$$

Conjunction: If $\varphi \equiv \psi \wedge \theta$ then:

$$i(\llbracket \varphi(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) = i(\llbracket \psi(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) \wedge i(\llbracket \theta(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) = \llbracket \psi(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}} \wedge \llbracket \theta(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}} = \llbracket \varphi(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}}$$

Existential: Let $\varphi \equiv \exists y \psi(y)$. We have that:

$$i(\llbracket \varphi(\nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) = \bigvee_{\chi \in M} i(\llbracket \psi(y/\chi, \nu) \rrbracket_{\mathsf{B}}^{\mathcal{M}}) = \bigvee_{\omega \in N} \llbracket \psi(y/\omega, \nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}} = \llbracket \varphi(\nu') \rrbracket_{\mathsf{C}}^{\mathcal{N}}$$

where second equality holds since $\langle i, \Phi \rangle$ is an isomorphism.

□

Now we want to generalize the Soundness Theorem 2.1.15 to boolean valued models. In order to do this we first need a:

Lemma 2.3.11. *Given a* B*-valued model* $\mathcal{M}$ *in the language* $\mathcal{L}$*, assume* $\varphi(x_1, \dots, x_n)$ *is an* $\mathcal{L}$*-formula and* $\overline{\tau} = (\tau_1, \dots, \tau_n), \overline{\sigma} = (\sigma_1, \dots, \sigma_n) \in M^n$*. Then:*

$$\llbracket \overline{\tau} = \overline{\sigma} \rrbracket \wedge \llbracket \varphi(\overline{\tau}) \rrbracket \le \llbracket \varphi(\overline{\sigma}) \rrbracket$$

Proof. The proof proceeds by induction on the complexity of φ. Given $\varphi(x_1, \dots, x_n)$, we can consider $\nu = (x_1/\tau_1, \dots, x_n/\tau_n)$ and $\nu' = (x_1/\sigma_1, \dots, x_n/\sigma_n)$, so that the thesis becomes:

$$\llbracket \overline{\tau} = \overline{\sigma} \rrbracket \wedge \llbracket \varphi(\nu) \rrbracket \le \llbracket \varphi(\nu') \rrbracket$$

Atomic formulae: If φ is an atomic formula the thesis follows from the definitions and from [5, Lemma 3, Chapter 8]. Here is the proof: first, consider a term t and $\mu \in M$. We will show, recursively on the complexity of the term, that

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![(t = \mu)(\nu)]\!] \leq [\![(t = \mu)(\nu')]\!]$$

If t is a constant or a variable, this follows from the definition of B-valued model. If $t = f(t_1, \ldots, t_n)$ and the thesis holds for $t_1, \ldots, t_n$, then we have:

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![(f(t_1, \ldots, t_n) = \mu)(\nu)]\!] =$$

$$\bigvee_{\overline{\chi} \in M^n} \left([\![(\overline{t} = \overline{\chi})(\nu)]\!] \wedge [\![\overline{\tau} = \overline{\sigma}]\!] \right) \wedge [\![f(\chi_1, \ldots, \chi_n) = \mu]\!] \leq$$

$$\bigvee_{\overline{\chi} \in M^n} [\![(\overline{t} = \overline{\chi})(\nu')]\!] \wedge [\![f(\chi_1, \ldots, \chi_n) = \mu]\!] = [\![(f(t_1, \ldots, t_n) = \mu)(\nu')]\!]$$

Consider now $\varphi \equiv t_1 = t_2$. We have then:

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![(t_1 = t_2)(\nu)]\!] = \bigvee_{\chi \in M} ([\![(t_1 = \chi)(\nu)]\!] \wedge [\![\overline{\tau} = \overline{\sigma}]\!]) \wedge ([\![(t_2 = \chi)(\nu)]\!] \wedge [\![\overline{\tau} = \overline{\sigma}]\!]) \leq$$

$$\bigvee_{\chi \in M} [\![(t_1 = \chi)(\nu')]\!] \wedge [\![(t_2 = \chi)(\nu')]\!] = [\![(t_1 = t_2)(\nu')]\!]$$

If $\varphi \equiv R(t_1, \ldots, t_n)$, then:

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![R(t_1, \ldots, t_n)(\nu)]\!] = \bigvee_{\overline{\chi} \in M^n} [\![(\overline{t} = \overline{\chi})(\nu)]\!] \wedge [\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![R(\chi_1, \ldots, \chi_n)]\!] \leq$$

$$\bigvee_{\overline{\chi} \in M^n} [\![(\overline{t} = \overline{\chi})(\nu')]\!] \wedge [\![R(\chi_1, \ldots, \chi_n)]\!] = [\![R(t_1, \ldots, t_n)(\nu')]\!]$$

Negation: If $\varphi \equiv \neg\psi$, by induction we have

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\psi(\nu)]\!] \leq [\![\psi(\nu')]\!]$$
$$[\![\overline{\sigma} = \overline{\tau}]\!] \wedge [\![\psi(\nu')]\!] \leq [\![\psi(\nu)]\!],$$

which means

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\psi(\nu)]\!] \wedge [\![\psi(\nu')]\!] = [\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\psi(\nu)]\!]$$

and

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\psi(\nu)]\!] \wedge [\![\psi(\nu')]\!] = [\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\psi(\nu')]\!].$$

Hence

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\psi(\nu')]\!] = [\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\psi(\nu)]\!]$$

It follows that:

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge \neg [\![\psi(\nu)]\!] = [\![\overline{\tau} = \overline{\sigma}]\!] \wedge \neg [\![\psi(\nu)]\!] \wedge ([\![\psi(\nu')]\!] \vee \neg [\![\psi(\nu')]\!]) =$$
$$([\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\psi(\nu')]\!] \wedge \neg [\![\psi(\nu)]\!]) \vee ([\![\overline{\tau} = \overline{\sigma}]\!] \wedge \neg [\![\psi(\nu')]\!] \wedge \neg [\![\psi(\nu)]\!]) =$$
$$([\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\psi(\nu)]\!] \wedge \neg [\![\psi(\nu)]\!]) \vee ([\![\overline{\tau} = \overline{\sigma}]\!] \wedge \neg [\![\psi(\nu')]\!] \wedge \neg [\![\psi(\nu)]\!]) =$$
$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge \neg [\![\psi(\nu')]\!] \wedge \neg [\![\psi(\nu)]\!],$$

which means

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge \neg [\![\psi(\nu)]\!] \leq \neg [\![\psi(\nu')]\!].$$

Conjunction: If $\varphi \equiv \psi \wedge \theta$ we have:

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\varphi(\nu)]\!] = ([\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\psi(\nu)]\!]) \wedge ([\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\theta(\nu)]\!]) \leq$$

$$[\![\psi(\nu')]\!] \wedge [\![\theta(\nu')]\!] = [\![\varphi(\nu')]\!]$$

Existential: If $\varphi \equiv \exists y \psi(y)$ we have that:

$$[\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![\varphi(\nu)]\!] = \bigvee_{\chi \in M} ([\![\psi(y/\chi, \nu)]\!] \wedge [\![\overline{\tau} = \overline{\sigma}]\!]) \leq \bigvee_{\chi \in M} [\![\psi(y/\chi, \nu')]\!] = [\![\varphi(\nu')]\!]$$

□

Definition 2.3.12. Let $\mathcal{M}$ be a B-valued model in a language $\mathcal{L}$. A formula φ with parameters in $\mathcal{M}$ is said to be **satisfied** or **valid** in $\mathcal{M}$ if $[\![\varphi]\!] = 1_{\mathsf{B}}$. A theory T is **valid** in $\mathcal{M}$ if every $\varphi \in T$ is valid in $\mathcal{M}$.

Remark 2.3.13. Given a boolean algebra B and two elements a, b in it, we can define $a \to b \equiv \neg a \vee b$, and we observe that

$$a \to b = 1_{\mathsf{B}} \Leftrightarrow a \leq b.$$

On the one hand assume $\neg a \vee b = 1_{\mathsf{B}}$. Then:

$$a = 1_{\mathsf{B}} \wedge a = a \wedge (\neg a \vee b) = (a \wedge \neg a) \vee (a \wedge b) = 0 \vee (a \wedge b) = a \wedge b$$

which is equivalent to $a \leq b$. On the other hand $a \leq b$ implies $1_{\mathsf{B}} = \neg a \vee a \leq \neg a \vee b$.

Theorem 2.3.14 (Soundness Theorem)**.** *Let $\mathcal{L}$ be a language, if φ is a $\mathcal{L}$-formula which is syntactically provable by a $\mathcal{L}$-theory T, and T is valid in a B-valued model $\mathcal{M}$, then $[\![\varphi(\nu)]\!]_{\mathcal{M}} = 1_{\mathsf{B}}$ for all valuations ν in $\mathcal{M}$.*

Proof. We refer to the system proof defined in [12, Chapter 2, Section 6]. We need to show that, for all valuations, all the *logical axioms*:

1. $\varphi \vee \neg\varphi$;
2. $x = x$;
3. $\varphi(a) \to \exists\varphi(x)$;
4. $(\overline{x} = \overline{y}) \to (f(\overline{x}) = f(\overline{y}))$
5. $(\overline{x} = \overline{y}) \to (\varphi(\overline{x}) \to \varphi(\overline{y}))$;

have boolean value equal to 1_{B}, and that for all *logical rules*:

6. $\varphi \vdash \varphi \vee \psi$;
7. $\varphi \vee \varphi \vdash \varphi$;
8. $(\varphi \vee (\psi \vee \chi)) \vdash ((\varphi \vee \psi) \vee \chi)$;
9. $(\varphi \vee \psi) \wedge (\neg\varphi \vee \chi) \vdash \psi \vee \chi$;
10. if x is not free in ψ, $\varphi(a) \to \psi \vdash \exists x(\varphi(x) \to \psi)$;

it holds that if $\varphi \vdash \psi$, then $[\![\varphi]\!] \leq [\![\psi]\!]$.

1. Since for $a \in \mathsf{B}$
$$a \vee \neg a = 1_{\mathsf{B}}$$
for every valuation ν we have that $[\![(\varphi \vee \neg\varphi)(\nu)]\!] = [\![(\varphi)(\nu)]\!] \vee \neg [\![(\varphi)(\nu)]\!] = 1_{\mathsf{B}}$.

2. From the definition of boolean valued models, for $\tau \in M$:
$$[\![\tau = \tau]\!] = 1_{\mathsf{B}}$$
hence for all valuations and variables $[\![(x = x)(\nu)]\!] = 1_{\mathsf{B}}$.

3. By definition we have
$$[\![\exists\varphi(x)]\!] = \bigvee_{\sigma \in M} [\![\varphi(\sigma)]\!] \geq [\![\varphi(\tau)]\!]$$
hence we can conclude using Remark 2.3.13.

4. Using the definition of boolean valued model and of boolean evaluation of a formula, suppose $\nu(\overline{x}) = \overline{\tau}$ and $\nu(\overline{y}) = \overline{\sigma}$. We have then:
$$[\![(f(\overline{x}) = f(\overline{y}))(\nu)]\!] = \bigvee_{\omega \in M} ([\![(f(\overline{x}) = \omega)(\nu)]\!] \wedge [\![(f(\overline{y}) = \omega)(\nu)]\!]) =$$
$$\bigvee_{\omega \in M} \left(\left(\bigvee_{\overline{\chi} \in M^n} [\![\overline{\tau} = \overline{\chi}]\!] \wedge [\![f(\overline{\chi}) = \omega]\!] \right) \wedge \left(\bigvee_{\overline{\chi} \in M^n} [\![\overline{\sigma} = \overline{\chi}]\!] \wedge [\![f(\overline{\chi}) = \omega]\!] \right) \right) \geq$$
$$\bigvee_{\omega \in M} (([\![\overline{\tau} = \overline{\sigma}]\!] \wedge [\![f(\overline{\sigma}) = \omega]\!]) \wedge ([\![\overline{\sigma} = \overline{\tau}]\!] \wedge [\![f(\overline{\tau}) = \omega]\!])) =$$
$$\bigvee_{\omega \in M} ([\![\tau = \sigma]\!]) \wedge \bigvee_{\omega \in M} ([\![f(\overline{\sigma}) = \omega]\!]) \wedge \bigvee_{\omega \in M} ([\![f(\overline{\tau}) = \omega]\!]) = [\![\tau = \sigma]\!] \wedge 1_{\mathsf{B}} \wedge 1_{\mathsf{B}} = [\![(x = y)(\nu)]\!]$$
using the property
$$\bigvee_{i \in I} (a_i \wedge b_i) = \left(\bigvee_{i \in I} a_i \right) \wedge \left(\bigvee_{i \in I} b_i \right)$$
for $a_i, b_i \in \mathsf{B}$. A proof of this fact can be found in [5, Corollary 2, Chapter 8].

5. It is sufficient to show that, for $\overline{\tau}, \overline{\sigma} \in M^n$:
$$[\![\overline{\tau} = \overline{\sigma}]\!] \leq [\![\varphi(\overline{\tau}) \to \varphi(\overline{\sigma})]\!] .$$
This is a consequence of Lemma 2.3.11, and of the fact that if $a, b, c \in \mathsf{B}$ are such that $a \wedge b \leq c$, then $a \leq \neg b \vee c$. This follows from
$$a \leq \neg b \vee a = (\neg b \vee a) \wedge (\neg b \vee b) = \neg b \vee (a \wedge b) \leq \neg b \vee c$$

6. It follows form $a \leq a \vee b$ for all $a, b \in \mathsf{B}$.

7. It follows from $a = a \vee a$ for all $a \in \mathsf{B}$.

8. It follows from $(a \vee b) \vee c = a \vee (b \vee c)$ for $a, b, c \in \mathsf{B}$.

9. Let be $[\![\varphi]\!] = a, [\![\psi]\!] = b, [\![\chi]\!] = c$. Then we have:
$$(a \vee b) \wedge (\neg a \vee c) = (a \wedge \neg a) \vee (a \wedge c) \vee (b \wedge \neg a) \vee (b \wedge c) \leq c \vee b \vee (b \vee c) = b \vee c.$$

10. This item is proved as follows:

$$[\![\varphi(\tau) \to \psi]\!] = [\![\neg\varphi(\tau) \vee \psi]\!] \leq \bigvee_{\sigma \in M} [\![\neg\varphi(\sigma) \vee \psi]\!] = [\![\exists x(\varphi(x) \to \psi)]\!] \,.$$

□

Remark 2.3.15. Since first order models are a subfamily of boolean valued models, we can infer with no additional effort the Completeness Theorem also for boolean valued semantic.

Theorem 2.3.16 (Soundness and Completeness). *Given a language $\mathcal{L}$, an $\mathcal{L}$-formula φ is syntactically provable from an $\mathcal{L}$-theory T if and only if $[\![\varphi(\nu)]\!] = 1_{\mathsf{B}}$ for all complete boolean algebras B, all B-valued model $\mathcal{M}$ in which T is valid, and all valuations ν in $\mathcal{M}$.*

We now introduce *quotients* of B-valued models by a filter on B. Given a B-valued model $\mathcal{M}$ in the language $\mathcal{L}$ and a filter F in B, we can define a B/F-valued model $\mathcal{M}/F$ whose domain will be defined through an equivalence relation built on M using F. Some difficulties arise due to the fact that B/F may not be necessarily complete, thus it might not be possible to satisfy all conditions in Definition 2.3.1 in order that $\mathcal{M}/F$ satisfies Definition 2.3.8 for the boolean algebra B/F. Nonetheless completeness is not strictly necessary, since these definitions work once B/F contains "enough" suprema and infima.

Definition 2.3.17. Given B a boolean algebra and $\mathcal{M}$ a tuple defined as in 1-5 of Definition 2.3.1, the couple $\langle \mathsf{B}, \mathcal{M} \rangle$ is a **boolean couple** in the language $\mathcal{L}$ if B contains all suprema and infima required for Definitions 2.3.1 and 2.3.8.

We can generalize Definition 2.3.1 saying that $\mathcal{M}$ is a B-valued model if and only if $\langle \mathsf{B}, \mathcal{M} \rangle$ is a boolean couple.

All the results we have presented so far for boolean valued model can be generalized to boolean couples. Nevertheless our analysis will focus on B valued models with B a complete boolean algebra.

Definition 2.3.18. Let $\langle \mathsf{B}, \mathcal{M} \rangle$ be a boolean couple in the language $\mathcal{L}$ and F a filter of B. Define the equivalence relation on M

$$\tau \equiv_F \sigma \Leftrightarrow [\![\tau = \sigma]\!] \in F$$

and M/F as the set of all equivalence classes $\{[\tau]_F : \tau \in M\}$. The B/F-model $\mathcal{M}/F = \langle M/F, =^{\mathcal{M}/F}, R_i^{\mathcal{M}/F} : i \in I, f_j^{\mathcal{M}/F} : j \in J, c_k^{\mathcal{M}/F} : k \in K \rangle$ is defined as follows:

- $=^{\mathcal{M}/F}$ is defined as:

$$\begin{aligned} =^{\mathcal{M}/F} \colon (M/F)^2 &\to \mathsf{B}/F \\ ([\tau]_F, [\sigma]_F) &\mapsto \left[[\![\tau = \sigma]\!]_{\mathsf{B}}^{\mathcal{M}} \right]_F \end{aligned}$$

- For any n-ary relation symbol R in $\mathcal{L}$, let:

$$\begin{aligned} R^{\mathcal{M}/F} : (M/F)^n &\to \mathsf{B}/F \\ ([\tau_1]_F, \ldots, [\tau_n]_F) &\mapsto \left[[\![R(\tau_1, \ldots, \tau_n)]\!]_{\mathsf{B}}^{\mathcal{M}} \right]_F \end{aligned}$$

- For any n-ary function symbol f in $\mathcal{L}$, let:

$$f^{\mathcal{M}/F} : (M/F)^{n+1} \to \mathsf{B}/F$$
$$([\tau_1]_F, \ldots, [\tau_n]_F, [\sigma]_F) \mapsto \left[\llbracket f_j(\tau_1, \ldots, \tau_n) = \sigma \rrbracket^{\mathcal{M}}_{\mathsf{B}}\right]_F$$

- For any constant symbol c in $\mathcal{L}$, let $c^{\mathcal{M}/F} = [c^{\mathcal{M}}]_F \in M/F$.

Whenever $\langle \mathsf{B}/F, \mathcal{M}/F \rangle$ is a boolean couple (except for vi, all points i-vii of Definition 2.3.1 are always trivially satisfied), we say that the B/F-valued model $\mathcal{M}/F$ is the **quotient** of $\mathcal{M}$ by F.

Remark 2.3.19. The functions $=^{\mathcal{M}/F}, R^{\mathcal{M}/F}, f^{\mathcal{M}/F}$ are all well-defined. For $=^{\mathcal{M}/F}$ and $R^{\mathcal{M}/F}$ the proof is the same and we shall see in detail the case of $=^{\mathcal{M}/F}$. Let $\tau, \tau', \sigma, \sigma' \in M$ such that $[\tau]_F = [\tau']_F$ and $[\sigma]_F = [\sigma']_F$, we want to show

$$[\llbracket \tau = \sigma \rrbracket]_F = [\llbracket \tau' = \sigma' \rrbracket]_F$$

Since both $a = \llbracket \tau = \tau' \rrbracket$ and $b = \llbracket \sigma = \sigma' \rrbracket$ belong to F we have:

$$[a]_F = 1_{\mathsf{B}/F}$$

$$[b]_F = 1_{\mathsf{B}/F}$$

and since $\mathcal{M}$ is a B-valued model it follows that:

$$[\llbracket \tau = \sigma \rrbracket]_F = [\llbracket \tau = \sigma \rrbracket]_F \wedge [a]_F \wedge [b]_F = [\llbracket \tau = \sigma \rrbracket \wedge a \wedge b]_F \leq [\llbracket \tau' = \sigma' \rrbracket]_F .$$

In the same way it can be shown that $[\llbracket \tau' = \sigma' \rrbracket]_F \leq [\llbracket \tau = \sigma \rrbracket]_F$.

For f an n-ary function symbol, the proof is similar but it requires Proposition 2.3.11: Let $(\tau_1, \ldots, \tau_n, \sigma)$ and $(\tau_1', \ldots, \tau_n', \sigma')$ in M^{n+1} such that $a = \llbracket \overline{\tau} = \overline{\tau}' \rrbracket^{\mathcal{M}}_{\mathsf{B}}$ and $b = \llbracket \sigma = \sigma' \rrbracket^{\mathcal{M}}_{\mathsf{B}}$ are in F. We have

$$[\llbracket f(\tau_1, \ldots, \tau_n) = \sigma \rrbracket]_F = [\llbracket f(\tau_1, \ldots, \tau_n) = \sigma \rrbracket]_F \wedge [a]_F \wedge [b]_F =$$
$$[\llbracket f(\tau_1, \ldots, \tau_n) = \sigma \rrbracket \wedge a \wedge b]_F \leq [\llbracket f(\tau_1', \ldots, \tau_n') = \sigma' \rrbracket]_F$$

The other way around can be shown in the same way.

We show now that morphisms of boolean valued models are preserved by quotients.

Proposition 2.3.20. *Let $\langle \mathsf{B}, \mathcal{M} \rangle$ and $\langle \mathsf{C}, \mathcal{N} \rangle$ be two boolean couples in the language $\mathcal{L}$. Let F be a filter in B and $i : \mathsf{B} \to \mathsf{C}$ an injective morphism of boolean algebras. Suppose both $\langle \mathsf{B}/F, \mathcal{M}/F \rangle$ and $\langle \mathsf{C}/i^*(F), \mathcal{N}/i^*(F) \rangle$ are boolean couples, and $\Phi \subseteq M \times N$ is such that $\langle i, \Phi \rangle$ is a morphism of boolean valued models. Let*

$$\Phi_F = \{(\alpha, \beta) \in M/F \times N/i^*(F) : \exists \sigma \in \alpha, \tau \in \beta \text{ such that } (\sigma, \tau) \in \Phi\}.$$

Then $\langle i_F, \Phi_F \rangle$ is a morphism between the boolean valued models $\mathcal{M}/F$ and $\mathcal{N}/i^*(F)$. Moreover, if $\langle i, \Phi \rangle$ is an injective morphism, embedding, or isomorphism of boolean valued models, then $\langle i_F, \Phi_F \rangle$ is respectively an injective morphism, embedding, or isomorphism of boolean valued models.

Proof. Given $(\alpha_j, \beta_j) \in \Phi_F$, we let $\sigma_j \in M$ and $\tau_j \in N$ be two elements such that $(\sigma_j, \tau_j) \in \Phi$ and $\alpha_j = [\sigma_j]_F, \beta_j = [\tau_j]_{i^*(F)}$.

1. Since $\mathrm{dom}(\Phi) = M$, it follows that Φ_F is everywhere defined.

2. Consider $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \Phi_F$. We have then:

$$i_F(\llbracket \alpha_1 = \alpha_2 \rrbracket) = i_F([\llbracket \tau_1 = \tau_2 \rrbracket]_F) = [i(\llbracket \tau_1 = \tau_2 \rrbracket)]_{i^*(F)} \leq [\llbracket \sigma_1 = \sigma_2 \rrbracket]_{i^*(F)} = \llbracket \beta_1 = \beta_2 \rrbracket$$

3. Let R be an n-ary relation symbol in $\mathcal{L}$ and $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \in \Phi_F$. We have that:

$$i_F(\llbracket R(\alpha_1, \ldots, \alpha_n) \rrbracket) = i_F([\llbracket R(\tau_1, \ldots, \tau_n) \rrbracket]_F) = [i(\llbracket R(\tau_1, \ldots, \tau_n) \rrbracket)]_{i^*(F)} \leq$$

$$[\llbracket R(\sigma_1, \ldots, \sigma_n) \rrbracket]_{i^*(F)} = \llbracket R(\beta_1, \ldots, \beta_n) \rrbracket$$

4. Consider f an n-ary function symbol in $\mathcal{L}$ and $(\alpha_1, \beta_1) \ldots, (\alpha_{n+1}, \beta_{n+1}) \in \Phi_F$. Then:

$$i_F(\llbracket f(\alpha_1, \ldots, \alpha_n) = \alpha_{n+1} \rrbracket) = i_F([\llbracket f(\tau_1, \ldots, \tau_n) = \tau_{n+1} \rrbracket]_F) =$$

$$[i(\llbracket f(\tau_1, \ldots, \tau_n) = \tau_{n+1} \rrbracket)]_{i^*(F)} \leq [\llbracket f(\sigma_1, \ldots, \sigma_n) = \sigma_{n+1} \rrbracket]_{i^*(F)} =$$

$$\llbracket f(\beta_1, \ldots, \beta_n) = \beta_{n+1} \rrbracket$$

5. Let c a constant symbol and $(\alpha, \beta) \in \Phi_F$. Then:

$$i_F(\llbracket \alpha = c \rrbracket) = i_F([\llbracket \tau = c \rrbracket]_F) = [i(\llbracket \tau = c \rrbracket)]_{i^*(F)} \leq [\llbracket \sigma = c \rrbracket]_{i^*(F)} = \llbracket \beta = c \rrbracket$$

It can be easily checked that whenever equality holds in 2-5 of Definition 2.3.4, equality holds as well in the respective points of this proof, and recalling Proposition 2.2.11, the proof is concluded. □

We also want to outline how some properties of a boolean valued model behave when passing to quotient models.

Definition 2.3.21. Given a boolean couple $\langle \mathsf{B}, \mathcal{M} \rangle$ we say that $\mathcal{M}$ is an **extensional** B-valued model if, given $\tau, \sigma \in M$:

$$\tau = \sigma \Leftrightarrow \llbracket \tau = \sigma \rrbracket = 1_{\mathsf{B}}$$

Remark 2.3.22. If $\langle i, \Phi \rangle$ is a morphism between the boolean couples $\langle \mathsf{B}, \mathcal{M} \rangle$ and $\langle \mathsf{C}, \mathcal{N} \rangle$ and $\mathcal{N}$ is extensional, then Φ is a function, in fact given $(\tau, \sigma_1), (\tau, \sigma_2) \in \Phi$:

$$1_{\mathsf{C}} = i(1_{\mathsf{B}}) = i(\llbracket \tau = \tau \rrbracket) \leq \llbracket \sigma_1 = \sigma_2 \rrbracket$$

hence $\sigma_1 = \sigma_2$.

Proposition 2.3.23. *Given a boolean couple $\langle \mathsf{B}, \mathcal{M} \rangle$ and a filter F in B, if $\langle \mathsf{B}/F, \mathcal{M}/F \rangle$ is a boolean couple, then $\mathcal{M}/F$ is an extensional B/F-valued model.*

Proof. Given $\tau, \sigma \in M$, we have that:

$$[\tau]_F = [\sigma]_F \Leftrightarrow \llbracket \tau = \sigma \rrbracket \in F$$

Recalling Remark 2.2.12, we can infer that:

$$[\tau]_F = [\sigma]_F \Leftrightarrow \llbracket \tau = \sigma \rrbracket_{\mathsf{B}} \in F \Leftrightarrow \llbracket [\tau]_F = [\sigma]_F \rrbracket_{\mathsf{B}/F} = 1_{\mathsf{B}/F}$$

□

Example 2.3.24. We now introduce an example of extensional model inspired by functional analysis. Let $\mathcal{L}^\infty(\mathbb{R})$ be the set of all measurable functions with domain $\mathbb{R}$. Let Meas be the boolean algebra of measurable subsets of $\mathbb{R}$, and Null be the ideal on Meas given by the null measure subsets of the real line. We shall denote with MALG the quotient boolean algebra $\mathsf{Meas}/\mathsf{Null}$. The family MALG with the operation of intersection, union, set complement, and with set inclusion, is a complete boolean algebra (for more details on measure algebras see [5, Chapter 31]). We define a structure of MALG-valued model on $\mathcal{L}^\infty(\mathbb{R})$ in the language $\{\leq, +, *\}$. The definition of the boolean interpretations are the following (identifying any $A \in \mathsf{Meas}$ with its equivalence class in MALG):

$$[\![f = g]\!] = \{x \in \mathbb{R} : f(x) = g(x)\}$$

$$[\![f \leq g]\!] = \{x \in \mathbb{R} : f(x) \leq g(x)\}$$

$$[\![f + g = h]\!] = \{x \in \mathbb{R} : f(x) + g(x) = h(x)\}$$

$$[\![f * g = h]\!] = \{x \in \mathbb{R} : f(x) * g(x) = h(x)\}$$

It is immediate to check that $\mathcal{L}^\infty(\mathbb{R})$ is an MALG-valued model with these interpretations. Let $F = \{\mathbb{R}\} = \{1_\mathsf{M}\}$ be the trivial filter on MALG, and consider the quotient M/F. The boolean algebras MALG and MALG/F are clearly isomorphic. Hence MALG/F is complete, and $\langle \mathsf{MALG}/F, \mathcal{L}^\infty(\mathbb{R})/F\rangle$ is a boolean couple with $\mathcal{L}^\infty(\mathbb{R})/F$ an extensional $\mathsf{MALG} = \mathsf{MALG}/F$-model. In $\mathcal{L}^\infty(\mathbb{R})/F$ it holds that:

$$[f]_F = [g]_F \Leftrightarrow [\![f = g]\!] \in F \Leftrightarrow [\![f = g]\!] = \mathbb{R}$$

This means that we are identifying all functions that differ only on a null measure set. $\mathcal{L}^\infty(\mathbb{R})/F$ is what is usually denoted with $L^\infty(\mathbb{R})$.

Repeating verbatim the procedure above for any B-valued model $\mathcal{M}$ gives an extensional model $\mathcal{M}/F$ with $F = \{1_\mathsf{B}\}$. If $\mathcal{M}$ is already extensional, then $\mathcal{M}/F = \mathcal{M}$.

The following property is fundamental when considering the quotient of a boolean valued model by a ultrafilter.

Definition 2.3.25. A B-valued model $\mathcal{M}$ for the language $\mathcal{L}$ is **full** if for every $\mathcal{L}$-formula $\varphi(x, \overline{y})$ and every $\overline{\tau} \in M^{|\overline{y}|}$ there is a $\sigma \in M$ such that

$$[\![\exists x \varphi(x, \overline{\tau})]\!] = [\![\varphi(\sigma, \overline{\tau})]\!]$$

Remark 2.3.26. Consider $\mathcal{M}$ an extensional full B-valued model in the language $\mathcal{L}$ and f an n-ary function symbol in $\mathcal{L}$. The boolean interpretation of f defines a function from M^n to M. We will denote it with the same name of the boolean interpretation of f, namely $f^\mathcal{M}$:

$$f^\mathcal{M} : M^n \to M$$
$$(\tau_1, \ldots, \tau_n) \to \sigma$$

where σ is such that $[\![f(\tau_1, \ldots, \tau_n) = \sigma]\!] = 1_\mathsf{B}$ (here we use the hypothesis that $\mathcal{M}$ is full). To see this observe that whenever $\sigma_1, \sigma_2 \in M$ both satisfy the above property, then

$$1_\mathsf{B} = [\![f(\tau_1, \ldots, \tau_n) = \sigma_1]\!] \wedge [\![f(\tau_1, \ldots, \tau_n) = \sigma_2]\!] \leq [\![\sigma_1 = \sigma_2]\!]$$

therefore, since $\mathcal{M}$ is extensional, it follows $\sigma_1 = \sigma_2$. When dealing with extensional full boolean valued models, we will always refer to this function when considering the interpretation of a function symbol in the language.

Theorem 2.3.27 (Boolean Valued Models Łoś's Theorem). *Assume $\mathcal{M}$ is a full B-valued model for the language $\mathcal{L}$. Let $G \in St(\mathsf{B})$. Then $\mathcal{M}/G$ is a first order model for $\mathcal{L}$ and for every formula $\varphi(x_1, \dots, x_n)$ in $\mathcal{L}$ and $(\tau_1, \dots, \tau_n) \in M^n$:*

$$\mathcal{M}/G \models \varphi([\tau_1]_G, \dots [\tau_n]_G) \Leftrightarrow [\![\varphi(\tau_1, \dots, \tau_n)]\!] \in G$$

Proof. As Remark 2.2.12 shows, $\mathsf{B}/G = \{[0_\mathsf{B}]_G, [1_\mathsf{B}]_G\}$, so that $\langle \mathsf{B}/G, \mathcal{M}/G \rangle$ is always a boolean couple. Moreover if $\mathcal{M}$ is full, $\mathcal{M}/G$ naturally defines a first order model (which will be denoted with the same name). Given an n-ary relation symbol R, $R^{\mathcal{M}/G}$ (as defined in Definition 2.3.18) is the characteristic function of the set (which we shall denote with the same name)

$$\begin{aligned} R^{\mathcal{M}/G} &= \{([\tau_1]_G, \dots, [\tau_n]_G) \in (M/G)^n : [\![R([\tau_1]_G, \dots, [\tau_n]_G)]\!] = 1_{\mathsf{B}/G}\} \\ &= \{([\tau_1]_G, \dots, [\tau_n]_G) \in (M/G)^n : [\![R(\tau_1, \dots, \tau_n)]\!] \in G\} \end{aligned}$$

Given f a function symbol, $f^{\mathcal{M}/G}$ can be defined according to Remark 2.3.26, since $\mathcal{M}/G$ is extensional (see Proposition 2.3.23) and full (since $\mathcal{M}$ is full).

From this definition follows that

$$\mathcal{M}/G \models f([\tau_1]_G, \dots, [\tau_n]_G) = [\sigma]_G \Leftrightarrow [\![f(\tau_1, \dots, \tau_n) = \sigma]\!] \in G$$

and

$$\mathcal{M}/G \models R([\tau_1]_G, \dots, [\tau_n]_G) \Leftrightarrow [\![R(\tau_1, \dots, \tau_n)]\!] \in G$$

We proof now the second part of the theorem. Denote with $\overline{\tau} = (\tau_1, \dots, \tau_n)$ a vector of parameters in M and with $\overline{[\tau]}_G = ([\tau_1]_G, \dots, [\tau_n]_G)$ the vector of equivalence classes. With x, y we denote variables and with c, c' constants (and respectively with $\overline{x}$ and $\overline{c}$ the vectors). We will prove the Theorem just for a special class of formulae, those whose atomic subformulae are of the form

$$x = y \quad c = y \quad c = c', \tag{1}$$

$$R(\overline{x}, \overline{c}), \tag{2}$$

$$f(\overline{x}, \overline{c}) = y \quad f(\overline{x}, \overline{c}) = c. \tag{3}$$

This is sufficient remarking that an atomic formula of the form

$$f(\overline{x}) = g(\overline{y})$$

(or similarly defined using some constants as arguments for the functions) is logically equivalent to the formula

$$\exists z (f(\overline{x}) = z \wedge g(\overline{y}) = z).$$

On the other hand an atomic formula with nested function symbols such as $f(g(\overline{x})) = y$ is logically equivalent to the formula $\exists z (g(\overline{x}) = z \wedge f(z) = y)$. Thus by an easy induction we can prove that each formula is logically equivalent to one in which all the atomic subformale are of the form (1), (2) or (3). Thus it suffices to prove the theorem for this class of formulae:

Atomic formulae: Consider atomic formulae of the form

$$x = y \quad c = y \quad c = c', \tag{1}$$

$$R(\overline{x}, \overline{c}), \tag{2}$$

$$f(\overline{x}, \overline{c}) = y \quad f(\overline{x}, \overline{c}) = c. \tag{3}$$

In this case the theorem easily follows from the preceding observations.

Negation: By induction, suppose the theorem holds for $\varphi(x_1, \ldots, x_n)$, then

$$\mathcal{M}/G \models \neg\varphi(\overline{[\tau]_G}) \Leftrightarrow \mathcal{M}/G \not\models \varphi(\overline{[\tau]_G}) \Leftrightarrow [\![\varphi(\overline{\tau})]\!] \notin G \Leftrightarrow [\![\neg\varphi(\overline{\tau})]\!] \in G$$

where last equivalence holds since $[\![\neg\varphi(\overline{\tau})]\!] = \neg [\![\varphi(\overline{\tau})]\!]$ and G is a ultrafilter.

Conjunction: Suppose the Theorem holds for $\psi(x_1, \ldots, x_n)$ and for $\theta(x_1, \ldots, x_n)$. Let

$$\varphi(x_1, \ldots, x_n) \equiv \psi_1(x_1, \ldots, x_n) \wedge \theta(x_1, \ldots, x_n)$$

Then we have

$$\begin{aligned} \mathcal{M}/G \models \varphi(\overline{[\tau]_G}) &\Leftrightarrow \mathcal{M}/G \models \psi(\overline{[\tau]_G}) \text{ and } \mathcal{M}/G \models \theta(\overline{[\tau]_G}) \\ &\Leftrightarrow [\![\psi(\overline{\tau})]\!] , [\![\theta(\overline{\tau})]\!] \in G \\ &\Leftrightarrow [\![\psi(\overline{\tau}) \wedge \theta(\overline{\tau})]\!] = [\![\psi(\overline{\tau})]\!] \wedge [\![\theta(\overline{\tau})]\!] \in G. \end{aligned}$$

Where the last equivalence holds since G is a filter.

Existential: Let $\varphi(x_1, \ldots, x_n) = \exists y \psi(x_1, \ldots, x_n, y)$.

$$\begin{aligned} \mathcal{M}/G \models \varphi(\overline{[\tau]_G}) &\Leftrightarrow \mathcal{M}/G \models \psi(\overline{[\tau]_G}, [\sigma]_G) \text{ for some } \sigma \in \mathcal{M} \\ &\Leftrightarrow [\![\psi(\overline{\tau}, \sigma)]\!] \in G \text{ for some } \sigma \in \mathcal{M} \\ &\Leftrightarrow [\![\exists y \psi(\overline{\tau}, y)]\!] \in G. \end{aligned}$$

where in the last equivalence $\Rightarrow$ is always true, while the opposite direction holds since $\mathcal{M}$ is full.

□

Remark 2.3.28. Whenever $\mathcal{L}$ is a relational language (i.e. with no function symbols), in order to have a first order model $\mathcal{M}/G$ when G is a ultrafilter in B, it is not necessary for $\mathcal{M}$ to be full. Nevertheless, without this request the previous theorem will not generally hold, as the following example will show.

Example 2.3.29. Consider $\mathsf{B} = \mathsf{RO}(\mathbb{R})$ and $\mathcal{C}^\omega(\mathbb{R})$ the space of analytic functions from $\mathbb{R}$ to $\mathbb{R}$, we will use this structure to produce a counterexample to the above Theorem. Let $\mathcal{L} = \{<, C\}$ be a relational language, where $<$ is binary and C is unary. The boolean value of the relations is defined as follows:

$$[\![f = g]\!] = \overset{\circ}{\overline{\{x \in \mathbb{R} : f(x) = g(x)\}}}$$

$$[\![f < g]\!] = \overset{\circ}{\overline{\{x \in \mathbb{R} : f(x) < g(x)\}}}$$

$$[\![C(f)]\!] = \{x \in \mathbb{R} : \exists I \text{ open interval such that } x \in I \text{ and } f \restriction_I \text{ is constant}\}$$

The set $[\![C(f)]\!]$ is always open, and it is regular for analytic functions (it is regular for continuous functions as well). Assume f is analytic on $\mathbb{R}$ such that it admits a non-empty interval on which it is constant. Then f must be constant everywhere (see for example [10, Theorem 8.5,Chapter 8]). Hence the only possibilities are $[\![C(f)]\!] = \emptyset$ and $[\![C(f)]\!] = \mathbb{R}$ for all functions $f \in \mathcal{C}^\omega(\mathbb{R})$.

With these definitions $\mathcal{C}^\omega(\mathbb{R})$ is a B-valued extension of the boolean couple $\langle\{0,1\}, \mathbb{R}\rangle$ ($\mathbb{R}$ is identified with the constant functions $c_r(x) = r$). Fix some $f \in \mathcal{C}^\omega(\mathbb{R})$ and consider the formula $\exists x(f < x \wedge C(x))$, its boolean value can be calculated as follows:

$$[\![\exists x(f < x \wedge C(x))]\!] = \bigvee_{g \in \mathcal{C}^\omega} [\![f < g \wedge C(g)]\!] \geq \bigvee_{r \in \mathbb{R}} [\![f < c_r \wedge C(c_r)]\!] .$$

Since $C(c_r) = \mathbb{R}$, it follows that

$$[\![f < c_r \wedge C(c_r)]\!] = [\![f < c_r]\!] \wedge [\![C(c_r)]\!] = [\![f < c_r]\!] \, ;$$

therefore

$$\bigvee_{r\in\mathbb{R}} [\![f < c_r \wedge C(c_r)]\!] = \bigvee_{r\in\mathbb{R}} [\![f < c_r]\!] \, .$$

Consider now $f \restriction_{(n,n+1)}$ with $n \in \mathbb{Z}$, and set $m_n = \sup_{x\in(n,n+1)}(f(x)) + 1$. It follows that:

$$\bigvee_{n\in\mathbb{Z}} [\![f < c_{m_n}]\!] \geq \bigvee_{n\in\mathbb{Z}} (n, n+1) = \mathbb{R}.$$

Hence $[\![\exists x(f < x \wedge C(x))]\!]$ is a valid formula in the B-valued model considered.

Now pick $G \in St(\mathsf{B})$ extending $H = \{(n, +\infty) : n \in \mathbb{Z}\}$ (such ultrafilter can be found since H satisfies the finite intersection property, hence it generates a filter), and let $\mathcal{N}$ be the quotient $\mathcal{C}^\omega(\mathbb{R})/G$. We will show that

$$\mathcal{N} \models [id_{\mathbb{R}}]_G > y$$

for each $y \in \mathcal{N}$ such that $\mathcal{N} \models C(y)$. If $C([g]_G)$ holds in $\mathcal{N}$, there is $r \in \mathbb{R}$ such that $g = c_r$ on $\mathbb{R}$ (since $\mathcal{N} \models C([g]_G)$ if and only if $[\![C(g)]\!] = \mathbb{R}$). On the other hand we have that

$$[\![id_{\mathbb{R}} > c_r]\!] = (r, +\infty)$$

and this interval is in G since G contains $(\lfloor r \rfloor + 1, +\infty) \in H$. This means that

$$\mathcal{N} \models [id_{\mathbb{R}}] > [g]_G$$

for all g such that $[\![C(g)]\!] \in G$.

In conclusion, even if $\exists x(id_{\mathbb{R}} < x \wedge C(x))$ is valid in $\mathcal{C}^\omega(\mathbb{R})$, this formula is not true in $\mathcal{N}$.

We conclude this chapter with the following Lemma which will be needed later:

Lemma 2.3.30. *Let $\mathcal{M}$ be a full B-valued model in the language $\mathcal{L}$ and φ an $\mathcal{L}$-formula with parameters in $\mathcal{M}$. For any $b \in \mathsf{B}$ the following are equivalent:*

1. *$b \leq [\![\varphi]\!]$,*

2. *there is D a dense subset of $\mathcal{O}_b$ such that for each $G \in D$ $\mathcal{M}/G \models \varphi$ holds.*

Proof. Assume $b \leq [\![\varphi]\!]$, the relevant implication immediately follows from Theorem 2.3.27 with $D = \mathcal{O}_b$.

For the vice versa suppose $b \not\leq [\![\varphi]\!]$. This implies that $b \wedge \neg [\![\varphi]\!] \neq 0_{\mathsf{B}}$. From $0_{\mathsf{B}} < b \wedge \neg [\![\varphi]\!] \leq b$ it follows that $\mathcal{O}_{b\wedge\neg[\![\varphi]\!]}$ is a non-empty open subset of $\mathcal{O}_b$. By Theorem 2.3.27 we have that $\mathcal{M}/G \models \neg\varphi$ for any $G \in \mathcal{O}_{b\wedge\neg[\![\varphi]\!]}$. In particular the set of G such that $\mathcal{M}/G \models \varphi$ is disjoint from $\mathcal{O}_{b\wedge\neg[\![\varphi]\!]}$ and thus cannot be a dense subset of $\mathcal{O}_b$. □

Chapter 3

Set Theory and Forcing

We present a compact introduction to set theory and the basic properties of the forcing method. We assume the reader is already acquainted with the basic properties of set theory as formalized in the first order axiomatization ZFC. Sections $1, 2$ and 3 will be a brief summary of classic results, references for those parts are [6], [8], [1], [3]. We will assume all over the chapter to work in a language $\mathcal{L}$ with just one binary relation symbol $\in$.

3.1 Basics

Let V be the universe of sets, the standard model for set theory, i. e. the collection of all sets.

Definition 3.1.1. The **cumulative hierarchy of sets** is defined by recursion on the ordinals:

$$V_0 = \emptyset$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$

$$V_\beta = \bigcup_{\alpha<\beta} V_\alpha \text{ if } \beta \text{ is a limit ordinal}$$

The Axiom of Regularity guarantees that every set belongs to some V_α (see [6, Lemma 6.3]), i. e.:

$$V = \bigcup_{\alpha\in\mathbf{ON}} V_\alpha$$

We define the **rank** of a set x as

$$\rho(x) = \text{the least } \alpha \text{ such that } x \in V_{\alpha+1}$$

Remark 3.1.2. The universe V is not a set, thus we can not speak directly of it in ZFC, where only one type of objects are defined, namely sets. However, when dealing with ZFC, it is sometimes useful to consider *proper classes*. In general, a class C is the extension in V of a certain first order formula $C(x)$ in one or more free variables:

$$C = \{x : C(x)\}$$

Examples of classes we will consider are

$$\mathbf{ON} = \{x : x \text{ is an ordinal}\}$$

$$V = \{x : x = x\}$$
$$\in = \{(x, y) : x \in y\}$$

Classes are generally used in informal contexts as abbreviations. Given a class M and a binary relation E on it, we generalize the definition

$$\langle M, E\rangle \models \varphi$$

where φ is a $\{\in\}$-formula. In the definition we gave of first order model we required the domain of the structure to be a set. With this assumption the relation $M \models \varphi$ between sets $M \in V$ and (Gödel numbers for) formulae φ turns out to be a definable class in ZFC. However this is not the case if we try to extend the relation $\models$ so to include classes as the first argument of its domain. This issue can be solved showing that for each fixed $\in$-formula φ and each pair of classes M and $E \subseteq M^2$ the statement $\langle M, E\rangle \models \varphi$ is a definable class relation $\varphi^{M,E}$ (in case $E = \in$ we will simply write φ^M). Recall that $\varphi^{M,E}$ is the formula obtained from φ substituting $\in$ with E and all quantifiers $\exists x$ and $\forall x$ with $\exists x \in M$ (i.e. $\exists x M(x) \wedge \ldots$) and $\forall x \in M$ (i.e. $\forall x M(x) \rightarrow \ldots$) respectively. Keeping this in mind we will informally generalize most of the notions defined for first order models. More about this in [8, Chapter 1, Section 9] and [6, Chapter 1, Chapter 12].

Chapters 6 and 12 of [6] contain all the details and the proofs we will omit in the remaining part of this section.

Definition 3.1.3. A class A is **transitive** if $x \in A$ implies $x \subset A$.

Definition 3.1.4. Consider a class P and a binary relation E on it. For each $x \in P$ the **extension** of x is the class

$$\mathrm{ext}_E(x) = \{z \in P : zEx\}$$

The relation E on P is:

- **well-founded** if every non-empty set $A \subset P$ has an E-minimal element (i.e an $x \in A$ such that zEx holds for no $z \in A$);
- **set-like** if $\mathrm{ext}_E(x)$ is a set for every $x \in P$;
- **extensional** if given $x, y \in P$:

$$\mathrm{ext}_E(x) \neq \mathrm{ext}_E(y) \Rightarrow x \neq y$$

  or equivalently, if

$$\langle P, E\rangle \models \text{Axiom of Extensionality}$$

Theorem 3.1.5 (Mostowski's Collapsing Theorem)**.** *Assume E is a well-founded, set-like and extensional relation on a class P. Then there is a transitive class M and an isomorphism π between $\langle P, E\rangle$ and $\langle M, \in\rangle$. The transitive class M and the isomorphism π (the Mostowski's Collapse of P) are unique. Moreover, if $E = \in$ and $T \subset P$ is transitive, then $\pi(x) = x$ for each $x \in T$.*

Proof. For a proof see, [6, Theorem 6.15]. □

Definition 3.1.6. Let $\langle M, \in\rangle$ and $\langle N, \in\rangle$ be two $\{\in\}$-structures (possibly classes) such that $M \subseteq N$. We say that a $\{\in\}$-formula $\varphi(x_1, \ldots, x_n)$ is **upward absolute** for M, N if for every $(a_1, \ldots, a_n) \in M^n$:

$$\langle M, \in\rangle \models \varphi(a_1, \ldots, a_n) \Rightarrow \langle N, \in\rangle \models \varphi(a_1, \ldots, a_n)$$

The formula $\varphi(x_1, \dots, x_n)$ is **downward absolute** for M, N if for every $(a_1, \dots, a_n) \in M^n$:

$$\langle M, \in \rangle \models \varphi(a_1, \dots, a_n) \Leftarrow \langle N, \in \rangle \models \varphi(a_1, \dots, a_n)$$

A formula is **absolute** for M, N if it is both upward and downward absolute for M, N. A formula is (upward, downward) **absolute** for M if it is (upward, downward) absolute for M, V.

Theorem 3.1.7 (Reflection Principle)**.** *Consider a $\{\in\}$-formula $\varphi(x_1, \dots, x_n)$ and a set A. There exists an ordinal α such that:*

- *$A \in V_\alpha$;*
- *φ is absolute for V_α.*

In this case we say that V_α reflects φ.

Proof. For a proof see [6, Theorem 12.14]. □

3.2 Boolean Valued Models for Set Theory

The aim of this section is to formalize the boolean generalization of the universe V. V can be built step by step iterating the operation of power set. Given a set X, $\mathcal{P}(X)$ can be identified with the set of characteristic functions of its element, i. e. as the functions from X to the boolean algebra $\{0, 1\}$. Roughly, the generalization of the power-set operation on a given set X we are looking for is obtained considering characteristic functions with domain X and range in an arbitrary complete boolean algebra instead of $\{0, 1\}$. See Chapter 14 of [6] for a complete analysis of this topic.

Definition 3.2.1. Consider a complete boolean algebra B. The class V^{B} is defined by induction on the ordinals as follows:

$$V_0^{\mathsf{B}} = \emptyset$$

$$V_{\alpha+1}^{\mathsf{B}} = \{f : X \to \mathsf{B} \mid X \subset V_\alpha^{\mathsf{B}}\}$$

$$V_\beta^{\mathsf{B}} = \bigcup_{\alpha<\beta} V_\alpha^{\mathsf{B}} \text{ if } \beta \text{ is a limit ordinal}$$

$$V^{\mathsf{B}} = \bigcup_{\alpha \in \mathbf{ON}} V_\alpha^{\mathsf{B}}$$

The **boolean rank** of a B-name τ of V^{B} is defined as:

$$\rho_{\mathsf{B}}(\tau) = \text{ the least } \alpha \text{ such that } \tau \in V_{\alpha+1}^{\mathsf{B}}$$

At first, it is easier to define the structure of B-valued model on V^{B} for $\mathcal{L} = \{\in, \subseteq\}$.

Definition 3.2.2. The interpretations of $\in, \subseteq$ and $=$ are defined by induction on the pairs $(\rho_{\mathsf{B}}(\tau), \rho_{\mathsf{B}}(\sigma))$:

- $[\![\tau \in \sigma]\!] = \bigvee_{\chi \in \mathrm{dom}(\sigma)} ([\![\tau = \chi]\!] \wedge \sigma(\chi))$;
- $[\![\tau \subseteq \sigma]\!] = \bigwedge_{\chi \in \mathrm{dom}(\tau)} (\tau(\chi) \to [\![\chi \in \sigma]\!])$;
- $[\![\tau = \sigma]\!] = [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!]$.

See [6, Lemmas 14.15 and 14.16] for a proof that the structure V^{B} with the maps

$$(\tau, \sigma) \mapsto [\![\tau \; R \; \sigma]\!]$$

for R in $\{=, \in, \subseteq\}$ as defined above is a B-valued model for the language $\{\in, \subseteq\}$.

Remark 3.2.3. Definitions 3.2.1 and 3.2.2 work starting from any *transitive* first order model $\mathcal{M} = \langle M, \in \rangle$ of ZFC (the request for the model to be transitive is redundant, but we are mainly interested in transitive well-founded models in order to be able to build *transitive* generic extensions of such models). In this case, in order to have a boolean couple $\langle \mathsf{B}, \mathcal{M}^{\mathsf{B}} \rangle$ we need $\mathsf{B} \in \mathcal{M}$ and

$$\mathcal{M} \models \mathsf{B} \text{ is a complete boolean algebra}$$

so that all subsets of B which belong to $\mathcal{M}$ admit a supremum in $\mathcal{M}$ itself. In this section we will just analyze the boolean valued model V^{B}, although our analysis can be easily generalized so that it applies to any first order model of ZFC.

Remark 3.2.4. Every element x of V has a canonical name $\check{x}$ in V^{B} which we define by induction on the rank:

- $\check{\emptyset} = \emptyset$;
- for $x \in V$, $\check{x}$ is the function whose domain is $\{\check{y} : y \in x\}$ and such that $\check{x}(\check{y}) = 1_{\mathsf{B}}$ for every $y \in x$.

This map, when restricted to sets, is an embedding of B-valued models. In fact, given $x, y \in V$:

$$[\![\check{x} \in \check{y}]\!] = \begin{cases} 1_{\mathsf{B}} \text{ if } x \in y \\ 0_{\mathsf{B}} \text{ if } x \notin y \end{cases} \quad [\![\check{x} \subseteq \check{y}]\!] = \begin{cases} 1_{\mathsf{B}} \text{ if } x \subseteq y \\ 0_{\mathsf{B}} \text{ if } x \not\subseteq y \end{cases} \quad [\![\check{x} = \check{y}]\!] = \begin{cases} 1_{\mathsf{B}} \text{ if } x = y \\ 0_{\mathsf{B}} \text{ if } x \neq y \end{cases}$$

For a proof see [6, Lemma 14.21]

Theorem 3.2.5. *Let* B *be a complete boolean algebra, then*

- V^{B} *is full;*
- *ZFC is valid in* V^{B}.

Proof. See [6, Lemma 14.17, Lemma 14.19, Theorem 14.24]. □

Remark 3.2.6. In ZFC the binary relation $x \subseteq y$ is introduced as an abbreviation for the formula in the language $\{\in\}$:

$$y \subseteq x \equiv \forall z(z \in y \to z \in x).$$

In order to show that V^{B} is a B-valued model in $\mathcal{L} = \{\in\}$ we need to show that:

$$[\![y \subseteq x]\!] = [\![\forall z(z \in y \to z \in x)]\!]$$

A proof of this is given in [1, Corollary 1.18].

Lastly we define a B-name which will be useful later.

Definition 3.2.7. $\dot{G}_{\mathsf{B}} \in V^{\mathsf{B}}$ is defined as:

$$\operatorname{dom}(\dot{G}_{\mathsf{B}}) = \{\check{b} : b \in \mathsf{B}\} \qquad \dot{G}_{\mathsf{B}}(\check{b}) = b$$

$\dot{G}_{\mathsf{B}}$ is the **canonical name for a generic ultrafilter**. If no confusion on the algebra B considered can arise we shall omit the subscript B and denote $\dot{G}_{\mathsf{B}}$ by $\dot{G}$.

3.3 Generic extensions

From now on we will be interested in a structure $\langle M, \in\rangle$ (we will refer to it as M), which is a transitive first order model of ZFC (possibly a class), and to a $\mathsf{B} \in M$ boolean algebra which M models to be complete.

Definition 3.3.1. Given a partial order $(P, <)$ we say that:

- $D \subseteq P$ is **dense** if for every $p \in P$ there is a $d \in D$ such that $d \leq p$;
- $D \subseteq P$ is **dense below** $q \in P$ if for every $p \leq q$ there is $d \in D$ such that $d \leq p$;
- $D \subseteq P$ is **open** if $p \in D$ and $q \leq p$ implies $q \in D$;
- $D \subseteq P$ is **predense** if its downward closure

$$\downarrow D = \{b \in \mathsf{B} : \exists d \in D(b \leq d)\}$$

  is dense.

We shall identify a boolean algebra B with the partial order $\mathsf{B}^+ = \mathsf{B} \setminus \{0_\mathsf{B}\}$ and we generalize the notion of filter to partially ordered sets as follows:

Definition 3.3.2. Given a partial order $(P, <)$ a subset $F \subseteq P$ is a **filter** if:

- F is non-empty;
- if $p \leq q$ and $p \in F$ then $q \in F$;
- if $p, q \in F$ then there exists $r \in F$ such that $r \leq p$ and $r \leq q$.

Definition 3.3.3. Given a partial order $(P, <)$, $G \subseteq P$ is **generic** over a class C (or C**-generic**) if:

- G is a filter;
- if $D \subseteq P$ is dense and $D \in C$, then $G \cap D \neq \emptyset$.

The following lemmas will be useful later. We will not omit the proof of the first one, as we will need the map defined in it in the next chapter.

Lemma 3.3.4. *Let B be a complete boolean algebra. There is a bijection between $\mathcal{D}$, the family of open dense subsets of B^+, and $\mathcal{E}$, the family of open dense subsets of $St(\mathsf{B})$.*

$$\Psi : \mathcal{E} \to \mathcal{D}$$
$$E \mapsto \{a \in \mathsf{B}^+ : \mathcal{O}_a \subseteq E\}$$

Proof. Well-defined: Assume E is open and dense in $St(\mathsf{B})$. Consider $b \in \mathsf{B}^+$, it follows that

$$\mathcal{O}_b \cap E \neq \emptyset$$

We can therefore find $a \in \mathsf{B}^+$ such that

$$\mathcal{O}_a \subseteq \mathcal{O}_b \cap E$$

Hence $a \leq b$ and $a \in \Psi(E)$ hold. The set $\Psi(E)$ is open since, given $a, b \in \mathsf{B}^+$

$$a \leq b \Rightarrow \mathcal{O}_a \subseteq \mathcal{O}_b$$

Injective: Let $E \neq E'$ be elements of $\mathcal{E}$. We can assume there exists $G \in E \setminus E'$. E is open, we can find therefore $a \in \mathsf{B}^+$ such that

$$G \in \mathcal{O}_a \subseteq E$$

On the other hand we have:

$$\mathcal{O}_a \nsubseteq E'$$

because $G \in \mathcal{O}_a$. In conclusion, $a \in \Psi(E) \setminus \Psi(E')$, hence the map is injective.

Surjective: Let D be a dense open subset of B^+, we will show that

$$D = \Psi\left(\bigcup_{a \in D} \mathcal{O}_a\right)$$

Consider $b \in \mathsf{B}^+$. There exists $a \in D$ such that $a \leq b$, so that

$$\emptyset \neq \mathcal{O}_a \subseteq \mathcal{O}_b$$

It follows that the intersection of $\bigcup_{a \in D} \mathcal{O}_a$ and $\mathcal{O}_b$ is non-empty. The thesis follows since $b \in \mathsf{B}^+$ was arbitrary.

□

Lemma 3.3.5. *Assume $(P, <)$ is a partial order and $\mathcal{D}$ is a countable family of dense subsets of P. Then there exists a $\mathcal{D}$-generic filter on P. Moreover, for every $p \in P$ there is a $\mathcal{D}$-generic filter on P containing p.*

Proof. For a proof see, for example, [6, Lemma 14.4] □

Definition 3.3.6. Let G be an M-generic ultrafilter over B. We define by induction on ρ_{B} the **interpretation** by G of the B-names as:

- $\emptyset^G = \emptyset$;
- $\tau^G = \{\sigma^G : \tau(\sigma) \in G\}$;

The **generic extension** of M by G is the class:

$$M[G] = \{\tau^G : \tau \in M^{\mathsf{B}}\}$$

Remark 3.3.7. The previous definition can be given for any $G \in St(\mathsf{B})$, and does not require $G \in M$. $M[G]$ (which will not be called generic extension if G is not generic) is always transitive. We will always assume G to be M-generic for B unless otherwise stated.

Lemma 3.3.8. *Assume G is an M-generic filter in B. Then for all $\tau, \sigma \in M^{\mathsf{B}}$:*

$\tau^G \in \sigma^G \Leftrightarrow [\![\tau \in \sigma]\!] \in G$;

$\tau^G \subseteq \sigma^G \Leftrightarrow [\![\tau \subseteq \sigma]\!] \in G$;

$\tau^G = \sigma^G \Leftrightarrow [\![\tau = \sigma]\!] \in G$.

Proof. For a proof see [6, Lemma 14.28] □

Now we want to show which is the relationship between $M[G]$ and M^{B}/G. Since M^{B} is a class, given $\tau \in M^{\mathsf{B}}$, $[\tau]_G$ might be a class itself. We can bypass this problem with the Scott's trick, defining $[\tau]_G$ as the set of $\sigma \in M^{\mathsf{B}}$ of minimal rank ρ such that $\sigma \equiv_G \tau$.

Theorem 3.3.9. *Let M be a transitive model of ZFC, B a boolean algebra which M models to be complete, and G an M-generic ultrafilter in B. The map*

$$\pi_G^M : M^{\mathsf{B}}/G \to M[G]$$
$$[\tau]_G \mapsto \tau^G$$

is an isomorphism between the two models, more precisely it is the Mostowsky's Collapse of the extensional well founded model $(M^{\mathsf{B}}/G, \in^{M^{\mathsf{B}}/G})$.

Proof. From Lemma 3.3.8 and Theorem 2.3.27 ($\langle M^{\mathsf{B}}, \in^{\mathsf{B}}\rangle$ is a full boolean couple due to Theorem 3.2.5) it follows that π_G^M is an $\mathcal{L}$-embedding. The map is surjective, since for any $x \in M[G]$, by definition $x = \tau^G$ for some $\tau \in M^{\mathsf{B}}$, hence

$$x = \pi_G^M([\tau]_G)$$

□

Theorem 3.3.10 (Cohen's Forcing Theorem)**.** *Assume M is a transitive first order model of ZFC, B a boolean algebra which M models to be complete and G an M-generic filter in B. Let $\varphi(x_1, \dots, x_n)$ be an $\mathcal{L}$-formula, then for every $\tau_1, \dots, \tau_n \in M^{\mathsf{B}}$:*

$$M[G] \models \varphi(\tau_1^G, \dots, \tau_n^G) \Leftrightarrow [\![\varphi(\tau_1, \dots, \tau_n)]\!] \in G$$

Proof. The map π_G^M defined in Theorem 3.3.9 is an isomorphism between M^{B}/G and $M[G]$, therefore

$$M[G] \models \varphi(\tau_1^G, \dots, \tau_n^G) \Leftrightarrow M^{\mathsf{B}}/G \models \varphi([\tau_1]_G, \dots, [\tau_n]_G)$$

M^{B} is full, therefore we can conclude the proof using Theorem 2.3.27. □

The last theorem we present in this section tells us that $M[G]$ is the smallest transitive first order model N such that $M \subset N$ and $G \in N$.

Theorem 3.3.11 (Generic Model Theorem)**.** *Let M be a transitive model of ZFC, B a boolean algebra which M models to be complete and G an M-generic ultrafilter in B. Then $\langle M[G], \in\rangle$ is a first order transitive model such that:*

- $M[G] \models$ *ZFC;*
- $M \subseteq M[G]$ *and* $G \in M[G]$*;*
- $\mathbf{ON}^M = \mathbf{ON}^{M[G]}$*;*
- *if N is a transitive first order model of ZFC such that $M \subseteq N$ and $G \in N$ then $M[G] \subseteq N$.*

Proof. To prove the first item we observe that by Theorem 3.2.5 all axioms of ZFC have boolean value equal to 1_{B}, which clearly belongs to any ultrafilter in $St(\mathsf{B})$. The thesis follows therefore from Theorem 3.3.10.

The second item holds since it can be shown that or each $x \in M$

$$\check{x}^G = x$$

and also that

$$\dot{G}^G = G$$

A detailed proof of this and of the other two items can be found in [6, Lemma 14.31].

□

3.4 More on Cohen's Forcing Theorem

The hypotheses of Cohen's Forcing Theorem are too strict, and on a first sight they make the theorem useless: on the one hand we assumed the existence of a transitive first order model of ZFC M, which is unprovable within ZFC because of Gödel Incompleteness Theorem. On the other hand, we assumed the existence of an M-generic ultrafilter. Unfortunately, the existence of such an ultrafilter can not generally be proved, unless M satisfies certain strict requirements, for example M-generic ultrafilters extists whenever M is a countable model (see Lemma 3.3.5). Therefore, one may ask how we can use Cohen's Forcing Theorem in order to show that a certain formula φ is consistent with ZFC. In this section we provide a method to bypass the problems exposed above. The first observation is that, when proving that φ is consistent with ZFC by means of Cohen's Forcing Theorem, by a compactness argument, we only need to prove that φ is consistent with an arbitrary finite set of ZFC-axioms. We can also argue that in order to show that in an M-generic extension $M[G]$ a finite set of ZFC-axioms hold, we just need M to be a model of a (possibly different) finite set of ZFC-axioms. Finally, we can use the Reflection Principle 3.1.7, the Downward Löwenheim−Skolem Theorem 2.1.20, and the Mostowski's Collapse 3.1.5 to find countable transitive models M of any given finite set of ZFC-axioms. Such models M will be the ones to which we can apply Cohen's Forcing Theorem.

We will need to work among different classes which may or may not be models of ZFC, hence we introduce some definitions regarding how a formula behaves when we consider it in different models

The behaviour of formulae between different structures is mainly determined by their logical *complexity*. A precise formalization of this concept can be given by means of the *Lévy Hierarchy* (see [6, Chapter 13]). For our purposes the following definitions are sufficient.

Definition 3.4.1. A $\{\in\} \cup \{a_1, \dots, a_n\}$-formula $\varphi(x_1, \dots, x_n, a_1, \dots, a_n)$ (with $a_1, \dots, a_n$ additional constant symbols) is Δ_0 if

- it has no quantifiers;
- it is of the form $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$ and both φ and ψ are Δ_0;
- it is of the form
$$(\exists x \in y)\psi(x, x_1, \dots, x_n, a_1, \dots, a_n)$$
or
$$(\forall x \in y)\psi(x, x_1, \dots, x_n, a_1, \dots, a_n)$$
and ψ is[1] Δ_0.

We will loosely say that φ is Δ_0 over a theory T if it is provable from T that φ is equivalent to a formula of the form above. A formula φ is Σ_1 (Π_1) in T if it is provable to be equivalent in T to one of the form $\exists x\psi(x)$ ($\forall x\psi(x)$) where ψ is a Δ_0-formula. A formula φ is Δ_1 in T if it is both Π_1 and Σ_1 in T. If not otherwise stated we will assume $T = \mathsf{ZFC}$ and we will just say that φ is Δ_0 (or $\Sigma_1, \Pi_1, \Delta_1$).

Proposition 3.4.2. *Given a transitive $\{\in\}$-structure M, every Δ_0-formula is absolute for M.*

Proof. For a proof see [6, Lemma 12.9] □

Remark 3.4.3. An easy consequence of this is that, when considering transitive classes, Σ_1-formulae are upward absolute, Π_1-formulae are downward absolute, and Δ_1 formulae are absolute.

[1] Recall that $(\exists x \in y)\psi(x)$ is a shorthand for $\exists x(x \in y \wedge \psi(x))$ and similarly $(\forall x \in y)\psi(x)$ is a shorthand for $\forall x(x \in y \to \psi(x))$.

Consider a complete boolean algebra B. We need to generalize the definition of V^{B} to transitive classes M which are not necessarily models of ZFC and are such that $\mathsf{B} \in M$. We will perform this generalization studying the formula which defines V^{B}, although this is not the quickest way to do it. Notice that the properties which define a boolean algebra can always be expressed with quantifiers bounded in B. Hence "B is a boolean algebra" and its operations and relations are absolute for transitive classes. Moreover, $\mathsf{B} \in M$ and M is transitive imply $\mathsf{B} \subset M$, so that if B is complete in V, it has to be complete in M as well (every supremum which is in V is also in M). Observe that the vice versa does not hold. In fact, if B is a complete boolean algebra in a transitive class M, and N is a transitive class containing M, there might exists some subset of B belonging to N but not to M, thus that set may not have a supremum.

Definition 3.4.4. Given a set X, the **transitive closure** of X is defined as:

$$TC(X) = \bigcup \left\{ \left(\bigcup X\right)^n : n \in \omega \right\}$$

i.e. the smallest transitive set containing X.

Definition 3.4.5. Given a complete boolean algebra B and $f : V \to \mathsf{B}$ a partial function (i. e. whose domain is contained in V) which is an element of V, we define:

$$\overset{*}{\bigcup} f = \bigcup \{\mathrm{dom}(x) : \exists b (x, b) \in f \wedge x \text{ is a partial function with range in } \mathsf{B}\}$$

and we define

$$TC_{\mathsf{B}}(f) = \bigcup \left\{ \left(\overset{*}{\bigcup} f\right)^n : n \in \omega \right\}$$

where

$$\left(\overset{*}{\bigcup} f\right)^0 = \mathrm{dom}(f)$$

$$\left(\overset{*}{\bigcup} f\right)^1 = \overset{*}{\bigcup} f$$

$$\left(\overset{*}{\bigcup} f\right)^{n+1} = \overset{*}{\bigcup} \left(\overset{*}{\bigcup} f\right)^n \quad \text{for } n \geq 1$$

We call $TC_{\mathsf{B}}(f)$ the **boolean transitive closure** of f on B.

Lemma 3.4.6. *Let B be a complete boolean algebra, then $\tau \in V^{\mathsf{B}}$ if and only if*

$$\Phi(\tau, \mathsf{B}) \equiv \tau \textit{ is a function } \wedge Im(\tau) \subseteq \mathsf{B} \wedge \forall \sigma \in TC_{\mathsf{B}}(\tau)(\sigma \textit{ is a function } \wedge Im(\sigma) \subseteq \mathsf{B})$$

Moreover $\Phi(x, \mathsf{B})$ is Δ_1 in the parameter B.

Proof. We first check the above equivalence. Let τ be a B-name, hence it is a function with range in B. Then we proceed by induction: if $\sigma \in \mathrm{dom}(\tau)$ then by definition $\sigma \in V^{\mathsf{B}}$ hence it is a function whose rank is contained in B. Consider now $\sigma \in \left(\overset{*}{\bigcup} \tau\right)^{n+1}$. The element σ is in the domain of some $\eta \in \left(\overset{*}{\bigcup} \tau\right)^n$, which by inductive hypothesis is in V^{B}, hence σ is a B-name.

Conversely, towards a contradiction let τ be of minimum rank such that $\Phi(\tau)$ holds but $\tau \notin V^{\mathsf{B}}$. Since

$$\forall \sigma \in TC_{\mathsf{B}}(\tau)(\sigma \text{ is a function } \wedge \operatorname{Im}(\sigma) \subseteq \mathsf{B})$$

holds, every $\eta \in \operatorname{dom}(\tau)$ is a function whose image is in B. Moreover

$$\forall \sigma \in TC_{\mathsf{B}}(\eta)(\sigma \text{ is a function } \wedge \operatorname{Im}(\sigma) \subseteq \mathsf{B})$$

hence $\Phi(\eta, \mathsf{B})$ holds. Since $x \in y$ implies $\rho(x) < \rho(y)$ and for a certain $b \in \mathsf{B}$

$$\eta \in \{\eta\} \in \{\{\eta\}, \{\eta, b\}\} = (\eta, b) \in \tau$$

we have $\rho(\eta) < \rho(\tau)$. Thus $\eta \in V^{\mathsf{B}}$ and from this follows that τ is a function in B whose domain is composed by B-names, hence it is a B-name itself.

For the proof that Φ is Δ_1 we will assume most of the basic facts about the Lévy Hierarchy. For those results see [3, Lemma 2.6, Theorem 2.7, Lemma 2.8] and [6, Lemma 12.10]. This means that we only need to show that $y = TC_{\mathsf{B}}(f)$ is Δ_1 with the parameter B. In order to do this, it is enough to prove that $x \in \overset{*}{\bigcup} f$ is Δ_1, since

$$y = TC_{\mathsf{B}}(f) \leftrightarrow \forall x \in y \exists z \in \omega (x \in (\overset{*}{\bigcup} f)^z)$$

and the fact that $x \in (\overset{*}{\bigcup} f)^n$ is Δ_1 easily follows once we have showed that $x \in \overset{*}{\bigcup} f$ is Δ_1 (trivial for $n = 0$, otherwise just substitute f with $(\overset{*}{\bigcup} f)^{n-1}$). On the one hand we have

$$x \in \overset{*}{\bigcup} f \leftrightarrow \exists z \exists b \in \mathsf{B}((z, b) \in f \wedge z \text{ is a function } \wedge \operatorname{Im}(z) \subseteq \mathsf{B} \wedge x \in \operatorname{dom}(z))$$

and the right side formula is Σ_1. On the other hand

$$y \supseteq \overset{*}{\bigcup} f \leftrightarrow \forall z\, [(z \text{ is a function } \wedge \operatorname{Im}(z) \subseteq \mathsf{B} \wedge \exists b \in \mathsf{B}((z, b) \in f)) \to \operatorname{dom}(z) \subseteq y]$$

tells us that $y \supseteq \overset{*}{\bigcup} f$ is Π_1, and since

$$x \in \overset{*}{\bigcup} f \leftrightarrow \forall y (y \supseteq \overset{*}{\bigcup} f \to x \in y)$$

we have the thesis. □

Remark 3.4.7. The B-names in a transitive class M are defined as those elements τ such that $\Phi(\tau, \mathsf{B})^M$ holds. With the previous lemma we have showed that $\Phi(x, \mathsf{B})$ is absolute for transitive structures, hence if $\tau \in M$

$$\Phi(\tau, \mathsf{B})^M \Leftrightarrow \Phi(\tau, \mathsf{B})$$

so that the class of B-names in M, which we shall call M^{B}, overlaps with $V^{\mathsf{B}} \cap M$. We need to check that M^{B} inherits the structure of B-valued model.

Proposition 3.4.8. *Let M be a transitive class and B complete boolean algebra which belongs to M. Then $M^{\mathsf{B}} = V^{\mathsf{B}} \cap M$ with relations defined for $\sigma, \tau \in M^{\mathsf{B}}$ by:*

$$[\![\sigma \in \tau]\!]^{M^{\mathsf{B}}} = ([\![\sigma \in \tau]\!]^{V^{\mathsf{B}}})^M$$

$$[\![\sigma = \tau]\!]^{M^{\mathsf{B}}} = ([\![\sigma = \tau]\!]^{V^{\mathsf{B}}})^M$$

is a B*-valued model for the language* $\{\in\}$*. Moreover if* $\varphi(x_1, \ldots, x_n)$ *is a formula and* $\tau_1, \ldots, \tau_n \in M^{\mathsf{B}}$ *then:*

$$[\![\varphi(\tau_1, \ldots, \tau_n)]\!]^{M^{\mathsf{B}}} = ([\![\varphi(\tau_1, \ldots, \tau_n)]\!]^{V^{\mathsf{B}}})^M$$

Proof. For the first part of the proof it suffices to show that $[\![\sigma \in \tau]\!]^{V^{\mathsf{B}}}$ and $[\![\sigma = \tau]\!]^{V^{\mathsf{B}}}$ are absolute for transitive models M such that $\mathsf{B} \in M$ and M models B to be a complete boolean algebra. We will only prove the $\in$-clause, the one with $\subseteq$ (from which the one with $=$ follows) can be inferred similarly. The proof proceeds by induction on the pairs $(\rho_{\mathsf{B}}(\sigma), \rho_{\mathsf{B}}(\tau))$. For the empty set:

$$([\![\emptyset \in \emptyset]\!]^{V^{\mathsf{B}}})^M = (0_{\mathsf{B}})^M = 0_{\mathsf{B}} = [\![\emptyset \in \emptyset]\!]^{V^{\mathsf{B}}}.$$

Given $\sigma, \tau \in M^{\mathsf{B}}$:

$$([\![\sigma \in \tau]\!]^{V^{\mathsf{B}}})^M = \bigvee_{\eta \in \mathrm{dom}(\tau) \cap M} ([\![\eta = \sigma]\!]^{V^{\mathsf{B}}})^M \wedge \tau(\eta).$$

By the inductive hypothesis $([\![\eta = \sigma]\!]^{V^{\mathsf{B}}})^M = [\![\eta = \sigma]\!]^{V^{\mathsf{B}}}$, and since $\tau \in M$, we have $\tau \subseteq M$, so that if $\eta \in \mathrm{dom}(\tau)$ then $(\eta, b) \in M$ for some $b \in \mathsf{B}$. By the definition of ordered couple

$$(\eta, b) = \{\{\eta\}, \{\eta, b\}\}$$

and from the transitivity of M it follows that $\eta \in M$, hence $\mathrm{dom}(\tau) \subseteq M$, which leads to the thesis.

The second part of the proposition is by induction on the complexity of φ. The case for φ an atomic formula holds by definition. The proof for conjunction and negation is immediate, lastly for $\varphi \equiv \exists x \psi(x)$ we have that:

$$([\![\varphi]\!]^{V^{\mathsf{B}}})^M = \bigvee_{\tau \in V^{\mathsf{B}} \cap M} ([\![\psi(\tau)]\!]^{V^{\mathsf{B}}})^M = \bigvee_{\tau \in M^{\mathsf{B}}} [\![\psi(\tau)]\!]^{M^{\mathsf{B}}} = [\![\varphi]\!]^{M^{\mathsf{B}}}.$$

□

Remark 3.4.9. Notice that defining $[\![\sigma \in \tau]\!]^{M^{\mathsf{B}}}$ and $[\![\sigma = \tau]\!]^{M^{\mathsf{B}}}$ as in the proposition or as in Definition 3.2.2 gives the same result.

We have defined M^{B} for any transitive class M which contains B. Since ZFC does not necessarily hold in M, we do not know which properties of V^{B} (fullness, validity of ZFC, etc.) hold in M^{B}.

Definition 3.4.10. Given a Boolean algebra B, $A \subseteq \mathsf{B} \setminus \{0_{\mathsf{B}}\}$ is an **antichain** if any two distinct $x, y \in A$, are **incompatible**, i. e. $x \wedge y = 0_{\mathsf{B}}$.

The following Lemma is crucial if we are looking for a full B-valued model, it holds for every model of ZFC (see Lemma 14.18 in [6]), and it implies that V^{B} is full.

Lemma 3.4.11 (Mixing Lemma for B)**.** *Assume* B *is a complete boolean algebra in* V*. Let* A *be an antichain of* B *and for any* $a \in A$ *let* τ_a *be an element of* V^{B}*. Then there exists some* $\tau \in V^{\mathsf{B}}$ *such that* $a \leq [\![\tau = \tau_a]\!]$ *for all* $a \in A$*.*

In [6, Lemma 14.19] is shown how the Mixing Lemma for B implies that V^{B} is full. In this proof some axioms of ZFC are used (Zorn's Lemma, the axiom schema of separation and of replacement). We are not interested in a full analysis of which of them are necessary and which redundant. For our purposes consider the following:

Proposition 3.4.12. *Let M be a transitive class which is a model of the axiom schema of separation, of replacement, and of the axiom of choice. Let moreover $\mathsf{B} \in M$ a boolean algebra which M models to be complete. If:*

$$M \models \textit{Mixing Lemma}$$

then M^{B} is full.

Proof. The proof is as in [6, Lemma 14.19]. □

Observe that in order to prove the proposition above for a single formula φ, i. e. that there exists $\tau \in V^{\mathsf{B}}$ such that $[\![\exists x \varphi(x)]\!]^{M^{\mathsf{B}}} = [\![\varphi(\tau)]\!]^{M^{\mathsf{B}}}$, by compactness only a finite number of axioms of ZFC are needed.

Remark 3.4.13. So far we have defined the class M^{B} for any transitive M such that $\mathsf{B} \in M$ and M models B to be a complete boolean algebra. Given an M-generic filter G in B, we can generalize to these models also Definition 3.3.6, i. e. the generic extension $M[G]$. We do not know if M models ZFC, hence we can not infer Theorems 3.3.9 and 3.3.10 for $M[G]$. However if we want these theorems to hold for a specific formula φ in $M[G]$, only a finite number of axioms of ZFC is required to hold in M. For instance we can require that Lemma 3.3.8 holds in M, and we can ask M^{B} to be full for φ (which can be obtained through specific instances of the Mixing Lemma as described above). By compactness, given a formula φ and a transitive class M with a complete boolean algebra $\mathsf{B} \in M$, by requiring that M satisfies a certain formula Θ_φ we can infer that Theorem 3.3.9 holds in M whenever G is M-generic, and that Cohen's Forcing Theorem 3.3.10 holds in M for φ. Θ_φ will depend on which instance φ of Cohen's Forcing Theorem we want to satisfy.

Now we can finally present the final result of this section. We state it for V a transitive ZFC-model. However with some more intricacies in its formulation, it can be inferred for arbitrary first order models of ZFC.

Proposition 3.4.14. *Let B be a complete boolean algebra in V. Let $b \in \mathsf{B}$, consider a formula $\varphi(x_1, \ldots, x_n)$ and $\tau_1, \ldots, \tau_n \in V^{\mathsf{B}}$. Let $\alpha > \omega$ be an ordinal (given by the Reflection Principle) such that $\tau_1, \ldots, \tau_n, b, \mathsf{B} \in V_\alpha$ and such that V_α reflects $([\![\varphi(\tau_1, \ldots, \tau_n)]\!]^{V^{\mathsf{B}}} \geq b) \wedge \Theta_\varphi$. Then the following are equivalent:*

1. *$[\![\varphi(\tau_1, \ldots, \tau_n)]\!]^{V^{\mathsf{B}}} \geq b$.*
2. *For every $M \in V$ such that $M \prec V_\alpha$ and M is a countable structure to which $\tau_1, \ldots, \tau_n, b, \mathsf{B}$ belong, if $\pi : M \to N$ is the Mostowski Collapse, and $G \subseteq \pi(B)$ is an N-generic ultrafilter such that $\pi(b) \in G$, then $N[G] \models \varphi(\pi(\tau_1)^G, \ldots, \pi(\tau_n)^G)$.*
3. *There exists some countable $M \prec V_\alpha$ with $M \in V$ and $\tau_1, \ldots, \tau_n, b, \mathsf{B} \in M$ such that letting $\pi : M \to N$ be the Mostowski Collapse, and $G \subseteq \pi(B)$ be any N-generic ultrafilter such that $\pi(b) \in G$, then $N[G] \models \varphi(\pi(\tau_1)^G, \ldots, \pi(\tau_n)^G)$.*

Proof. 1 ⇔ 2. Fix $M \prec V_\alpha$. Since being a B-name is absolute for transitive models and π is an isomorphism, we have that $\pi(\tau_1), \ldots, \pi(\tau_n) \in N^{\pi(\mathsf{B})}$. Moreover, since V_α reflects $[\![\varphi(\tau_1, \ldots, \tau_n)]\!]^{V^{\mathsf{B}}} \geq b$, M is an elementary substructure of V_α and π is an isomorphism, we can infer that:

$$\begin{aligned}
[\![\varphi(\tau_1, \ldots, \tau_n)]\!]^{V^{\mathsf{B}}} \geq b &\Leftrightarrow ([\![\varphi(\tau_1, \ldots, \tau_n)]\!]^{V^{\mathsf{B}}} \geq b)^{V_\alpha} \\
&\Leftrightarrow ([\![\varphi(\tau_1, \ldots, \tau_n)]\!]^{V^{\mathsf{B}}} \geq b)^{M} \\
&\Leftrightarrow [\![\varphi(\pi(\tau_1), \ldots, \pi(\tau_n))]\!]^{N^{\pi(\mathsf{B})}} \geq \pi(b).
\end{aligned}$$

For the same reasons Θ_φ holds in N. Moreover, Lemma 3.3.5 tells us that N-generic ultrafilter are dense in $St(\pi(\mathsf{B}))$. Then one direction of the thesis follows from Theorem 3.3.10, the other from Lemma 2.3.30.

$1 \Leftrightarrow 3$. The existence of a countable model M such that $M \prec V_\alpha$ is guaranteed by Downward Löwenheim−Skolem Theorem 2.1.20. With this observation the proof proceeds as in the previous point. □

Remark 3.4.15. Let φ be a formula. Suppose that ZFC proves that for some boolean algebra B $[\![\varphi]\!]_\mathsf{B} > 0_\mathsf{B}$. By compactness only a finite number of axioms of ZFC are needed to prove this derivation. We let Ξ_φ denote the conjunction of these axioms. By the Reflection Principle 3.1.7 we get a V_α which reflects $\Xi_\varphi \wedge \Theta_\varphi$. By the Downward Löwenheim−Skolem Theorem 2.1.20, and the Mostowski's Collapse 3.1.5, we can find a countable transitive set N which is a model of $\Xi_\varphi \wedge \Theta_\varphi$. Since N is a countable model of Ξ_φ, there exists a boolean algebra $\mathsf{B} \in N$ which N models to be complete and such that $[\![\varphi]\!]_\mathsf{B} > 0_\mathsf{B}$. By Lemma 3.3.5 there exists an N-generic ultrafilter G with $[\![\varphi]\!]_\mathsf{B} \in G$. We conclude, by means of Cohen's Forcing Theorem applied to φ, that $N[G] \models \varphi$ whenever G is N-generic for B with $[\![\varphi]\!]_\mathsf{B} \in G$ (this is the case since N models Θ_φ).

Since we can repeat this procedure for any finite set of axioms of ZFC which includes $\Xi_\varphi \wedge \Theta_\varphi$, we obtain that φ is finitely consistent with ZFC, and thus, by compactness, that it is consistent with ZFC.

This explains why set theorists can use Cohen's Forcing Theorem without being worried by the non-existence of a V-generic ultrafilter.

3.5 Absoluteness results

The previous sections provided the basic tools needed in order to obtain consistency results using forcing. Now we will present some theorems which show how the forcing techniques can be used to derive properties within ZFC. The key of this process will be Shoenfield's Absoluteness Theorem. An exhaustive description of this topic can be found in Chapter 25 of [6]. Our description will be slightly different, and it will be closer to the introduction of [13].

Definition 3.5.1. Let κ be a cardinal. We define the set H_κ as

$$H_\kappa = \{x : |TC(x)| < \kappa\}$$

Proposition 3.5.2. *Assume $\kappa > \omega$ is regular. Then H_κ is a model of all axioms of ZFC except for the power set axiom.*

Proof. See [8, Theorem 6.5] □

Theorem 3.5.3 (Cohen's Generic Absoluteness)**.** *Assume $\varphi(x, a)$ is a Δ_0-formula and $a \subseteq \omega$. The following are equivalent:*

1. $\langle H_{\aleph_1}, \in\rangle \models \exists x\varphi(x, a)$;
2. *there is a complete boolean algebra* B *such that* $[\![\exists x\varphi(x, \check{a})]\!]^{V^\mathsf{B}} > 0_\mathsf{B}$.

Proof. Suppose $\langle H_{\aleph_1}, \in\rangle \models \exists x\varphi(x, a)$, since $H_{\aleph_1} \subset V$ and Σ_1-formulae are upward absolute it holds

$$V \models \exists x\varphi(x, a)$$

hence, considering $\mathsf{B} = \{0, 1\}$, we have $[\![\exists x\varphi(x, \check{a})]\!]^{V^\mathsf{B}} = 1_\mathsf{B}$.

Conversely, assume there is a complete boolean algebra B such that $[\![\exists x\varphi(x,\check{a})]\!]^{V^{\mathsf{B}}} > 0_{\mathsf{B}}$. By Remark 3.4.15 there exists a countable transitive model N and an N-generic ultrafilter G such that $N[G] \models \varphi(s,a)$ for a certain $s \in N[G]$ (notice that $\pi(a) = a$ since $a \subseteq \omega$). The set $N[G]$ is transitive and countable, hence it is contained in $H_{\aleph_1}$ and so does a. The formula φ is Δ_0, so it is absolute for transitive models, hence $H_{\aleph_1} \models \varphi(s,a)$ (every H_κ is clearly transitive), which leads to $H_{\aleph_1} \models \exists x\varphi(x,a)$. □

Remark 3.5.4. Cohen's Absoluteness Theorem deserves some comments. We know that $(H_{\aleph_1})^V \subseteq V$ for any $V \models \mathsf{ZFC}$, and Σ_1-formulae are upward absolute. Thus Cohen's Absoluteness Theorem tells us that in order to proof $\exists x\varphi(x,a)$ (φ any Δ_0-formula) in ZFC, it is not necessary to show that this formula holds in every model of ZFC, but just that $[\![\exists x\varphi(x,\check{a})]\!] > 0_{\mathsf{B}}$ holds for some B. This can be performed finding one appropriate model through forcing. This technique can also be applied to Σ^1_2-formulae, which are a specific class of formulae in the second order arithmetic (see [6, Chapter 25] for more details), since these formulae can be translated into Σ_1-formulae in $H_{\aleph_1}$ (see Lemma 25.25 in [6]).

Chapter 4

C^*-algebras and B-names for complex numbers

In this chapter we will finally draw a bridge between the theory of commutative C^*-algebras and the theory of boolean valued models of set theory. First we will show how to define (given B a complete boolean algebra) a structure of B-valued extension of $\mathbb{C}$ on the C^*-algebra $\mathcal{C}(St(\mathsf{B}))$, the set of continuous functions from $St(\mathsf{B})$ to $\mathbb{C}$. In the second part we will consider C^*-algebras $\mathcal{A}$ whose spectrum is extremely disconnected, and we will show that

$$\mathcal{A} \cong \mathcal{C}(St(\mathsf{B}))$$

for a specific complete boolean algebra B. In the last section we will provide an embedding of these C^*-algebras in the B-names for complex numbers of V^{B}. From this we will get that the quotient $\mathcal{C}(St(\mathsf{B}))/G$, where G is a V-generic ultrafilter of B, is an algebraically closed field which extends $\mathbb{C}$, and which preserves the truth value of Σ_2-formulae of $\mathbb{C}$.

4.1 A boolean valued extension of $\mathbb{C}$

We introduce some definitions from topology.

Definition 4.1.1. Let X be a topological space.

- $A \subseteq X$ is **nowhere dense** if its complement contains an open dense set;
- $A \subseteq X$ is **meager** if it is the union of countably many nowhere dense sets;
- $A \subseteq X$ is **comeager** if its complement is meager;
- $A \subseteq X$ has the **Baire property** if there exists an open set $U \subseteq X$ such that $A \Delta U$ is meager.

The family of **Borel sets** of X is the σ-algebra generated by the open subsets of X, i. e. the smallest family $\mathcal{F}$ of subsets of X such that:

1. if $A \subseteq X$ is open then $A \in \mathcal{F}$;
2. if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$;

3. if $\{A_n\}_{n\in\omega} \subseteq \mathcal{F}$ then $\bigcup_{n\in\omega} A_n \in \mathcal{F}$.

We need a couple of properties before being able to present the boolean extension of $\mathbb{C}$.

Proposition 4.1.2. *Assume X is a compact Hausdorff space. Then every Borel set B of X has the Baire property and there exists a unique regular open set U such that $B\Delta U$ is meager.*

Proof. For a proof see [5, Chapter 29, Lemma 2]. □

Proposition 4.1.3. *Let X, Y be two topological spaces and $f : X \to Y$ a continuous map. If $B \subseteq Y$ is a Borel set, then $f^{-1}[B]$ is a Borel set of X.*

Proof. This is the case since the preimage of open sets through f is open, and:

$$f^{-1}\left[\bigcup_{n\in\omega} A_n\right] = \bigcup_{n\in\omega} f^{-1}[A_n]$$

$$f^{-1}[A_n^c] = f^{-1}[A_n]^c$$

□

The following example shows how to obtain a boolean extension of a topological space X when the language is composed by symbols which are interpreted as Borel subsets of X^n.

Example 4.1.4. Fix a complete boolean algebra B, a topological space X, and $R \subseteq X \times X$ a binary Borel relation on X. Consider $M = \mathcal{C}(St(\mathsf{B}), X)$ the set of continuous functions from $St(\mathsf{B})$ to X. We can define a structure of B-valued extension of X on M. Given $f, g \in M$, the set

$$W = \{G \in St(\mathsf{B}) : f(G)Rg(G)\} = (f \times g)^{-1}(R)$$

is a Borel subset of $St(\mathsf{B})$ since both f and g are continuous. By Proposition 4.1.2 W has the Baire property and

$$\overline{\overset{\circ}{\{G \in St(\mathsf{B}) : f(G)Rg(G)\}}}$$

is the unique regular open set $U \subseteq St(\mathsf{B})$ such that $W\Delta U$ is meager. If we identify B and $\mathsf{RO}(St(\mathsf{B}))$ (B is complete), we have that

$$R^M(f, g) = \overline{\overset{\circ}{\{G \in St(\mathsf{B}) : f(G)Rg(G)\}}}$$

is a well defined element of B. We can repeat verbatim the procedure above in order to define the boolean interpretation of any n-ary Borel relation on X and of any function

$$F : X^n \to X$$

whose graph is a Borel subset of X^{n+1} (we will simply say that F is a Borel function). The boolean interpretation of the equality can be defined as long as the set

$$\Delta_X = \{(x, x) \in X \times X : x \in X\}$$

is a Borel set in $X \times X$. With these definitions it can be checked that M is a B-valued model. Moreover the set $\{c_x \in M : x \in X\}$, where c_x is the constant function with value x, is a copy of X in M in the sense that, for $x, y \in X$:

$$xRy \Leftrightarrow [\![c_x R c_y]\!] = 1_{\mathsf{B}}$$

$$\neg xRy \Leftrightarrow [\![c_x R c_y]\!] = 0_{\mathsf{B}}$$
$$x = y \Leftrightarrow [\![c_x = c_y]\!] = 1_{\mathsf{B}}$$
$$x \neq y \Leftrightarrow [\![c_x = c_y]\!] = 0_{\mathsf{B}}$$

Thus we can infer that M is the boolean extension of an isomorphic copy of X seen as a 2-valued model.

Remark 4.1.5. If $X = \mathbb{C}$ we have that $\Delta_{\mathbb{C}}$ is closed ($\mathbb{C}$ is Hausdorff). Hence, fixed a language $\mathcal{L}$ whose elements are Borel relations and functions in $\mathbb{C}$, we can define in the C^*-algebra $\mathcal{C}(St(\mathsf{B}))$ a structure of B-valued extension of $\mathbb{C}$ for the language $\mathcal{L}$.

4.2 C^*-algebras as B-valued models

Definition 4.2.1. Let X be a 0-dimensional compact Hausdorff space. Consider the function

$$\varphi_X : X \to St(\mathsf{CL}(X))$$
$$x \mapsto G_x$$

with $G_x = \{A \in \mathsf{CL}(X) : x \in A\}$ the ultrafilter defined in Proposition 2.2.17. We shall call ψ_X the inverse map of φ_X and, for $G \in St(\mathsf{B})$

$$\psi_X(G) = x_G$$

We recall that $\{x_G\} = \bigcap G$.

Proposition 4.2.2. *Let $\mathcal{A}$ be a commutative unital C^*-algebra, $X = \sigma(\mathcal{A})$ its spectrum and $\mathsf{B} = \mathsf{RO}(X)$. Assume furthermore that X is extremely disconnected. Then X is homeomorphic to $St(\mathsf{B})$ and the map*

$$\Phi_{\mathcal{A}} : \mathcal{C}(X) \to \mathcal{C}(St(\mathsf{B}))$$
$$f \mapsto \hat{f}$$

where $\hat{f}(G) = f(x_G)$, is an isometric $*$-isomorphism of C^*-algebras.

Proof. Proposition 1.2.8 implies that X is regular, hence it admits a basis of regular open sets, and since $\mathsf{RO}(X) = \mathsf{CL}(X)$ we have that X is 0-dimensional. By Proposition 2.2.17, and the fact that $\mathsf{B} = \mathsf{CL}(X)$, we infer that $X \cong St(\mathsf{CL}(X)) = St(\mathsf{B})$. Thus the map

$$\Phi_{\mathcal{A}} : \mathcal{C}(X) \to \mathcal{C}(St(\mathsf{B}))$$
$$f \mapsto \hat{f}$$

is an homeomorphism. By the definition of $\hat{f}$ it easily follows that $\Phi_{\mathcal{A}}$ is an isometric $*$-isomorphism. □

Remark 4.2.3. The previous proposition connects abelian unital C^*-algebras whose spectrum is extremely disconnected with boolean extensions of the complex field. In fact, assume that $\mathcal{A}$ is a C^*-algebra with such properties, X its spectrum and $\mathsf{B} = \mathsf{RO}(X)$. If we combine the previous result with the Gelfand-Naimark Theorem 1.2.10, we obtain :

$$\mathcal{A} \cong \mathcal{C}(X) \cong \mathcal{C}(St(\mathsf{B}))$$

Now we want to see what happens when we consider a commutative unital C^*-algebra whose spectrum is not extremely disconnected. In order to do this we will generalize the map $\Phi_{\mathcal{A}}$ defined in Proposition 4.2.2. First we need to generalize Definition 4.2.1.

Lemma 4.2.4. *Let X be a compact Hausdorff topological space and $\mathsf{B} = \mathsf{RO}(X)$. The map*

$$\psi_X : St(\mathsf{B}) \to X$$
$$G \mapsto x_G$$

where x_G is the only element in

$$C_G = \bigcap \{\overline{U} : U \in G\}$$

is well-defined, continuous and surjective. This map is the function ψ_X given by Definition 4.2.1 if and only if X is extremely disconnected.

Proof. Well-defined: We have to show that, given $G \in St(\mathsf{B})$, C_G is a singleton. The set is non-empty since C_G inherits the finite intersection property from G, and X is compact. The space X is regular, hence if $x \neq y \in C_G$ there exists a regular open set A such that $x \in A$ and $y \notin \overline{A}$. From this and

$$A \in G \Rightarrow (X \mathring{\setminus} A) \notin G$$

follows that

$$x \in C_G \Rightarrow y \notin C_G$$

which is absurd.

Surjective: For each $x \in X$ consider a ultrafilter G which extends the filter of the regular open nieghborhoods of the point. In this case $x_G = x$.

Continuous: Let A be an open set in X, and let G be such that $x_G \in A$. Let B a regular open set such that $x_G \in B$ and $\overline{B} \subseteq A$. B is open and $x_G \notin B^c$, so that $x_G \notin \overline{\neg B}$. This means that $B \in G$ (otherwise $\neg B \in G$, which is absurd since $x_G \notin \overline{\neg B}$), and from $\overline{B} \subseteq A$ follows $\mathcal{O}_B \subseteq \psi^{-1}(A)$.

□

The following theorem generalizes Proposition 4.2.2 to any commutative unital C^*-algebra.

Theorem 4.2.5. *Fix a commutative unital C^*-algebra $\mathcal{A}$, let X be its spectrum, and let $\mathsf{B} = \mathsf{RO}(X)$. The function*

$$\Phi_{\mathcal{A}} : \mathcal{C}(X) \to \mathcal{C}(St(\mathsf{B}))$$
$$f \mapsto \hat{f}$$

where $\hat{f}(G) = f(x_G)$, is an injective $*$-homomorphism of C^*-algebras. Moreover, the map $\Phi_{\mathcal{A}}$ is an isometric $*$-isomorphism if and only if X is extremely disconnected.

Proof. The map clearly preserves sum, product and involution.

Continuous: These inequalities guarantee continuity:

$$\|\Phi_{\mathcal{A}}(f)\| = \max_{G \in St(\mathsf{B})} \left|\hat{f}(G)\right| = \max_{G \in St(\mathsf{B})} |f(x_G)| \leq \|f\|$$

Injective: Consider $f \neq 0$, hence there exists $x \in X$ such that $f(x) \neq 0$. From Lemma 4.2.4 we know that there exists $G \in St(\mathsf{B})$ such that $x = x_G$, thus $\hat{f}(G) \neq 0$.

Surjective: As we have showed earlier, if X is extremely disconnected the map $\Phi_{\mathcal{A}}$ is an isometric $*$-isomorphism. Suppose X is not extremely disconnected and consider the surjective continuous map ψ_X in Lemma 4.2.4. Since both X and $St(\mathsf{B})$ are compact Hausdorff spaces, the map can not be bijective, otherwise it would be an homeomorphism, which is impossible since $St(\mathsf{B})$ is extremely disconnected. Thus injectivity of ψ_X must fail, and there exist two ultrafilters G and H such that $x_G = x_H$. We conclude that for every $\hat{f} \in \mathrm{Im}(\Phi_{\mathcal{A}})$

$$\hat{f}(G) = \hat{f}(H)$$

(even more, $\hat{f}$ is constant over the fiber of ψ_X at any point $x \in X$). Since $St(\mathsf{B})$ is a normal space we can always find an $h \in \mathcal{C}(St(\mathsf{B}))$ which assumes different values on different points which are on the same fiber of ψ_X. We conclude that h can not belong to $\mathrm{Im}(\Phi_{\mathcal{A}})$.

□

Remark 4.2.6. Summing up, given $\mathcal{A}$ a unital commutative C^*-algebra and $\mathsf{B} = \mathsf{RO}(\sigma(\mathcal{A}))$, the map $\Phi_{\mathcal{A}} \circ \Gamma_{\mathcal{A}}$ is a $*$-homomorphism which embeds $\mathcal{A}$ in $\mathcal{C}(St(\mathsf{B}))$, a B-valued extension of $\mathbb{C}$. We have also showed that this map is a $*$-isomorphism if and only if $\sigma(\mathcal{A})$ is extremely disconnected. We may ask whether this request is too strong and if only exotic examples of C^*-algebras satisfy it. This is not the case, and we will provide an example.

Example 4.2.7. Let $\mathcal{A} = L^\infty(\mathbb{C})$ and X be its spectrum. Working directly with $\mathsf{B} = \mathsf{RO}(X)$ is a bit laborious. We consider then MALG, the complete boolean algebra of measurable sets in $\mathbb{C}$ modulo null measure sets. We will work in this more familiar context, and show that $X \cong St(\mathsf{MALG})$. This will be enough, in fact this will guarantee that X is extremely disconnected, hence $X \cong St(\mathsf{RO}(X))$, and by Corollary 2.2.20 (which implies $\mathsf{B} \cong \mathsf{MALG}$) we conclude that $L^\infty(\mathbb{C})$ is isomorphic to $C(X) \cong C(St(\mathsf{RO}(X)) \cong C(St(\mathsf{MALG}))$.

Linear combinations of characteristic functions of measurable sets are dense in $L^\infty(\mathbb{C})$. Thus any character k in the spectrum X of $L^\infty(\mathbb{C})$ is univocally determined by its value on characteristic functions $\{\chi_A\}_{A \in \mathsf{MALG}}$. In addition, since $k \in X$ is a non zero homomorphism, given any $A \in \mathsf{MALG}$:

$$k(1) = k(\chi_A + \chi_{A^c}) = 1$$

$$k(0) = k(\chi_A \chi_{A^c}) = 0$$

This means that every characteristic function is mapped by k or in 0 or in 1, and if $k(\chi_A) = 1$ then $k(\chi_{A^c}) = 0$, and vice versa. Assume $G \in St(\mathsf{MALG})$, then the function k_G defined as

$$k_G(\chi_A) = \begin{cases} 1 \text{ if } A \in G \\ 0 \text{ if } A \notin G \end{cases}$$

and extended by linearity and continuity to $L^\infty(\mathbb{C})$ belongs to the spectrum. Conversely, given $k \in X$, we can consider the set $G_k = \{A : k(\chi_A) = 1\}$ which can be easily verified to a ultrafilter. The maps

$$\Theta : X \to St(\mathsf{MALG})$$
$$k \mapsto G_k$$

$$\begin{aligned}\Xi : St(\mathsf{MALG}) &\to X \\ G &\mapsto k_G\end{aligned}$$

are one the inverse of the other. If we show that one of them is continuous we obtain $X \cong St(\mathsf{MALG})$, since X and $St(\mathsf{MALG})$ are both compact Hausdorff spaces. We will work on Θ. Consider $\mathcal{O}_A$ a clopen set in $St(\mathsf{MALG})$. Its preimage is the set of all $k \in X$ such that $k(\chi_A) = 1$, which is closed with respect to pointwise convergence, hence closed in X. In particular we get that the preimage of $\mathcal{O}_{A^c}$ is open in X, thus we conclude that Θ is continuous. We can therefore write the isomorphism Λ (which is $\Gamma_{\mathcal{L}^\infty(\mathbb{C})}$ composed with the isomorphism between $\mathcal{C}(X)$ and $\mathcal{C}(St(\mathsf{MALG}))$):

$$\begin{aligned}\Lambda : L^\infty(\mathbb{C}) &\to \mathcal{C}(St(\mathsf{MALG})) \\ f &\mapsto \tilde{f}\end{aligned}$$

where $\tilde{f}(G) = k_G(f)$.

4.3 B-names for complex numbers

Through this section we will assume V to be a transitive model of ZFC and $\mathsf{B} \in V$ a boolean algebra which V models to be complete. If G is a V-generic ultrafilter in B, $V[G]$ will denote the generic extension of Definition 3.3.6.

Definition 4.3.1. $\sigma \in V^{\mathsf{B}}$ is a **B-name for a complex number** if

$$[\![\sigma \text{ is a complex number}]\!] = 1_{\mathsf{B}}$$

We denote with $\mathbb{C}^{\mathsf{B}}$ the set of all B-names for complex numbers modulo the equivalence relation:

$$\sigma \equiv \tau \Leftrightarrow [\![\sigma = \tau]\!] = 1_{\mathsf{B}}$$

Remark 4.3.2. Let

$$\mathcal{B} = \{U_n : n \in \omega\}$$

be the family of the open balls in $\mathbb{C}$ whose radius and centre coordinates are rational. Every Borel subset of $\mathbb{C}$ is obtained, in fewer than $\aleph_1$ steps, from the elements of $\mathcal{B}$ by taking countable unions and complements. It is possible to code these operations with a real number r. A detailed explanation of this procedure is given in section "Borel Codes" of [6, Chapter 25]. For our purposes it is enough to say that if R is a Borel subset of $\mathbb{C}$, there is a real number $r \in \mathbb{R}$ and a (ZFC provably) Δ_1-property $P(x, y)$ such that

$$x \in R \Leftrightarrow P(x, r)$$

Lemmas 25.24 and 25.25 in [6] show that these Borel codes and the inclusion relation between these sets are absolute for transitive models. Consider two Borel sets R, S associated to $r, s \in \mathbb{R}$ respectively, and V a transitive model of ZFC such that $r, s \in V$. Let B be a complete boolean algebra in V and assume there could be G a V-generic filter for B. The generic extension $V[G]$ would be a transitive model of ZFC which contains V, hence the absoluteness properties between V and $V[G]$ would guarantee that

$$R^V \subseteq S^V \Leftrightarrow R^{V[G]} \subseteq S^{V[G]}$$

where

$$R^V = \{x \in V : P(x, r)\}$$
$$R^{V[G]} = \{x \in V[G] : P(x, r)\}$$

and similarly for S. Guided by these considerations we define in V the following B-name:

$$\dot{R} = \{(\tau, [\![P(\tau, \check{r})]\!]) : \tau \text{ is a } B\text{-name for a complex number}\}$$

It is easy to see that:

$$\dot{R}^G = R^{V[G]}$$

whenever G is a V-generic filter for B. In particular $\dot{R} \in V^{\mathsf{B}}$ is a canonical name to interpret the Borel relation R in any generic extension of V by a generic filter G.

We return now to $\mathbb{C}^{\mathsf{B}}$. Observe that, given $\tau \in \mathbb{C}^{\mathsf{B}}$ and a first order formula $\varphi(x)$, the element $[\![\varphi(\tau)]\!]^{V^{\mathsf{B}}}$ is well-defined in B, even if τ is an equivalence class. This follows from Lemma 2.3.11. Keeping this in mind we give the following:

Definition 4.3.3. Given R a Borel n-ary relation on $\mathbb{C}$ we define, for $\sigma_1, \ldots, \sigma_n \in \mathbb{C}^{\mathsf{B}}$:

$$[\![R(\sigma_1, \ldots, \sigma_n)]\!]^{\mathbb{C}^{\mathsf{B}}} = \left[\!\!\left[(\sigma_1, \ldots, \sigma_n) \in \dot{R}\right]\!\!\right]^{V^{\mathsf{B}}}$$

and similarly for Borel functions.

Lemma 2.3.11 and Soundness Theorem 2.3.14 applied in V^{B} guarantee that items i-vii in Definition 2.3.1 hold for the structure

$$\langle \mathbb{C}^{\mathsf{B}}, R_1^{\mathbb{C}^{\mathsf{B}}}, \ldots, R_k^{\mathbb{C}^{\mathsf{B}}}, F_1^{\mathbb{C}^{\mathsf{B}}}, \ldots, F_l^{\mathbb{C}^{\mathsf{B}}} \rangle$$

where each R_i (F_j) is an arbitrary Borel relation (function) on $\mathbb{C}^{n_i}$ (from $\mathbb{C}^{m_j}$ to $\mathbb{C}$). As a consequence, $\mathbb{C}^{\mathsf{B}}$ is B-valued extension of $\mathbb{C}$ when we consider languages composed by Borel relations and functions. We have seen that this is the case also for $\mathcal{C}(St(\mathsf{B}))$, and we will be able to build an embedding of this space in $\mathbb{C}^{\mathsf{B}}$. Unfortunately these two spaces are not isomorphic as B-valued models. In order to have an isomorphism we need to consider a larger set of functions from $St(\mathsf{B})$.

Definition 4.3.4. Let X be a compact Hausdorff extremely disconnected space. $\mathcal{C}^+(X)$ is the set of continuous functions f from X to the one point compactification of the complex field $\mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$ such that $f^{-1}[\{\infty\}]$ is nowhere dense.

We can define on $\mathcal{C}^+(St(\mathsf{B}))$ a structure of B-valued extension of $\mathbb{C}$ as we did for $\mathcal{C}(St(\mathsf{B}))$ in Example 4.1.4. Our aim is the following:

Theorem 4.3.5. *Fix a set*

$$\mathcal{L} = \{R_i : i \in I\} \cup \{F_j : j \in J\}$$

where:

- *for $i \in I$, R_i is a Borel subset of $\mathbb{C}^{n_i}$;*
- *for $j \in J$, F_j is a Borel function from $\mathbb{C}^{m_j}$ to $\mathbb{C}$.*

Then the B-valued models $\mathcal{C}^+(St(\mathsf{B}))$ and $\mathbb{C}^{\mathsf{B}}$ in the language $\mathcal{L}$ (as defined in Example 4.1.4 and Definition 4.3.3 respectively) are isomorphic.

With some more effort we will be able to show how the study of complex numbers in a generic extension can be reduced to the study of $\mathcal{C}(St(\mathsf{B}))$, and we will derive the following:

Theorem 4.3.6. *Let V be a transitive model of ZFC, $\mathsf{B} \in V$ which V models to be a complete boolean algebra, and G a V-generic filter in B. Assume $R_1, \dots, R_s$ are n_i-ary Borel relations and $f_1, \dots, f_t$ m_j-ary Borel functions on $\mathbb{C}$. Then*

$$\langle \mathbb{C}, R_1, \dots, R_s, f_1, \dots, f_t \rangle \prec_{\Sigma_2} \langle \mathcal{C}(St(\mathsf{B}))/G, R_1/G, \dots, R_s/G, f_1/G, \dots, f_t/G \rangle$$

Before getting to these theorems, we need several intermediate results.

Something similar to Remark 4.3.2 can be performed in $St(\mathsf{B})$. In this case the parameters for the "code" have to be taken among real numbers (to code the complexity of the Borel relation) and elements of B (to code the basic open sets), since a basis for $St(\mathsf{B})$ is

$$\{\mathcal{O}_a : a \in \mathsf{B}\}$$

We will show explicitly the absoluteness of the inclusion relation for generic extensions in the case of open and closed sets, since it will be needed later.

Lemma 4.3.7. *Let V be a transitive models of ZFC, $\mathsf{B} \in V$ a boolean algebra which V models to be complete, and G a V-generic filter over B. Assume R^V, S^V are two open or closed sets in $St(\mathsf{B})^V$. Then*

$$R^V \subseteq S^V \Leftrightarrow R^{V[G]} \subseteq S^{V[G]}$$

Proof. <u>Clopen $\subseteq$ Clopen:</u> Given $a, b \in \mathsf{B}$:

$$\mathcal{O}_a^V \subseteq \mathcal{O}_b^V \Leftrightarrow a \leq b$$

The order relation in B is absolute for generic extensions, therefore the thesis follows.

<u>Open $\subseteq$ Closed:</u> Let $A, B \subseteq \mathsf{B}$. Then:

$$\begin{aligned}\bigcup_{a \in A} \mathcal{O}_a^V \subseteq \bigcap_{b \in \mathsf{B}} \mathcal{O}_b^V &\Leftrightarrow \forall a \in A \forall b \in B (\mathcal{O}_a^V \subseteq \mathcal{O}_b^V) \\ &\Leftrightarrow \forall a \in A \forall b \in B (\mathcal{O}_a^{V[G]} \subseteq \mathcal{O}_b^{V[G]}) \\ &\Leftrightarrow \bigcup_{a \in A} \mathcal{O}_a^{V[G]} \subseteq \bigcap_{b \in \mathsf{B}} \mathcal{O}_b^{V[G]}\end{aligned}$$

<u>Closed $\subseteq$ Open:</u> Given $A, B \subseteq \mathsf{B}$, assume that we have

$$\bigcap_{a \in A} \mathcal{O}_a^V \subseteq \bigcup_{b \in B} \mathcal{O}_b^V$$

We define the set

$$\neg B = \{\neg b : b \in B\}$$

The hypothesis guarantees that $A \cup \neg B$ does not have the finite intersection property. Since this property of $A \cup \neg B$ as a subset of B is absolute for generic extensions, this is true in $V[G]$ as well, therefore there is no $H \in \bigcap_{a \in A} \mathcal{O}_a^{V[G]}$ such that

$$H \in \bigcap_{b \in B} \mathcal{O}_{\neg b}^{V[G]} = \left(\bigcup_{b \in B} \mathcal{O}_b^{V[G]} \right)^c$$

Now assume that

$$\bigcap_{a\in A}\mathcal{O}_a^V \not\subseteq \bigcup_{b\in B}\mathcal{O}_b^V$$

Then the family $A\cup\neg B$ has the finite intersection property in V as well as in $V[G]$, thus we can find $H\in St(\mathsf{B})^{V[G]}$ such that

$$H\in\bigcap_{a\in A}\mathcal{O}_a^{V[G]}\cap\bigcap_{b\in B}\mathcal{O}_{\neg b}^{V[G]}$$

<u>Open $\subseteq$ Open:</u> Given $A, B\subseteq\mathsf{B}$, consider

$$\bigcup_{a\in A}\mathcal{O}_a^V, \bigcup_{b\in B}\mathcal{O}_b^V$$

Since every $\mathcal{O}_a^V\in\Delta_1^0\subseteq\Pi_1^0$, from previous items we have that:

$$\mathcal{O}_a^V\subseteq\bigcup_{b\in B}\mathcal{O}_b^V\Leftrightarrow\mathcal{O}_a^{V[G]}\subseteq\bigcup_{b\in B}\mathcal{O}_b^{V[G]}$$

thus:

$$\bigcup_{a\in A}\mathcal{O}_a^V\subseteq\bigcup_{b\in B}\mathcal{O}_b^V\Leftrightarrow\bigcup_{a\in A}\mathcal{O}_a^{V[G]}\subseteq\bigcup_{b\in B}\mathcal{O}_b^{V[G]}$$

□

Remark 4.3.8. In the following part, given B a complete boolean algebra, we will often confuse it with $\mathsf{RO}(St(\mathsf{B}))$. If U is a regular open set of $St(\mathsf{B})$ and $G\in St(\mathsf{B})$, we may write equivalently

$$G\in U, U\in G$$

depending if we are considering U as an element of $\mathsf{RO}(St(\mathsf{B}))$ or as the correspondent element in B.

We recall the definition of $\mathcal{B}$ in Remark 4.3.2, the countable basis of $\mathbb{C}$ whose elements are the open balls with rational radius and whose centre has rational coordinates:

$$\mathcal{B}=\{U_n : n\in\omega\}$$

The following lemma characterizes the elements of $\mathcal{C}^+(St(\mathsf{B}))$.

Lemma 4.3.9. *Assume $f\in V$ is an element of $\mathcal{C}^+(St(\mathsf{B}))$. For $H\in St(\mathsf{B})$ we define*

$$\Sigma_f^H=\{\overline{U}_n : \overline{\mathring{f^{-1}[U_n]}}\in H\}$$

Then, for $H\in St(\mathsf{B})$, we have:

$$f(H)=\sigma_f^H$$

where σ_f^H it is the only element in $\bigcap\Sigma_f^H$ if Σ_f^H is non-empty, and $\sigma_f^H=\infty$ otherwise.

Proof. Assume Σ_f^H is empty. If $f(H)\in U_n$ for some $n\in\omega$ it follows that:

$$H\in f^{-1}[U_n]\subseteq\overline{\mathring{f^{-1}[U_n]}}$$

hence $\overline{\mathring{f^{-1}[U_n]}}\in\Sigma_f^H$, which is absurd. Suppose now that Σ_f^H is non-empty.

Claim 4.3.9.1. *Assume Σ_f^H is non-empty. Then $\bigcap \Sigma_f^H$ is a singleton.*

Proof. Let $m \in \omega$ be such that $\overline{U}_m \in \Sigma_f^H$.

Existence: The family

$$\hat{\Sigma}_f^H = \{\overline{U}_m \cap \overline{U}_n : \overset{\circ}{\overline{f^{-1}[U_n]}} \in H\}$$

is a family of closed subsets of $\overline{U}_m$. Σ_f^H inherits the finite intersection property from H, hence so does $\hat{\Sigma}_f^H$. We can conclude that

$$\emptyset \neq \bigcap \hat{\Sigma}_f^H \subseteq \bigcap \Sigma_f^H$$

Uniqueness: Suppose there are two different points $x, y \in \bigcap \Sigma_f^H$. There exists $p \in \omega$ such that $x \in U_p, y \notin \overline{U}_p$. The last relation implies $\overline{U}_p \notin \Sigma_f^H$. Now we show that if an element in $\bigcap \Sigma_f^H$ is in a certain U_n, then $\overset{\circ}{\overline{f^{-1}[U_n]}} \in H$. Therefore $x \in U_p$ implies $\overline{U}_p \in \Sigma_f^H$, which is absurd. Suppose $\overset{\circ}{\overline{f^{-1}[U_p]}} \notin H$, we have that:

$$H \in \neg \overset{\circ}{\overline{f^{-1}[U_p]}} \cap \overset{\circ}{\overline{f^{-1}[U_m]}} \subseteq f^{-1}[\overline{U}_m \setminus U_p]$$

For each $z \in \overline{U}_m \setminus U_p$ there exists U_{n_z} such that

$$z \in U_{n_z} \wedge x \notin \overline{U}_{n_z}$$

This family of open balls covers the compact space $\overline{U}_m \setminus U_p$, so that there are $z_1, \cdots, z_k \in \overline{U}_m \setminus U_p$ which verify the following chain of inclusions:

$$f^{-1}[\overline{U}_m \setminus U_p] \subseteq \bigcup_{1 \leq i \leq k} f^{-1}[U_{n_{z_i}}] \subseteq \bigcup_{1 \leq i \leq k} \overset{\circ}{\overline{f^{-1}[U_{n_{z_i}}]}}$$

There is therefore a z_j such that $\overset{\circ}{\overline{f^{-1}[U_{n_{z_j}}]}} \in H$, hence $\overline{U}_{z_j} \in \Sigma_f^H$. This is absurd since $x \notin \overline{U}_{z_j}$.

□

Suppose $f(H) \neq \sigma_f^H$ and consider two open balls U_1, U_2 in $\mathcal{B}$ such that

$$\overline{U}_1 \cap \overline{U}_2 = \emptyset$$

$$f(H) \in U_1$$

$$\sigma_f^H \in U_2$$

It easily follows that both $\overset{\circ}{\overline{f^{-1}[U_1]}}$ and $\overset{\circ}{\overline{f^{-1}[U_2]}}$ are in H (the second assertion, can be shown along the same lines of the uniqueness proof in Claim 4.3.9.1). These two sets are disjoint, a contradiction follows. □

Remark 4.3.10. Previous lemma shows that in ZFC, given $f \in \mathcal{C}^+(St(\mathsf{B}))$, it holds

$$f(H) = x \Leftrightarrow x = \sigma_f^H$$

The latter is a (ZFC provably) Δ_1-property with ω and $\{a_n = \overline{\mathring{f^{-1}[U_n]}} : n \in \mathbb{N}\}$ as parameters. Thus, given V a transitive model of ZFC, B a complete boolean algebra in V, G a V-generic filter in B, any $f \in V$ element of $\mathcal{C}^+(St(\mathsf{B}))^V$ can be extended in an absolute manner to $V[G]$ by the rule:

$$f^{V[G]} : St(\mathsf{B})^{V[G]} \to \mathbb{C}^{V[G]}$$
$$H \mapsto \sigma_f^H$$

where σ_f^H is defined as in the previous lemma through the set

$$\Sigma_f^H = \{\overline{U}_n : a_n \in H\}$$

Remark 4.3.11. The next proofs will often use Proposition 3.4.14 in order to show that for a certain formula φ and for some $b \in \mathsf{B}$

$$[\![\varphi]\!] \geq b$$

In that proposition we work with M, a countable set which reflects enough axioms of ZFC as well as the formula $[\![\varphi]\!] \geq b$ (precisely M reflects $\Theta_\varphi \wedge ([\![\varphi]\!] \geq b)$), and with N, its transitive image through the Mostwoski's Collapse. From now on, when using Proposition 3.4.14, we will work under the assumption that there exists a countable $M \prec V$ with $M \in V$ for V a model of ZFC containing the relevant parameters of the formula $[\![\varphi]\!] \geq b$. If the reader is disturbed by this assumption (which is consistencywise stronger than ZFC), she/he can apply some compactness arguments in combination with Proposition 3.4.14 to remove this assumption, keeping in mind that in each of the following proofs we need to reflect only a finite number of properties of V, thus we could always reduce our attention to a countable $M \prec V_\alpha$ for a certain ordinal α.

Lemma 4.3.12. *Fix V a transitive model of ZFC and $\mathsf{B} \in V$ a boolean algebra which V models to be complete. Let $f \in \mathcal{C}^+(St(\mathsf{B}))$ and consider*

$$\mathcal{B} = \{U_n : n \in \omega\}$$

the countable basis of $\mathbb{C}$ defined in Remark 4.3.2. For each $n \in \omega$ let

$$a_n = \overline{\mathring{f^{-1}[U_n]}}$$

There exists a unique $\tau_f \in \mathbb{C}^{\mathsf{B}}$ such that

$$\left[\!\!\left[\tau_f \in \dot{U}_n\right]\!\!\right] = a_n$$

Proof. Consider the B-name

$$\Sigma_f = \{(\dot{U}_n, a_n) : n \in \omega\}$$

We start showing that

$$\left[\!\!\left[\exists! x (x \in \bigcap \Sigma_f)\right]\!\!\right] = 1_{\mathsf{B}}$$

which, by Proposition 3.4.14, is a consequence of the following:

Claim 4.3.12.1. *Assume $M \prec V$ is a countable model of ZFC such that $\omega \cup \{a_n : n \in \omega\} \cup \{\mathsf{B}, f\} \subseteq M$, and $\pi : M \to N$ is the Mostowski's Collapse. Let G be an N-generic filter for $\pi(\mathsf{B})$. Then:*

$$N[G] \models \exists! x \left(x \in \bigcap \Sigma^G_{\pi(f)}\right)$$

where $\Sigma^G_{\pi(f)} = \{\overline{U}_n^{N[G]} : \pi(a_n) \in G\}$.

Proof. Notice that since $\omega \subseteq M$ is transitive, rational and complex numbers (the power-set of a transitive set is transitive) are preserved by π and that $\mathbb{C}^N = \mathbb{C} \cap N$. First, we prove that $\Sigma^G_{\pi(f)}$ is non-empty (observe that $\pi(f)$ preserves all the properties of f since π is an isomorphism). The preimage of $\mathbb{C}^N$ through $\pi(f)$ contains an open dense subset of $St(\pi(\mathsf{B}))^N$, hence (observe that $\pi(f)^{-1}[U^N_n] = \pi(f^{-1}[U_n])$) it follows that

$$\bigcup_{n\in\omega} \pi(a_n)$$

is an open dense subset of $St(\pi(\mathsf{B}))$ as well. By Lemma 3.3.4 we can infer that

$$D = \{a \in \pi(\mathsf{B})^+ : \exists n \in \omega(a \leq \pi(a_n))\}$$

is open dense in $\pi(\mathsf{B})^+$. Since G is N-generic, $G \cap D \neq \emptyset$. There exists therefore $m \in \omega$ such that $\pi(a_m) \in G$, thus $\overline{U}^{N[G]}_m \in \Sigma^G_{\pi(f)}$. The proof that $\bigcap \Sigma^G_{\pi(f)}$ is a singleton can be carried in N as in Claim 4.3.9.1. ☐

V^{B} is full, there is therefore a B-name τ_f such that

$$\left[\!\!\left[\tau_f \in \bigcap \Sigma_f\right]\!\!\right] = 1_{\mathsf{B}}$$

This is a B-name for a complex number. Moreover, if τ is a B-name for a complex number and

$$\left[\!\!\left[\tau \in \bigcap \Sigma_f\right]\!\!\right] = 1_{\mathsf{B}}$$

then, from

$$(\tau_f \in \bigcap \Sigma_f) \wedge (\tau \in \bigcap \Sigma_f) \wedge (\exists! x(x \in \bigcap \Sigma_f)) \rightarrow \tau = \tau_f$$

it follows that (by the Soundness Theorem 2.3.14):

$$[\![\tau = \tau_f]\!] = 1_{\mathsf{B}}$$

Now we focus on the proof of the thesis of the lemma. We will use again Proposition 3.4.14. Let $M \prec V$ be a countable structure as in Claim 4.3.12.1, $\pi : M \to N$ its Mostowski's Collapse, and G an N-generic filter for $\pi(\mathsf{B})$. On the one hand we have (using the same proof of the uniqueness part in Claim 4.3.9.1) that if $\tau^G_{\pi(f)} \in U^{N[G]}_n$ then $\pi(a_n) \in G$, which means by Proposition 3.4.14

$$\neg a_n \leq \left[\!\!\left[\tau_f \notin \dot{U}_n\right]\!\!\right]$$

or equivalently

$$\left[\!\!\left[\tau_f \in \dot{U}_n\right]\!\!\right] \leq a_n$$

On the other hand

$$G \in \pi(f)^{N[G]^{-1}}[U^{N[G]}_n] \Rightarrow \pi(f)^{N[G]}(G) = \tau^G_{\pi(f)} \in U^{N[G]}_n \Rightarrow \left[\!\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\!\right]^{N^{\pi(\mathsf{B})}} \in G$$

which means, interpreting $\left[\!\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\!\right]^{N^{\pi(\mathsf{B})}}$ as a clopen subset of $St(\pi(\mathsf{B}))^{N[G]}$, that

$$\pi(f)^{N[G]^{-1}}[U^{N[G]}_n] \subseteq \left(\left[\!\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\!\right]^{N^{\pi(\mathsf{B})}}\right)^{N[G]}$$

Lemmas 4.3.7 and 4.3.9 guarantee that this is equivalent to

$$\pi(f)^{-1}[U_n^N] \subseteq \left[\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\right]^{N^{\pi(\mathsf{B})}}$$

Since $\left[\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\right]^{N^{\pi(\mathsf{B})}}$ is clopen this implies that

$$\pi(a_n) \leq \left[\!\left[\tau_{\pi(f)} \in \dot{U}_n\right]\!\right]^{N^{\pi(\mathsf{B})}}$$

and therefore:

$$\left(a_n \leq \left[\!\left[\tau_f \in \dot{U}_n\right]\!\right]\right)^M$$

which guarantees the thesis in V since $M \prec V$. □

Corollary 4.3.13. *With the hypotheses of Lemma 4.3.12, if G is a V-generic filter in B then:*

$$f^{V[G]}(G) = \tau_f^G$$

Moreover, fix R (F) an n-ary Borel relation (function) over $\mathbb{C}$ and $f_1, \ldots, f_{n+1} \in \mathcal{C}^+(St(\mathsf{B}))$. Then:

$$[\![R(\tau_{f_1}, \ldots, \tau_{f_n})]\!]^{\mathbb{C}^{\mathsf{B}}} \in G \Leftrightarrow R^{V[G]}(f_1^{V[G]}(G), \ldots, f_n^{V[G]}(G))$$

$$\left[\!\left[F(\tau_{f_1} \ldots, \tau_{f_n}) = \tau_{f_{n+1}}\right]\!\right]^{\mathbb{C}^{\mathsf{B}}} \in G \Leftrightarrow F^{V[G]}(f_1^{V[G]}(G), \ldots, f_n^{V[G]}(G)) = f_{n+1}^{V[G]}(G)$$

Proof. This follows from the previous result and from the definition of boolean interpretation of Borel relations and functions in $\mathbb{C}^{\mathsf{B}}$ (Definition 4.3.3). □

The map we have built in the lemma above is surjective.

Lemma 4.3.14. *Assume $\tau \in \mathbb{C}^{\mathsf{B}}$. Consider*

$$f_\tau : St(\mathsf{B}) \to \mathbb{C} \cup \{\infty\}$$
$$H \mapsto \sigma_f^H$$

where, given

$$\Sigma_f^H = \{\overline{U}_n : \left[\!\left[\tau \in \dot{U}_n\right]\!\right] \in H\}$$

σ_f^H is the only element in $\bigcap \Sigma_f^H$ if Σ_f^H is non-empty, $\sigma_f^H = \infty$ otherwise. The function f belongs to $\mathcal{C}^+(St(\mathsf{B}))$ and $\tau_{f_\tau} = \tau$.

Proof. The proof that if Σ_f^H is non-empty then its intersection has one single point can be carried as in Claim 4.3.9.1 substituting $\overline{\mathring{f^{-1}[U_n]}}$ with $\left[\!\left[\tau \in \dot{U}_n\right]\!\right]$.

Preimage of $\{\infty\}$ is nowhere dense: We show that the preimage of $\mathbb{C}$ through f_τ contains an open dense set. We use the following abbreviation

$$a_n = \left[\!\left[\tau \in \dot{U}_n\right]\!\right]$$

and consider the set $A = \{a_n : n \in \omega\}$. First we show that:

$$\bigvee_{n \in \omega} a_n = 1_{\mathsf{B}}$$

We use Proposition 3.4.14. Let $M \prec V$ be a countable structure such that $\mathsf{B}, \tau \in M$, $\omega, A \subseteq M$, and as usual let $\pi : M \to N$ be the Mostowski's Collapse. Since τ is a B-name for a complex number in M, $\pi(\tau)$ is a $\pi(\mathsf{B})$-name for a complex number in N. Let G be an N-generic filter over $\pi(\mathsf{B})$, we have therefore:

$$N[G] \models \pi(\tau)^G \in \mathbb{C}$$

we can infer

$$N[G] \models \exists n \in \omega(\pi(\tau)^G \in U_n)$$

Since generic extensions and π preserve ω, we get that $\check{\omega}^G = \omega^{N[G]}$. Thus, by Proposition 3.4.14:

$$\bigvee_{n\in\omega} a_n = \left[\!\left[\exists n \in \check{\omega}(\tau \in \dot{U}_n) \right]\!\right] \geq 1_{\mathsf{B}}$$

This means that A is predense because if $b \in \mathsf{B}^+$:

$$b = b \wedge \bigvee_{n\in\omega} a_n = \bigvee_{n\in\omega} b \wedge a_n$$

so that $0_{\mathsf{B}} \neq a_k \wedge b \leq b$ for some $k \in \omega$. We can conclude with Lemma 3.3.4.

Continuous: Let $H \in St(\mathsf{B})$ be in the preimage of $\mathbb{C}$, hence there exists $m \in \omega$ such that $a_m \in H$ and as a consequence

$$f_\tau(H) \in \overline{U}_m$$

Let U be an open subset of $\mathbb{C}$ containing $f_\tau(H)$, and consider $U_k \in \mathcal{B}$ such that

$$f_\tau(H) \in U_k$$

$$\overline{U}_k \subseteq U_m \cap U$$

Since

$$f_\tau(H) \in U_k \Rightarrow a_k \in H \tag{1}$$

which can be showed as in the uniqueness part in Claim 4.3.9.1 substituting $\overset{\circ}{\overline{f^{-1}[U_n]}}$ with $\left[\!\left[\tau \in \dot{U}_n\right]\!\right]$, and since the following inclusion holds:

$$\mathcal{O}_{a_k} \subseteq f_\tau^{-1}(U)$$

the continuity of f_τ for points in the preimage of $\mathbb{C}$ is proved.

Consider now $H \in f_\tau^{-1}(\{\infty\})$. Let A be an open neighborhood of ∞, and let U_k be an open ball centred in zero with rational radius such that

$$\overline{U}_k^c \subseteq A$$

We also consider U_l such that

$$\overline{U}_k \subseteq U_l$$

By definition of f_τ we have that $H \in \mathcal{O}_{a_l}^c$, and by (1) the image of any element in the open set $\mathcal{O}_{a_l}^c$ can not belong to U_l. Thus

$$\mathcal{O}_{a_l}^c \subseteq f_\tau^{-1}[U_l^c] \subseteq f_\tau^{-1}[\overline{U}_k^c] \subseteq f_\tau^{-1}[A]$$

$\underline{\tau_{f_\tau} = \tau}$: We already know that (see (1)):

$$f_\tau^{-1}[U_n] \subseteq \mathcal{O}_{a_n}$$

The second set is clopen, thus:

$$\left[\!\left[\tau_{f_\tau} \in \dot{U}_n\right]\!\right] = \overline{\overset{\circ}{f_\tau^{-1}[U_n]}} \subseteq \mathcal{O}_{a_n} \tag{2}$$

Toward a contradiction, assume $[\![\tau = \tau_{f_\tau}]\!] \neq 1_{\mathsf{B}}$. By Proposition 3.4.14 we can find $M \prec V$ a countable structure with $\mathsf{B}, \tau, f \in M$, $\omega \subseteq M$, and where $\pi : M \to N$ is the Mostowski's Collapse, such that there is an N-generic filter G which verifies

$$N[G] \models \pi(\tau)^G \neq \pi(\tau_{f_\tau})^G$$

Thus there exists $n \in \omega$ such that:

$$\pi(\tau_{f_\tau})^G \in U_n^{N[G]}$$

$$\pi(\tau)^G \notin U_n^{N[G]}$$

The inclusion relation (2) implies

$$\left[\!\left[\pi(\tau_{f_\tau}) \in \dot{U}_n\right]\!\right] \leq \left[\!\left[\pi(\tau) \in \dot{U}_n\right]\!\right]$$

and by Cohen's Forcing Theorem $\left[\!\left[\pi(\tau_{f_\tau}) \in \dot{U}_n\right]\!\right] \in G$. This is a contradiction.

□

Finally we can show that the map $f \to \tau_f$ is an isomorphism of B-valued models between $\mathbb{C}^{\mathsf{B}}$ and $\mathcal{C}^+(St(\mathsf{B}))$.

Theorem 4.3.15. *Fix a set*

$$\mathcal{L} = \{R_i : i \in I\} \cup \{F_j : j \in J\}$$

where:

- *for $i \in I$, R_i is a Borel subset of $\mathbb{C}^{n_i}$;*
- *for $j \in J$, F_j is a Borel function from $\mathbb{C}^{m_j}$ to $\mathbb{C}$.*

and consider $\mathcal{C}^+(St(\mathsf{B}))$ and $\mathbb{C}^{\mathsf{B}}$ as B-valued models in the language $\mathcal{L}$ as defined in Example 4.1.4 and Definition 4.3.3 respectively. The map

$$\Omega : \mathcal{C}^+(St(\mathsf{B})) \to \mathbb{C}^{\mathsf{B}}$$
$$f \mapsto \tau_f$$

is an isomorphism of B-valued models.

Proof. We will first consider the case of $R \subseteq \mathbb{C}$ a unary Borel relation in $\mathbb{C}$. Given $f \in \mathcal{C}^+(St(\mathsf{B}))$, consider $[\![R(f)]\!]$ and $\left[\!\left[\tau_f \in \dot{R}\right]\!\right]$ as regular open subsets of $St(\mathsf{B})$. By Proposition 4.1.2, in order to show that they overlap, it is sufficient to prove that their symmetric difference is meager. By definition, we already know that $[\![R(f)]\!]$ has meager difference with the set

$$\{H \in St(\mathsf{B}) : f(H) \in R\} = f^{-1}[R]$$

Therefore it suffices to prove that $\left[\!\left[\tau_f \in \dot{R}\right]\!\right]$ and $f^{-1}[R]$ have meager difference. The proof proceeds step by step on the hierarchy of Borel sets Σ^0_α, Π^0_α, for α a countable ordinal, as defined in [6, Chapter 11, Section 1].

$\underline{\Sigma^0_1}$: Let R be an element of the basis

$$\mathcal{B} = \{U_n : n \in \omega\}$$

defined in Remark 4.3.2. The thesis follows from Lemma 4.3.12, in fact

$$\left[\!\left[\tau_f \in \dot{U}_n\right]\!\right] = \overset{\circ}{\overline{f^{-1}[U_n]}}$$

which has meager difference with $f^{-1}[U_n]$. Consider now

$$R = \bigcup_{i \in \mathcal{I}} U_i$$

where $\mathcal{I}$ is a countable set of indexes. In this case we have that

$$f^{-1}[R] = \bigcup_{i \in \mathcal{I}} f^{-1}[U_i]$$

and

$$\left[\!\left[\tau_f \in \dot{R}\right]\!\right] = \bigvee_{i \in \mathcal{I}} \left[\!\left[\tau_f \in \dot{U}_i\right]\!\right] = \overset{\circ}{\overline{A}}$$

where $A = \bigcup_{i \in \mathcal{I}} \left[\!\left[\tau_f \in \dot{U}_i\right]\!\right]$. For each $i \in \mathcal{I}$, the sets $f^{-1}[U_i]$ and $\left[\!\left[\tau_f \in \dot{U}_i\right]\!\right]$ have meager difference, thus $f^{-1}[R]\Delta A$ is meager. This is true since meager sets are closed for countable union, and the following property holds:

$$\left(\bigcup_{k \in \mathcal{K}} B_k\right) \Delta \left(\bigcup_{k \in \mathcal{K}} C_k\right) \subseteq \bigcup_{k \in \mathcal{K}} (B_k \Delta C_k)$$

The proof is therefore concluded because $A\Delta\ \overset{\circ}{\overline{A}}$ is meager.

$\underline{\Sigma^0_\alpha \Rightarrow \Pi^0_\alpha}$: Suppose $R \in \Pi^0_\alpha$, and that the thesis holds for Borel sets in Σ^0_α. By definition $R^c \in \Sigma^0_\alpha$, therefore:

$$f^{-1}[R^c]\Delta \left[\!\left[\tau_f \in \dot{R}^c\right]\!\right] \text{ is meager}$$

Since

$$B\Delta C = B^c \Delta C^c$$

we have that

$$f^{-1}[R]\Delta \left[\!\left[\tau_f \in \dot{R}^c\right]\!\right]^c \text{ is meager}$$

Thus we can conclude, in fact

$$\left[\!\left[\tau_f \in \dot{R}\right]\!\right] = \neg \left[\!\left[\tau_f \in \dot{R}^c\right]\!\right]$$

and the last set is the interior of $\left[\!\left[\tau_f \in \dot{R}^c\right]\!\right]^c$, therefore they differ by a meager set.

$\Pi^0_\alpha \Rightarrow \Sigma^0_{\alpha+1}$: This item can be proved as the second part of the case $\alpha = 1$, substituting the U_n with Borel sets in Π^0_α.

Σ^0_β for β limit ordinal: If the thesis holds for $\alpha < \beta$, then the proof can be carried similarly to the case $\Pi^0_\alpha \Rightarrow \Sigma^0_{\alpha+1}$.

This proves the theorem for unary Borel relations. The m-ary case (for both relation and function symbols) can be shown similarly starting from the open basis of $\mathbb{C}^m$

$$\{U_{n_1} \times \cdots \times U_{n_m} : n_1, \ldots, n_m \in \omega\}$$

Base step holds due to Lemma 4.3.12, and then we proceed by induction as we did earlier. Lemma 4.3.14 guarantees that Ω is surjective, thus the proof is concluded. □

The morphism Ω had to be defined on $\mathcal{C}^+(St(\mathsf{B}))$ in order to be surjective. Nevertheless, when passing to a generic extension of V, it is enough to consider the equivalence classes of the element of the C^*-algebra $\mathcal{C}(St(\mathsf{B}))$.

Proposition 4.3.16. *Assume V is a model of ZFC, B a complete boolean algebra in V and G a V-generic filter in B. Then*

$$\mathcal{C}^+(St(\mathsf{B}))/G \cong \mathcal{C}(St(\mathsf{B}))/G$$

Proof. We need to show that for each $f \in \mathcal{C}^+(St(\mathsf{B}))$ we can find a $\tilde{f} \in \mathcal{C}(St(\mathsf{B}))$ such that

$$\left[\!\left[f = \tilde{f}\right]\!\right] \in G$$

which, by Corollary 4.3.13, is equivalent to

$$f^{V[G]}(G) = \tilde{f}^{V[G]}(G)$$

We denote again:

$$a_n = \overline{\mathring{f^{-1}[U_n]}}$$

Proceeding as in Claim 4.3.12.1, we can find $m \in \omega$ such that $a_m \in G$. For each $H \in \mathcal{O}_{a_m}$ we have that:

$$f(H) \in \overline{U}_m$$

by Lemma 4.3.9. We can therefore consider the restriction of f to $\mathcal{O}_a$ and extend it to a $\tilde{f} \in \mathcal{C}(St(\mathsf{B}))$[1] The implication

$$f \restriction_{\mathcal{O}^V_{a_m}} = \tilde{f} \restriction_{\mathcal{O}^V_{a_m}} \Rightarrow f^{V[G]} \restriction_{\mathcal{O}^{V[G]}_{a_m}} = \tilde{f}^{V[G]} \restriction_{\mathcal{O}^{V[G]}_{a_m}}$$

guarantees the thesis since $G \in \mathcal{O}^{V[G]}_{a_m}$. □

[1]This is a fact of general topology, known as Tietze's Extension Theorem. For a proof of the latter see [15, Theorem 15.8].

Remark 4.3.17. Once we fix a language of Borel relations and functions in $\mathbb{C}$, the theorems proved so far guarantee that the following two first order models are isomorphic whenever G is V-generic for B:

$$\mathcal{C}(St(\mathsf{B}))/G \cong \mathbb{C}^{\mathsf{B}}/G$$

If we consider for instance $\mathcal{L} = \{+, *, ^{-}\}$, we obtain that $\mathcal{C}(St(\mathsf{B}))/G$ is an algebraically closed field which extends $\mathbb{C}$ (modulo isomorphism).

We conclude combining the results of this section with Cohen's Generic Absoluteness Theorem 3.5.3.

Theorem 4.3.18. *Let V be a transitive model of ZFC, $\mathsf{B} \in V$ which V models to be a complete boolean algebra, and G a ultrafilter on B. Assume $R_1, \dots, R_s$ are n_i-ary Borel relations and $f_1, \dots, f_t$ m_j-ary Borel functions on $\mathbb{C}$. Then*

$$\langle \mathbb{C}, R_1, \dots, R_s, f_1, \dots, f_t \rangle \prec_{\Sigma_2} \langle \mathcal{C}^+(St(\mathsf{B}))/G, R_1/G, \dots, R_s/G, f_1/G, \dots, f_t/G \rangle$$

Moreover, if G is V-generic, then the following holds:

$$\langle \mathbb{C}, R_1, \dots, R_s, f_1, \dots, f_t \rangle \prec_{\Sigma_2} \langle \mathcal{C}(St(\mathsf{B}))/G, R_1/G, \dots, R_s/G, f_1/G, \dots, f_t/G \rangle$$

Proof. Cohen's Generic Absoluteness Theorem 3.5.3 implies that

$$V \prec_{\Sigma_1} V^{\mathsf{B}}/G$$

Thus the thesis is a consequence of

$$\mathcal{C}^+(St(\mathsf{B}))/G \cong \mathbb{C}^{\mathsf{B}}/G$$

and of Lemma 25.25 in [6], which provides a translation of Σ^1_2-formulae to Σ_1-formulae in V. The second part follows from Proposition 4.3.16. □

This is the last theorem we present. The results of this chapter might be a starting point and useful tools in order to prove properties of the complex field using ideas arising from the theory of C^*-algebras, and vice versa. Applications of these results are topic of present research.

Bibliography

[1] John L. Bell, *Set theory*, third ed., Oxford Logic Guides, vol. 47, The Clarendon Press, Oxford University Press, Oxford, 2005, Boolean-valued models and independence proofs. MR 2257858 (2007d:03087)

[2] Haim Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. MR 2759829 (2012a:35002)

[3] F.R. Drake, *Set theory: An introduction to large cardinals*, Studies in logic and the Foundations of Mathematics, North-Holland, 1974.

[4] Gerald B. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR 1397028 (98c:43001)

[5] Steven Givant and Paul Halmos, *Introduction to Boolean algebras*, Undergraduate Texts in Mathematics, Springer, New York, 2009. MR 2466574 (2009j:06001)

[6] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513 (2004g:03071)

[7] Sabine Koppelberg, *Handbook of Boolean algebras. Vol. 1*, North-Holland Publishing Co., Amsterdam, 1989, Edited by J. Donald Monk and Robert Bonnet. MR 991565 (90k:06002)

[8] Kenneth Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam-New York, 1980, An introduction to independence proofs. MR 597342 (82f:03001)

[9] David Marker, *Model theory*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002, An introduction. MR 1924282 (2003e:03060)

[10] Walter Rudin, *Principles of mathematical analysis*, third ed., McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976, International Series in Pure and Applied Mathematics. MR 0385023 (52 #5893)

[11] Bryan P. Rynne and Martin A. Youngson, *Linear functional analysis*, second ed., Springer Undergraduate Mathematics Series, Springer-Verlag London, Ltd., London, 2008. MR 2370216 (2008i:46001)

[12] Joseph R. Shoenfield, *Mathematical logic*, Association for Symbolic Logic, Urbana, IL; A K Peters, Ltd., Natick, MA, 2001, Reprint of the 1973 second printing. MR 1809685 (2001h:03003)

[13] Matteo Viale, *Martin's maximum revisited*, 2015, Not yet published; http://www.personalweb.unito.it/matteo.viale/.

[14] ———, *Notes on forcing*, 2015, Not yet published; http://www.personalweb.unito.it/matteo.viale/.

[15] Stephen Willard, *General topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970. MR 0264581 (41 #9173)