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Ultraproducts of finite partial orders and some of their applications in model theory and set theory

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Introduction

This thesis analyzes the gap structures of ultraproducts of linear orders. Many of the results that we present appear in a recent article due to Malliaris and Shelah [5], but the proofs in Section 1.4 and Chapter 2 follow a presentation due to Professor J. Steprans [10]. The principal aims of the thesis are two: we prove that the two cardinal invariants $\mathfrak{p}$ and $\mathfrak{t}$ are equal and we study Keisler's order on the countable complete first order theories. In particular we give some conditions that ensure maximality in Keisler's order and prove that a large class of theories, called SOP_2 theories, is maximal with respect to this order. Both results were originally proved in [5]. We conclude the thesis with a consistency result: we prove assuming Martin's axiom that the first order theory of the random graphs is not maximal in Keisler's order.

In Chapter 1 we introduce the key definitions and prove the harder technical results. First of all we define the notion of (κ, θ) gap on a linear order $(L, <)$, that is a couple of sequences $(a_\alpha)_{\alpha\in\kappa}$, $(b_\beta)_{\beta\in\theta}$ with the property that for every $\alpha \in \kappa$ and $\beta \in \theta$ we have $a_\alpha < b_\beta$ and that no $x \in L$ separates the two sequences. For a given ultrafilter $\mathcal{U}$, we study the relation between the existence of certain gaps in ultraproducts of finite linear orders modulo $\mathcal{U}$ and two specific cardinals $\mathfrak{p}(\mathcal{U}), \mathfrak{t}(\mathcal{U})$ which are defined as follows: the first represents the minimal size of a gap in some ultraproduct of finite *linear orders* modulo $\mathcal{U}$ and the second represents the minimal size of an unbounded increasing sequence in some ultraproduct of finite *pseudo-trees* modulo $\mathcal{U}$. In the last part of the chapter, we prove the main technical result of this thesis i.e. that $\mathfrak{p}(\mathcal{U}) = \mathfrak{t}(\mathcal{U})$, hence that there exists no (κ, θ) gaps for $\kappa + \theta < \mathfrak{t}(\mathcal{U})$ on any ultraproduct of finite linear orders modulo $\mathcal{U}$. Our presentation expands on Steprans' handout [10].

In Chapter 2 we define two cardinal invariants $\mathfrak{p}$ and $\mathfrak{t}$: $\mathfrak{p}$ is the minimal size of a family $\mathcal{F} \subseteq [\omega]^{\aleph_0}$ such that every finite subfamily of $\mathcal{F}$ has infinite intersection and there exists no A such that $A \subseteq^* F$ for every $F \in \mathcal{F}$; $\mathfrak{t}$ is the minimal size of a family $\mathcal{F} \subseteq [\omega]^{\aleph_0}$ such that the order $\supseteq^*$ is a well-order on $\mathcal{F}$ and there exists no A such that $A \subseteq^* F$ for every $F \in \mathcal{F}$. In the main theorem of this chapter, we prove that $\mathfrak{p} = \mathfrak{t}$. This result appears in [5], but we follow the proof given in [10] and which can be deduced in a rather straightforward manner from the results of the first chapter. In order to obtain it, we study the relation between $\mathfrak{p}, \mathfrak{t}, \mathfrak{p}(G)$ and $\mathfrak{t}(G)$, when G is a V-generic ultrafilter over the notion of forcing $([\omega]^{\aleph_0}, \subseteq^*)$.

In Chapter 3 we first prove the existence of a special class of ultrafilters, the λ-*good* ultrafilters, and we show that for a λ-good countably incomplete ultrafilter over a set

I every ultraproduct $\prod_{i\in I}\mathcal{M}_i/\mathcal{U}$ of first order structures is λ-saturated. In the second part of the chapter we introduce the notion of *strong chain* of $\mathcal{L}$-structures, that is a sequence of structures $(\mathcal{M}_\alpha : \alpha < \kappa)$ indexed by an inaccessible cardinal κ such that: at successor steps, $\mathcal{M}_{\alpha+1}$ is an ultrapower of the structure $\mathcal{M}_\alpha$ modulo an $|\mathcal{M}_\alpha|^+$-good ultrafilter and at a limit step, $\mathcal{M}_\alpha$ is the direct limit of the already constructed structures. In the last part of the chapter we deduce a characterization of elementarily equivalent structures using the existence of strong chains of arbitrarily large size. This will require us to assume the existence of a proper class of inaccessible cardinals to obtain our characterization, more precisely we will show that: if there are class many inaccessible cardinals, two $\mathcal{L}$-structures $\mathcal{M},\mathcal{N}$ are elementarily equivalent if and only if there exists two isomorphic (and saturated) structures $\mathcal{M}',\mathcal{N}'$ (of inaccessible size $\kappa > |\mathcal{M}|,|\mathcal{N}|$) such that $\mathcal{M}\prec\mathcal{M}'$ and $\mathcal{N}\prec\mathcal{N}'$.

In Chapter 4, we define and analyze Keisler's order on the class of countable complete theories, that is we write $T_1 \trianglelefteq_\lambda T_2$, if for all models $\mathcal{M}_1,\mathcal{M}_2$ of T_1,T_2, respectively, and each regular ultrafilter $\mathcal{U}$ on λ, if $\mathcal{M}_2^\lambda/\mathcal{U}$ is λ^+ saturated, then so is $\mathcal{M}_1^\lambda/\mathcal{U}$. In section 4.1 we prove the basic properties of this order, and in section 4.2 we give a condition equivalent to maximality, more precisely we prove that a theory is maximal if and only if for all cardinals λ the only ultrafilters which saturate the ultrapowers of models of T are λ^+-good. We continue the study of maximal theories proving in section 4.3 that every SOP theory, that is a theory which can define a partial order with infinite chains, is maximal in Keisler's order. This result appears in Shelah's book "Classification theory", but here we present a simpler proof. In section 4.4 we use the theory of gaps studied in Chapter 1 to characterize the λ^+-good ultrafilter. In particular we obtain a characterization of good ultrafilters in terms of gaps that we can find in an ultrapowers of the linear order $(\omega,<)$. We next introduce the notion of *treetops*. This notion generalizes the idea of unbounded chain given in Chapter 1 and is useful to analyze the properties of unbounded increasing chains on ultraproducts of *arbitrary* pseudo-trees (i.e. the pseudo trees appearing as factors of the ultraproduct can now be infinite). We conclude this section showing that the existence of certain treetops on a given ultraproduct of pseudo-trees is equivalent to the goodness of the ultrafilter by which the ultraproduct is taken. In the last two sections (4.5, 4.6) of this chapter we first define what is the SOP_2 property, that is: T has the SOP_2-property if in some model $\mathcal{M}$ of T and for some formula $\psi(x,\bar{y})$ in the language of T, there is an interpretation of a tree $(\{\bar{a}_s \mid s\in\mu^{<\kappa}\},\trianglelefteq)$ in $\mathcal{M}$ with the property that a ψ-type with parameters in T is consistent if an only if the parameters are $\trianglelefteq$-compatible. We next show that every SOP theory has SOP_2 property and we conclude the chapter proving that every SOP_2-theory is maximal in Keisler's order, which (together with the proof that $\mathfrak{p}=\mathfrak{t}$) is one of the main results of [5].

In Chapter 5 we continue the study of Keisler's order showing that the theory of the random graph is not maximal in this order if we assume Martin's axiom. To this aim we first introduce two-step iterated ultrapowers and recall some basic facts on the first order theory of random graphs. Finally, under the assumption that Martin's axiom holds, we construct an ultrafilter $\mathcal{U}$ on $\aleph_1$ such that $\mathcal{U}$ is not $\aleph_2$-good, but each ultrapower $\mathcal{M}^{\aleph_1}/\mathcal{U}$

is $\aleph_2$-saturated for every random graph $\mathcal{M}$. In this way we deduce that the statement *the theory of random graphs is not maximal in Keisler's order* is consistent with ZFC.

In Appendix A we give a short introduction to the method of forcing sufficient to understand its use in Chapter 2. In the last part of the appendix, we introduce Martin's axiom and prove some of its consequences.

In Appendix B, we give a brief introduction to model theory recalling some classical results such as: the Compactness Theorem, the Löwenheim-Skolem's Theorem, and Łoś's Theorem. In the last part of this appendix, we prove that the theory of discrete linear orders with minimum element and without maximum has quantifier elimination in the language $\{<, s, 0\}$.

Notations

$X^{<M}$	when M is a linear order, this is the set of all functions from an initial segment of M to X
$\mathrm{pred}(p)$	the set of the predecessors of p
0_L	the minimum element L
1_L	the maximum element L
$x+1$	the immediate successor of x in a dicrete linear order
$x+n$	in a discrete linear order this is the element obtained by x applying n times the function successor
$x-1$	the immediate predecessor of x in a dicrete linear order
$x-n$	in a discrete linear order this is the element obtained by x applying n times the function predecessor
$\mathcal{P}(X)$	the set of all subsets of X
$\mathrm{dom}(p)$	the domain of the function p
$\mathrm{range}(p)$	the range of the function p
$p=(p^0,p^1)\colon A\to B\times C$	p^0 and p^1 are the projection of p into B and C, respectively
$cof(\alpha)$	the cofinality of α
$[\kappa]^\lambda$	the set of all subsets of κ of cardinality λ
$\dot{a}$	the name of a set $a\in V[G]$
$\check{a}$	the canonical name of a set $a\in V$
$\mathcal{M}^{\mathbb{P}}$	the class of $\mathbb{P}$-names
$\mathcal{M}[G]$	the generic extension of a transitive model $\mathcal{M}$
Γ	the canonical name of a $\mathcal{M}$-generic filter
$\Vdash$	the forcing relation
$f<^* g$	holds if $f(n)<g(n)$ for all but finitely many $n\in\omega$
$S_\omega(X)$	the set of all finite subsets of X
$Th(\mathcal{M})$	the set of all sentences ϕ such that $\mathcal{M}\models\phi$
$\models$	the satisfaction relation
$\mathcal{M}\equiv\mathcal{N}$	the structures $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent
$\mathcal{M}\preccurlyeq\mathcal{N}$	$\mathcal{M}$ is an elementary substructure of $\mathcal{N}$
$\varinjlim_{i\in I}\mathcal{M}_i$	the direct limit of the family $\{\mathcal{M}_i \mid i\in I\}$
$[F]$	the filter generated by F
$\trianglelefteq$	Keisler's order
MA	Martin's axiom
$\mathrm{Fn}(I,J)$	the set of all finite function from I to J
T_{rg}	the theory of random graphs
$\mathcal{U}\otimes\mathcal{V}$	the tensor ultrafilter of $\mathcal{U}$ and $\mathcal{V}$
$\mathrm{TC}(X)$	the transitive closure of X

Chapter 1

Gaps in infinite ultraproducts

In this chapter we introduce the notion of (κ, θ) *gap* in linear orders, that is a couple of sequences in a linear order such that there is no element that separates the sequences. Many of the results, that we present, appear in a recent article by Malliaris and Shelah [5], that studies the theory of gaps in more general way. Here we concentrate our attention on the existence of gaps in ultraproducts of finite linear orders modulo a fixed ultrafilter $\mathcal{U}$. To this aim we define two cardinals $\mathfrak{p}(\mathcal{U})$ and $\mathfrak{t}(\mathcal{U})$: the first cardinal represents the minimal size of a gap in some ultraproduct of finite linear orders modulo $\mathcal{U}$ and the second represents the minimal size of an unbounded increasing sequence in some ultraproduct of finite *pseudo-trees*.

In Section 1.1, we concentrate our attention on ultraproducts of finite partial orders modulo an ultrafilter $\mathcal{U}$ and we give the definitions of gap and of cardinals $\mathfrak{p}(\mathcal{U})$ and $\mathfrak{t}(\mathcal{U})$. In the last part of this section, we recall the notion of *internal* subset of an ultraproduct and we study some basic properties that these sets have.

In Section 1.2, we show the relation between the size of *symmetric* gaps, that is (κ, κ) gaps, and the cardinals $\mathfrak{p}(\mathcal{U}), \mathfrak{t}(\mathcal{U})$. In particular, we construct symmetric gaps when $\kappa = \mathfrak{t}(\mathcal{U})$ and we prove that there exists no (κ, κ) gaps for "small" κ.

In Section 1.3, we show that the structure of certain gap is rigid, that is, for some κ, the existence of (κ, θ) gaps characterizes uniquely θ. Moreover, for these cardinals κ, the (κ, θ) gap appears in every ultraproduct of linear orders modulo $\mathcal{U}$.

The results of Sections 1.4 appear for the first time in [5], but the proofs, that we present, are due to J. Steprans and appear in a manuscript not published. After proving some technical lemmas, we give the proof of the main theorem of this chapter. In particular, we show that $\mathfrak{t}(\mathcal{U}) = \mathfrak{p}(\mathcal{U})$, hence there are no (κ, θ) gaps for $\kappa + \theta < \mathfrak{t}(\mathcal{U})$.

1.1 Some basic definitions

In the rest of the chapter, we concentrate our attention on ultraproducts of finite orders, see Appendix B for a brief introduction. From now on, we fix an infinite cardinal λ and a non-principal ultrafilter $\mathcal{U}$ on λ.

Definition 1.1.1. Denote by $\mathbb{L}(\mathcal{U})$ the class of all ultraproducts on λ modulo $\mathcal{U}$ of finite linear orders with cardinality not uniformly bounded (i.e. ultraproducts $\prod_{i\in I} L_i/\mathcal{U}$ such that

$$\{i \in I : |L_i| > n\} \in \mathcal{U}$$

for all $n \in \mathbb{N}$).

Definition 1.1.2. Let M be a linear order and X be a set. We write $X^{<M}$ to indicate the set of all functions from an initial segment of M to X. For a subset P of $X^{<M}$ we say that $(P, \subseteq)$ is a *pseudo-tree*, if it is closed under initial segments: that is, if $t \in P$ and $s \subseteq t$, then $s \in P$. When the sets X, M are finite, we say that $(P, \subseteq)$ is a *finite pseudo-tree.*

Note that a pseudo-tree $(P, \subseteq)$ has always a unique root $\emptyset$ and, for each $p \in P$, the set $\mathrm{pred}(p) = \{s \in P \mid s \subseteq p\}$ is linearly ordered by inclusion.

Definition 1.1.3. Let $\mathbb{P}(\mathcal{U})$ be the class of ultraproducts $\prod_{i\in\lambda}(P_i, \subseteq)/\mathcal{U}$, where $(P_i, \subseteq)$ is a finite pseudo-tree and the cardinality of the sets is not uniformly bounded.

Remark 1.1.4. By Łoś's Theorem B.1.13 and the hypothesis that the cardinality of the sets is not uniformly bounded, we have that the ultraproducts of $\mathbb{L}(\mathcal{U})$ and $\mathbb{P}(\mathcal{U})$ are not finite.

Now we fix some notation.

Notation. *Every L in $\mathbb{L}(\mathcal{U})$ has a minimum and a maximum, that we indicate with 0_L, 1_L, respectively. By Łoś's Theorem B.1.13, if L is in $\mathbb{L}(\mathcal{U})$, then every element $x \neq 0$, $x \neq 1$ of L has an immediate successor and predecessor, that we indicate with $x + 1$, $x - 1$, respectively. Moreover, for $n \in \omega$ we write $x + n$, $x - n$ to indicate $x + \underbrace{1 + \ldots + 1}_{k \text{ times}}$ and $x - \underbrace{1 - \ldots - 1}_{k \text{ times}}$, respectively, unless it is well defined.*

Definition 1.1.5. In our notations, we say that x is *near* 1, if there exists $n \in \omega$ such that $x + n = 1$.

Now we define the objects of our studies.

Definition 1.1.6. Let $(X, <)$ be an infinite partial order and κ be an infinite regular cardinal. We say that a sequence $(c_\alpha)_{\alpha\in\kappa}$ is *unbounded* in X, if there exists no $c \in X$ such that $c_\alpha < c$ for all $\alpha \in \kappa$.

Let $(X, <)$ be an infinite linear order and κ_1, κ_2 be an infinite regular cardinals. We say that two sequences $(x_\alpha)_{\alpha\in\kappa_1}$, $(y_\alpha)_{\alpha\in\kappa_2}$ represent a (κ_1, κ_2) *gap* in X, if the following properties hold:

(i) $x_\beta < x_\alpha < y_{\alpha'} < y_{\beta'}$, for all $\beta \in \alpha \in \kappa_1$ and $\beta' \in \alpha' \in \kappa_2$.

(ii) There exists no $z \in X$ such that $x_\alpha \leq z \leq y_{\alpha'}$ for all $\alpha \in \kappa_1$ and $\alpha' \in \kappa_2$.

Now we let

$$\mathfrak{C}(\mathcal{U}) = \{(\kappa_1, \kappa_2) \mid \text{there exists a } (\kappa_1, \kappa_2) \text{ gap in some linear order } L \in \mathbb{L}(\mathcal{U})\}.$$

In order to study the size of the gaps of a linear orders of $\mathbb{L}(\mathcal{U})$, it is natural define these cardinals: $\mathfrak{p}(\mathcal{U})$ is the minimum of the set

$$\{\kappa \mid \text{ there exists } (\kappa_1, \kappa_2) \in \mathfrak{C}(\mathcal{U}) \text{ such that } \kappa_1 + \kappa_2 = \kappa\}$$

and $\mathfrak{t}(\mathcal{U})$ is the minimum of the set

$$\{\kappa \geq \aleph_0 \mid \kappa \text{ is regular and there is an increasing unbounded sequence } (x_\alpha)_{\alpha\in\kappa} \text{ in some } P \text{ of } \mathbb{P}(\mathcal{U})\ \}.$$

Finally set

$$CSP(\mathcal{U}) = \{(\kappa_1, \kappa_2) \in \mathfrak{C}(\mathcal{U}) \mid \kappa_1 + \kappa_2 < \mathfrak{t}(\mathcal{U})\}.$$

Remark 1.1.7. If L is in $\mathbb{L}(\mathcal{U})$, then no infinite sequence $(x_\alpha)_{\alpha\in\kappa}$ is cofinal in L, since L has a maximum element.

An elementary fact about the ultraproducts of finite partial orders is that lots of their bounded subsets have neither minimum nor maximum. But there are special subsets that have many of the properties that we want.

Definition 1.1.8. Let $X = \prod_{i\in\lambda} X_i/\mathcal{U}$ be an ultraproduct of $\mathcal{L}$-structures. A subset Y of X is *internal*, if there exists a sequence $(Y_i)_{i\in\lambda}$ such that the following holds for all $x \in X$ and $i \in \lambda$:

1) $Y_i \subseteq X_i$;

2) $x \in Y$ if and only if $\{i \in \lambda \mid x(i) \in Y_i\} \in \mathcal{U}$.

In a similar way, a map $f \colon X^n \to X$ is *internal*, if there exists a sequence $(f_i)_{i\in\lambda}$ such that

1) $f_i \colon X_i^n \to X_i$;

2) $f(x_1, \ldots, x_n) = y$ if and only if $\{i \in \lambda \mid f_i(x_1(i), \ldots, x_n(i)) = y(i)\} \in \mathcal{U}$, for all $(x_1, \ldots, x_n) \in X^n$.

Now it is obvious that a non-empty internal set Y of P in $\mathbb{P}(\mathcal{U})$ has minimum and maximum, where $\min(Y) = [(\min(Y_i))_{i\in\lambda}]$ and $\max(Y) = [(\max(Y_i))_{i\in\lambda}]$.

Lemma 1.1.9. *Let $X = \prod_{i\in\lambda} X_i/\mathcal{U}$ be an ultraproduct of $\mathcal{L}$-structures. The set of all internal subsets of X is closed under finite unions, finite intersections and complements. Moreover, for every $\mathcal{L}$-formula $\psi(x)$, the set*

$$\psi(X) = \{[a] \in X \mid X \models \psi([a])\}$$

is internal.

Proof. Assume that A, B are internal subsets of X, that is there exists two sequences $(A_i)_{i\in\lambda}$ and $(B_i)_{i\in\lambda}$ such that the clauses $1), 2)$ of the above definition hold. We have

$$\begin{aligned} x \in A \cup B &\iff x \in A \text{ or } x \in B \\ &\iff \{i \in \lambda \mid x(i) \in A_i\} \in \mathcal{U} \text{ or } \{i \in \lambda \mid x(i) \in B_i\} \in \mathcal{U} \\ &\iff \{i \in \lambda \mid x(i) \in A_i \cup B_i\} \in \mathcal{U}, \end{aligned}$$

where in last equivalence we have used that $\mathcal{U}$ is an ultrafilter, and in the same way

$$\begin{aligned} x \in X \setminus A &\iff x \notin A \\ &\iff \{i \in \lambda \mid x(i) \in A_i\} \notin \mathcal{U} \\ &\iff \{i \in \lambda \mid x(i) \notin A_i\} \in \mathcal{U}. \end{aligned}$$

To prove the second part of the lemma, put

$$\psi(X_i) = \{a \in X_i \mid X_i \models \psi(a)\}.$$

By Loś's Theorem B.1.13, the sequence $((\psi(X_i))_{i\in\lambda}$ witnesses that $\psi(X)$ is internal, in fact

$$\begin{aligned} [a] \in \psi(X) &\iff X \models \psi([a]) \\ &\iff \{i \in \lambda \mid X_i \models \psi(a(i))\} \in \mathcal{U} \\ &\iff \{i \in \lambda \mid a(i) \in \psi(X_i)\} \in \mathcal{U}. \end{aligned}$$

□

1.2 On the existence of (κ, κ) gaps

Notation. *Given a cartesian product $B \times C$ we indicate with π_B and π_C the projections into B and C, respectively. When p is a function from A to $B \times C$, we shall write often $p = (p^0, p^1)$, where $p^0 = \pi_B \circ p$ and $p^1 = \pi_C \circ p$.*

Theorem 1.2.1 (Lemma 6.1 [5])**.** *Let κ be a regular cardinal such that $\kappa < \mathfrak{t}(\mathcal{U})$ and $\kappa \le \mathfrak{p}(\mathcal{U})$. Then we have that $(\kappa, \kappa) \notin \mathfrak{C}(\mathcal{U})$.*

Proof. Assume for contradiction that $(L, \le) = \prod_{i\in\lambda}(L_i, \le_i)/\mathcal{U}$ has a (κ, κ) gap, witnessed by the sequences $(a_\alpha)_{\alpha\in\kappa}$ and $(b_\alpha)_{\alpha\in\kappa}$. Put P_i the set of all function $p\colon L_i \to L_i^2$ such that:

1. $\mathrm{dom}(p)$ is an initial segment of L_i.
2. $p^0(d) <_i p^0(d') <_i p^1(d') <_i p^1(d)$, where $p = (p^0, p^1)$ and $d <_i d'$ belong to $\mathrm{dom}(p)$.

Put $(P, \sqsubseteq) = \prod_{i\in\lambda}(P_i, \subseteq)/\mathcal{U}$ and note that $(P, \sqsubseteq)$ belongs to $\mathbb{P}(\mathcal{U})$. We construct inductively an increasing sequence $(c_\alpha)_{\alpha\in\kappa}$ of P such that $c_\alpha(d_\alpha) = (c^0_\alpha(d_\alpha), c^1_\alpha(d_\alpha)) = (a_\alpha, b_\alpha)$, where d_α is the maximal element of $\mathrm{dom}(c_\alpha)$, and it is not near 1. Put $c_0 = [(p_i)]$, where

p_i has domain the minimum of L_i and range $\{(a_0, b_0)\}$. In the successor step, put $c_{\alpha+1} = c_\alpha \cup \{(d_\alpha + 1, (a_{\alpha+1}, b_{\alpha+1}))\}$ and note that $d_{\alpha+1} = d_\alpha + 1$ is well defined, since d_α is not near 1. In the limit step, assume that $(c_\beta)_{\beta\in\alpha}$ is defined, then there exists $c \in P$ such that $c_\beta \sqsubseteq c$ for any $\beta \in \alpha$, since $\alpha \in \kappa < \mathfrak{t}(\mathcal{U})$. Let d_c be the maximal element of the dom(c). Put

$$d_\alpha = \max\{e \le d_c \mid c^0(e) < a_\alpha < b_\alpha < c^1(e)\},$$

and note that d_α is well defined, since it is the maximum of an internal not empty set. If d_α is not near 1, we complete the construction defining $c_\alpha = c\big|_{\{e<d_\alpha\}} \cup \{(d_\alpha, (a_\alpha, b_\alpha))\}$. Otherwise note that the sequences $(d_\beta)_{\beta\in\alpha}$, $(d_\alpha - n)_{n\in\omega}$ don't represent a $(cof(\alpha), \aleph_0)$ gap, since $cof(\alpha)+\aleph_0 = cof(\alpha) < \mathfrak{p}(\mathcal{U})$. Hence there exists $\tilde{d}_\alpha$ such that $d_\beta < \tilde{d}_\alpha < d_\alpha - n$ for any $\beta \in \alpha$, $n \in \omega$. The element $\tilde{d}_\alpha$ is not near 1 and

$$c^0(\tilde{d}_\alpha) < c^0(d_\alpha) < a_\alpha < b_\alpha < c^1(d_\alpha) < c^1(\tilde{d}_\alpha),$$

hence we can define $c_\alpha = c\big|_{\{e<\tilde{d}_\alpha\}} \cup \{(\tilde{d}_\alpha, (a_\alpha, b_\alpha))\}$. This completes the construction. By hypothesis there exists $c \in P$ such that $c_\alpha \sqsubseteq c$ for each $\alpha \in \kappa$. Let d be the maximal element of dom(c), we have

$$a_\alpha = c^0(d_\alpha) < c^0(d) < c^1(d) < c^1(d_\alpha) = b_\alpha,$$

for all $\alpha \in \kappa$, contradiction. □

Theorem 1.2.2 (Lemma 6.2 [5])**.** *If $\kappa = \mathfrak{t}(\mathcal{U})$, then $(\kappa,\kappa) \in \mathfrak{C}(\mathcal{U})$.*

Proof. Let $(P, \sqsubseteq) = \prod_{i\in\lambda}(P_i, \subseteq)/\mathcal{U}$ be an ultraproduct of finite pseudo-tree, such that there exists an increasing unbounded sequence $(c_\alpha)_{\alpha\in\kappa}$ in $(P, \sqsubseteq)$. Assume that $P_i \subseteq X_i^{<M_i}$, where M_i is a finite linear order. For $i \in \lambda$, choose a linear order $<_i$ on X_i. Note that if $p, q \in P_i$ are $\subseteq$-incompatible, there exists a maximal $s_{pq} \in P_i$ and $n_p, n_q \in X_i$ such that $s_{pq} \subseteq p, q$, $s_{pq}^\frown n_p \subseteq p$ and $s_{pq}^\frown n_q \subseteq q$. Define a binary relation $\prec_i$ on $Q_i = P_i \times \{0, 1\}$:

1) If $p = q$, then $(p, 0) \prec_i (q, 1)$.

2) If $p \subseteq q$, then $(p, 0) \prec_i (q, 0) \prec_i (q, 1) \prec_i (p, 1)$.

3) If p, q are $\subseteq$-incompatible and $n_p <_i n_q$, then $(p, h) \prec_i (q, j)$ for $h, j \in \{0, 1\}$.

Claim 1.2.3. *The binary relation $\prec_i$ on Q_i is linear order.*

Proof. By definition, the relation $\prec_i$ is clearly irreflexive and for every two elements $(p, l), (q, j)$ we have $(p, l) \prec_i (q, j)$ or $(q, j) \prec_i (p, l)$. Hence it is sufficient to prove that $\prec_i$ is transitive. Assume that $(p, l) \prec_i (q, j) \prec_i (r, k)$. There are four possible cases:

(i) Assume that $p \subseteq q \subseteq r$. By definition, we have necessarily $l = 0$ and $p \subseteq r$. Hence $(p, 0) \prec_i (r, k)$ holds for each $k \in \{0, 1\}$.

(ii) Assume that $p \subseteq q$ and q, r are $\subseteq$-incompatible. Hence we have $l = 0$. If $p \subseteq r$, then $(p, 0) \prec_i (r, k)$ for each $k \in \{0, 1\}$. Otherwise, p and r are $\subseteq$-incompatible, hence $n_{qr} = n_{pr}$. Since we have $(q, j) \prec_i (r, k)$, we conclude that $(p, l) \prec_i (r, k)$.

(iii) Assume that p, q are $\subseteq$-incompatible and $q \subseteq r$. Clearly p, r are $\subseteq$-incompatible and $s_{pr} = s_{pq}$. By clause 3), we conclude that $(p, l) \prec_i (r, j)$.

(iv) Assume that p, q are $\subseteq$-incompatible and q, r are $\subseteq$-incompatible. If $s_{pq} = s_{qr}$, then we have $n_p <_i n_q <_i n_r$. Hence we conclude $(p, l) \prec_i (r, k)$. If $s_{pq} \subseteq s_{qr}$, we have $s_{pq} = s_{pr}$, hence $(p, l) \prec_i (r, k)$. Finally, if $s_{qr} \subseteq s_{pq}$, we obtain $s_{qr} = s_{pr}$, hence the thesis.

□

Put $(Q, \prec) = \prod_{i\in\lambda}(Q_i, \prec_i)/\mathcal{U}$. For $j = 0$ or $j = 1$, denote by $f^j \in \{0,1\}^\lambda$ the constant function j. We show that the sequences $(c^0_\alpha)_{\alpha\in\kappa}$ and $(c^1_\alpha)_{\alpha\in\kappa}$ represent a (κ, κ) gap in Q, where $c^j_\alpha = (c_\alpha, f^j)$. For $\beta \le \alpha \in \lambda$, the set

$$\{i \in \lambda \mid c_\beta(i) \subseteq c_\alpha(i)\} \in \mathcal{U}$$

is contained in $\{i \in \lambda \mid c^0_\beta(i) \prec_i c^0_\alpha(i) \prec_i c^1_\alpha(i) \prec_i c^1_\beta(i)\}$, hence $c^0_\beta \prec c^0_\alpha \prec c^1_\alpha \prec c^1_\beta$. Now suppose for a contradiction that there exists $(c, f) \in Q$ such that $c^0_\alpha \prec (c, f) \prec c^1_\beta$ for any $\alpha, \beta \in \kappa$. Without loss of generality we can assume $f = f^0$ or $f = f^1$. If $f = f^0$, there exists $\alpha \in \kappa$ such that $c_\alpha \not\subseteq c$, since $(c_\alpha)_{\alpha\in\kappa}$ is unbounded. We have $c^0_\alpha \prec (c, f^0)$, hence for almost all $i \in \lambda$ the clause 3) holds, then we obtain $(c^1_\alpha) \prec (c, f^0) \prec (c^1_\alpha)$, contradiction. In a similar way we conclude if $f = f^1$. □

Corollary 1.2.4. *If $\mathcal{U}$ is an ultrafilter on λ, then $\mathfrak{p}(\mathcal{U}) \le \mathfrak{t}(\mathcal{U})$.*

Proof. By Theorem 1.2.2, there exists a $(\mathfrak{t}(\mathcal{U}), \mathfrak{t}(\mathcal{U}))$ gap. □

1.3 On the existence of (κ, θ) gaps

Given a (κ, θ) gap in L, it is easy to see that there is a linear order of $\mathbb{L}(\mathcal{U})$ with a (θ, κ) gap. In fact, it is sufficient consider the set L with the dual order. The next simple claim shall be very useful to characterizes the structure of the (κ, θ) gaps.

Lemma 1.3.1. *Let L be a linear order of $\mathbb{L}(\mathcal{U})$. Assume that the sequences $(a_\xi)_{\xi\in\kappa}$, $(b^0_\xi)_{\xi\in\theta_0}$ and $(a_\xi)_{\xi\in\kappa}$, $(b^1_\xi)_{\xi\in\theta_1}$ witness a (κ, θ_0) gap and a (κ, θ_1) gap in L, respectively. Then $\theta_0 = \theta_1$.*

Proof. Assume for a contradiction that $\theta_0 < \theta_1$. Consider the map $f \colon \theta_0 \to \theta_1$ such that

$$f(\xi) = \min\{\gamma \in \theta_1 \mid b^1_\gamma < b^0_\xi\}.$$

The map f is well defined, since for every b^0_ξ there exists a b^1_γ such that $b^1_\gamma < b^0_\xi$. Moreover, the map f is cofinal in θ_1. In fact, if $\gamma \in \theta_1$, then there exists $b^0_\xi < b^1_\gamma$, since $(a_\xi)_{\xi\in\kappa}$, $(b^0_\xi)_{\xi\in\theta_0}$ witness a gap; hence we conclude $f(\xi) > \gamma$, absurd. □

Theorem 1.3.2. *Let κ be a regular infinite cardinal, such that $\kappa < \mathfrak{t}(\mathcal{U})$ and $\kappa \leq \mathfrak{p}(\mathcal{U})$. In every infinite linear order L of $\mathbb{L}(\mathcal{U})$, there exist a (κ, θ_0) and a (θ_1, κ) gap, for some infinite regular cardinals θ_0, θ_1.*

Proof. Fix an ultraproduct $(L, \leq) = \prod_{i\in\lambda}(L_i, \leq_i)/\mathcal{U}$ of linear finite orders. We prove the existence of a (κ, θ) in L for some regular cardinal θ. Let $(c_\alpha)_{\alpha\in\kappa}$ be a sequence such that every c_α is not near 1. Choose $c_0 = 0_L$ and $c_{\alpha+1} = c_\alpha + 1$. If α limit ordinal, there exists a d_α not near 1 such that $d_\beta < d_\alpha$. Otherwise the sequences $(d_\beta)_{\beta\in cof(\alpha)}$, $(1-n)_{n\in\omega}$ represent a $(cof(\alpha), \aleph_0)$ gap, but $cof(\alpha)+\aleph_0 = cof(\alpha) < \mathfrak{p}(\mathcal{U})$, contradiction. When the construction is completed, note that the sequence $(c_\alpha)_{\alpha\in\kappa}$ is not unbounded in L. Hence the set $A = \{a \in L \mid c_\alpha \leq a \text{ for all } \alpha \in \kappa\}$ is not empty. Let θ_0 be the cofinality of A, considered with the dual order. Note that θ_0 is not finite, otherwise the set A has a minimum a, hence the sequence $(c_\alpha)_{\alpha\in\kappa}$ is cofinal below a. Since a has an immediate predecessor, we obtain a contradiction. We conclude that there exists a (κ, θ_0) gap in L.

To prove the existence of a (θ_1, κ) gap in L, put $c_0 = 1$ and $c_{\alpha+1} = c_\alpha - 1$. In the limit step, use the hypothesis $\kappa \leq \mathfrak{t}(\mathcal{U})$, to choose c_α such that $c_\alpha < c_\beta$ for all $\beta \in \alpha$ and $c_\alpha - n \neq 0$ for all $n \in \omega$. As above it easy to conclude that there exists a (θ_1, κ) in L. □

Theorem 1.3.3 (Theorem 3.2 [5])**.** *Let κ a regular infinite cardinal, such that $\kappa < \mathfrak{t}(\mathcal{U})$ and $\kappa \leq \mathfrak{p}(\mathcal{U})$. Then there exists a unique regular cardinal θ such that $(\kappa, \theta) \in \mathfrak{C}(\mathcal{U})$.*

Proof. By Theorem 1.3.2, it is sufficient to prove that the cardinal θ is unique. Assume for a contradiction that $\theta_0 < \theta_1$ and the sequences $(a^0_\xi)_{\xi\in\kappa}$, $(b^0_\xi)_{\xi\in\theta_0}$ and $(a^1_\xi)_{\xi\in\kappa}$, $(b^1_\xi)_{\xi\in\theta_1}$ represent a (κ, θ_0) and (κ, θ_1) gap in $(M, \leq_M)$ and $(N, \leq_N)$, respectively, where $M = \prod_{i\in\lambda}(M_i, \leq_{M_i})/\mathcal{U}$ and $N = \prod_{i\in\lambda}(N_i, \leq_{N_i})/\mathcal{U}$. Let P_i be the set of all functions p with the following properties:

1. Domain of p is an initial segment of $M_i \sqcup N_i$ and range of p is a subset of $M_i \times N_i$.

2. If $d < d'$ belong to dom(p), then $p^0(n) < p^0(n')$ and $p^1(n) < p^1(n')$, where $p = (p^0, p^1)$.

Put $(P, \sqsubseteq) = \prod_{i\in\lambda}(P_i, \subseteq)/\mathcal{U}$. We construct inductively an increasing sequence $(c_\alpha)_{\alpha\in\kappa}$ in P such that $c_\alpha(d_\alpha) = (c^0(d_\alpha), c^1(d_\alpha)) = (a^0_\alpha, a^1_\alpha)$, where d_α the maximal element of dom(c_α) and it is not near 1. For $\alpha = 0$, put $c_0 = [(p_i)_{i\in\lambda}]$, where p_i has domain the minimum of $M_i \sqcup N_i$ and range $\{(a^0_0, a^1_0)\}$. For $\alpha + 1$, put $c_{\alpha+1} = c_\alpha \cup \{(d_\alpha + 1, (a^0_{\alpha+1}, a^1_{\alpha+1}))\}$ and note that $d_{\alpha+1}$ is well defined since d_α is not near 1. For α limit ordinal, assume that $(c_\beta)_{\beta\in\alpha}$ is defined, then there exist a $c \in P$ such that $c_\beta \sqsubseteq c$ for any $\beta \in \alpha$, since $\alpha \in \kappa < \mathfrak{t}(\mathcal{U})$. Let d_c be the maximal element of dom(c). Put

$$d_\alpha = \max\{e \leq d_c \mid c^0(e) < a^0_\alpha,\ c^1(e) < a^1_\alpha\}$$

and note that d_α is well defined since it is the maximum of an internal not empty set. If d_α is not near 1, define $c_\alpha = c\big|_{\{e<d_\alpha\}} \cup \{(d_\alpha, (a^0_\alpha, a^1_\alpha))\}$. Otherwise choose $\tilde{d}_\alpha$ such that

$d_\beta < \tilde{d}_\alpha < d_\alpha - n$ for any $\beta \in \alpha$, $n \in \omega$, it is possible since $cof(\alpha) + \aleph_0 = cof(\alpha) < \mathfrak{t}(\mathcal{U})$. Hence define $c_\alpha = c|_{\{e<\tilde{d}_\alpha\}} \cup \{(\tilde{d}_\alpha, (a^0_\alpha, a^1_\alpha))\}$. This completes the construction. By hypothesis there exists $c \in P$ such that $c_\alpha \sqsubseteq c$ for each $\alpha \in \kappa$. Let d be the maximal element of $\mathrm{dom}(c)$. To complete the proof, we shall construct two sequences $(d^0_\xi)_{\xi\in\theta_0}$ and $(d^1_\xi)_{\xi\in\theta_1}$ such that $(d_\xi)_{\xi\in\kappa}$, $(d^0_\xi)_{\xi\in\theta_0}$ and $(d_\xi)_{\xi\in\kappa}$, $(d^1_\xi)_{\xi\in\theta_1}$ witness a (κ,θ_0) gap and a (κ,θ_1) gap in $\prod_{i\in\lambda}(M_i \sqcup N_i, \leq_{M_i\sqcup N_i})/\mathcal{U}$. When the construction is complete we obtain a contradiction by Lemma 1.3.1. The construction of the sequences is similar, hence it is sufficient construct the sequence $(d^0_\xi)_{\xi\in\theta_0}$. If d^0_ξ is defined, put

$$d^0_{\xi+1} = \max(\{d \in \mathrm{dom}(c) \mid c^0(d) \leq_M b^0_{\xi+1},\ d < d^0_\xi\}).$$

Note that the maximum exists since the set is not empty and internal, moreover $d_\eta < d^0_{\xi+1} < d^0_\xi$ for all $\eta \in \kappa$. Suppose ξ limit ordinal and that the sequence $(d^0_\eta)_{\eta\in\xi}$ has the property that $c^0(d^0_\eta) \leq_M b^0_\eta$ for all $\eta \in \xi$. We say that there is d with the following properties:

1. $d \in \mathrm{dom}(c)$.
2. $c^0(d) \leq b^0_\xi$.
3. $d_\gamma < d$ for each $\gamma \in \kappa$.
4. $d < d^0_\eta$ for each $\eta \in \xi$.

Assume for a contradiction that such an element d don't exists. Let $f\colon \xi \to \theta_0$ be such that

$$f(\eta) = \min\{\gamma \in \theta_0 \mid c^0(d^0_\eta) > b^0_\gamma\}.$$

Claim 1.3.4. *The map f is well defined and cofinal in θ_0.*

Proof. . The map f is well defined since $(a^0_\xi)_{\xi\in\kappa}$, $(b^0_\xi)_{\xi\in\theta_0}$ represent a (κ,θ_0) gap in M and $c^0(d^0_\eta) > c^0(d_\xi) = a^0_\xi$ for any $\xi \in \kappa$. Finally, we show that f is cofinal. Fix $\gamma \in \theta_0$. If $\gamma \in \xi$, then we conclude that $f(\gamma) \geq \gamma$. In the other case, we know that

$$e = \max\{d \in \mathrm{dom}(c) \mid c^0(e) \leq_M b^0_\gamma\}$$

has the properties $1), 2), 3)$, since $b^0_\gamma < b^0_\xi$. Then $e \geq d^0_\beta$ for some $\beta \in \xi$ and $b^0_\gamma \geq c^0(e) \geq c^0(d^0_\beta)$ holds; hence we conclude $f(\beta) \geq \gamma$. □

By the Claim 1.3.4 and the regularity of θ_0, we obtain a contradiction, then there exists d with the properties $1), 2), 3), 4)$. This completes the construction of sequence $(d^0_\xi)_{\xi\in\theta_0}$. Finally, we have to prove that the sequences $(d_\xi)_{\xi\in\kappa}$ and $(d^0_\xi)_{\xi\in\theta_0}$ witness a (κ,θ) gap. By construction we have

$$d_\xi < d_\eta < d^0_{\eta_0} < d^0_{\xi_0},$$

for all $\xi \in \eta \in \kappa$ and $\xi_0 \in \eta_0 \in \theta_0$. If there exists a x such that

$$d_\xi < x < d^0_{\xi_0}$$

for all $\xi \in \kappa$ and $\xi_0 \in \theta_0$, then

$$a^0_\xi = c^0(d_\xi) < c^0(x) < c^0(d^0_{\xi_0}) \leq b^0_{\xi_0}$$

for all $\xi \in \kappa$ and $\xi_0 \in \theta_0$. We obtain a contradiction since the sequences $(a^0_\xi)_{\xi\in\kappa}$ and $(b^0_\xi)_{\xi\in\theta_0}$ witness a (κ, θ_0) in $(M, \leq_M)$. □

Corollary 1.3.5. *Let κ be an infinite regular cardinal such that $\kappa < \mathfrak{t}(\mathcal{U})$ and $\kappa \leq \mathfrak{p}(\mathcal{U})$. If there is no (κ, θ) gap in some linear order L of $\mathbb{L}(\mathcal{U})$, then $(\kappa, \theta) \notin \mathfrak{C}(\mathcal{U})$.*

Proof. By theorems 1.3.2 and 1.3.3, if (κ, θ) belongs to $\mathfrak{C}(\mathcal{U})$, then there exists a (κ, θ) gap in every L. □

1.4 $\mathfrak{p}(\mathcal{U}) = \mathfrak{t}(\mathcal{U})$

The aim of this section is to prove that for all ultrafilter $\mathcal{U}$ that the cardinals $\mathfrak{p}(\mathcal{U}), \mathfrak{t}(\mathcal{U})$ are equal. This result is proved by Malliaris and Shelah in [5], but here we present the proof due to Steprans [10]. In order to prove the main theorem, we need the following technical lemmas.

Lemma 1.4.1. *Let $\{(X_i, \leq_i)\}_{i\in\lambda}$ be a family of linear orders and $\mathcal{U}$ be a non-principal ultrafilter on λ. Put*

$$(X, \leq) = \prod_{i\in\lambda}(X_i, \leq_i)/\mathcal{U}.$$

Assume that there exist an infinite set $U \subseteq X$ and a family $\mathcal{Z}$ of internal sets of X such that $|U|, |\mathcal{Z}| < \mathfrak{p}(\mathcal{U}), \mathfrak{t}(\mathcal{U})$ and $U \subseteq Z$ for all $Z \in \mathcal{Z}$. Then there is an internal set Y such that $U \subseteq Y \subseteq \bigcap \mathcal{Z}$.

Proof. Let $\mathcal{Z} = (Z_\xi)_{\xi\in\kappa}$ be an enumeration of the family $\mathcal{Z}$. Let Q_i be the set of the functions f with the following properties:

1. $\mathrm{dom}(f)$ is an initial segment of X_i.

2. $\mathrm{range}(f) \subseteq \mathcal{P}(X_i)$.

3. $f(y) \subseteq f(x)$, if $x \leq_i y$.

Put $(Q, \sqsubseteq) = \prod_{i\in\lambda}(Q_i, \subseteq)/\mathcal{U}$. We construct inductively an increasing sequence $(q_\alpha)_{\alpha\in\kappa}$ in Q, such that, called d_α the maximal element of $\mathrm{dom}(q_\alpha)$, the following hold:

(i) d_α is not near 1.

(ii) $U \subseteq q_\alpha(z)$ for all $d \leq d_\alpha$.

(iii) $q_\alpha(d_\alpha) \subseteq Z_\alpha$.

Put $q_0 = \emptyset$. Assume that q_α is defined. Put $q_{\alpha+1} = q_\alpha \cup \{(d_\alpha + 1, Z_{\alpha+1} \cap q_\alpha(d_\alpha))\}$, which is an element of Q, since the set $Z_{\alpha+1}$ is an internal set. Now we suppose α limit ordinal. The sequence $(q_\beta)_{\beta\in\alpha}$ is increasing and $|\alpha| < \mathfrak{t}(\mathcal{U})$, so there exists $q \in Q$ such that $q_\beta \sqsubseteq q$, for all $\beta \in \alpha$. For $u \in U$, put

$$e_u = \max\{d \in \mathrm{dom}(q) \mid u \in q(d)\}$$

and note that $d_\beta \leq e_u$ for any $\beta \in \alpha$. By hypothesis $|U|, |\alpha| < \mathfrak{p}(\mathcal{U})$, hence there exists d_α such that $d_\beta \leq d_\alpha \leq e_u$ for all $u \in U$ and $\beta \in \alpha$. We conclude the limit step defining $q_\alpha = q\big|_{d_\alpha} \cup \{(d_\alpha, Z_\alpha \cap q(d_\alpha))\}$. Finally the set $Y = q_k(d_k)$ has the required properties. □

Lemma 1.4.2. *Let $\{(X_i, \leq_i)\}_{i\in\lambda}$ be a family of finite linear orders, $\mathcal{U}$ be an ultrafilter on λ and $D = \{d_\alpha\}_{\alpha\in\kappa}$ be a decreasing chain in*

$$(X, \leq) = \prod_{i\in\lambda}(X_i, \leq_i)/\mathcal{U},$$

where $\kappa < \mathfrak{t}(\mathcal{U})$. For any $F\colon D^2 \to X$ there exists an internal function $H\colon X^2 \to X$ such that $F \subseteq H$.

Proof. Firstly we prove the one-dimensional case of the lemma, that is we assume that $F\colon D \to X$. Let P_i be the set of all partial functions f such that $\mathrm{dom}(f) = \{x \in X_i \mid x >_i b_i\}$ for some $b_i \in X_i$ and $\mathrm{range}(f) \subseteq X_i$, that is we consider the finite pseudo-tree of the functions from X_i with the order $>_i$, to X_i. Put $(P, \sqsubseteq) = \prod_{i\in\lambda}(P_i, \subseteq)/\mathcal{U}$. We construct inductively an increasing chain $(c_\alpha)_{\alpha\in\kappa}$ such that:

1. $\mathrm{dom}(c_\alpha) = \{x \in X \mid x > d_\alpha\}$;
2. $c_\alpha(d_\beta) = F(d_\beta)$ for $\beta \in \alpha$.

Let c_0 be a function with domain $\{x \in X \mid x > d_0\}$. Given c_α, put $c_{\alpha+1}$ so that:

$$c_{\alpha+1}(x) = \begin{cases} c_\alpha(x) & \text{if } x \in \mathrm{dom}(c_\alpha); \\ F(d_\alpha) & \text{if } d_\alpha \geq x > d_{\alpha+1}. \end{cases}$$

Now suppose α limit ordinal, then there exists c such that $c_\beta \sqsubseteq c$ for all $\beta \in \alpha$. Define $c_\alpha = c\big|_{\{x\in X\mid x>d_\alpha\}}$. When the construction is completed, use the hypothesis $\kappa < \mathfrak{t}(\mathcal{U})$ to find c such that $c_\alpha \sqsubseteq c$ for any $\alpha \in \kappa$. If necessary extend c on X and this completes the one-dimensional case.

Now we prove the lemma. Let P_i be the set of all partial functions f such that $\mathrm{dom}(f) = (\{x \in X_i \mid x >_i b_i\})^2$ for some $b_i \in X_i$ and $\mathrm{range}(f) \subseteq X_i$. As in the one-dimensional case, construct an increasing chain $(c_\alpha)_{\alpha\in\kappa}$ with the following properties:

1. $\mathrm{dom}(c_\alpha) = (\{x \in X \mid x > d_\alpha\})^2$;
2. $c_\alpha(d_\beta, d_\gamma) = F(d_\beta, d_\gamma)$ for $\beta, \gamma \in \alpha$.

Obtained the sequence we conclude as in one-dimensional case. If α is limit ordinal, we can proceed as previously said. In order to define $c_{\alpha+1}$, we know that there exist two internal functions f, g such that $f(d_\eta) = F(d_\alpha, d_\eta)$ and $g(d_\eta) = F(d_\eta, d_\alpha)$. Extend c_α on $\{x \in X \mid x > d_{\alpha+1}\}^2$ so that

$$c_{\alpha+1}(x,y) = \begin{cases} c_\alpha(x,y) & \text{if } (x,y) \in \mathrm{dom}(c_\alpha); \\ f(y) & \text{if } d_\alpha = x; \\ g(x) & \text{otherwise} \end{cases}$$

and this concludes. □

To prove the main theorem, we need a result of cardinal combinatorics due to Todorčević.

Theorem 1.4.3 (Corollary 15.8 and Remark 15.10 [11]). *If κ is a regular infinite cardinal, then*

$$\kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}.$$

In other words, there exists a map

$$f\colon [k^+]^2 \to k^+,$$

such that $f([A]^2) = \kappa^+$ for any $A \subseteq \kappa^+$ of cardinality κ^+.

In the next proof we shall use an easy corollary of this theorem.

Corollary 1.4.4. *If κ is a regular infinite cardinal, then there exists a function*

$$f\colon [\kappa^+]^2 \to \kappa$$

such that $|f([A]^2)| = \kappa$, for all cofinal subset $A \subseteq \kappa^+$.

Proof. Let $\tilde{f}\colon [k^+]^2 \to k^+$ be the function of Theorem 1.4.3. Define $f\colon [k^+]^2 \to k$ in such a way that

$$f(\gamma) = \begin{cases} \tilde{f}(\gamma) & \text{if } \tilde{f}(\gamma) \in \kappa; \\ 0 & \text{otherwise.} \end{cases}$$

For each cofinal $A \subseteq \kappa^+$, we have $\tilde{f}([A]^2) = \kappa^+$, hence $f([A]^2) = \kappa$. □

The next theorem is the main result on gaps of linear orders appearing in [5]. We give the (unpublished) proof provided by Steprans [10] of this result.

Theorem 1.4.5 (Theorem 8.5 [5]). *If $\mathcal{U}$ is an ultrafilter on λ, then $\mathfrak{p}(\mathcal{U}) = \mathfrak{t}(\mathcal{U})$ holds.*

Proof. By Corollary 1.2.4, we have $\mathfrak{p}(U) \leq \mathfrak{t}(\mathcal{U})$, hence it is sufficient to show that $\mathfrak{t}(\mathcal{U}) \leq \mathfrak{p}(\mathcal{U})$ holds. Let $(X, \leq) = \prod_{i\in\lambda}(X_i, \leq_i)/\mathcal{U}$ be a linear order of $L(\mathcal{U})$, such that there exists a (κ, θ) gap, where $\theta \leq \kappa = \mathfrak{p}(\mathcal{U})$. If $\theta = \kappa$, then we conclude $\mathfrak{t}(\mathcal{U}) \leq \kappa =$

$\mathfrak{p}(\mathcal{U})$, by Theorem 1.2.1. Now suppose that $\theta < \kappa = p(\mathcal{U}) < t(\mathcal{U})$ and the sequences $\{x^1_\xi\}_{\xi\in\kappa}$, $\{x^0_\xi\}_{\xi\in\theta}$ witness the existence of a (κ,θ)-gap in X, that is

$$x^1_{\beta^1} \le x^1_{\alpha^1} \le x^0_{\alpha^0} \le x^0_{\beta^0}$$

for all $\beta^1 \le \alpha^1 \in \kappa$, $\beta^0 \le \alpha^0 \in \theta$ and there exists no $x \in X$ such that

$$x^1_\alpha \le x \le x^0_\beta$$

for all $\alpha \in \kappa$, $\beta \in \theta$. We will reach a contradiction. For each $x \in X_i$, put $X_i\big|_x = \{x' \in X_i \mid x' \le_i x\}$. Let P_i be the set of all partial functions with domain D^2, for some $D \subseteq X_i$, and range included in X_i. Define Q_i the set of the functions ψ with the following properties:

1. $\mathrm{dom}(\psi) = X_i\big|_x$ for some $x \in X_i$.

2. $\mathrm{range}(\psi) \subseteq X_i \times P_i$ and $\psi(z) = (\psi^1(z), \psi^2(z))$.

3. $\psi^2(z)(a,b) \ge_i \psi^1(z)$ for any $z \le_i x$ e $(a,b) \in \mathrm{dom}(\psi^2)$.

4. $\psi^1(z) \le_i \psi^1(z')$, if $z \le_i z' \le_i x$.

5. If $z \le_i z' \le_i x$ and $\{a,b\}$ is a subset of $\mathrm{dom}(\psi^2(w))$ for any $z \le_i w \le_i z'$, then $\psi^2(z)(a,b) = \psi^2(w)(a,b) = \psi^2(z')(a,b)$ for all such $z \le_i w \le_i z'$.

Put $(Q,\sqsubseteq) = \prod_{i\in\lambda}(Q_i,\subseteq)/\mathcal{U}$. If c is an element of Q, we denote by d_c the maximal element of $\mathrm{dom}(c)$. For $z \le d_c$, we write $c(z) = (c^1(z), c^2(z))$ and $(D_c(z))^2 = \mathrm{dom}(c^2(z))$. Define $c^1 = c^1(d_c), c^2 = c^2(d_c)$ and $D_c = D_c(d_c)$.

By Corollary 1.4.4, there exists a function $G_0\colon [\theta^+]^2 \to \theta$ such that, if $A \subseteq \theta^+$ is cofinal in θ^+, then $|G_0(A^2)| = \theta$. First of all we extend G_0 trivially to a function $G : [\kappa]^2 \to \theta$ such that $G \restriction [\theta^+]^2 = G_0$. Now construct a sequence $(c_\alpha)_{\alpha\in\kappa}$ in Q with the following properties:

1. $c_\alpha = (c^1_\alpha, c^2_\alpha)$.

2. $c_\alpha \sqsubseteq c_\beta$ if $\alpha \in \beta$ and d_{c_α} is not near 1.

3. There exists $y_\beta \in D_{c_\beta}$, so that if $\beta \in \alpha \in \kappa$, then:

   (A) $y_\alpha \le y_\beta$;

   (B) $c^2_\alpha(y_\beta, y_\alpha) = x^0_{G(\alpha,\beta)}$;

   (C) $y_\beta \in D_{c_\alpha}(z)$, if $d_{c_\beta} \le z \le d_{c_\alpha}$.

4. $c^1_\alpha > x^1_\alpha$.

To conclude the proof it is sufficient to prove that such a sequence $(c_\alpha)_{\alpha\in\kappa}$ exists and it is unbounded: if this is the case we would get $\mathfrak{t}(\mathcal{U}) \le \kappa = \mathfrak{p}(\mathcal{U}) < \mathfrak{t}(\mathcal{U})$, a contradiction. First we show that such a sequence $(c_\alpha)_{\alpha\in\kappa}$ is unbounded. Suppose that there is c such that $c_\alpha \sqsubseteq c$ for all $\alpha \in \kappa$. Now for $\eta \in \theta^+$, let z_η be the maximal element of the set

$$A_\eta = \{z \in \mathrm{dom}(c) \mid \forall z'[d_{c_\eta} \le z' \le z \to y_\eta \in D_c(z')]\}.$$

By condition 3(C), we have $z_\eta \ge d_{c_\alpha}$ for all $\alpha \in \kappa$. By condition 4, we have $c^1(z_\eta) \ge c^1(d_{c_\alpha}) = c^1_\alpha > x^1_\alpha$ for all $\alpha \in \kappa$. Hence there exists $F(\eta) \in \theta$, such that $c^1(z_\eta) > x^0_{F(\eta)}$. Let $A \subseteq \theta^+$ be a cofinal set such that $F(\eta) = \gamma$ for some $\gamma \in \theta$ and any $\eta \in A$. Note that such an A exists since θ^+ is regular and

$$\theta^+ = \bigcup\{F^{-1}(\alpha) \mid \alpha \in \theta\}.$$

Choose ζ, η in A such that $G(\zeta, \eta) > \gamma$. Put $z^* = \min\{z_\eta, z_\zeta\}$, we have $\{y_\eta, y_\zeta\} \subseteq D_c(z^*)$, since $d_{c_\eta}, d_{c_\zeta} \le z^* \le z_\eta, z_\zeta$ holds. Let μ the maximum between η, ζ, then $\{y_\eta, y_\zeta\} \subseteq D_c(z')$ for all $d_{c_\mu} \le z' \le z^*$. Hence we have

$$c^2(z^*)(y_\eta, y_\zeta) = c^2(d_{c_\mu})(y_\eta, y_\zeta) = x^0_{G(\eta,\zeta)} < x^0_\gamma$$

and

$$c^2(z^*)(y_\eta, y_\zeta) \ge c^1(z^*) > x^0_{F(\eta)} = x^0_\gamma$$

contradiction.

So, it is sufficient to prove that the sequence $(c_\alpha)_{\alpha\in\kappa}$ exists. We need a technical claim.

Claim 1.4.6. *Assume $\xi \in \kappa$ and*

- *$U = \{u_\alpha\}_{\alpha\in\xi} \subseteq X$ is decreasing in the order $\le$.*
- *$F\colon \xi^2 \to X\big|_{\{x\in X \mid x > w\}}$ is a function.*
- *$\bar\rho \in \prod_{i\in\lambda} P_i/\mathcal{U}$ is such that $\bar\rho(u_\alpha, u_\beta) = F(\alpha, \beta)$ for all α, β such that $(u_\alpha, u_\beta) \in$ dom$(\bar\rho)$ hold.*

Then there exists $\rho \in \prod_{i\in\lambda} P_i/\mathcal{U}$ such that:

1. *$\rho(u_\alpha, u_\beta) = F(\alpha, \beta)$ for all α, β.*
2. *$\rho(x, y) \ge w$ for all $(x, y) \in$ dom(ρ).*
3. *If $(x, y) \in$ dom$(\rho) \cap$ dom$(\bar\rho)$, then $\bar\rho(x, y) = \rho(x, y)$.*

Proof claim. By Lemma 1.4.2, there exists an internal function $\rho_1\colon X^2 \to X$ that extends F. Define ρ_2 so that:

$$\rho_2(x, y) = \begin{cases} \bar\rho(x, y) & \text{if } (x, y) \in \mathrm{dom}(\bar\rho); \\ \rho_1(x, y) & \text{otherwise.} \end{cases}$$

Note that also ρ_2 extends F, so ρ_2 satisfies 1 and 3 above, but 2 is as yet unclear. For $u \in U$, put $Z_u = \{x \in X \mid \rho_2(x,u) > w\}$. Hence $U \subseteq Z_u$ and Z_u is an internal set. More precisely, since the function ρ_2 is internal, we have $\rho_2 = \prod_{i\in\lambda} \rho_2^i/\mathcal{U}$, where $\rho_2^i \in P_i$ for each $i \in \lambda$. Set

$$Z_u^i = \{x_i \in X_i \mid \rho_2^i(x_i, u(i)) > w(i)\},$$

for each $i \in \lambda$, we obtain that Z_u is internal, since

$$x \in Z_u \Longleftrightarrow x(i) \in Z_u^i.$$

By Lemma 1.4.1, there exists an internal set Y such that $U \subseteq Y \subseteq Z_u$ for $u \in U$. Put

$$Y^* = Y \setminus \{y \in Y \mid (\exists y' \in Y)\rho_2(y,y') < w\}$$

and observe that $U \subseteq Y^*$ and $\rho_2(x,y) \geq w$ for any $x, y \in Y^*$. Then $\rho = \rho_2\big|_{(Y^*)^2}$ has all required properties. □

Now we construct c_α for $\alpha \in \kappa < t(\mathcal{U})$ as follows: In the successor case, we define $c_{\alpha+1}$ as follows: We choose $y_{\alpha+1}$ below y_α. By Claim 1.4.6, we can find ρ such that:

- $\rho(y_\gamma, y_\delta) = x^0_{G(\gamma,\delta)}$, if $\gamma \leq \delta \leq \alpha + 1$.
- $\rho(x,y) \geq x^1_{\alpha+1} + 1$ for all $x, y \in \mathrm{dom}(\rho)$.
- $\rho(x,y) = c^2_\alpha(x,y)$, if $x, y \in \mathrm{dom}(\rho) \cap \mathrm{dom}(c^2_\alpha)$.

Then we define $c_{\alpha+1} = c_\alpha \cup \{(d_{c_\alpha} + 1, (x^1_{\alpha+1} + 1, \rho))\}$.

If $\alpha \in \kappa$ is a limit ordinal, we also have that $cof(\alpha) < \mathfrak{t}(\mathcal{U})$, hence we can find a c such that $c_\beta \sqsubseteq c$ for all $\beta \in \alpha$. For $\beta \in \alpha$, put

$$e_\beta = \max\{z \in \mathrm{dom}(c) \mid \forall z'[d_{c_\beta} \leq z' \leq z \rightarrow y_\beta \in D_c(z')]\}.$$

The set $\{e_\beta\}_{\beta\in\alpha}$ is entirely above every d_{c_ξ} for all $\xi < \alpha$ and $|\alpha| < \kappa = \mathfrak{p}(\mathcal{U})$, hence there exists d_α such that $d_{c_\beta} \leq d_\alpha \leq e_\beta$ for all $\beta \in \alpha$. Otherwise $\{d_{c_\beta} \mid \beta \in \alpha\}$ and $\{e_\beta \mid \beta \in \alpha\}$ is a $(cof(\alpha), \xi)$ gap where ξ is the coinitiality of the set $\{e_\beta \mid \beta \in \alpha\}$ in the linear order $(X, \leq)$ and is thus a regular cardinal smaller or equal than $|\alpha|$. But

$$cof(\alpha), \xi \leq |\alpha| < \kappa = \mathfrak{p}(\mathcal{U}),$$

this contradicts with the very definition of $\mathfrak{p}(\mathcal{U})$. Put $c' = c\big|_{d_\alpha}$, then we can find y_α which is below y_η for all $\eta \in \alpha$, since $cof(\alpha) < \kappa < \mathfrak{t}(\mathcal{U})$. Now we can proceed to find the required ρ as in the successor case applied to the function c' (instead of c_α) and the element y_α (instead of $y_{\alpha+1}$). We can now let $c_\alpha = c' \cup \{(d_\alpha, (x_\alpha + 1, \rho))\}$. □

Corollary 1.4.7. *If $\mathcal{U}$ is a non-principal ultrafilter on λ, then $CSP(\mathcal{U}) = \emptyset$.*

Proof. By Theorem 1.4.5, we have $\mathfrak{t}(\mathcal{U}) = \mathfrak{p}(\mathcal{U})$, hence there are no (κ, θ) gap for $\kappa + \theta < \mathfrak{t}(\mathcal{U}) = \mathfrak{p}(\mathcal{U})$. □

Chapter 2

The cardinals $\mathfrak{p}$ and $\mathfrak{t}$

The aim of this chapter is to prove that two cardinal invariants $\mathfrak{p}, \mathfrak{t}$ are equal. This result is proved recently by Malliaris and Shelah in [5], but we follow a presentation due to Professor J. Steprans that appears in a manuscript not published [10].

In Section 2.1, we work in a generic extension $V[G]$, where G is a V-generic ultrafilter over $([\omega]^{\aleph_0}, \subseteq^*)$, to investigate the existence of certain gaps in some ultraproducts of ω and we study the relation between cardinals $\mathfrak{p}, \mathfrak{t}, \mathfrak{p}(G), \mathfrak{t}(G)$. Working in V and $V[G]$, we show that $\mathfrak{t} \leq \mathfrak{t}(G)$ and, assuming $\mathfrak{p} < \mathfrak{t}$, we obtain a contradiction, by a theorem due to Shelah.

2.1 $\mathfrak{p} = \mathfrak{t}$

We recall some central definitions.

Definition 2.1.1. For $A, B \subseteq \omega$ we write $A \subseteq^* B$ to indicate that $|A \setminus B| \in \omega$.

Let $\mathcal{F}$ be a subset of $[\omega]^{\aleph_0}$. We say that:

- $\mathcal{F}$ has the *strong finite intersection property* , abbreviated as *s.f.i.p.*, if each finite family of sets of $\mathcal{F}$ has infinite intersection.
- $\mathcal{F}$ has the *infinite pseudo-intersection property*, abbreviated as *p.i.p.*, if there exists a infinite set $A \subseteq \omega$ such that $A \subseteq^* F$ for all $F \in \mathcal{F}$. This set A is called *pseudo-finite intersection.*
- We say that a set $\{X_\alpha \in [\omega]^{\aleph_0} \mid \alpha \in \kappa\}$ is a *tower*, if $X_\alpha \supseteq^* X_\beta$ for all $\alpha \in \beta \in \kappa$. In particular the family $\{X_\alpha \in [\omega]^{\aleph_0} \mid \alpha \in \kappa\}$ is well ordered by $\supseteq^*$.

Now define

$$\mathfrak{t} = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq [\omega]^{\aleph_0} \text{ is a tower and has not i.p.i.p.}\},$$

$$\mathfrak{p} = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq [\omega]^{\aleph_0} \text{has s.f.i.p., but not i.p.i.p.}\}.$$

Lemma 2.1.2. *Cardinals* $\mathfrak{p}$ *and* $\mathfrak{t}$ *are regular and* $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{t}$.

Proof. The regularity of $\mathfrak{t}$ follows by definition. For a proof of the regularity of $\mathfrak{p}$ see Theorem 7.15 in [1]. Obviously $\mathfrak{p} \leq \mathfrak{t}$, since any tower has the strong finite intersection property. Finally, we conclude proving that $\aleph_1 \leq \mathfrak{p}$ holds. Let $\mathcal{F} = \{F_i \in [\omega]^{\aleph_0} \mid i \in \omega\}$ be a countable family with s.f.i.p. We construct an infinite set A such that $A \subseteq^* F_i$ for all $i \in \omega$. Choose $a_0 \in F_0$ and $a_n \in F_0 \cap \ldots \cap F_n$, such that $a_i \neq a_j$ for all $i, j \in \omega$. Conclude putting $A = \{a_i \mid i \in \omega\}$. □

The next theorems required some basic notions on the method of forcing, see Section A.1 of the appendix to a summary of all facts that we use.

Remark 2.1.3. The notion of forcing $([\omega]^{\aleph_0}, \subseteq^*, \omega)$ is a $\mathfrak{t}$-closed, hence any cardinal less or equal to $\mathfrak{t}$ is preserved by Corollary A.1.18.

Now we concentrate our attention on ultraproducts of finite orders modulo a non-principal ultrafilter on ω.

Lemma 2.1.4. *Let $G \subseteq [\omega]^{\aleph_0}$ be an ultrafilter V-generic, then $V[G] \models \mathfrak{t} \leq \mathfrak{t}(G)$.*

Proof. In $V[G]$, let $(Q, \leq) = \prod_{i\in\omega}(Q_n, \leq_n)/G$ be the ultraproduct of finite pseudo-trees $(Q_n, \leq_n)$ modulo G. Without loss of generality we can suppose that the sets Q_n are pairwise disjoint. Assume for a contradiction that there exists $\kappa < \mathfrak{t}$ such that

$$V[G] \models (q_\xi)_{\xi\in\kappa} \text{ is unbounded increasing chain in } Q.$$

By Forcing Theorem A.1.14, there exist $B \in G$ such that

$$B \Vdash (\dot{q_\xi})_{\check{\xi}\in\check{\kappa}} \text{ is unbounded increasing chain in } \dot{Q},$$

that is

$$B \Vdash (\dot{q_\xi})_{\check{\xi}\in\check{\kappa}} \text{ is an unbounded and } \{\dot{n} \in \dot{\omega} \mid \dot{q_\xi}(\dot{n}) < \dot{q_\eta}(\dot{n})\} \in \Gamma \text{ for all } \check{\xi} < \check{\eta} \in \check{\kappa}\ ,$$

where $(q_\alpha(n))_{n\in\omega}$ is an element of the equivalence class of q_ξ and Γ is the canonical name of G. Note that $\{(q_\alpha(n))_{n\in\omega} \mid \alpha \in \kappa\} \in V$. In fact, fixed $n \in \omega$ and $\alpha \in \kappa$, we have $q_\alpha(n) \in V$, since Q_n is finite, then, using Transfinite Recursion many times, we obtain that the sequence $(q_\alpha(n))_{\alpha\in\omega}$ is in V for each $\alpha \in \kappa$, hence $\{(q_\alpha(n))_{n\in\omega} \mid \alpha \in \kappa\}$ is a set of V. Put

$$A_\xi = \bigcup_{n\in B}\{q \in Q_n \mid q_\xi(n) \leq_n q\},$$

this give that $\{A_\xi \mid \xi \in \kappa\} \in V$. We have $A_\eta \subseteq^* A_\xi$ for any $\xi \leq \eta$, otherwise the set $A_\eta \setminus A_\xi$ is infinite: hence there exists an infinite set $C \subseteq B$, such that $q_\xi(n) \not< q_\eta(n)$ for all $n \in C$. Now let H be a V-generic ultrafilter on $[\omega]^{\aleph_0}$ such that $C \in H$. By Forcing Theorem A.1.14, we have

$$V[H] \models \ \{n \in \omega \mid q_\xi(n) < q_\eta(n)\} \in H \text{ for all } \xi < \eta \in \kappa$$

and

$$V[H] \models \{n \in \omega \mid q_\xi(n) \not< q_\eta(n)\} \in H,$$

contradiction. Hence we conclude that $A_\eta \subseteq^* A_\xi$ for any $\xi \leq \eta$. Since $\kappa < \mathfrak{t}$, there exists an infinite set $A \subseteq^* A_\xi$ for all $\xi \in \kappa$. Now we can construct a countable set $B^* \subseteq^* B$ and a sequence $(q^*(n))_{n \in B^*}$ such that $q(n)_\xi \leq_n q^*(n)$ for any $\xi \in \kappa$ and for all but finitely many $n \in B^*$. In fact, it is sufficient to choose $q^*(n) \in A$ for all but finitely many $n \in B$. Extend arbitrarily the sequence on ω and put $q^* = [(q^*(n))_{n \in \omega}]$. Let H be a V-generic ultrafilter on $[\omega]^{\aleph_0}$ such that $B^* \in H$, then

$$V[H] \models (q_\xi)_{\xi \in \kappa} \text{ is unbounded in } Q \text{ and } q_\xi \leq q^* \text{ for all } \xi \in \kappa,$$

contradiction. □

On ω^ω we can consider the partial order $<^*$ defined as:

$$f <^* g \iff \text{there exists } N \in \omega \text{ such that } f(n) < g(n) \text{ for all } n \geq N.$$

Definition 2.1.5. Let κ_1, κ_2 be infinite regular cardinals. We say that two sequences $(f_\alpha)_{\alpha \in \kappa_1}$, $(g_\alpha)_{\alpha \in \kappa_2}$ represent a (κ_1, κ_2) *tight gap* on $(\omega^\omega, <^*)$, if they witness a (κ_1, κ_2) gap in $(\omega^\omega, <^*)$, that is following properties hold:

- $f_j <^* f_i$ holds, if $i < j < \kappa_1$.
- $g_i <^* g_j$ holds, if $i < j < \kappa_2$.
- $g_j <^* f_i$ holds, if $i < \kappa_1$ and $j < \kappa_2$.
- If $f \in \omega^\omega$ is such that $f \leq^* f_i$ for all $i < \kappa_1$, then $f \leq^* g_j$ for some $j \in \kappa_2$.
- If $f \in \omega^\omega$ is such that $g_j \leq^* f$ for all $j \in \kappa_2$, then $f_i \leq^* f$ for some $i \in \kappa_1$.

Lemma 2.1.6. *If there exists a (κ_1, κ_2) tight gap, then there exists a (κ_2, κ_1) tight gap.*

Proof. Assume that the sequences $(f_\alpha)_{\alpha \in \kappa_1}$, $(g_\alpha)_{\alpha \in \kappa_2}$ represent a (κ_1, κ_2) tight gap in $(\omega^\omega, <^*)$. Define two new sequences $(f_0 - g_\alpha)_{\alpha \in \kappa_2}$ and $(f_0 - f_\alpha)_{\alpha \in \kappa_1}$, where we assume that $(f_0 - g_\alpha)(n) = 0$ and $(f_0 - f_\alpha)(n) = 0$ if $g_\alpha(n) > f_0(n)$ and $f_\alpha(n) > f_0(n)$, respectively. Now it is clear that the sequences $(f_0 - f_\alpha)_{\alpha \in \kappa_1}$ and $(f_0 - g_\alpha)_{\alpha \in \kappa_2}$ are increasing and decreasing, respectively. Now we prove that no function separate the sequences. Assume for a contradiction there exists $h \in \omega^\omega$ such that

$$f_0 - f_\alpha <^* h <^* f_0 - g_\beta$$

for all $\alpha \in \kappa_1$ and $\beta \in \kappa_2$. Consider the sequence $f_0 - h$, defined zero when f_0 is less than h. Then, we have

$$g_\beta <^* f_0 - h <^* f_\alpha,$$

for all $\alpha \in \kappa_1$ and $\beta \in \kappa_2$, absurd. So we conclude that the sequences represent a (κ_2, κ_1) tight gap in $(\omega^\omega, <^*)$. □

The next is a technical result due to Shelah.

Theorem 2.1.7 (Shelah, Theorem 1.12 [8]). *If $\mathfrak{p} < \mathfrak{t}$, then there exists a $(\kappa, \mathfrak{p})$ tight gap in $(\omega^\omega, <^*)$ for some regular cardinal $\kappa < \mathfrak{p}$.*

Lemma 2.1.8. *If $\mathfrak{p} < \mathfrak{t}$ and $G \subseteq [\omega]^{\aleph_0}$ is a V-generic ultrafilter, then $V[G] \models \mathfrak{p}(G) \leq \mathfrak{p}$.*

Proof. By Theorem 2.1.7, there exist a regular cardinal $\kappa < \mathfrak{p}$ and a $(\kappa, \mathfrak{p})$ tight gap in $(\omega^\omega, <^*)$. Let $(f_\xi)_{\xi \in \mathfrak{p}}$, $(g_\xi)_{\xi \in \kappa}$ be the sequences that represent the tight gap in ω^ω. Put

$$(X, \leq) = \prod_{i \in \omega} (g_0(n), \leq_n)/G,$$

where $\leq_n$ is the standard order on ω. For any $\xi \in \mathfrak{p}$ and $\xi' \in \kappa$ we have $f_\xi(n), g_{\xi'}(n) < g_0(n)$ for all but finitely many $n \in \omega$. Hence we can assume that $\operatorname{range}(f_\xi), \operatorname{range}(g_{\xi'}) \subseteq \operatorname{range}(g_0)$ for any $\xi \in \mathfrak{p}$, $\xi' \in \kappa$, provided that the function ranges are modified in a finite number of points. Put $[f_\xi]$ and $[g_{\xi'}]$ the equivalence classes in $X \in V[G]$ of the sequences f_ξ, $g_{\xi'}$, respectively. Finally we have to show that the sequences $([f_\xi])_{\xi \in \mathfrak{p}}, ([g_\xi])_{\xi \in \kappa}$ represent a $(\mathfrak{p}, \kappa)$ gap in $X \in V[G]$. The monotonicity of the sequences is obvious. To conclude the proof it is sufficient to show that there is no $[h]$ such that

$$V[G] \models [f_\xi] < [h] < [g_\eta] \text{ for all } \xi \in \mathfrak{p} \text{ and } \eta \in \kappa.$$

Assume that there exists $[h] \in X$ such that

$$V[G] \models [h] \leq [g_\xi] \text{ for all } \xi \in \kappa,$$

that is for some $A \in G$ we have

$$A \Vdash \{\dot{n} \in \dot{\omega} \mid \dot{h}(\dot{n}) \leq_n \dot{g}_\xi(\dot{n})\} \in \Gamma \text{ for all } \xi \in \kappa,$$

by Forcing Theorem A.1.14, where Γ is the canonical name of G. Put

$$A_\xi = \{n \in \omega \mid h(n) \leq_n g_\xi(n)\}$$

and note that $A_\xi \in G$ for all $\xi \in \kappa$. For each $\xi \in \kappa$, we have $A \subseteq^* A_\xi$, otherwise the set $A \setminus A_\xi$ is infinite, hence there exists an infinite set $A' \subseteq A$ such that $g_\xi(n) <_n h(n)$ for all $n \in A'$. Now let H be a V-generic ultrafilter over $[\omega]^{\aleph_0}$ such that $A' \in H$. Since $A' \subseteq^* A$, we have

$$V[H] \models [h] \leq [g_\xi]$$

and

$$V[H] \models [g_\xi] < [h],$$

contradiction. Hence we obtain that $A \subseteq^* A_\xi$ for all $\xi \in \kappa$. Let $\tilde{h} \in \omega^\omega$ be so that

$$\tilde{h}(n) = \begin{cases} h(n) & \text{if } n \in A; \\ 0 & \text{otherwise;} \end{cases}$$

We have $\tilde{h} \leq^* g_\xi$ for all $\xi \in \kappa$, hence there exists $\gamma \in \mathfrak{p}$ such that $\tilde{h} <^* f_\gamma$. Since $A \in G$, we conclude

$$V[G] \models [h] = [\tilde{h}] <^* [f_\gamma].$$

□

Theorem 2.1.9. *We have* $V \models \mathfrak{p} = \mathfrak{t}$.

Proof. We know that $\mathfrak{p} \leq \mathfrak{t}$. Assume for contradiction that $\mathfrak{p} < \mathfrak{t}$. By lemmas 2.1.8, 2.1.4, 1.4.5, we have

$$V[G] \models \mathfrak{t} \leq \mathfrak{t}(G) \leq \mathfrak{p}(G) \leq \mathfrak{p} < \mathfrak{t},$$

where $G \subseteq [\omega]^{\aleph_0}$ is a V-generic ultrafilter, contradiction. □

Chapter 3

A characterization of elementary classes

In this chapter we give an algebraic characterization of elementarily equivalent structures under the assumption that there exists proper class of inaccessible cardinals. In particular, we show that two $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$ are elementarily equivalent if and only if there exists two isomorphic $\mathcal{L}$-structures $\mathcal{M}'$ and $\mathcal{N}'$ (saturated and of inaccessible size) such that $\mathcal{M} \prec \mathcal{M}'$ and $\mathcal{N} \prec \mathcal{N}'$.

In Section 3.1, we concentrate our attention to a class of ultrafilters, called λ-*good*. We prove the existence of λ^+-good ultrafilters on λ and we show that these ultrafilters ensure a certain saturation of all ultraproducts of $\mathcal{L}$-structures.

In Section 3.2, we recall the definition of direct limit of a family of $\mathcal{L}$-structures and prove some basic properties.

In Section 3.3, we define a *strong chain* for a structure $\mathcal{M}$, that is a sequence of structures such that: at the successor step, $\mathcal{M}_{\alpha+1}$ is an ultraproduct of the structures $\mathcal{M}_\alpha$ modulo a good ultrafilter and, at the limit step, we keep the direct limit of the already constructed structures. Finally, we deduce a characterization of elementarily equivalent structures by the existence of strong chains.

3.1 Good ultrafilters and saturated structures

Definition 3.1.1. Let $\mathcal{U}$ be an ultrafilter on I and λ an infinite cardinal. We say that $\mathcal{U}$ is λ-*regular*, if there exists a λ-*regularizing family* $\mathcal{E} \subseteq \mathcal{U}$, that is a family of sets such that $|\mathcal{E}| = \lambda$ and for any $i \in I$

$$|\{E \in \mathcal{E} \mid i \in E\}| < \omega.$$

We write *regular* if $\mathcal{U}$ is $|I|$-regular. We say that an ultrafilter is $\aleph_1$-*incomplete* or *countably incomplete*, if there exists a countable family $\mathcal{E} \subseteq \mathcal{U}$ such that $\bigcap \mathcal{E} \notin \mathcal{U}$.

By definition follows immediately that a λ-regular ultrafilter $\mathcal{U}$ is μ-regular for each $\mu < \lambda$.

Proposition 3.1.2. *Let $\mathcal{U}$ be an ultrafilter on I. Then the following properties are equivalent:*

1. *$\mathcal{U}$ is $\aleph_0$-regular.*
2. *$\mathcal{U}$ is countably incomplete.*
3. *There exists a countable family $\{I_n \mid n \in \omega\} \subseteq \mathcal{U}$ such that $I_0 = I$, $I_n \supseteq I_{n+1}$ and $\bigcap I_n = \emptyset$.*

Proof.

$1 \Rightarrow 2$ If the family $\{E_n \mid n \in \omega\}$ regularizes $\mathcal{U}$, then $\bigcap_{n\in\omega} E_n = \emptyset$. Hence we conclude that $\bigcap_{n\in\omega} E_n \notin \mathcal{U}$.

$2 \Rightarrow 3$ Let $\mathcal{E} = \{E_i \mid i \in \omega\} \subseteq \mathcal{U}$ be a family such that $\bigcap \mathcal{E} \notin \mathcal{U}$. Put $E = I \setminus \bigcap \mathcal{E}$ and note that $E \in \mathcal{U}$. Define $I_0 = I$ and $I_{n+1} = E_0 \cap \dots E_n \cap E$ for $n \in \omega$. Then the family $\{I_n \mid n \in \omega\}$ has the required properties.

$3 \Rightarrow 1$ Obvious.

□

Remark 3.1.3. For an ultrafilter $\mathcal{U}$ on ω, we have that $\mathcal{U}$ is non-principal if and only if $\mathcal{U}$ is regular. In fact, if $\mathcal{U}$ is non-principal, then the set $I_n = \omega \setminus \{1, \dots, n\}$ belongs to $\mathcal{U}$ for every $n \in \omega$ and $\bigcap_{n\in\omega} I_n = \emptyset$. If $\mathcal{U}$ is principal, then $\mathcal{U}$ is generated by some $n \in \omega$ and we conclude that $n \in \bigcap \mathcal{U}$.

From now on, we assume that each ultrafilter is non-principal.

Notation. *For a set I, we indicate with $S_\omega(I)$ the set of all finite subsets of I.*

We show the existence of λ-regular ultrafilter on λ.

Lemma 3.1.4. *Let I be a set of cardinality λ. There exists a regular ultrafilter $\mathcal{U}$ on I.*

Proof. Since $|S_\omega(\lambda)| = \lambda$, it is sufficient to prove this lemma when $I = S_\omega(\lambda)$. For $\alpha \in \lambda$, put

$$X_\alpha = \{u \in S_\omega(\lambda) \mid \alpha \in u\}.$$

The family $X = \{X_\alpha \subseteq S_\omega(\lambda) \mid \alpha \in \lambda\}$ has the finite intersection property, in fact, for all $\alpha_1, \dots, \alpha_n \in \lambda$ we have

$$X_{\alpha_1} \cap \dots \cap X_{\alpha_n} \ni \{\alpha_1, \dots, \alpha_n\}.$$

Then there exists an ultrafilter $\mathcal{U}$ that extends the family X and clearly X is a λ-regularizing family for $\mathcal{U}$, since every $u \in S_\omega(\lambda)$ is finite. □

Lemma 3.1.5. *If $\mathcal{U}$ is an ultrafilter on λ, then $\mathcal{U}$ is not λ^+-regular.*

Proof. Assume for a contradiction that the family $\{X_\alpha \mid \alpha \in \lambda^+\}$ regularizes $\mathcal{U}$. For every $\alpha \in \lambda^+$, choose $i_\alpha \in \lambda$ such that $i_\alpha \in X_\alpha$. Then $\lambda^+ = \bigcup_{\beta\in\lambda}\{\alpha \in \lambda^+ \mid i_\alpha = \beta\}$, hence there exists $\beta \in \lambda$ such that $|\{\alpha \in \lambda^+ \mid i_\alpha = \beta\}| = \lambda^+$. We conclude that $\beta \in X_\alpha$ for all α such that $i_\alpha = \beta$, absurd. □

Definition 3.1.6. Let I be a set and $f, g\colon S_\omega(I) \to \mathcal{U}$ be two functions. We write $f \leq g$, if $f(A) \subseteq g(A)$ for all $A \in S_\omega(I)$. We say that f is *monotone*, if for any A, B in $S_\omega(I)$:

$$A \subseteq B \text{ implies } f(B) \subseteq f(A).$$

We say that g is *additive*, if

$$g(B \cup A) = g(A) \cap g(B).$$

Definition 3.1.7. Let α be an infinite cardinal. An ultrafilter $\mathcal{U}$ on I is called *α-good* if for every cardinal $\beta < \alpha$ and every monotone function $f\colon S_\omega(\beta) \to \mathcal{U}$, there exists an additive function $g\colon S_\omega(\beta) \to \mathcal{U}$, such that $g \leq f$.

Lemma 3.1.8. *Let $\mathcal{U}$ be a countably incomplete ultrafilter over a set I. If $\mathcal{U}$ is λ^+-good, then $\mathcal{U}$ is λ-regular.*

Proof. Let $\{I_n \in \mathcal{U} \mid n \in \omega\}$ be a family such that $I_n \supseteq I_{n+1}$ and $\bigcap_{n\in\omega} I_n = \emptyset$. Put $f\colon S_\omega(\lambda) \to \mathcal{U}$ such that $f(u) = I_{|u|}$. The map f is monotone, hence there exists g additive such that $g \leq f$. For $t \in g(\{\alpha_1\}) \cap \ldots \cap g(\{\alpha_n\})$, we have $t \in f(\{\alpha_1, \ldots, \alpha_n\}) = I_n$. Hence the family $\{g(\{\alpha\}) \mid \alpha \in \lambda\}$ is λ-regularizing. □

The next lemma give an equivalent condition to λ^+-goodness.

Lemma 3.1.9. *An ultrafilter $\mathcal{U}$ over a set I is λ^+-good if and only if the following statement holds:*

For every monotne map $f\colon S_\omega(\lambda) \to \mathcal{U}$ there exists an additive map $g\colon S_\omega(\lambda) \to \mathcal{U}$ such that $g \leq f$.

Proof. It is sufficient to show that the implication from right to left holds. For $\gamma \leq \lambda$, let $f\colon S_\omega(\gamma) \to \mathcal{U}$ be monotone. Put $\bar{f}\colon S_\omega(\lambda) \to \mathcal{U}$ such that $\bar{f}(u) = f(u \cap \gamma)$. The map $\bar{f}$ is clearly monotone, hence there exists an additive map $\bar{g} \leq \bar{f}$. We conclude noting that the restriction of $\bar{g}$ to $S_\omega(\gamma)$ has all required properties. □

The next aim is to prove the existence of λ^+-good ultrafilters on λ. To do this we need some technical lemmas.

Lemma 3.1.10 (Theorem 6.1.6 [2])**.** *Let λ be an infinite cardinal and $\{Y_\gamma \mid \gamma \in \lambda\}$ be a family of subsets of λ, such $|Y_\gamma| = \lambda$ for every $\gamma \in \lambda$. There exists a family $\{Z_\gamma \mid \gamma \in \lambda\}$ with the following properties for each $\gamma, \eta \in \lambda$:*

(i) $Z_\gamma \subseteq Y_\gamma$ and $|Z_\gamma| = \lambda$.

(ii) $Z_\gamma \cap Z_\eta = \emptyset$, *if* $\gamma \neq \eta$.

Proof. For each ordinal $\gamma \leq \lambda$, put

$$X_\gamma = \{(\alpha, \beta) \mid \alpha \leq \beta \text{ and } \beta < \gamma\} \subseteq \gamma \times \gamma.$$

We have $X_\lambda = \bigcup_{\gamma \in \lambda} X_\gamma$, since λ is limit ordinal. We construct inductively a family $\{f_\gamma \mid \gamma \leq \lambda\}$ of injective functions such that

(i) the function f_γ has domain contained in X_γ and $f_\eta \subseteq f_\gamma$ for every $\eta \leq \gamma \leq \lambda$.

(ii) If $\alpha \leq \beta < \gamma$, then $f_\gamma(\alpha, \beta) \in Y_\alpha$.

At the successor step, assume that f_γ is defined. Note that $|X_\gamma| < \lambda$ and $|Y_\alpha| = \lambda$ for all $\alpha < \lambda$. We can define

$$f_{\gamma+1}((\alpha, \beta)) = \begin{cases} f_\gamma((\alpha, \beta)) & \text{if } \alpha \leq \beta < \gamma; \\ h((\alpha, \gamma)) & \text{if } \alpha \leq \gamma < \gamma + 1. \end{cases},$$

where h is an injective map such that $\mathrm{dom}(h) \cap \mathrm{dom}(f_\gamma) = \emptyset$ and $h(\alpha, \gamma) \in Y_\alpha$ for all $\alpha \leq \gamma$. At the limit step, choose $f_\gamma = \bigcup_{\eta \in \gamma} f_\eta$. When the construction is completed, put

$$Z_\alpha = \{f_\lambda(\alpha, \beta) \mid \alpha \leq \beta < \lambda\}.$$

Since the map f_λ is injective, follows that the clauses (i) and (ii) hold. □

Definition 3.1.11. Let Π be a non-empty family of partitions of a cardinal λ, such that every $P \in \Pi$ has exactly λ many sets. For a filter F on λ, we say that the couple (Π, F) is *consistent*, if the following assertion holds:

$$\text{for each } X \in F,\ n \in \omega \text{ and } X_1 \in P_1 \in \Pi_1, \ldots, X_n \in P_n \in \Pi_n,$$
$$\text{if } P_1, \ldots, P_n \text{ are distinct, then } X \cap \bigcap_{i \leq n} X_i \neq \emptyset$$

Notation. *Let I be a set and $F \subseteq \mathcal{P}(I)$ be a family with the finite intersection property, we indicate with $[F]$ the filter over I generated by the family F.*

Definition 3.1.12. An ultrafilter $\mathcal{U}$ on λ is *uniform*, if every set of $\mathcal{U}$ has cardinality λ.

Lemma 3.1.13 (Theorem 6.1.7 *i*) [2]). *Let F be an uniform filter on λ generated by a family $E \subseteq F$ such that $|E| \leq \lambda$. Then there exists a family Π of partitions of λ such that $|\Pi| = 2^\lambda$, every $P \in \Pi$ has exactly λ many sets and the couple (Π, F) is consistent.*

Proof. Let $\{J_\gamma \mid \gamma \in \lambda\}$ be the family of all finite intersections of members of E. Note that J_γ has cardinality λ for all $\gamma < \lambda$. By Lemma 3.1.10, there exists a family $\{I_\gamma \mid \gamma \in \lambda\}$ such that $|I_\gamma| = \lambda$, $I_\gamma \subseteq J_\gamma$ and $I_\gamma \cap I_\eta = \emptyset$ for all $\gamma \neq \eta$. Put

$$B = \{(s, r) \mid s \in S_\omega(\lambda) \text{ and } r\colon \mathcal{P}(s) \to \lambda\}$$

and note that $|B| = \lambda$. Consider an enumeration $B = \{(s_\xi, r_\xi) \mid \xi \in \lambda\}$ with possible repetitions in such a way that for all $\gamma \in \lambda$

$$B = \{(s_\xi, r_\xi) \mid \xi \in I_\gamma\}.$$

For each non-empty $J \subseteq \lambda$, define the map $F_J \colon \lambda \to \lambda$ such that

$$F_J(\xi) = \begin{cases} r_\xi(J \cap s_\xi) & \text{if } \xi \in \bigcup_{\gamma\in\lambda} I_\gamma; \\ 0 & \text{otherwise.} \end{cases}$$

Claim 3.1.14. *Every map F_J is surjective and $F_{J_1} \neq F_{J_2}$, for every distinct sets J_1, J_2.*

Proof. First of all we prove that for every $J \subseteq \lambda$ the map F_J is surjective. Fix $\gamma \in \lambda$ and $x \in J$. Put $s = \{x\}$ and $r = \{(s, \gamma)\}$. We have $(s, r) = (s_\xi, r_\xi)$ for some $\xi \in \lambda$, hence we conclude

$$F_J(\xi) = r_\xi(J \cap s_\xi) = r_\xi(s_\xi) = \gamma.$$

Finally, we prove that $F_{J_1} \neq F_{J_2}$ for $J_1 \neq J_2$. Without loss of generality we can assume that there exists $x \in J_1 \setminus J_2$. Set $s = \{x\}$ and $r = \{(s, 0), (\emptyset, 1)\}$. Since $(s, r) \in B$, there exists $\xi \in \lambda$ such that $(s, r) = (s_\xi, r_\xi)$. Then we conclude

$$F_{J_1}(\xi) = r_\xi(J_1 \cap s_\xi) = r_\xi(s_\xi) = 0$$

and

$$F_{J_2}(\xi) = r_\xi(J_2 \cap s_\xi) = r_\xi(\emptyset) = 1.$$

□

By the claim we obtain the family

$$\Pi = \{\{F_J^{-1}(\gamma) \mid \gamma \in \lambda\} \mid J \subseteq \lambda\}$$

has cardinality 2^λ and every partition has exactly λ equivalence classes. To conclude the proof it is sufficient to show that (Π, F) is consistent. Fix distinct $J_1 \dots, J_n \subseteq \lambda$ and $\gamma, \gamma_1, \dots, \gamma_n \in \lambda$, we have to find $\xi \in I_\gamma$ such that

$$F_{J_i}(\xi) = \gamma_i \quad \text{for } 1 \leq i \leq n.$$

Let $s \in S_\omega(\lambda)$ be such that

$$s \cap J_i \neq s \cap J_h \quad \text{for } 1 \leq i < h \leq n.$$

Now define $r \colon S_\omega(\lambda) \to \lambda$ in the following way:

$$r(J_i \cap s) = \gamma_i \quad \text{for } 1 \leq i \leq n.$$

For some $\xi \in I_\gamma$, we have $(s, r) = (s_\xi, r_\xi)$, hence

$$F_{J_i}(\xi) = r_\xi(J_i \cap s_\xi) = \gamma_i.$$

□

Lemma 3.1.15 (Theorem 6.1.7 *ii*) [2]). *Let* (Π, F) *be consistent and* $J \subseteq \lambda$*, where* λ *is a cardinal. Then for some cofinite* $\Pi' \subseteq \Pi$ *either* $(\Pi, [F \cup \{J\}])$ *is consistent or* $(\Pi', [F \cup \{\lambda \setminus J\}])$ *is.*

Proof. There are two cases:

(i) $F \cup \{J\}$ has not the finite intersection property, then there exists $X \in F$ such that $X \cap J = \emptyset$. Clearly $F \cup \{\lambda \setminus J\}$ has the finite intersection property, hence it is sufficient to show that $(\Pi, [F \cup \{\lambda \setminus J\}])$ is consistent. For $X' \in F$ and $X_1 \in P_1 \in \Pi, \dots, X_n \in P_n \in \Pi$, we have

$$X' \cap (\lambda \setminus J) \cap \bigcap_{i \leq n} X_i \supseteq X' \cap X \cap \lambda \setminus J \cap \bigcap_{i \leq n} X_i = X' \cap X \cap \bigcap_{i \leq n} X_i \neq \emptyset.$$

(ii) The families $F \cup \{J\}$ and $F \cup \{\lambda \setminus J\}$ both have the finite intersection property. Now assume that $(\Pi, [F \cup \{J\}])$ is not consistent, hence for distinct $P_1, \dots P_n \in \Pi$, there exist $X \in F$ and $X_1 \in P_1, \dots, X_n \in P_n$ such that

$$* \quad X \cap J \cap \bigcap_{i \leq n} X_i = \emptyset.$$

Put $\Pi' = \Pi \setminus \{P_1, \dots, P_n\}$. Now we prove that $(\Pi', [F \cup \{\lambda \setminus J\}])$ is consistent. For distinct $P'_1, \dots P'_m \in \Pi'$, consider $X'_1 \in P'_1, \dots X'_m \in P'_m$ and $X' \in F$. We have

$$X' \cap (\lambda \setminus J) \cap \bigcap_{i \leq m} X'_i \supseteq X' \cap X \cap (\lambda \setminus J) \cap \bigcap_{i \leq m} X'_i \cap \bigcap_{i \leq n} X_i \neq \emptyset,$$

since $*$ holds and

$$X' \cap X \cap \bigcap_{i \leq m} X'_i \cap \bigcap_{i \leq n} X_i \neq \emptyset.$$

□

Lemma 3.1.16 (Theorem 6.1.7 *iii*) [2]). *For a cardinal* λ*, let* (Π, F) *be consistent,* $p \colon S_\omega(\lambda) \to F$ *be monotone and* $P \in \Pi$*. Then there exist an extension* F' *of* F *and an additive function* $q \colon S_\omega(\lambda) \to F'$ *such that* $q \leq p$ *and* $(\Pi \setminus \{P\}, F')$ *is consistent.*

Proof. Let $P = \{X_\gamma \mid \gamma \in \lambda\}$ and $S_\omega(\lambda) = \{t_\gamma \mid \gamma \in \lambda\}$ be some enumerations. For $\gamma \in \lambda$, define a function $q_\gamma \colon S_\omega(\lambda) \to \mathcal{P}(\lambda)$ such that

$$q_\gamma(s) = \begin{cases} p(t_\gamma) \cap X_\gamma & \text{if } s \subseteq t_\gamma; \\ \emptyset & \text{if } s \not\subseteq t_\gamma. \end{cases}$$

Put $q(s) = \bigcup_{\gamma \in \lambda} q_\gamma(s)$.

Claim 3.1.17. *The function* q *is additive,* $q \leq p$ *and* $F \cup \mathit{range}(q)$ *has the finite intersection property.*

Proof. For $s \in S_\omega(\lambda)$ and $\gamma \in \lambda$ such that $s \subseteq t_\gamma$, we have

$$q_\gamma(s) = p(t_\gamma) \cap X_\gamma \subseteq p(s) \cap X_\gamma \subseteq p(s),$$

hence $q(s) = \bigcup_{\gamma\in\lambda} q_\gamma(s) \subseteq p(s)$. Now we prove that q is additive. For $\gamma \in \lambda$ the map q_γ is additive, since

$$s \cup s' \subseteq t_\gamma \iff s \subseteq t_\gamma \text{ and } s' \subseteq t_\gamma.$$

We obtain

$$q(s \cup s') = \bigcup_{\gamma\in\lambda} q_\gamma(s \cup s') = \left(\bigcup_{\gamma\in\lambda} q_\gamma(s)\right) \cap \left(\bigcup_{\gamma'\in\lambda} q_{\gamma'}(s')\right) = q(s) \cap q(s'),$$

where in the second equality we have used that $q_\gamma(s) \cap q_{\gamma'}(s') = \emptyset$ for $\gamma \neq \gamma'$. Then we conclude $q(s \cup s') = q(s) \cap q(s')$. To conclude the proof, we prove that the family $F \cup \mathrm{range}(q)$ has the finite intersection property. Fix $Y_1, \ldots, Y_n \in F$ and $s_1, \ldots, s_m \in S_\omega(\lambda)$. We have

$$Y_1 \cap \ldots \cap Y_n \cap q(s_1) \cap \ldots \cap q(s_m) = Y_1 \cap \ldots \cap Y_n \cap q(s_1 \cup \ldots \cup s_m)$$

since q is additive. Note that $s_1 \cup \ldots \cup s_m \in S_\omega(\lambda)$, hence $s_1 \cup \ldots \cup s_m = t_\gamma$ for some $\gamma \in \lambda$. We obtain

$$Y_1 \cap \ldots \cap Y_n \cap q(t_\gamma) \supseteq Y_1 \cap \ldots \cap Y_n \cap q_\gamma(t_\gamma) = \underbrace{Y_1 \cap \ldots \cap Y_n \cap p(t_\gamma)}_{\in F} \cap X_\gamma$$

where the last set is not empty since (Π, F) is consistent. □

Finally, we have to prove that $(\Pi \setminus \{P\}, F')$ is consistent. Fix distinct $P_1, \ldots, P_n \in \Pi \setminus \{P\}$, $X \in F$ $s \in S_\omega(\lambda)$ and $X_1 \in P_1, \ldots, X_n \in P_n$. There exists $\gamma \in \lambda$ such that $s = t_\gamma$, then

$$X \cap q(t_\gamma) \cap \bigcap_{i\leq n} X_i \supseteq X \cap q_\gamma(t_\gamma) \cap \bigcap_{i\leq n} X_i = \underbrace{X \cap p(t_\gamma)}_{\in F} \cap X_\gamma \cap \bigcap_{i\leq n} X_i \neq \emptyset,$$

since (Π, F) is consistent and $X_\gamma \in P$. □

Now we prove the existence of λ^+-good ultrafilter.

Theorem 3.1.18 (Theorem 6.1.4 [2]). *Let I be a set of cardinality λ. Then there exists an λ^+-good countably incomplete ultrafilter $\mathcal{U}$ over I.*

Proof. Without loss of generality we can assume $I = \lambda$. First of all, let $\{p_\eta \mid \eta \in 2^\lambda\}$ be an enumeration of all monotone maps form $S_\omega(\lambda)$ to $\mathcal{P}(\lambda)$ and $\{J_\eta \mid \eta \in 2^\lambda\}$ be an enumeration of $\mathcal{P}(\lambda)$. For $n \in \omega$, consider the set

$$I_n = \lambda \setminus \bigcap_{i\leq n} \{\gamma + i \mid \gamma \in \lambda \text{ is limit ordinal}\}.$$

We have $I_n \supseteq I_{n+1}$, $|I_n| = \lambda$ and $\bigcap_{n\in\omega} I_n = \emptyset$. Let U_0 be the uniform filter generated by the family $\{I_n \mid i \in \omega\}$. By Lemma 3.1.13, we can find a family of partitions Π_0 of λ such that $|\Pi_0| = 2^\lambda$ and (Π_0, U_0) is consistent. We shall construct two families $\{\Pi_\gamma \mid \gamma \in 2^\lambda\}$ and $\{U_\gamma \mid \gamma \in 2^\lambda\}$ with the following properties:

(i) $\Pi_\xi \supseteq \Pi_\gamma$ and $U_\xi \subseteq U_\gamma$, for all $\xi \leq \gamma \in 2^\lambda$.

(ii) $|\Pi_\gamma| = 2^\lambda$, $|\Pi_\gamma \setminus \Pi_{\gamma+1}| \in \omega$ and $\Pi_\gamma = \bigcap_{\eta\in\gamma} \Pi_\eta$ for γ limit ordinal.

(iii) (Π_γ, U_γ) is consistent for each $\gamma \in 2^\lambda$.

At the limit step, put $\Pi_\gamma = \bigcap_{\eta\in\gamma} \Pi_\eta$ and $U_\gamma = \bigcup_{\eta\in\gamma} U_\eta$ and note that the clauses $(i), (iii)$ are trivial. To check that $ii)$ holds, observe that $|\Pi_0 \setminus \Pi_\gamma| = |\gamma \cdot \omega|$. Now assume that U_γ and Π_γ are defined. If $\gamma + 1$ is odd, let J_η be the first element of $\mathcal{P}(\lambda)$ not already in U_γ. By Lemma 3.1.15, there exist $U_{\gamma+1}$ and $\Pi_{\gamma+1}$ such that neither $J_\eta \in U_{\gamma+1}$ nor $\lambda \setminus J_\eta \in U_{\gamma+1}$ and clauses $(i), (ii), (iii)$ hold. If $\gamma + 1$ is even, let $p_\eta \colon S_\omega(\lambda) \to U_\gamma$ be the first function which we have not already dealt with. By Lemma 3.1.16, we can find $U_{\gamma+1}, \Pi_{\gamma+1}$ and an additive map $q \colon S_\omega(\lambda) \to U_{\gamma+1}$ such that $q \leq p_\eta$, $U_{\gamma+1} = [U_\gamma \cup \{\text{range}(q)\}]$ and clauses $(i), (ii), (iii)$ holds. When the construction is completed define $\mathcal{U} = \bigcup_{\gamma\in 2^\lambda} U_\gamma$. Clearly $\mathcal{U}$ is a countably incomplete ultrafilter on λ. Moreover, if $p \colon S_\omega(\lambda) \to \mathcal{U}$ is monotone, there exists $\gamma \in 2^\lambda$ such that $\text{range}(p) \subseteq U_\xi$ for all $\xi \geq \gamma$, since $cof(\lambda) < cof(\lambda^\lambda) = cof(2^\lambda)$. Hence, by construction of $\mathcal{U}$, we conclude that $\mathcal{U}$ is λ^+-good. □

Note that this result of existence is the best possible, in fact, by lemmas 3.1.5, 3.1.8, there exists no λ^{++}-good countably incomplete ultrafilter on λ.

To continue our studies we introduce a class of rich structures, called *saturated*, but first of all we recall the following definition.

Definition 3.1.19. Let T be a theory and $\mathcal{M}$ be a model of T. A *type* $p(x)$ with parameters in $A \subseteq \mathcal{M}$ is a set of formulas of T with parameters in A. We say that $p(x)$ is *finitely satisfiable* in $\mathcal{M}$, if for every $\phi_1(x), \ldots, \phi_n(x) \in p(x)$ we have

$$\mathcal{M} \models \exists x\ \phi_1(x) \wedge \ldots \wedge \phi_n(x).$$

Definition 3.1.20. Let λ be an infinite cardinal and $\mathcal{M}$ be an $\mathcal{L}$-structure. We say that $\mathcal{M}$ is λ-*saturated*, if for every $A \subseteq \mathcal{M}$, with $|A| < \lambda$, $\mathcal{M}$ realizes every type $p(x)$ such that

1. $p(x)$ has parameters in A;

2. $p(x)$ is finitely satisfiable in $\mathcal{M}$.

We say that $\mathcal{M}$ is *saturated*, if it is $|M|$-saturated.

Remark 3.1.21. Assume that $\mathcal{M}$ is an infinite κ-saturated $\mathcal{L}$-structure, then $|\mathcal{M}| \geq \kappa$. Otherwise, $\mathcal{M} = \{a_\gamma \mid \gamma \in \lambda\}$ for some $\lambda < \kappa$ and the type

$$p(x) = \{\neg(x = a_\gamma) \mid \gamma \in \lambda\}$$

should have a realization.

Definition 3.1.22. Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{L}$-structures. A partial map $f \colon \mathcal{M} \to \mathcal{N}$ is called *elementary* if for every $\mathcal{L}$-formula $\psi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in \mathrm{dom}(f)$ we have

$$\mathcal{M} \models \psi(a_1, \dots, a_n) \Longleftrightarrow \mathcal{N} \models \psi(f(a_1), \dots, f(a_n))$$

Remark 3.1.23. Let $f \colon \mathcal{M} \to \mathcal{N}$ be an elementary map and $p(x, (a)_{a \in \mathrm{dom}(f)})$ be a finitely satisfiable type in $\mathcal{M}$ with parameters in $\mathrm{dom}(f)$. It is easy to check that the type $p(x, (f(a))_{a \in \mathrm{dom}(f)})$ is finitely satisfiable in $\mathcal{N}$. In fact, for $\psi_1(x, f(\bar{a}_1)), \dots, \psi_n(x, f(\bar{a}_n))$ in $p(x, (f(a))_{a \in \mathrm{dom}(f)})$ we have

$$\mathcal{N} \models \exists x \bigwedge_{i=1}^{n} \psi_i(x, f(\bar{a}_i)) \Longleftrightarrow \mathcal{M} \models \exists x \bigwedge_{i=1}^{n} \psi_i(x, \bar{a}_i)$$

Lemma 3.1.24. *Let $\mathcal{M}$ be a λ-saturated $\mathcal{L}$-structure, where $|\mathcal{L}| \leq \lambda$. For $A \subseteq \mathcal{M}$ of cardinality less than λ, let $p((x_\alpha)_{\alpha \in \kappa})$ be a type with parameters in A and with infinite variables $\{x_\alpha \mid \alpha \in \kappa\}$ for some $\kappa \leq \lambda$. If $p((x_\alpha)_{\alpha \in \kappa})$ is finitely satisfiable in $\mathcal{M}$, then it is realized.*

Proof. By Corollary B.1.3, there exists an elementary extension $\mathcal{M}'$ of $\mathcal{M}$ that realizes the type $p((x_\alpha)_{\alpha \in \kappa})$. We indicate by $(b_\alpha)_{\alpha \in \kappa}$ the realization of $p((x_\alpha)_{\alpha \in \kappa})$ in $\mathcal{M}'$. By Löwenheim-Skolem Theorem, there exists an $\mathcal{L}$-structure $\mathcal{N} \preccurlyeq \mathcal{M}'$ such that $b_\alpha \in \mathcal{N}$ for all $\alpha \in \kappa$, $A \subseteq \mathcal{N}$ and $|\mathcal{N}| = \gamma \leq \lambda$. Fix $\mathcal{N} = \{c_\alpha \mid \alpha \in \gamma\}$ an enumeration of $\mathcal{N}$. We construct a set of partial functions $\{f_\alpha \mid \alpha \leq \gamma\}$ with the following properties:

(i) $f_\alpha \colon \mathcal{N} \to \mathcal{M}$ is partial elementary map and fixes A.

(ii) $c_\alpha \in \mathrm{dom}(f_{\alpha+1})$.

(iii) $f_\beta \subseteq f_\alpha$ holds for all $\beta \in \alpha \leq \gamma$.

When the construction is completed, the sequence $(f_\gamma(b_\alpha))_{\alpha \in \kappa}$ is a realization of the type $p((x_\alpha)_{\alpha \in \kappa})$. Put $f_0 = id_A$ and note that f_0 is elementary since $\mathcal{M} \preccurlyeq \mathcal{M}'$ and $\mathcal{N} \preccurlyeq \mathcal{M}'$. In the successor step, assume that f_α is defined and $c_\alpha \notin \mathrm{dom}(f_\alpha)$. Put

$$q(x, (a)_{a \in \mathrm{dom}(f_\alpha) \cup A}) = \{\psi(x) \mid \psi(x) \text{ has parameters in } A \cup \mathrm{dom}(f_\alpha) \text{ and } \mathcal{N} \models \psi(c_\alpha)\}.$$

By inductive hypothesis, the type

$$q(x, (f_\alpha(a))_{a \in \mathrm{dom}(f_\alpha) \cup A})$$

with parameters in $\mathrm{range}(f_\alpha) \cup A$ is finitely satisfiable in $\mathcal{M}$, hence this is realized by some $c \in \mathcal{M}$. Put $f_{\alpha+1} = f_\alpha \cup \{(c_\alpha, c)\}$. In the limit step, put $f_\alpha = \bigcup_{\beta \in \alpha} f_\beta$.

□

The next result is simple application of *back and forth* constructions.

Theorem 3.1.25. *Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures elementarily equivalent. Assume that $\mathcal{M}, \mathcal{N}$ have the same cardinality λ. If $\mathcal{M}, \mathcal{N}$ are saturated, then $\mathcal{M} \cong \mathcal{N}$.*

Proof. Let $\mathcal{M} = \{a_\gamma \mid \gamma \in \lambda\}$ and $\mathcal{N} = \{b_\gamma \mid \gamma \in \lambda\}$ be an enumeration of $\mathcal{M}, \mathcal{N}$, respectively. We construct inductively a family of partial functions $\{f_\gamma \mid \gamma \leq \lambda\}$ with the following properties:

1. the map $f_{\gamma+1} \colon \mathcal{M} \to \mathcal{N}$ is elementary, $a_\gamma \in \mathrm{dom}(f_{\gamma+1})$ and $b_\gamma \in \mathrm{range}(f_{\gamma+1})$.
2. $f_\eta \subseteq f_\gamma$ holds for all $\eta \leq \gamma \leq \lambda$.

For $\gamma = 0$, we put $f_0 = \emptyset$. Note that f_0 is elementary, since $\mathcal{M} \equiv \mathcal{N}$. Assume that f_γ is defined, $a_\gamma \notin \mathrm{dom}(f_\gamma)$ and $b_\gamma \notin \mathrm{range}(f_\gamma)$. Put

$$p(x, (a)_{a \in \mathrm{dom}(f_\gamma)}) = \{\psi(x) \mid \psi(x) \text{ has parameters in } \mathrm{dom}(f_\gamma) \text{ and } \mathcal{M} \models \psi(a_\gamma)\}.$$

The type $p(x, (f_\gamma(a))_{a \in \mathrm{dom}(f_\gamma)})$ is finitely satisfiable in $\mathcal{N}$, since f_γ is elementary, so there exists a realization b in $\mathcal{N}$. Now consider the map $g = f_\gamma^{-1} \cup \{(b, a_\gamma)\}$ and repeat the preceding argument to find c such that $g \cup \{(b_\gamma, c)\}$ is an elementary map. Finally, put $f_{\gamma+1} = f_\gamma \cup \{(a_\gamma, b)\} \cup \{(c, b_\gamma)\}$. For γ limit ordinal, define $f_\gamma = \bigcup_{\eta \in \gamma} f_\eta$. When the construction is complete, the map f_λ is an isomorphism. □

Notation. *In the sequel of this thesis if $p(x)$ is a type, we write $S_\omega(p)$ to indicate $S_\omega(|p|)$.*

The following result shows how the existence of α-good ultrafilter allows to construct saturated structures.

Theorem 3.1.26 (Theorem 6.1.8 [2])**.** *Let $\mathcal{U}$ be a λ-good countably incomplete ultrafilter on I. Assume that $\{\mathcal{M}_i \mid i \in I\}$ is a family of $\mathcal{L}$-structures with $|\mathcal{L}| < \lambda$. Then the $\mathcal{L}$-structure $\prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ is λ-saturated.*

Proof. Let $p(x, (a_\beta)_{\beta \in \alpha})$ be a finitely satisfiable type in $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ with parameters $(a_\beta)_{\beta \in \alpha}$ for $\alpha \in \lambda$. We can expand the language $\mathcal{L}$ to $\mathcal{L}'$, adding α new constants $\{c_\beta\}_{\beta \in \alpha}$, such that $c_\beta^{\mathcal{M}} = a_\beta$. Note that the language $\mathcal{L}'$ has cardinality less than λ. Moreover, the type $\tilde{p}(x) = p(x, (c_\beta)_{\beta \in \alpha})$ is finitely satisfiable without parameters in $\mathcal{M}$. Hence it is sufficient to show that every finitely satisfiable type $p(x)$ without parameters has a realization in $\mathcal{M}$. Let $\{I_n \mid n \in \omega\} \subseteq \mathcal{U}$ be a family such that

$$I = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_n \supseteq \ldots$$

and $\bigcap I_n = \emptyset$. Note that $|p(x)| < \lambda$ since $|p(x)| \leq |\{\psi(x) \mid \psi(x) \text{ is } \mathcal{L}\text{-formula}\}| = |\mathcal{L}| < \lambda$. Define the map

$$f \colon S_\omega(p) \to \mathcal{U}$$

such that

$$f(u) = I_{|u|} \cap \{i \in I \mid \mathcal{M}_i \models \exists x \bigwedge u(x)\}.$$

The map f is monotone. In fact, if $u \subseteq u'$, then

$$I_{|u|} \supseteq I_{|u'|}$$

and

$$\{i \in I \mid \mathcal{M}_i \models \exists x \bigwedge u(x)\} \supseteq \{i \in I \mid \mathcal{M}_i \models \exists x \bigwedge u'(x)\}.$$

Hence there exists an additive map g such that $g \leq f$. For $i \in I$, define

$$\pi(i) = \bigcup \{\phi(x) \in p(x) \mid i \in g(\{\phi(x)\})\}.$$

We prove that $|\pi(i)| \in \omega$ for all $i \in I$. Otherwise, for all $n \in \omega$ we have

$$\begin{aligned} |\pi(i)| \geq n \Rightarrow & \{\phi_1(x), \ldots, \phi_n(x)\} \in \pi(i) \Rightarrow \\ & \{\phi_1(x)\} \cup \ldots \cup \{\phi_n(x)\} \in \pi(i) \Rightarrow \\ & i \in g(\{\phi_1(x)\}) \cap \ldots \cap g(\{\phi_n(x)\}) \Rightarrow \\ & i \in g(\{\phi_1(x)\} \cup \ldots \cup \{\phi_n(x)\}) \Rightarrow \\ & i \in f(\{\phi_1(x), \ldots, \phi_n(x)\}) \subseteq I_n, \end{aligned}$$

that is $\bigcap_{n \in \omega} I_n \neq \emptyset$, contradiction. For $i \in I$ we have $\pi(i) \in S_\omega(p)$ and

$$i \in \bigcap \{g(\{\phi(x)\}) \mid \phi(x) \in \pi(i)\} = g(\pi(i)) \subseteq f(\pi(i)).$$

For $i \in I$ such that $\pi(i) \neq \emptyset$, we can choose $h(i) \in \mathcal{M}_i$ such that

$$\mathcal{M}_i \models \bigwedge \pi(i)(h(i)).$$

We complete the proof showing that $[(h(i))_{i \in I}]$ is a realization of $p(x)$. Let $\phi(x)$ be a formula of $p(x)$. Since $i \in g(\{\phi(x)\})$ implies $\phi(x) \in \pi(i)$, we have

$$\mathcal{U} \ni g(\{\phi(x)\}) \subseteq \{i \in I \mid \mathcal{M}_i \models \phi(h(i))\}.$$

We conclude by Loś's Theorem. □

3.2 Direct limit of $\mathcal{L}$-structures

Definition 3.2.1. A partial order $(I, \leq)$ is a *directed set*, if for any $i, j \in I$ there exists a $k \in I$ such that $i, j \leq k$. A *directed system* $\{(\mathcal{M}_i, e_{ij}) \mid i, j \in I,\ \ i \leq j\}$ of $\mathcal{L}$-structures consists in a directed set $(I, \leq)$, a family $\{\mathcal{M}_i \mid i \in I\}$ of $\mathcal{L}$-structures and a family $\{e_{ij} \mid i, j \in I,\ \ i \leq j\}$ of morphisms of $\mathcal{L}$-structures such that:

1. $e_{ij} \colon \mathcal{M}_i \to M_j$;

2. $e_{ii} = Id_{\mathcal{M}_i}$ for all $i \in I$;

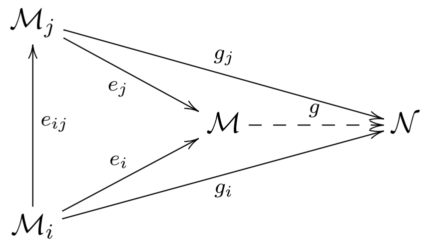
3. $e_{ik} = e_{jk} \circ e_{ij}$ for all $i \leq j \leq k$.

Lemma 3.2.2. *Let $\{(\mathcal{M}_i, e_{ij}) \mid i, j \in I,\ i < j\}$ be a directed system of $\mathcal{L}$-structures, where the morphisms e_{ij} are elementary maps. There exists an $\mathcal{L}$-structure $\mathcal{M}$ and a family $\{e_i \mid i \in I\}$ of elementary morphisms such that*

1) $e_i \colon \mathcal{M}_i \to \mathcal{M}$;

2) $e_i = e_j \circ e_{ij}$ for all $i \leq j$.

3) If $\{g_i \mid i \in I\}$ is a family of morphisms such that $g_i \colon \mathcal{M}_i \to \mathcal{N}$ for some $\mathcal{L}$-structures $\mathcal{N}$ and for all $i \leq j$ the following diagram commutes



there exists a unique $g \colon \mathcal{M} \to \mathcal{N}$ such that $g_i = g \circ e_i$.

The $\mathcal{L}$-structure $\mathcal{M}$ is called direct limit of the family $\{\mathcal{M}_i \mid i \in I\}$ and it is denoted by $\varinjlim_{i \in I} \mathcal{M}_i$.

Proof. First of all we construct the $\mathcal{L}$-structure $\mathcal{M}$. Define an equivalence relation $\sim$ over $\bigcup_{i \in I} M_i$ such that, if $x \in M_i$, $y \in M_j$ and $i \leq j$,

$$x \sim y \iff e_{ik}(x) = e_{jk}(y) \text{ for some } k \geq i, j.$$

Let $M = \bigcup_{i \in I} M_i / \sim$ be the domain of the structure $\mathcal{M}$. Now we have to define the interpretation of the symbols of constant, relation and function. If c in $\mathcal{L}$ is a symbol of constant, put $c^{\mathcal{M}} = [c^{\mathcal{M}_i}]$ and note that $c^{\mathcal{M}_i} \sim c^{\mathcal{M}_j}$ for every $i, j \in I$. Let $R(x_1, \ldots, x_n)$ be a relation symbol of $\mathcal{L}$ and $a_i \in \mathcal{M}_{h_i}$ for $1 \leq i \leq n$, we say that

$$R^{\mathcal{M}}([a_1], \ldots, [a_n])$$

holds if there exists a $k \geq h_i$ for any i, such that

$$R^{\mathcal{M}_k}(e_{h_1 k}(a_1), \ldots, e_{h_n k}(a_n)).$$

In a similar way, suppose that $f(x_1, \ldots, x_n)$ is a function symbol and $a_i \in \mathcal{M}_{h_i}$ for $1 \leq i \leq n+1$, we say that

$$f^{M}([x_1], \ldots, [x_n]) = [x_{n+1}]$$

holds, if there exists a $k \geq h_i$ for any i, such that

$$f^{M_k}(e_{h_1 k}(a_1), \ldots, e_{h_n k}(a_n)) = e_{h_{n+1} k}(a_{n+1}).$$

By the definition of $\sim$, it is easy to see that the interpretations are well defined. Now put

$$e_i\colon \mathcal{M}_i \to \mathcal{M}$$

such that $e_i(a) = [a]$ and clearly $1), 2)$ hold. We prove that the clause 3) holds. Assume that there exist $g_i\colon \mathcal{M}_i \to \mathcal{N}$ for each $i \in I$, such that $g_i = e_{ij} \circ g_j$ for any $i \leq j$. Define $g\colon \mathcal{M} \to \mathcal{N}$ such that $g([a]) = g_i(a)$ if $a \in \mathcal{M}_i$.

Claim 3.2.3. *The map g is a well defined morphism such that $g \circ e_i = g_i$ for every $i \in I$.*

Proof. We prove that g is well defined. Assume that $a \sim b$, $a \in \mathcal{M}_i$ and $b \in \mathcal{M}_j$, then for some $k \in I$ such that $i, j \leq k$ we have $e_{ik}(a) = e_{jk}(b)$. We have

$$g([a]) = g_i(a) = g_k(e_{ik}(a)) = g_k(e_{jk}(b)) = g_j(b) = g([b]),$$

thus g is well defined. By definition we have $g \circ e_i = g_i$ for every $i \in I$. Now we prove that g is a morphism, that is the clauses of Definition B.1.10 hold. If $[c^{\mathcal{M}_i}] \in \mathcal{M}$ is the interpretation of a constant, we have $g([c^{\mathcal{M}_i}]) = f_i(c^{\mathcal{M}_i})$. Now consider a term $f^{\mathcal{M}}([a_1], \dots, [a_n])$. We can assume that $a_1, \dots, a_n \in \mathcal{M}_k$ for some $k \in I$, then

$$\begin{aligned} g(f^{\mathcal{M}}([a_1], \dots, [a_n])) &= g_k(f^{\mathcal{M}_k}(a_1, \dots, a_n)) \\ &= f^{\mathcal{N}}(g_k(a_1), \dots, g_k(a_n)) = \\ &= f^{\mathcal{N}}(g([a_1]), \dots, g([a_1])). \end{aligned}$$

In a similar way we can prove that if $R^{\mathcal{M}}([a_1], \dots, [a_n])$, then $R^{\mathcal{N}}(g([a_1]), \dots, g([a_n]))$; hence we conclude that g is a morphism.

□

Finally, we prove that the maps e_i are elementary, that is, for any $\mathcal{L}$-formula $\psi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in M_i$, we have

$$\mathcal{M} \models \psi([a_1], \dots, [a_n]) \iff \mathcal{M}_i \models \psi(a_1, \dots, a_n).$$

We proceed by induction on the complexity of the formula $\psi(x_1, \dots, x_n)$. When $\psi(x_1, \dots, x_n)$ is atomic, we conclude since the maps e_{ij} are elementary. Assume $\psi(x_1, \dots, x_n) = \psi_1(x_1, \dots, x_n) \wedge \psi_2(x_1, \dots, x_n)$, then

$$\begin{aligned} &\mathcal{M} \models \psi([a_1], \dots, [a_n]) \iff \\ &\mathcal{M} \models \psi_1([a_1], \dots, [a_n]) \wedge \psi_2([a_1], \dots, [a_n]) \iff \\ &\mathcal{M} \models \psi_1([a_1], \dots, [a_n]) \text{ and } \mathcal{M} \models \psi_2([a_1], \dots, [a_n]) \iff \\ &\mathcal{M}_i \models \psi_1(a_1, \dots, a_n) \text{ and } \mathcal{M}_i \models \psi_2(a_1, \dots, a_n) \iff \\ &\mathcal{M}_i \models \psi_1(a_1, \dots, a_n) \wedge \psi_2(a_1, \dots, a_n) \iff \\ &\mathcal{M}_i \models \psi(a_1, \dots, a_n). \end{aligned}$$

Assume $\psi(x_1, \ldots, x_n) = \neg\phi(x_1, \ldots, x_n)$, then

$$\begin{aligned} \mathcal{M} \models \psi([a_1], \ldots, [a_n]) \iff & \mathcal{M} \models \neg\phi([a_1], \ldots, [a_n]) \iff \\ & \mathcal{M} \not\models \phi([a_1], \ldots, [a_n]) \iff \\ & \mathcal{M}_i \not\models \phi(a_1, \ldots, a_n) \iff \\ & \mathcal{M}_i \models \neg\phi(a_1, \ldots, a_n) \iff \\ & \mathcal{M}_i \models \psi(a_1, \ldots, a_n). \end{aligned}$$

Finally, assume $\psi(x_1, \ldots, x_n) = \exists x\phi(x, x_1, \ldots, x_n)$, then

$$\begin{aligned} \mathcal{M}_i \models \psi(a_1, \ldots, a_n) \implies & \mathcal{M}_i \models \exists x\phi(x, a_1, \ldots, a_n) \implies \\ & \mathcal{M}_i \models \phi(a, a_1, \ldots, a_n) \text{ for some } a \in M_i \implies \\ & \mathcal{M} \models \phi([a], [a_1], \ldots, [a_n]) \text{ for some } a \in M_i \implies \\ & \mathcal{M} \models \exists x\phi(x, [a_1], \ldots, [a_n]) \implies \\ & \mathcal{M} \models \psi([a_1], \ldots, [a_n]). \end{aligned}$$

Now we show the reverse implication:

$$\begin{aligned} & \mathcal{M} \models \psi([a_1], \ldots, [a_n]) \implies \\ & \mathcal{M} \models \exists x\phi(x, [a_1], \ldots, [a_n]) \implies \\ & \mathcal{M} \models \phi([a], [a_1], \ldots, [a_n]) \text{ for some } a \in M_j \implies \\ & \mathcal{M} \models \phi([e_{jk}(a)], [e_{ik}(a_1)], \ldots, [e_{ik}(a_n)]) \text{ for some } a \in M_j \text{ and } k \geq i, j \implies \\ & \mathcal{M}_k \models \phi(e_{jk}(a), e_{ik}(a_1), \ldots, e_{ik}(a_n)) \text{ for some } a \in M_j \text{ and } k \geq i, j \implies \\ & \mathcal{M}_k \models \exists x\phi(x, e_{ik}(a_1), \ldots, e_{ik}(a_n)) \text{ for some } k \geq i, j, \end{aligned}$$

By hypothesis the map e_{ik} is elementary, hence $\mathcal{M}_i \models \exists x\phi(x, a_1, \ldots, a_n)$. □

3.3 The strong chains

Definition 3.3.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and κ be an infinite cardinal. We say that a family $\{\mathcal{M}_\alpha \mid \alpha \in \kappa\}$ of $\mathcal{L}$-structures is a *strong chain* for $\mathcal{M}$, if $\mathcal{M}_0 = \mathcal{M}$ and the following properties hold:

- for every $\alpha \in \kappa$ there exists a cardinal $\lambda \geq \alpha$ such that $\mathcal{M}_{\alpha+1} = \mathcal{M}_\alpha^\lambda/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on λ.
- If $\alpha \in \kappa$ is limit ordinal, then $\mathcal{M}_\alpha = \varinjlim_{\beta\in\alpha} \mathcal{M}_\beta$.

Remark 3.3.2. Note that a strong chain is a directed system of $\mathcal{L}$-structures. By Lemma 3.2.2 and Corollary B.1.14, there exist some elementary maps $e_{\beta\alpha} \colon \mathcal{M}_\beta \to \mathcal{M}_\alpha$ for every $\beta \leq \alpha$. Hence, given a strong chain $\{\mathcal{M}_\alpha \mid \alpha \in \kappa\}$, is well defined the direct limit of the structures $\mathcal{M}_\alpha$ and the maps $e_\alpha \colon \mathcal{M}_\alpha \to \varinjlim_{\alpha\in\kappa} \mathcal{M}_\alpha$ are elementary, by Lemma 3.2.2.

Theorem 3.3.3. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Assume that κ is an inaccessible cardinal such that $|\mathcal{L}|, |\mathcal{M}| < \kappa$. There exists a strong chain $\{\mathcal{M}_\alpha \mid \alpha \in \kappa\}$ for $\mathcal{M}$, such that $\varinjlim_{\alpha\in\kappa} \mathcal{M}_\alpha$ has cardinality κ and it is saturated.*

Proof. Let $(\lambda_i)_{i\in\kappa}$ be an increasing sequence of cardinal such that $\lambda_0 = \max\{|\mathcal{M}|^+, |\mathcal{L}|^+\}$ and $\lambda_i \nearrow \kappa$. Note that each increasing cardinal sequence that converges to κ has cardinality κ, since κ is regular. We construct inductively a strong chain for $\mathcal{M}$ such that

1) $|M_\alpha| < \kappa$ for all $\alpha \in \kappa$;

2) $\mathcal{M}_{\alpha+1}$ is λ_α^+-saturated for all $\alpha \in \kappa$.

Put $\mathcal{M}_0 = \mathcal{M}$. At the successor step assume that $\mathcal{M}_\alpha$ is defined. Let $\mathcal{U}$ be a λ_α^+-good countably incomplete ultrafilter on λ_α and put $\mathcal{M}_{\alpha+1} = M_\alpha^{\lambda_\alpha}/\mathcal{U}$. Note that $\mathcal{U}$ exists by Theorem 3.1.18. The condition 2) holds by Theorem 3.1.26. Moreover, $|M_{\alpha+1}| \leq |M_\alpha^{\lambda_\alpha}| < \kappa$, since κ is inaccessible. At the limit step, define $\mathcal{M}_\alpha = \varinjlim_{\beta\in\alpha} M_\beta$. Note that $|\mathcal{M}_\alpha| \leq \sum_{\beta\in\alpha} |M_\beta| < \kappa$, by regularity of κ. When the construction is completed, put

$$\mathcal{M}^* = \varinjlim_{\alpha\in\kappa} \mathcal{M}_\alpha$$

and note that the maps $e_\alpha \colon \mathcal{M}_i \to \mathcal{M}^*$ are elementary for all $\alpha \in \kappa$, by Lemma 3.2.2. Obviously $|\mathcal{M}^*| \leq \kappa$ holds. To conclude the proof it is sufficient to prove that $\mathcal{M}^*$ is κ-saturated. For $\lambda < \kappa$, let $p(x, ([a_\gamma])_{\gamma\in\lambda})$ be a type of $\mathcal{M}^*$, with parameters $([a_\gamma])_{\gamma\in\lambda}$. By regularity of κ, we can assume that there exists $\alpha \in \kappa$ such that $a_\gamma \in \mathcal{M}_\alpha$ for any $\gamma \in \lambda$. Note that $p(x, (a_\gamma)_{\gamma\in\lambda})$ is a type finitely satisfiable in $\mathcal{M}_\alpha$, since the map e_α is an elementary. Let $\beta \geq \alpha$ be such that $\lambda_\beta \geq \lambda$, then we have that $\mathcal{M}_{\beta+1}$ is λ_β^+-saturated. Hence there exists a realization $a \in M_{\beta+1}$ of the type $p(x, (e_{\alpha\beta+1}(a_\gamma))_{\gamma\in\lambda})$. Since $e_{\beta+1}$ is elementary, we conclude that $[a]$ realizes the type $p(x, ([a_\gamma])_{\gamma\in\lambda})$. $\square$

Theorem 3.3.4. *Assume the existence of a proper class of inaccessible cardinals. Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures, then the following are equivalent:*

(i) $\mathcal{M} \equiv \mathcal{N}$*;*

(ii) for some inaccessible cardinal κ larger than $|\mathcal{L}|+|\mathcal{M}|+|\mathcal{N}|$, there exist two isomorphic and saturated $\mathcal{L}$-structures $\mathcal{M}'$ and $\mathcal{N}'$ of cardinality κ such that $\mathcal{M} \prec \mathcal{M}'$, $\mathcal{N} \prec \mathcal{N}'$.

Proof.

$(i) \Rightarrow (ii)$ Put

$$\kappa = \min\{\lambda > |\mathcal{L}| + | + \mathcal{M}| + |\mathcal{N}| \mid \lambda \text{ is an inaccessible cardinal}\}.$$

By Theorem 3.3.3, there exists two strong chains $\{\mathcal{M}_\alpha \mid \alpha \in \kappa\}$ and $\{\mathcal{N}_\alpha \mid \alpha \in \kappa\}$ for $\mathcal{M}, \mathcal{N}$, respectively, such that $\mathcal{M}' = \varinjlim_{\alpha\in\kappa} \mathcal{M}_\alpha$ and $\mathcal{N}' = \varinjlim_{\alpha\in\kappa} \mathcal{N}_\alpha$ have the same cardinality κ and they are saturated. Moreover, we have $\mathcal{M} \equiv \mathcal{M}'$ and $\mathcal{N} \equiv \mathcal{N}'$, by Lemma 3.2.2. By Theorem 3.1.25, we obtain $\mathcal{M}' \cong \mathcal{N}'$.

$(ii) \Rightarrow (i)$ By Lemma B.1.11, we have $\mathcal{M}' \equiv \mathcal{N}'$. Hence we conclude $\mathcal{M} \equiv \mathcal{M}' \equiv \mathcal{N}' \equiv \mathcal{N}$.

$\square$

Chapter 4

Keisler's order

In this chapter, we define Keisler's order on the class of complete theories on a countable language. Our aim is to characterize of the maximal theories in combinatorial terms. We will link the theory of gaps, studied in Chapter 1, with the properties of Keisler's order.

In Section 4.1, we define a binary relation between the complete countable theories: we write $T_1 \trianglelefteq_\lambda T_2$, if for all models $\mathcal{M}_1, \mathcal{M}_2$ of T_1, T_2, respectively, and each regular ultrafilter $\mathcal{U}$ on λ, if $\mathcal{M}_2^\lambda/\mathcal{U}$ is λ^+ saturated, then so $\mathcal{M}_1^\lambda/\mathcal{U}$. We show that this relation is a preorder, that we call *Keisler's order.*

In Section 4.2, we show that a theory is maximal if and only if the only for all cardinals λ the only ultrafilters which saturate the ultrapowers of models of T are λ^+-good.

In Section 4.3, we prove the maximality of every theory with the SOP property, that is a theory in which we can define a partial order with infinite chains.

In Section 4.4, we show that every gap of some ultrapower of finite linear orders appears in an ultrapower of $(\omega, <)$. Hence, we obtain a characterization of good ultrafilters in terms of gaps that we can find in an ultrapowers of the linear order $(\omega, <)$. We introduce the notion of *treetops*, that generalizes the idea of unbounded chain given in Chapter 1. In fact, now we study unbounded increasing chains in ultraproducts of arbitrary pseudo-trees. We conclude the section showing that the existence of certain treetops is equivalent to the goodness of the ultrafilter.

In Section 4.5, we define a theory with the SOP_2 property, that is: in some model $\mathcal{M}$ and for some formula $\psi(x, \bar{y})$, there is an interpretation of a tree $(\{\bar{a}_s \mid s \in \mu^{<\kappa}\}, \trianglelefteq)$ in $\mathcal{M}$ with the property that a ψ-type with parameters in T is consistent if an only if the parameters are $\trianglelefteq$-compatible. For a model $\mathcal{M}$ of a SOP_2-theory, we prove that the realization of ψ-types is an ultrapower of $\mathcal{M}$ is equivalent to the existence of special functions, called *distributions.*

In Section 4.6, we show that every SOP_2-theory is maximal in Keisler's order. In particular, an ultrapower of a model $\mathcal{M}$ of a SOP_2-theory is λ^+-saturated if and only if $\mathcal{U}$ has the λ^+-treetops property.

4.1 Basic properties

We shall assume that every language is countable and every theory is complete, that is any two models of a theory T are elementarily equivalent.

Definition 4.1.1. Let λ be an infinite cardinal and T_1, T_2 theories. We say that $T_1 \trianglelefteq_\lambda T_2$ if and only if for every $\mathcal{M}_1, \mathcal{M}_2$ models of T_1, T_2, respectively, and any regular ultrafilter $\mathcal{U}$ on λ we have

$$\text{if } \mathcal{M}_2^\lambda/\mathcal{U} \text{ is } \lambda^+\text{-saturated, then } \mathcal{M}_1^\lambda/\mathcal{U} \text{ is } \lambda^+\text{-saturated.}$$

We write $T_1 \trianglelefteq T_2$, if $T_1 \trianglelefteq_\lambda T_2$ holds for every infinite cardinal λ.

Lemma 4.1.2 (Keisler, Theorem 2.1 [3])**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be elementarily equivalent $\mathcal{L}$-structures and I be a set of cardinality λ. If $\mathcal{U}$ is a regular ultrafilter on I and $\mathcal{N}^I/\mathcal{U}$ is λ^+-saturated, then $\mathcal{M}^I/\mathcal{U}$ is λ^+-saturated.*

Proof. Let $\gamma \leq \lambda$ be a cardinal and $p(x, (a_\alpha)_{\alpha\in\gamma})$ be a complete finitely satisfiable type in $\mathcal{M}^I/\mathcal{U}$. By hypothesis $\mathcal{L}$ is countable, then the type has cardinality γ. Let

$$\{\phi_\alpha(x, \bar{a}_\alpha) \mid \alpha \in \gamma\}$$

be an enumeration of the formulas of the type, where $\bar{a}_\alpha$ is a finite tuple for any $\alpha \in \gamma$. Let $X = \{X_\alpha \mid \alpha \in \lambda\}$ be a λ-regularizing family of $\mathcal{U}$ that is $\bigcap_{\alpha\in u} X_\alpha = \emptyset$ for all infinite $u \subseteq \lambda$. For $i \in I$, put

$$\Sigma(i) = \{\phi_\alpha(x, \bar{a}_\alpha(i)) \mid i \in X_\alpha\}$$

and note that $\Sigma(i)$ is finite.

Claim 4.1.3. *Fix $i \in I$. For every $\alpha \in \gamma$, we can choose $b_\alpha(i) \in \mathcal{N}$ such that for any subset $\{\phi_{\alpha_1}(x, \bar{a}_{\alpha_1}(i)), \ldots, \phi_{\alpha_n}(x, \bar{a}_{\alpha_n}(i))\}$ of sentences of $\Sigma(i)$ we have*

$$(*) \quad \mathcal{M} \models \exists x \bigwedge_{j=1}^{n} \phi_{\alpha_j}(x, \bar{a}_{\alpha_j}(i)) \Longleftrightarrow \mathcal{N} \models \exists x \bigwedge_{j=1}^{n} \phi_{\alpha_j}(x, \bar{b}_{\alpha_j}(i)).$$

Proof. Consider the set $\Lambda(i)$ of all formulas:

$$\exists x \bigwedge_{j=1}^{n} \phi_{\alpha_j}(x, \bar{x}_{\alpha_j}(i))$$

if

$$\mathcal{M} \models \exists x \bigwedge_{j=1}^{n} \phi_{\alpha_j}(x, \bar{a}_{\alpha_j}(i))$$

and otherwise

$$\neg\exists x \bigwedge_{j=1}^{n} \phi_{\alpha_j}(x, \bar{x}_{\alpha_j}(i)),$$

for every subset $\{\phi_{\alpha_1}(x, \bar{a}_{\alpha_1}(i)), \ldots, \phi_{\alpha_n}(x, \bar{a}_{\alpha_n}(i))\}$ of sentences of $\Sigma(i)$. The set $\Lambda(i)$ is finite, hence let $\psi(\bar{x})$ be its conjunction, where we indicate by $\bar{x}$ a finite tuple of variables. Since $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent and

$$\mathcal{M} \models \exists \bar{x}\, \psi(\bar{x}),$$

we have

$$\mathcal{N} \models \exists \bar{x}\, \psi(\bar{x}).$$

In $\mathcal{N}$ a realization $\{b_\alpha(i) \mid \alpha \in \gamma\}$ of the formula $\psi(\bar{x})$ is the required set. □

Now, for every $\alpha \in \gamma$, let b_α the class of equivalence modulo $\mathcal{U}$ of the sequence $(b_\alpha(i))_{i \in I}$. We claim that the type $\tilde{p}(x, (\bar{b}_i)_{i\in\gamma})$ is finitely satisfiable in $\mathcal{N}^I/\mathcal{U}$. In fact, if $\phi_{\alpha_1}(x, \bar{b}_{\alpha_1}), \ldots, \phi_{\alpha_n}(x, \bar{b}_{\alpha_n})$ are formulas of the type, then

$$X_{\alpha_1} \cap \ldots X_{\alpha_n} \cap \{i \in I \mid \mathcal{M} \models \exists x \bigwedge_{j=1}^{n} \phi_{\alpha_j}(x, \bar{a}_{\alpha_j}(i))\} \in \mathcal{U}$$

is a subset of $\{i \in I \mid \mathcal{N} \models \exists x \bigwedge_{j=1}^{n} \phi_{\alpha_j}(x, \bar{b}_{\alpha_j}(i))\}$ since $(*)$ holds. We obtain that

$$\mathcal{N}^I/\mathcal{U} \models \exists x \bigwedge_{j=1}^{n} \phi_{\alpha_j}(x, \bar{b}_{\alpha_j}(i))).$$

Let b be a realization of $\tilde{p}(x, (b_i)_{i\in\gamma})$ and put

$$\Gamma(i) = \{\phi_\alpha(x, \bar{a}_\alpha(i)) \in \Sigma(i) \mid \mathcal{N} \models \phi_\alpha(b(i), \bar{b}_\alpha(i))\}.$$

Note that $\Gamma(i)$ is finite and $\mathcal{M} \models \exists x \bigwedge \Gamma(i)(x)$, since $(*)$ holds. We may choose a realization $a(i) \in \mathcal{M}$ of the sentence $\exists x \bigwedge \Gamma(i)(x)$. Put $a = [(a(i))_{i\in I}]$. Now we show that a is a realization of the type $p(x, (a_\alpha)_{\alpha\in\gamma})$ in $\mathcal{M}^\lambda/\mathcal{U}$. If $\phi_\alpha(x, \bar{a}_\alpha)$ is a formula of $p(x, (a_\alpha)_{\alpha\in\gamma})$, then

$$\mathcal{U} \ni \{i \in I \mid \mathcal{N} \models \phi_\alpha(b(i), \bar{b}_\alpha(i))\} \cap X_\alpha \subseteq \{i \in I \mid \mathcal{M} \models \phi_\alpha(a(i), \bar{a}_\alpha(i))\},$$

hence we conclude that $\mathcal{M}^I/\mathcal{U} \models \phi_\alpha(a, \bar{a}_\alpha)$. □

Lemma 4.1.4. *The relations $\trianglelefteq_\lambda$ and $\trianglelefteq$ are preorders, that is, the reflexive and transitive properties hold.*

Proof. It is sufficient to show that $\trianglelefteq_\lambda$ is a preorder for all cardinal λ. Obviously the transitive property holds. The reflexivity follows by Lemma 4.1.2. □

The preorder $\trianglelefteq$ is called *Keisler's order.*

4.2 A characterization of the maximal theories

Lemma 4.2.1. *A theory T is maximal in Keisler's order $\trianglelefteq$ if and only if it is maximal in $\trianglelefteq_\lambda$ for any cardinal λ.*

Proof. Follows by definitions. □

For every cardinal γ, let
$$(S_\omega(\gamma), \subseteq, P)$$
be a structure such that
$$P^{S_\omega(\gamma)}(u) \iff u \neq \emptyset.$$
The next result is a useful criterion to check the goodness of an ultrafilter.

Lemma 4.2.2 (Theorem 2.2 Chapther VI [9])**.** *Let $\mathcal{U}$ be an ultrafilter on I. If for each $\gamma < \lambda$ and every λ-saturated elementary extension $\mathcal{M}$ of $(S_\omega(\gamma), \subseteq, P)$, the ultrapower $\mathcal{M}^I/\mathcal{U}$ realizes all types of the form*

$$p(x, (a_\alpha)_{\alpha\in\gamma}) = \{x \subseteq a_\alpha \mid \alpha \in \gamma\} \cup \{P(x)\}$$

that are finitely satisfiable in $\mathcal{M}^I/\mathcal{U}$, then $\mathcal{U}$ is λ-good.

Proof. Let $\gamma < \lambda$ and
$$f\colon S_\omega(\gamma) \to \mathcal{U}$$
be a monotonic function. For $i \in I$, consider the type $p_i((x_\alpha)_{\alpha\in\gamma})$ in $\mathcal{M}$ such that for every $u \in S_\omega(\gamma)$ the following holds:

$$\text{if } i \in f(u), \text{ then } \varphi_u \in p_i((x_\alpha)_{\alpha\in\gamma}),$$

$$\text{if } i \notin f(u), \text{ then } \neg\varphi_u \in p_i((x_\alpha)_{\alpha\in\gamma}),$$

where
$$\varphi_u = \left\{\exists x \left[\bigwedge_{\alpha\in u} x \subseteq x_\alpha \wedge P(x)\right]\right\}$$
and
$$\neg\varphi_u = \left\{\neg\exists x \left[\bigwedge_{\alpha\in u} x \subseteq x_\alpha \wedge P(x)\right]\right\}.$$

Claim 4.2.3. *For every $i \in I$, the type $p_i((x_\alpha)_{\alpha\in\gamma})$ is realized in $\mathcal{M}$.*

Proof. Fix $i \in I$. By Lemma 3.1.24, it is sufficient to show that $p_i((x_\alpha)_{\alpha\in\gamma})$ is finitely satisfiable in $\mathcal{M}$. Let $\{u_1, \ldots, u_n, v_1, \ldots, v_m\} \subseteq S_\omega(\gamma)$ be such that $i \in f(u_j)$ and $i \notin f(v_l)$ for every $j \leq n$ and $l \leq m$. Put $\{\alpha_1, \ldots, \alpha_k\} = \bigcup_{j,l} u_j \cup v_l$. To complete the proof, we find some finite subsets $A_1, \ldots, A_k$ of ω such that

$$\bigcap_{j\in u_l} A_j \neq \emptyset \text{ and } \bigcap_{j\in v_h} A_j = \emptyset$$

for all $l \leq n$ and $h \leq m$, then $x_{\alpha_j} = A_j$ for $j \leq k$ provides a realization of

$$\{\varphi_{u_1}, \dots, \varphi_{u_n}, \varphi_{v_1}, \dots, \varphi_{v_m}\}.$$

We construct the sets $A_1, \dots, A_k$ as follows: consider an injective map

$$h\colon \{u_j \mid j \leq n\} \to k$$

such that $h(u_j) = l_{u_j}$. For every $\alpha \in \{\alpha_1, \dots, \alpha_k\}$, put

$$F(\alpha) = \{l_{u_j} \mid \alpha \in u_j\}.$$

Now we show that the sets $F(\alpha)$ for $\alpha \in \{\alpha_1, \dots, \alpha_k\}$ have the required properties. For $j \leq n$ we have $\bigcap_{\alpha \in u_j} F(\alpha) \ni l_{u_j}$. Now assume for a contradiction that there exists l such that $l \in \bigcap_{\alpha \in v_l} F(\alpha)$ for some $l \leq m$. For some u_j, we have $l = l_{u_j}$, then $v_l \subseteq u_j$. By monotonicity of f, we conclude $i \in f(u_j) \subseteq f(v_l)$, contradiction. □

For every $i \in I$, let $\{a_\alpha(i) \mid \alpha \in \gamma\} \subseteq \mathcal{M}$ be a realization of the type $p((x_\alpha)_{\alpha \in \gamma})$, that is for every finite set $u \in S_\omega(\gamma)$ we have

$$\mathcal{M} \models \exists x \left[\bigwedge_{\alpha \in u} x \subseteq a_\alpha(i) \wedge P(x) \right] \Longleftrightarrow i \in f(u).$$

Now, in $\mathcal{M}^I/\mathcal{U}$ consider the type

$$q(x, (a_\alpha)_{\alpha \in \gamma})) = \{x \subseteq a_\alpha \mid \alpha \in \gamma\} \cup \{P(x)\},$$

where $a_\alpha = [(a_\alpha(i))_{i \in I}]$ for every $\alpha \in \gamma$. We show that $q(x, (a_\alpha)_{\alpha \in \gamma}))$ is finitely satisfiable in $\mathcal{M}^I/\mathcal{U}$. For every finite set $u \subseteq \gamma$ we have

$$\{i \in I \mid \mathcal{M} \models \exists x \bigwedge_{\alpha \in u} x \subseteq a_\alpha(i) \wedge P(x)\} = \{i \in I \mid i \in f(u)\} \in \mathcal{U},$$

hence

$$\mathcal{M}^I/\mathcal{U} \models \exists x \bigwedge_{\alpha \in u} x \subseteq a_\alpha \wedge P(x).$$

Now let $b \in \mathcal{M}^I/\mathcal{U}$ be a realization of $q(x, (a_\alpha)_{\alpha \in \gamma}))$ and define

$$g(u) = \{i \in I \mid \mathcal{M} \models \bigwedge_{\alpha \in u} b(i) \subseteq a_\alpha(i) \wedge P(b(i))\} \in \mathcal{U}.$$

Obviously g is an additive function on $S_\omega(\gamma)$, range$(g) \subseteq \mathcal{U}$ and $g \leq f$. □

Theorem 4.2.4 (Lemma 4.2 Chapter VI [9]). *A theory T is maximal in the preorder $\trianglelefteq_\lambda$ if and only if for any model $\mathcal{M}$ and any regular ultrafilter $\mathcal{U}$ on λ, the following property holds:*

$\mathcal{M}^\lambda/\mathcal{U}$ is λ^+-saturated if and only if $\mathcal{U}$ is λ^+-good. $*$

Proof. We first assume that the property $*$ holds. Let T' be a countable complete theory and $\mathcal{N}$ be a model of T'. Assume that $\mathcal{M} \models T$. If $\mathcal{M}^\lambda/\mathcal{U}$ is λ^+-saturated, then $\mathcal{U}$ is λ^+-good. By Theorem 3.1.26, we conclude that $\mathcal{N}^\lambda/\mathcal{U}$ is λ^+-saturated, hence $T' \trianglelefteq_\lambda T$. Hence T is maximal in the preorder $\trianglelefteq_\lambda$.

Note that in $*$ the implication from right to left holds for all theories T, by Theorem 3.1.26. Now assume that T is maximal. Let $\mathcal{M}$ be a model of T, such that $\mathcal{M}^\lambda/\mathcal{U}$ is λ^+-saturated, for some regular ultrafilter $\mathcal{U}$. By maximality of T, we have

$$Th((S_\omega(\gamma), \subseteq, P)) \trianglelefteq_\lambda T,$$

for every $\gamma \in \lambda^+$, that is $S_\omega(\gamma)^\lambda/\mathcal{U}$ is λ^+-saturated. By Theorem 4.1.2, we conclude that $\mathcal{N}^\lambda/\mathcal{U}$ is λ^+-saturated, for every elementary extension $\mathcal{N}$ of $(S_\omega(\gamma), \subseteq, P)$. Thus the ultrafilter $\mathcal{U}$ is λ^+-good, by Lemma 4.2.2. □

4.3 SOP-theories

Definition 4.3.1. A complete $\mathcal{L}$-theory T has the *strict order property*, abbreviated as SOP, if in some model $\mathcal{M}$ of T there is an $\mathcal{L}$-formula $\psi(x, y)$ that defines a partial order with infinite chains.

Remark 4.3.2. Note that if $\mathcal{M}$ and $\psi(x, y)$ have the properties of the definition above, then $\psi(x, y)$ define a partial order in every structure $\mathcal{N}$ such that $\mathcal{N} \equiv \mathcal{M}$.

Theorem 4.3.3. *For every complete theory T with the strict order property, we have $Th((\mathbb{Q}, <)) \trianglelefteq T$.*

Proof. Let λ be an infinite cardinal, $\mathcal{U}$ be a regular ultrafilter on λ and $\{X_\alpha \mid \alpha \in \lambda\}$ be a λ-regularizing family for $\mathcal{U}$. Assume that for some model $\mathcal{M}$ of T, the ultrapower $\mathcal{M}^\lambda/\mathcal{U}$ is λ^+-saturated. By Lemma 4.1.2, it is sufficient to prove that $\mathbb{Q}^\lambda/\mathcal{U}$ is λ^+-saturated. Let $p(x)$ be a finitely consistent type in $\mathbb{Q}^\lambda/\mathcal{U}$ with parameters in A, where $|A| = \gamma \leq \lambda$. By quantifier elimination of dense linear orders without endpoints, every formula $\psi(x)$ with parameters $a_1, \ldots, a_n$ is equivalent, modulo the theory, to a finite conjunction of formulas of the form

$$x > a_{i_1} \vee x < a_{i_1}.$$

Hence we can assume without loss of generality that $p(x) = \{\phi_\alpha(x, a_\alpha) \mid \alpha \in \gamma\}$ is a set of atomic formulas. Put

$$A_1 = \{a \mid \{x > a\} \in p(x)\}$$

and

$$A_2 = \{a \mid \{x < a\} \in p(x)\}.$$

If $A_2 = \emptyset$, for every $\xi \in \lambda$ consider the finite set $\Sigma(\xi) = \{\phi_\alpha(x, a_\alpha) \mid \xi \in X_\alpha\}$. For every $\xi \in \lambda$, choose $a(\xi) = \max\{a_\alpha(\xi) \mid \xi \in X_\alpha\}$ and note that

$$\mathbb{Q} \models \bigwedge_{\phi_\alpha(x, a_\alpha) \in \Sigma(\xi)} \phi_\alpha(a(\xi), a_\alpha(\xi)).$$

Not put $a = [(a(\xi))_{\xi\in\lambda}]$. For every formulas $\phi_\alpha(x, a_\alpha) \in p(x)$ we have

$$\mathcal{U} \ni X_\alpha \subseteq \{\xi \in \lambda \mid \mathbb{Q} \models \phi_\alpha(a(\xi), a_\alpha(\xi))\},$$

hence a is a realization of the type $p(x)$. Using a similar argument we conclude that $p(x)$ is realized, when $A_1 = \emptyset$. To conclude the proof, assume that A_1 and A_2 are not empty. Hence for some cardinals κ_1, κ_2, we want to find an element a that separates two sequences $(a^1_\alpha)_{\alpha\in\kappa_1}$ and $(a^2_\alpha)_{\alpha\in\kappa_2}$. Since $\kappa_1 + \kappa_2 \leq \lambda$, we can find a subfamily

$$\{X^1_\alpha \mid \alpha \in \kappa_1\} \cup \{X^2_\alpha \mid \alpha \in \kappa_2\}$$

of the λ-regularizing family $\{X_\alpha \mid \alpha \in \lambda\}$. For $\xi \in \lambda$, consider the finite set

$$\Sigma(\xi) = \{a^1_\alpha(\xi) \mid \xi \in X^1_\alpha\} \cup \{a^2_\alpha(\xi) \mid \xi \in X^2_\alpha\}.$$

Since $\mathcal{M}$ has an infinite chain, we can copy $\Sigma(\xi)$ in a finite subset

$$\{b^1_\alpha(\xi) \mid \xi \in X^1_\alpha\} \cup \{b^2_\alpha(\xi) \mid \xi \in X^2_\alpha\}$$

of $\mathcal{M}$ so that the order is preserved, that is

$$\mathbb{Q} \models a^i_\alpha(\xi) < a^j_\beta(\xi) \iff \mathcal{M} \models b^i_\alpha(\xi) < b^j_\beta(\xi).$$

In this way we have obtained two sequences $(b^1_\alpha)_{\alpha\in\kappa_1}$ and $(b^2_\alpha)_{\alpha\in\kappa_2}$ in $\mathcal{M}^\lambda/\mathcal{U}$ such that for every $\alpha_1 < \alpha_2 \in \kappa_1$ and $\beta_1 < \beta_2 \in \kappa_2$ we have

$$\mathcal{M}^\lambda/\mathcal{U} \models b^1_{\alpha_1} < b^1_{\alpha_2} < b^2_{\beta_2} < b^2_{\beta_1}.$$

Since $\mathcal{M}^\lambda/\mathcal{U}$ is λ^+-saturated, there exists b that separates the two sequences in $\mathcal{M}^\lambda/\mathcal{U}$. Now consider

$$\Gamma(\xi) = \{a^1_\alpha(\xi) \mid \xi \in X^1_\alpha,\ \mathcal{M} \models b^1_\alpha(\xi) < b(\xi)\} \cup \{a^2_\alpha(\xi) \mid \xi \in X^2_\alpha,\ \mathcal{M} \models b(\xi) < b^2_\alpha(\xi)\}.$$

It is possible to find $a(\xi) \in \mathbb{Q}$ such that

$$\mathbb{Q} \models a^1_\alpha(\xi) < a(\xi) < a^2_\beta(\xi),$$

for all $a^1_\alpha(\xi), a^2_\beta(\xi) \in \Gamma(\xi)$. Let a be the class of equivalence modulo $\mathcal{U}$ of the sequence $(a(\xi))_{\xi\in\lambda}$. For every a^1_α and a^2_β, the set

$$X^1_\alpha \cap X^2_\beta \cap \{\xi \in \lambda \mid \mathcal{M} \models b^1_\alpha(\xi) < b(\xi) < b^2_\beta(\xi)\} \in \mathcal{U}$$

is a subset of

$$\{\xi \in \lambda \mid a^1_\alpha(\xi) < a(\xi) < a^2_\beta(\xi)\}.$$

We conclude that a separates the two sequences $(a^1_\alpha)_{\alpha\in\kappa_1}$ and $(a^2_\alpha)_{\alpha\in\kappa_2}$ in $\mathbb{Q}^\lambda/\mathcal{U}$. □

Theorem 4.3.4. *The theory $Th((\mathbb{Q}, <))$ is maximal in Keisler's order.*

Proof. Let λ be an infinite cardinal and $\mathcal{U}$ be a regular ultrafilter on λ such that the ultrapower $\mathbb{Q}^\lambda/\mathcal{U}$ is λ^+-saturated. We have to prove that $\mathcal{U}$ is λ^+-good. By Theorem 4.2.2, it is sufficent to prove that if $\mathcal{M}$ is a λ^+-saturated elementary extension of

$$(S_\omega(\lambda), \subseteq, \emptyset),$$

then $\mathcal{M}^\lambda/\mathcal{U}$ realizes all types

$$p(x, (a_\alpha)_{\alpha\in\lambda}) = \{x \subseteq a_\alpha \mid \alpha \in \lambda\} \cup \{\neg(x = \emptyset)\},$$

that are finitely satisfiable in $\mathcal{M}^\lambda/\mathcal{U}$. Consider a new binary relation symbol R and expand the language with R in such a way that

$$\mathcal{M} \models R(a, b) \iff \mathcal{M} \models a \subseteq b \wedge a \neq \emptyset.$$

Set

$$\Gamma = \{s \in \mathcal{M}^{<\omega} \mid \mathcal{M} \models \exists x \left[\bigwedge\{R(x, s(i)) \mid i < |s|\}\right]\}$$

and note that Γ has the power of $\mathcal{M}$, hence there exists a bijection $g\colon M \to \Gamma$. We expand the language, defining the following new functions and relations on $\mathcal{M}$:

- $c^* = g^{-1}(\emptyset)$.
- $a \leq b$ if and only if $g(a)$ is extended by $g(b)$.
- $F_1\colon \mathcal{M} \to \mathcal{M}$ is such that $F_1(a)$ is a witness of

$$\mathcal{M} \models \exists x \left[\bigwedge\{R(x, g(a)(i)) \mid i < |g(a)|\}\right]$$

- $P(b, \bar{a})$ if and only if $\bar{a} \in \Gamma$ is a sequence extended by $g(b)$.
- $Q(b, \bar{a})$ if and only if $g(b)^\frown\bar{a} \in \Gamma$.
- $F_2(b, \bar{a}) = c$ if and only if $\mathcal{M} \models Q(b, \bar{a})$ and $g(c) = g(b)^\frown\bar{a}$ or $\mathcal{M} \models \neg Q(b, \bar{a})$ and $c = b$.
- $F_3(b, \bar{a}) = c$ if and only if $g(c) \in \Gamma$ is extended by $g(b)$ and $g(c)$ is the longest initial segment of $g(b)$ such that $g(c)^\frown\bar{a}$ is in Γ (notice that $g(c)$ is always an initial segment of $g(b)$).

Let $\mathcal{N}_1 = (\mathcal{M}, R, \leq, F_1, F_2, F_3, P, Q, c^*)$ and $\mathcal{M}_1$ be the reduct of $\mathcal{N}_1$ to $L = \{R\}$. For the sake of simplicity let us denote by their usual names in the language the interpretations in $\mathcal{N}_1^\lambda/\mathcal{U}$ of the relation and function symbols of the signature of $\mathcal{N}_1^\lambda/\mathcal{U}$. We just note that for example

$$\mathcal{N}_1^\lambda/\mathcal{U} \models Q([f], [\bar{a}]) \iff \{i < \lambda \mid \mathcal{N}_1 \models g(f(i))^\frown\bar{a}(i) \in \Gamma\} \in \mathcal{U},$$

and similarly for the other relation and function symbols. Fix a family $\{\bar{f}_\alpha \mid \alpha \in \lambda\} \subseteq \mathcal{N}_1^\lambda/\mathcal{U}$ and a type

$$p(x, (\bar{f}_\alpha)_{\alpha\in\lambda}) = \{\phi_\alpha(x, \bar{f}_\alpha) \mid \alpha \in \lambda\},$$

of size λ and finitely satisfiable in $\mathcal{N}_1^\lambda/\mathcal{U}$, where every formula $\phi_\alpha(x, \bar{y})$ is a finite conjunction of xRy. Without loss of generality we can assume that the type is closed under finite conjunction. Before showing how to realize the type $p(x, (\bar{f}_\alpha)_{\alpha\in\lambda})$ we need two claims.

Claim 4.3.5. *1) Every $a \in \mathcal{N}_1/\mathcal{U}$ has an immediate successor b. Moreover, every element has infinitely many immediate successors.*

2) For every $a, b \in \mathcal{N}_1^\lambda/\mathcal{U}$ there exists a maximal element $c \in \mathcal{N}^\lambda/\mathcal{U}$ such that $c \leq a, b$.

Proof. 1) Fix $a \in \mathcal{N}_1^\lambda/\mathcal{U}$. For every $\xi \in \lambda$, the element $a(\xi) \in \mathcal{N}_1$ has an immediate successor $b(\xi) \in \mathcal{N}_1$, since the partial order $(\mathcal{N}_1, <)$ is isomorphic to the partial order $(\Gamma, \subseteq)$. Now the element $[(b(\xi))_{\xi\in\lambda}]$ has the required properties. The second observation follows noting that in $(N, \leq)$ every element has infinitely many immediate successors.

2) For every $\xi \in \lambda$, we can find a maximal $c(\xi) \in \mathcal{N}_1$ such that $c(\xi) \leq a(\xi), b(\xi)$, since the order $(\mathcal{N}_1, \leq)$ is isomorphic to the order $(\Gamma, \subseteq)$. Now the element $[(c(\xi))_{\xi\in\lambda}]$ has the required properties.

□

Claim 4.3.6. *For $\kappa, \kappa_1, \kappa_2 \leq \lambda$ we have:*

(i) If $((c_\alpha)_{\alpha\in\kappa})$ is an increasing chain in $\prod(N_1^\lambda, \leq)/\mathcal{U}$, then there exists $c \in \mathcal{N}_1^\lambda/\mathcal{U}$ such that $c_\alpha \leq c$ for all $\alpha \in \kappa$.

(ii) If $((a_\alpha)_{\alpha\in\kappa_1})$ and $((b_\alpha)_{\alpha\in\kappa_2})$ are sequence in $(\mathcal{N}_1^\lambda/\mathcal{U}, \leq)$ such that

$$a_{\alpha_1} < a_{\alpha_2} < b_{\beta_2} < b_{\beta_1}$$

for all $\alpha_1 < \alpha_2 < \kappa_1$ and $\beta_1 < \beta_2 < \kappa_2$. Then there exists an element $c \in N_1^\lambda/\mathcal{U}$ that separates the two sequences in $\prod(N_1^\lambda, \leq)/\mathcal{U}$.

Proof. For every $c \in \mathcal{N}_1^\lambda/\mathcal{U}$, the set S_c of immediate successors of c is infinite, by Claim 4.3.5 1). Choose a dense linear order $<_c$ on S_c and consider the following binary relation $\prec$ on $\mathcal{N}_1 \times \{0, 1\}$: $(a, i) \prec (b, j)$ if and only if

(A) $a = b$ and $i < j$.

(B) If $a < b$, then $(a, 0) \prec (b, 0) \prec (b, 1) \prec (a, 1)$.

(C) If a, b are not $<$-compatible, there exists a maximal $c \in \mathcal{N}_1^\lambda/\mathcal{U}$ such that $c < a, b$, by Claim 4.3.5 2). Now let $a', b' \in \mathcal{N}_1^\lambda/\mathcal{U}$ be the first elements such that $c < a' \leq a$ and $c < b' \leq b$. Then $a' <_c b'$.

By Claim 1.2.3, $\prec$ is a linear order, moreover it is easy to check that $\prec$ is a dense order with first and last element $(c^*, 0)$ and $(c^*, 1)$, respectively. Hence the structure

$$((\mathcal{N}_1 \times \{0,1\})^\lambda/\mathcal{U}, \prec) = (\mathcal{N}_1^\lambda/\mathcal{U} \times \{0,1\}, \prec)$$

is a dense linear order with first and last element and thus is a model of the theory of $(\mathbb{Q}, <)$. We conclude the proof of (i) and (ii) of the claim as follows:

(i) In $(\mathcal{N}_1^\lambda/\mathcal{U} \times \{0,1\}, \prec)$, consider the sequences $(c_\alpha, 0), (c_\alpha, 1)$. Since we are assuming that $\mathbb{Q}^\lambda/\mathcal{U}$ is λ^+-saturated, we have that

$$\mathcal{N}_1^\lambda/\mathcal{U} \times \{0,1\} \setminus \{(c^*, 0), (c^*, 1)\}$$

is λ^+-saturated dense linear order. Hence there exists (c, j) such that $(c_\alpha, 0) < (c, j) < (c_\beta, 1)$. We conclude proving that $c_\alpha < c$ for all α. Since the argument is similar we can assume that $j = 0$. Assume for a contradiction that we have $c_\alpha \not< c$, for some $\alpha \in \lambda$. Hence condition (C) holds and we obtain $(c_\alpha, 1) \prec (c, 0) \prec (c_\alpha, 1)$, contradiction.

(ii) Consider the sequences $(a_\alpha, 0)$, $(a_\alpha, 1)$, $(b_\alpha, 0)$ and $(b_\alpha, 1)$. By hypothesis we have

$$(a_\alpha, 0) \prec (b_\beta, 0) \prec (b_\beta, 1) \prec (a_\alpha, 1).$$

Since

$$\mathcal{N}_1^\lambda/\mathcal{U} \times \{0,1\} \setminus \{(c^*, 0), (c^*, 1)\}$$

is λ^+-saturated dense linear order, there exists (d, j) such that

$$(a_\alpha, 0) \prec (d, j) \prec (b_\beta, 0) \prec (b_\beta, 1) \prec (a_\alpha, 1).$$

Since

$$(a_\alpha, 0) \prec (d, j) \prec (a_\alpha, 1),$$

we obtain $a_\alpha < d$ for all $\alpha \in \lambda$, by properties (A), (B), (C) in the definition of the linear order $\prec$. If $d < b_\beta$ for all $\beta \in \lambda$, we conclude the proof, otherwise there exists a $\beta \in \lambda$ such that $d \not< b_\beta$. Let $d' \in \mathcal{N}_1^\lambda/\mathcal{U}$ be the maximal element such that $d' < b_\beta, d$. Now we prove that d' has the required properties. Clearly we have $a_\alpha \leq d'$ for all $\alpha \in \lambda$. Now observe tha for each $\alpha \in \lambda$, the elements d', b_α are $<$-compatible in $\mathcal{N}_1^\lambda/\mathcal{U}$ since b_α is in the thread of $\mathcal{N}_1^\lambda/\mathcal{U}$ determined by b_β and d' is also in that thread. Observe also that $(d', j) \prec (b_\alpha, 0)$ for all α, hence $d' < b_\alpha$ for all α is forced to be the case by properties (A), (B), (C) in the definition of the linear order $\prec$.

□

In order to realize the type $p(x, (\bar{f}_\alpha)_{\alpha\in\lambda})$ in $\mathcal{N}_1^\lambda/\mathcal{U}$, we build a family

$$\{c_i \mid i \leq \lambda\}$$

in $\mathcal{N}_1^\lambda/\mathcal{U}$ with the following properties:

i) $i \leq j$ if and only if $\mathcal{N}_1^\lambda/\mathcal{U} \models c_i \leq c_j$.

ii) $\mathcal{N}_1^\lambda/\mathcal{U} \models Q(c_i, \bar{f}_\alpha)$, for all $\alpha \in \lambda$.

iii) $\mathcal{N}_1^\lambda/\mathcal{U} \models P(c_{i+1}, \bar{f}_i)$, for all $i \in \lambda$.

case $i = 0$**:** Put $c_0 = [(c^*)_{l\in\lambda}] \in \mathcal{N}_1^\lambda/\mathcal{U}$. The conditions $i), iii)$ are clearly satisfied for c_0. The clause $ii)$ holds, since $g(c_0) = \emptyset$.

case $i = j + 1$**:** Put $c_{j+1} = F_2(c_j, \bar{f}_j) \in \mathcal{N}_1^\lambda/\mathcal{U}$. By inductive hypothesis we have

$$\mathcal{N}_1^\lambda/\mathcal{U} \models Q(c_j, \bar{f}_j),$$

hence $g(c_{j+1}) = g(c_j)^\frown \bar{f}_j$, by definition of the function F_2. Clearly the clause $i)$ and $iii)$ hold, by definition of c_{j+1}. To check the second requirement we have to prove that

$$\mathcal{N}_1^\lambda/\mathcal{U} \models Q(c_{j+1}, \bar{f}_\eta),$$

for all $\eta \in \lambda$. Clearly we have the following equivalences:

$$\begin{aligned}\mathcal{N}_1^\lambda/\mathcal{U} \models Q(c_{j+1}, \bar{f}_\eta) &\iff \{l \in \lambda \mid \mathcal{N}_1 \models g(c_{j+1})^\frown \bar{f}_\eta(l) \in \Gamma\ \} \in \mathcal{U} \\ &\iff \{l \in \lambda \mid \mathcal{N}_1 \models g(c_j)^\frown \bar{f}_j^\frown \bar{f}_\eta(l) \in \Gamma\ \} \in \mathcal{U}.\end{aligned}$$

Since the type $p(x, (\bar{f}_\alpha)_{\alpha\in\lambda})$ is closed under finite conjunction and for all $\zeta \in \lambda$

$$\mathcal{N}_1^\lambda/\mathcal{U} \models Q(c_j, \bar{f}_\zeta),$$

that is

$$\{l \in \lambda \mid \mathcal{N}_1 \models g(c_j)^\frown \bar{f}_\zeta(l) \in \Gamma\} \in \mathcal{U},$$

we conclude that

$$\{l \in \lambda \mid \mathcal{N}_1 \models g(c_j)^\frown \bar{f}_j^\frown \bar{f}_\eta(l) \in \Gamma\ \} \in \mathcal{U}.$$

case i **limit:** By Claim 4.3.6, there is $f \in \mathcal{N}_1^\lambda/\mathcal{U}$ such that $c_j \leq f$ for all $j < i$, but f may not satisfy the second requirement on c_i. To solve this problem we need the following claim.

Claim 4.3.7. *There exists a sequence* $\{d_\xi \mid \xi \leq \lambda\}$ *such that* $c_j < d_\xi \leq d_\eta$ *for all* $j \in i$ *and* $\eta \leq \xi \leq \lambda$. *Moreover, for* $\eta < \lambda$ *we have*

$$\mathcal{N}_1^\lambda/\mathcal{U} \models Q(d_\lambda, \bar{f}_\eta).$$

Proof. For $\eta = 0$, put $d_0 = f$. At the successor step, set $d_{\eta+1} = F_3(d_\eta, \bar{f}_\eta)$. We have to prove that for all $j < i$ we have $c_i < d_{\eta+1} \leq d_\eta$. By definition of $d_{\eta+1}$ follows that $d_{\eta+1} \leq d_\eta$. Now remember that $d_{\eta+1}$ is the largest initial segment of d_η such that

$$\{l \in \lambda \mid \mathcal{N}_1 \models g(d_{\eta+1})^\frown \bar{f}_\eta(l) \in \Gamma\} \in \mathcal{U}.$$

Since the inductive hypothesis holds and

$$\mathcal{N}_1^\lambda/\mathcal{U} \models Q(c_j, \bar{f}_\eta)$$

for all $j < i$, we have $c_j < d_\eta$ and

$$\{l \in \lambda \mid \mathcal{N}_1 \models g(c_j)^\frown \bar{f}_\eta(l) \in \Gamma\} \in \mathcal{U}$$

for all $j < i$, hence $c_j < d_{\eta+1}$ for every $j < i$. At the limit step use Claim 4.3.6 to fill the gap $(c_j)_{j\in i}$ $(d_\eta)_{\eta\in\xi}$ in $(\mathcal{N}_1^\lambda/\mathcal{U}, \leq)$. To complete the claim, we prove by induction on $\xi \leq \lambda$ that

$$\mathcal{N}_1^\lambda/\mathcal{U} \models Q(d_\xi, \bar{f}_\eta)$$

for all $\eta < \xi$. At the successor step, we have $\mathcal{N}_1^\lambda/\mathcal{U} \models Q(d_{\eta+1}, \bar{f}_\eta)$, by construction of $d_{\eta+1}$. Moreover, $\mathcal{N}_1^\lambda/\mathcal{U} \models Q(d_\xi, \bar{f}_\eta)$ for $\eta < \xi$ by inductive hypothesis. Since $d_{\xi+1}$ is an initial segment of d_ξ, we conclude that $\mathcal{N}_1^\lambda/\mathcal{U} \models Q(d_{\xi+1}, \bar{f}_\eta)$ for all $\eta < \xi$. At the limit step the thesis holds since d_ξ is an initial segment of d_ζ for all $\zeta \in \xi$. □

To conclude the limit step, put $c_i = d_\lambda$, where $(d_\xi)_{\xi\leq\lambda}$ is the sequence of Claim 4.3.7. We have to check that c_i satisfies all three prescribed requirements. The clause (i) follows by construction, (iii) is trivial and (ii) follows by Claim 4.3.7.

Finally, assume that the construction of the sequence $(c_i)_{i\leq\lambda}$ is completed. We have to prove that $c = F_1(c_\lambda)$ realizes the type $p(x, (\bar{f}_\alpha)_{\alpha\in\lambda})$. By clause $ii)$ we have

$$\{l < \lambda \mid \mathcal{N}_1 \models g(c(l))^\frown \bar{f}_\alpha(l) \in \Gamma\} \in \mathcal{U},$$

for each $\alpha \in \lambda$. By clauses $(i), (iii)$ we have that $g(c)$ extends $g(c_\xi)$ for $\xi \in \lambda$ and $g(c_{\xi+1})$ extend $\bar{f}_\xi$ for every $\xi \in \lambda$. By definition of the function F_1 we obtaint that

$$\{l \in \lambda \mid \mathcal{N}_1 \models \exists x \left[\bigwedge \{R(x, g(c(l))(i) \mid i < |g(c(l))|\}\right]\} \in \mathcal{U}$$

and this concluded the proof.

□

Corollary 4.3.8 (Shelah, Theorem 4.3 Chapter VI [9]). *Every theory with the strict order property is maximal in Keisler's order.*

Proof. Follows by Theorems 4.3.4 and 4.3.3. □

4.4 Treetops

Definition 4.4.1. Let $\mathcal{U}$ be a regular ultrafilter over a set I. We define

$$\mathcal{C}(\mathcal{U}) = \{(\kappa_1, \kappa_2) \in \ |I|^+ \times |I|^+ \mid \kappa_1, \kappa_2 \text{ are regular and } \omega^I/\mathcal{U} \text{ has a } (\kappa_1, \kappa_2) \text{ gap}\}.$$

Remark 4.4.2. Assume that κ_1 is an infinite regular cardinal and $\kappa_2 > 0$ is finite. Since every $a \neq 0$ of $\omega^I/\mathcal{U}$ has an immediate successor and an immediate predecessor, follows that $(\kappa_1, \kappa_2), (\kappa_2, \kappa_1) \notin \mathcal{C}(\mathcal{U})$.

Lemma 4.4.3. *For any infinite regular cardinals $\kappa_1, \kappa_2 \leq |I|$, we have $(\kappa_1, 0) \notin \mathcal{C}(\mathcal{U})$ and $(0, \kappa_2) \notin \mathcal{C}(\mathcal{U})$.*

Proof. Clearly we have $(0, \kappa_2) \notin \mathcal{C}(\mathcal{U})$, since $\omega^I/\mathcal{U}$ has a minimum element. Now assume for contradiction that $(\kappa_1, 0) \in \mathcal{C}(\mathcal{U})$, hence there exists a cofinal sequence $(a_\alpha)_{\alpha\in\kappa_1}$ in $\omega^I/\mathcal{U}$. Let $\{X_\alpha \mid \alpha \in |I|\} \subseteq \mathcal{U}$ be a family that regularizes $\mathcal{U}$. For every $i \in I$, the set $Y_i = \{a_\alpha(i) \mid i \in X_\alpha\}$ is finite. Put $a(i) = \max(Y_i)$, if Y_i is not empty, otherwise $a(i) = 0$. For any $\alpha \in \kappa_1$, we have

$$\mathcal{U} \ni X_\alpha \subseteq \{i \in I \mid a_\alpha(i) < a(i)\},$$

hence $[((a(i))_{i\in I}]$ is an upper bound of the sequence $(a_\alpha)_{\alpha\in\kappa_1}$, contradiction. □

The next result shows that an ultrapower $\omega^I/\mathcal{U}$ is enough saturated if there are no gaps of a certain size. In order to obtain this, we use that the theory of discrete linear orders with minimum element and without maximum has quantifier elimination in the language $\mathcal{L} = \{0, s, <\}$, see Theorem B.2.5 for a proof.

Theorem 4.4.4. *Let $\mathcal{U}$ be a regular ultrafilter on λ. Then $\mathcal{C}(\mathcal{U}) = \emptyset$ if and only if $\mathcal{U}$ is λ^+-good.*

Proof. If $\mathcal{U}$ is λ^+-good, then $\omega^\lambda/\mathcal{U}$ is λ^+-saturated, by Theorem 3.1.26. Hence we conclude that $\mathcal{C}(\mathcal{U})$ is empty. Now assume that $\mathcal{C}(\mathcal{U})$ is empty.

Claim 4.4.5. *The ultrapower $\omega^\lambda/\mathcal{U}$ is λ^+-saturated.*

Proof. We can expand the language $\{<\}$ adding an unary function symbol s and a constant symbol 0, that interpret the successor function and the minimum element, respectively. Note that s and 0 can be defined by $<$, in fact

$$s(x) = y \iff x < y \wedge \neg\exists z(x < z \wedge z < y)$$

and

$$x = 0 \iff \forall y(\ x = y \vee x < y).$$

Let T be the theory of discrete linear order with minimum and without maximum and $p(x)$ be a finitely satisfiable type in $\omega^\lambda/\mathcal{U}$ in the expanded language $\{0, s, <\}$ with parameters in some $A \subseteq \omega^\lambda/\mathcal{U}$ of cardinality less than λ^+. By Theorem B.2.5, $p(x)$

is equivalent modulo T to a set quantifier-free formulas. Modulo T, a quantifier-free formula $\psi(x)$ with parameters $a_1, \ldots, a_n$ is a finite conjunction of formulas of the form

$$x > a_{i_1} \vee x < a_{i_2} \vee x = s^{n_3}(a_{i_3}) \vee s^{n_4}(x) = a_{i_4}.$$

Hence we can assume that the type $p(x)$ is a set of atomic formulas with parameters in A. Put

$$A_1 = \{a \in A \mid \{a < x\} \in p(x)\} \text{ and } A_2 = \{a \in A \mid \{x < a\} \in p(x)\}.$$

There are six cases:

1) For some $a \in A_1$ and $n \in \omega$ we have $\{s^n(a) = x\} \in p(x)$. Then $p(x)$ is realized be $s^n(a)$.

2) For some $a \in A_2$ and $n \in \omega$ we have $\{s^n(x) = a\} \in p(x)$. Then $p(x)$ is realized by $s^{-n}(a)$.

3) The sets A_1 and A_2 are infinite. Then we find a realization a of $p(x)$, since $\mathcal{C}(\mathcal{U}) = \emptyset$.

4) At least one of A_1 and A_2 is empty. Then we find a realization a of $p(x)$, by the proof of Lemma 4.4.3.

5) A_1 is finite and A_2 is infinite. By cases 1) and 2), we can assume that there exists no $a \in A_1 \cup A_2$ and $n \in \omega$ such that neither $\{s^n(a) = x\} \in p(x)$ or $\{s^m(x) = a\} \in p(x)$. Let $a_1 \in A_1$ be the maximum. Since $\mathcal{C}(\mathcal{U}) = \emptyset$, the sequences $(s^n(a_1))_{n\in\omega}$ and $(a_2)_{a_2\in A_2}$ don't represent a gap, hence there exists $a \in \omega^I/\mathcal{U}$ such that $s^n(a_1) < a < a_2$ for all $n \in \omega$ and $a_2 \in A_2$. Then a realizes the type $p(x)$.

6) A_2 is finite and A_1 is infinite. We conclude as in 5).

We conclude that $p(x)$ is realized in $\omega^\lambda/\mathcal{U}$, hence $\omega^\lambda/\mathcal{U}$ is λ^+-saturated. □

By Corollary 4.3.8, the theory $Th((\omega, <))$ is maximal in Keisler's order, hence $\mathcal{U}$ is λ^+-good, by Theorem 4.2.4. □

Lemma 4.4.6 (Claim 10.17 [5]). *Let $\mathcal{U}$ be a regular ultrafilter on I. There exists a sequence $(n_i)_{i\in I} \in \omega^I$, such that for all regular cardinals $\kappa_1, \kappa_2 \leq |I| = \lambda$, the following are equivalent:*

(i) $(\kappa_1, \kappa_2) \in \mathcal{C}(\mathcal{U})$.

(ii) The linear order $\prod_{i\in I}(n_i, \leq_i)/\mathcal{U}$ has a (κ_1, κ_2) gap, where $<_i$ is the standard order on ω restricted to n_i.

Proof.

$(ii) \Rightarrow (i)$ Obvious.

$(i) \Rightarrow (ii)$ Let $\{X_i \mid i \in \lambda\}$ be a λ-regularizing family of sets of $\mathcal{U}$. Put

$$n_i = |\{\alpha \in \lambda \mid i \in X_\alpha\}| + 1$$

and note that $n_i \in \omega$ for all $i \in I$. Now assume that $(a_\alpha)_{\alpha \in \kappa_1}$ and $(b_\alpha)_{\beta \in \kappa_2}$ represent a (κ_1, κ_2) gap in $\omega^I/\mathcal{U}$ for $\kappa_1 + \kappa_2 \leq \lambda$. Let $h\colon \kappa_1 \times \{0\} \cup \kappa_2 \times \{1\} \to \lambda$ be an injective map. Define

$$f\colon \kappa_1 \times \{0\} \cup \kappa_2 \times \{1\} \to \mathcal{U}$$

that $f(x) = X_{h(x)}$ and note that $|\{x \in \mathrm{dom}(f) \mid i \in f(x)\}| < n_i$ for all $i \in I$. Set

$$Y_i = \{a_\alpha(i) \mid i \in f((\alpha, 0))\} \cup \{b_\alpha(i) \mid i \in f((\alpha, 1))\}$$

and note that $|Y_i| < n_i$. Choose an injective map $h_i\colon Y_i \to n_i$ such that range(h_i) is an interval of n_i and h_i preserves the order. Put $h = \prod_{i \in I} h_i$. We conclude showing that the sequences $(h(a_\alpha))_{\alpha \in \kappa_1}$, $(h(b_\alpha))_{\alpha \in \kappa_2}$ represent a (κ_1, κ_2) gap in $\prod_{i \in I}(n_i, <_i)/\mathcal{U}$. If $\beta \in \alpha \in \kappa_1$, then the set

$$f((\alpha, 0)) \cap f((\beta, 0)) \cap \{i \in I \mid a_\beta(i) < a_\alpha(i)\} \in \mathcal{U}$$

is contained in $\{i \in I \mid h_i(a_\beta(i)) < h_i(a_\alpha(i))\}$. Hence the sequence $(h(a_\alpha))_{\alpha \in \kappa_1}$ is increasing in $\prod_{i \in I}(n_i, <_i)/\mathcal{U}$ and in the same way we can prove that $(h(b_\alpha))_{\alpha \in \kappa_2}$ is decreasing. Assume for contradiction that there there exists c such that $h(a_\alpha) < c < h(b_\beta)$ for any $\alpha \in \kappa_1$, $\beta \in \kappa_2$. Since h_i preserves the order and range of h_i is an interval, we can construct a d such that $h(d) = c$ and $a_\alpha < d < b_\beta$, absurd.

□

Definition 4.4.7. Let κ be a regular infinite cardinal and $\mathcal{U}$ be a regular ultrafilter on I, where $|I| = \lambda$. We say that $\mathcal{U}$ has the *κ-treetops property* if: for every family $\{(P_i, \subseteq) \mid i \in \lambda\}$ of pseudo-trees and every regular cardinal $\gamma < \kappa$, if $(a_i)_{i \in \gamma}$ is an increasing sequence in $(P, \sqsubseteq) = \prod_{i \in I}(P_i, \subseteq)/\mathcal{U}$, then there exists $a \in P$ such that $a_i \sqsubseteq a$ for all $i \in \gamma$.

Lemma 4.4.8 (Claim 10.25 [5])**.** *Let $\mathcal{U}$ be a regular ultrafilter on I, where $|I| = \lambda$ and $\kappa \leq \lambda$ be a regular cardinal. The following are equivalent:*

(i) $\mathcal{U}$ has the κ^+-treetops property.

(ii) $\kappa^+ \leq \mathfrak{t}(\mathcal{U})$.

Proof.

$(i) \Rightarrow (ii)$ Obvious.

(ii) ⇒ *(i)* Assume for a contradiction that there exists an ultraproduct of pseudo-trees

$$(P, \sqsubseteq) = \prod_{i \in I} (P_i, \subseteq)/\mathcal{U}$$

such that the sequence $(a_\alpha)_{\alpha\in\gamma}$ is increasing with no upper bound, for some regular cardinal $\gamma \leq \kappa$. Let $\{X_i \mid i \in \lambda\}$ be a λ-regularizing family of $\mathcal{U}$. Define

$$n_i = |\{\alpha \in \lambda \mid i \in X_\alpha\}| + 1$$

and note that $n_i \in \omega$ for all $i \in I$. Let

$$f : \gamma \to \mathcal{U}$$

be such that $f(\alpha) = X_\alpha$ and put $Y_i = \{a_\alpha(i) \mid i \in f(\alpha)\}$. For every $i \in I$ let Q_i be a finite pseudo-tree of cardinality n_i and $h_i : Y_i \to Q_i$ be an injective order preserving map. Now set $(Q, \sqsubseteq) = \prod_{i\in I}(Q_i, \subseteq)/\mathcal{U}$ and $h = \prod_{i\in I} h_i/\mathcal{U}$. Note that $(Q, \sqsubseteq)$ belongs to $\mathbb{P}(\mathcal{U})$. The sequence $(h(a_\alpha))_{\alpha\in\gamma}$ is an increasing sequence in $(Q, \sqsubseteq)$, since for $\alpha < \beta$ the set

$$\{i \in I \mid a_\alpha(i) \subseteq a_\beta(i)\} \cap f(\{\alpha\}) \cap f(\{\beta\}) \in \mathcal{U}$$

is a subset of $\{i \in I \mid h_i(a_\alpha(i)) \subseteq h_i(a_\beta(i))\}$. Hence there exists b such that $h(a_\alpha) \sqsubseteq b$ for all $\alpha \in \kappa$. Let

$$f' : \gamma \to \mathcal{U}$$

be a map such that $f'(\alpha) = f(\alpha) \cap \{i \in I \mid h_i(a_\alpha(i) \subseteq b(i)\}$. Put $Z_i = \{h_i(a_\alpha(i)) \mid i \in f'(\alpha)\}$ and note that Z_i is finite and linearly ordered, hence there exists a maximum element $c(i)$. Now the element $a = \prod_{i\in I} h_i^{-1}(c(i))/\mathcal{U}$ is well defined and it is an upper bound of the sequence $(a_\alpha)_{\alpha\in\gamma}$, constradiction.

□

Lemma 4.4.9. *Let $\mathcal{U}$ be a regular ultrafilter on I, where $|I| = \lambda$ is an infinite cardinal. If $\mathcal{U}$ has the λ^+-treetops property, then $\mathcal{U}$ is λ^+-good.*

Proof. By Lemma 4.4.8, we have $\lambda^+ \leq \mathfrak{t}(\mathcal{U})$. By Theorem 4.4.4, it is sufficient to prove that $\mathcal{C}(\mathcal{U})$ is empty. Assume for contradiction that there exist two infinite regular cardinals $\kappa_1, \kappa_2 \leq \lambda$ such that there is a (κ_1, κ_2) gap in $\omega^I/\mathcal{U}$. By Lemma 4.4.6, there exists a linear order L in $\mathbb{L}$ in which there is a (κ_1, κ_2) gap. By Corollary 1.4.7, we have $CSP(\mathcal{U}) = \emptyset$ and $\kappa_1 + \kappa_2 \leq \lambda < \lambda^+ \leq \mathfrak{t}(\mathcal{U})$, contradiction. □

Lemma 4.4.10. *Let $\mathcal{U}$ be a regular ultrafilter on I, where $|I| = \lambda$. Then the following are equivalent:*

(i) If $\kappa \leq \lambda$, then $(\kappa, \kappa) \notin \mathcal{C}(\mathcal{U})$.

(ii) $\mathcal{U}$ has the λ^+-treetops property.

Proof.

$(ii) \Rightarrow (i)$ By Lemma 4.4.9, $\mathcal{U}$ is λ^+-good. We conclude that $\mathcal{C}(\mathcal{U}) = \emptyset$, by Theorem 4.4.4.

$(i) \Rightarrow (ii)$ Assume for contradiction that $\mathcal{U}$ has not the λ^+-treetops property. By Lemma 4.4.8, we have $\lambda^+ > \mathfrak{t}(\mathcal{U})$, hence there exists a $(\mathfrak{t}(\mathcal{U}), \mathfrak{t}(\mathcal{U}))$ gap in some linear order of $\mathbb{L}(\mathcal{U})$, by Theorem 1.2.2. By Lemma 4.4.6, we conclude that $(\mathfrak{t}(\mathcal{U}), \mathfrak{t}(\mathcal{U})) \in \mathcal{C}(\mathcal{U})$, contradiction.

□

Theorem 4.4.11. *Let* $\mathcal{U}$ *be a regular ultrafilter on* I*, where* $\lambda = |I|$*. Then the following are equivalent:*

(i) $\mathcal{U}$ *is* λ^+*-good.*

(ii) $\mathcal{C}(\mathcal{U}) = \emptyset$*.*

(iii) $\mathcal{U}$ *has the* λ^+*-treetops property.*

(iv) If $\kappa \leq \lambda$ *is regular, then* $(\kappa, \kappa) \notin \mathcal{C}(\mathcal{U})$*.*

Proof.

$(i) \Leftrightarrow (ii)$ By Theorem 4.4.4.

$(iii) \Leftrightarrow (iv)$ By Lemma 4.4.10.

$(i) \Rightarrow iii)$ By Theorem 3.1.26, every ultraproduct is λ^+-saturated.

$(ii) \Rightarrow (iv)$ Obvious.

$(iii) \Rightarrow (i)$ By Lemma 4.4.9.

□

4.5 The SOP_2-types

Definition 4.5.1. A theory T has the SOP_2 property, if there exist a formula $\psi(x, \bar{y})$ such that in all models $\mathcal{M}$ of T there is a copy of the tree $(\mu^{<\kappa}, \subseteq)$ $(\{\bar{a}_s \mid s \in \mu^{<\kappa}\}, \trianglelefteq)$ inside $\mathcal{M}$ such that:

(i) if $s, t \in \mu^{<\kappa}$ are incompatible sequences, then the the formula $\psi(x, \bar{a}_s) \wedge \psi(x, \ \bar{a}_t)$ is not realizable in $\mathcal{M}$.

(ii) For $s \in \mu^{\kappa}$, the ψ-type $\{\psi(x, \bar{a}_{s(i)}) \mid i \in \kappa\}$ is such that all its finite subsets can be realized in $\mathcal{M}$.

The tree $(\{\bar{a}_s \mid s \in \mu^{<\kappa}\}, \trianglelefteq)$ is a SOP_2-tree for $\psi(x, \bar{y})$ in $\mathcal{M}$.

Lemma 4.5.2. *If a theory T has the* SOP_2 *property, then in some model $\mathcal{N}$ of T there exists a* SOP_2*-tree* $(\{\bar{a}_s \mid s \in 2^{<\omega}\}, \trianglelefteq)$ *for a formula* $\psi(x, \bar{y})$.

Proof. Assume that in some model $\mathcal{M}$ of T there exists a SOP_2-tree $(\{\bar{a}_s \mid s \in \mu^{<\kappa}\}, \trianglelefteq)$ for $\psi(x, \bar{y})$. Expand the language $\mathcal{L}$ adding a set of finite tuples of contants $\{\bar{a}_s \mid s \in 2^{<\omega}\}$ and a binary relation symbol $\trianglelefteq$. Let T' be the theory obtained by T adding the following axioms:

(i) For all $s, t \in 2^{<\omega}$

$$\bar{a}_s \trianglelefteq \bar{a}_t \Longleftrightarrow s \subseteq t.$$

(ii) If s, t are $\subseteq$-incompatible, then

$$\neg\exists x[\psi(x, \bar{a}_s) \wedge \psi(x, \bar{a}_t)].$$

(iii) If $s_1, \ldots, s_n$ are compatible, then

$$\exists x[\psi(x, \bar{a}_{s_1}) \wedge \ldots \wedge \psi(x, \bar{a}_{s_n})]$$

By Compactness Theorem B.1.1, T' is consistent and a model $\mathcal{N}$ of T' has the required properties. □

Lemma 4.5.3. *If T is a complete $\mathcal{L}$-theory with the strict order property, then T has the* SOP_2 *property.*

Proof. Let $\mathcal{M}$ be a model of T and $\psi(x, y)$ be a formula that defines a partial order with an infinite chain $(a_i)_{i\in\omega}$ in $\mathcal{M}$. To simplify the notation we abbreviate $\psi(x, y)$ as $x < y$. By Compactness Theorem B.1.1, we can assume that there exists a sequence $(a_i)_{i\in\mathbb{Q}}$ such that

$$\mathcal{M} \models a_i < a_j \Longleftrightarrow \mathbb{Q} \models i < j.$$

In fact, expand $\mathcal{L}$ adding countable symbols of constant $\{a_i \mid i \in \mathbb{Q}\}$ and consider the theory $T' = T \cup \{a_i < a_j \mid \; i, j \in \mathbb{Q} \text{ and } i < j\}$. The theory T' is finitely consistent, since the model $\mathcal{M}$ of T satisfies every finite $\Sigma \subseteq T'$. Hence T' is consistent and a model $\mathcal{M}'$ has the required properties. Now we construct inductively an SOP_2-tree

$$(\{\bar{b}_s \mid s \in 2^{<\omega}\}, \trianglelefteq)$$

for the formula $\phi(x, \bar{y}) = y_1 < x \wedge x < y_2$. Put $b_\emptyset = (a_0, a_1)$. In the second step put $b_{<0>} = (a_0, a_{\frac{1}{2}})$ and $b_{<1>} = (a_{\frac{1}{2}}, a_1)$. Now assume that $b_s = (a_{s_0}, a_{s_1})$ has been defined for all $|s| = n$ in such a way $s_1 - s_0 = \frac{1}{2^n}$. Set

$$b_{s^\frown<0>} = (a_{s_0}, a_{s_0 + \frac{1}{2^{n+1}}}) \text{ and } b_{s^\frown<1>} = (a_{s_0 + \frac{1}{2^{n+1}}}, a_{s_1}).$$

By construction it follows that the above tree has all the properties required. □

Definition 4.5.4. Let $\mathcal{M}$ be a model of a theory T with the SOP_2-property and $\mathcal{U}$ be a regular ultrafilter over a set I of cardinality λ. Assume that $(T, \trianglelefteq)$ is a SOP_2-tree for $\psi(x, \bar{y})$ in $\mathcal{M}$. A type $p(x) = \{\psi(x, a_\alpha) \mid \alpha \in \gamma\}$ is a SOP_2-type for $(T, \trianglelefteq)$ in $\mathcal{M}^I/\mathcal{U}$, if $\gamma \leq \lambda$ and the set

$$X = \{j \in I \mid a_\alpha(j) \text{ belongs to } (T, \trianglelefteq) \text{ for all } \alpha \in \gamma\} \;\; \star$$

belongs to $\mathcal{U}$.

From now we call ψ-type a type of the form $p(x) = \{\psi(x, \bar{a}_i) \mid i \in \gamma\}$ and we just say that "$p(x)$ is a SOP_2-type in $\mathcal{M}^I/\mathcal{U}$" instead of "$p(x)$ is a SOP_2-type in $\mathcal{M}^I/\mathcal{U}$ for $(T, \trianglelefteq)$" if the reference to the tree $(T, \trianglelefteq)$ is not needed in our arguments.

Definition 4.5.5. Let $\mathcal{U}$ be a regular ultrapower on I and $p(x) = \{\psi(x, a_i) \mid i \in \gamma)\}$ be a ψ-type of cardinality $\gamma \leq |I|$. Assume that $p(x)$ is finitely satisfiable in $\mathcal{M}^I/\mathcal{U}$. A *distribution* for $p(x)$ is a map

$$d_p \colon S_\omega(\gamma) \to \mathcal{U}$$

such that:

(i) for every $u \in S_\omega(\gamma)$, we have

$$d_p(u) \subseteq \{i \in I \mid \mathcal{M} \models \exists x \bigwedge_{j \in u} \psi(x, a_j(i))\} \in \mathcal{U}.$$

(ii) the range of d_p is a γ-regularizing family of $\mathcal{U}$.

(iii) if $u \subseteq v$, then $d_p(v) \subseteq d_p(u)$.

Given an $\mathcal{L}$-structure $\mathcal{M}$ with a SOP_2 tree $(T, \trianglelefteq)$, we can add to the language $\mathcal{L}$ a binary relation $\trianglelefteq$ that we interpret as the partial order on the SOP_2-tree.

Lemma 4.5.6 (Lemma 11.6 [5]). *Let $\mathcal{U}$ be a regular ultrafilter on I, where $|I| = \lambda$, and $\mathcal{M}$ be a model of theory with the SOP_2 property and $(T, \trianglelefteq)$ be a tree contained in M witnessing the SOP_2-property for T in $\mathcal{M}$ relative to the formula $\psi(x, y)$. The following are equivalent for every ψ-type $p(x)$ on $\mathcal{M}^I/U$ which is a SOP_2-type for $(T, \trianglelefteq)$:*

(i) $p(x)$ has a realization in $\mathcal{M}^I/\mathcal{U}$.

(ii) $p(x)$ has a distribution d_p such that

$$i \in d_p(\{\alpha\}) \cap d_p(\{\beta\}) \Longrightarrow (a_\alpha(i) \trianglelefteq a_\beta(i)) \vee (a_\beta(i) \trianglelefteq a_\alpha(i)). \quad *$$

Proof.

$(i) \Rightarrow (ii)$ Let $p(x)$ be as in the hypothesis. By hypothesis there exists a realization $a \in \mathcal{M}^I/\mathcal{U}$ of $p(x)$ and

$$X = \{i \in I \mid a_\alpha(i) \text{ belongs to } T \text{ for all } \alpha \in \gamma\} \in \mathcal{U}$$

by property $\star$ of the SOP_2-type $p(X)$. Put

$$d_p(\{\alpha\}) = \{i \in I \mid \mathcal{M} \models \psi(a(i), a_\alpha(i))\} \cap X_\alpha \cap X \in \mathcal{U},$$

where $\{X_\alpha \mid \alpha \in \gamma\}$ is a γ-regularizing family of $\mathcal{U}$ and

$$d_p(u) = \bigcap_{\alpha \in u} d(\{\alpha\}),$$

for $|u| > 1$. The map $d_p \colon S_\omega(p) \to \mathcal{U}$ is a distribution of $p(x)$, hence it is sufficient to show that the condition $*$ holds. If $i \in d_p(\{\alpha\}) \cap d_p(\{\beta\})$, then the sentence $\exists x\, \psi(x, a_\alpha(i)) \wedge \psi(x, a_\beta(i))$ is satisfied in $\mathcal{M}$ as witnessed by a and $a_\alpha(i), a_\beta(i)$ belong to T. Hence we conclude that $(a_\alpha(i) \trianglelefteq a_\beta(i)) \vee (a_\beta(i) \trianglelefteq a_\alpha(i))$.

$(ii) \Rightarrow (i)$ Let $p(x)$ be as in the hypothesis. Choose a distribution d_p of p for which condition $*$ holds. Define the map

$$d'_p \colon S_\omega(p) \to \mathcal{U}$$

such that $d'_p(\{\alpha\}) = d_p(\{\alpha\})$ and $d'_p(u) = \bigcap_{\alpha \in u} d'_p(\{\alpha\})$ for $|u| > 1$. Obviously the range of d'_p is a regularizing family of $\mathcal{U}$ and $d'_p(u \cup v) = d'_p(u) \cap d'_p(v)$ for all $u, v \in S_\omega(p)$. Now for $i \in I$, put $\Sigma(i) = \bigcup\{u \in S_\omega(p) \mid i \in d'_p(u)\}$ and note that $\Sigma(i)$ is always finite. If $\Sigma(i) = \bigcup\{u_1, \ldots, u_h\}$, we have

$$i \in\ d'_p(u_1) \cap \ldots \cap d'_p(u_h) = d'_p(\bigcup_{l \le h} u_l) = d'_p(\Sigma(i))$$

We show that

$$d'_p(\Sigma(i)) \subseteq \{j \in I \mid \mathcal{M} \models \exists x \bigwedge_{\alpha \in \Sigma(i)} \psi(x, a_\alpha(j))\}$$

as follows: Let $\Sigma(i) = \{\alpha_1, \ldots, \alpha_n\}$, then for every $j \in d'_p(\Sigma(i)) = \bigcap_{i=1}^n d_p(\{\alpha_i\})$ the elements $a_{\alpha_1}(j), \ldots, a_{\alpha_n}(j)$ are $\trianglelefteq$-compatible since for all $l \neq k \in \{1, \ldots, n\}$ $j \in d_p(\{\alpha_l\}) \cap d_p(\{\alpha_k\})$ and thus $a_{\alpha_l}(j)$ and $a_{\alpha_k}(j)$ are $\trianglelefteq$-compatible by condition $*$ on d_p. This gives that the formula

$$\exists x \bigwedge_{\alpha \in \Sigma(i)} \psi(x, a_\alpha(j))$$

is realizable in M.

Hence we can choose $a(i) \in \mathcal{M}$ such that

$$\mathcal{M} \models \bigwedge_{\alpha \in \Sigma(i)} \psi(a(i), a_\alpha(i)).$$

Finally, we prove that $[(a(i))_{i\in\lambda}]$ is a realization of $p(x)$. If $\psi(x, a_\alpha)$ is a formula of $p(x)$, we have

$$\mathcal{U} \ni d'_p(\{\alpha\}) \subseteq \{i \in I \mid \mathcal{M} \models \psi_\alpha(a(i), a_\alpha(i))\},$$

since $i \in d'_p(\{\alpha\})$ implies $\alpha \in \Sigma(i)$.

□

4.6 The maximality of theories with SOP_2 property

Lemma 4.6.1 (Lemma 11.6 [5]). *Let $\mathcal{U}$ be a regular ultrafilter on I, where $|I| = \lambda$, and $\mathcal{M}$ be a model of theory with the SOP_2 property and $(T, \trianglelefteq)$ be a tree with $T \subseteq M$ witnessing it. The following are equivalent:*

(i) Every ψ-type $p(x)$ in $\mathcal{M}^I/\mathcal{U}$ of cardinality less or equal to λ which is a SOP_2-type for $(T, \trianglelefteq)$, has a distribution d_p such that

$$i \in d_p(\{\alpha\}) \cap d_p(\{\beta\}) \Longrightarrow (a_\alpha(i) \trianglelefteq a_\beta(i)) \vee (a_\beta(i) \trianglelefteq a_\alpha(i)). \quad *$$

(ii) If $(S, \trianglelefteq_S)$ is a tree and $(c_\alpha)_{\alpha\in\gamma}$ is an increasing sequence of $S^I/\mathcal{U}$, where $\gamma \leq \lambda$, then the sequence has an upper bound in $(S, \trianglelefteq_S)^I/\mathcal{U}$.

Proof.

(ii)⇒(i) Let $p(x) = \{\psi(x, a_\alpha) \mid \alpha \in \gamma\}$ be a SOP_2-type in $\mathcal{M}^I/\mathcal{U}$. Then the sequence $(a_\alpha)_{\alpha\in\gamma}$ is linearly ordered by $\trianglelefteq$, hence there exists an upper bound a applying (ii) to the tree $(T, \trianglelefteq_T)$. Define the map $d_p\colon S_\omega(p) \to \mathcal{U}$ such that

$$d_p(\{\alpha\}) = \{i \in I \mid \mathcal{M} \models a_\alpha(i) \trianglelefteq a(i)\} \cap X_\alpha \cap X$$

where $\{X_\alpha \mid \alpha \in \gamma\}$ is a γ-regularizing family of $\mathcal{U}$ and

$$X = \{i \in I \mid a_\alpha(i) \text{ belongs to a } \mathsf{SOP}_2\text{-tree for all } \alpha \in \gamma\} \in \mathcal{U}.$$

For $|u| > 1$, put $d_p(u) = \bigcap_{\alpha\in u} d_p(\{\alpha\})$. Clearly the range of d_p is a γ-regularizing family of $\mathcal{U}$, and $d_p(v) \subseteq d_p(u)$, when $u \subseteq v$. Now if $i \in d_p(\{\alpha\}) \cap d_p(\{\beta\})$, then $a_\alpha(i), a_\beta(i) \trianglelefteq a(i)$, hence $a_\alpha(i)$ and $a_\beta(i)$ are $\trianglelefteq$-compatible. By the same argument we can also conclude that

$$d_p(u) \subseteq \{i \in \mathcal{U} \mid \mathcal{M} \models \bigwedge_{\alpha\in u} \psi(x, a_\alpha(i))\}.$$

Thus d_p is the required distribution satisfying the conclusion of (i).

(i)⇒(ii) Let $(T, \trianglelefteq_T)$ be the subtree of $\mathcal{M}$ as in the hypothesis of the Lemma whichwitnesses the SOP_2-property of $\psi(x, y)$. Let $(S, \trianglelefteq_S)$ be a tree and $(c_\alpha)_{\alpha\in\gamma}$ be a $\trianglelefteq_S$-increasing sequence in $S^I/\mathcal{U}$, for some $\gamma \leq \lambda$. We show that the type $p(x) = \{x > c_\alpha \mid \alpha \in \gamma\}$ is realized in $S^I/\mathcal{U}$. Fix a distribution $d_p\colon S_\omega(p) \to \mathcal{U}$ of the type $p(x)$. For any $i \in I$, the set $\Sigma(i) = \{u \in S_\omega(\gamma) \mid i \in d_p(u)\}$ is finite. Now, if c_α, c_β are parameters of some $u, v \in \Sigma(i)$, then we choose $a_\alpha(i), a_\beta(i) \in T$ such that

$$c_\alpha(i) \triangleleft_S c_\beta(i) \iff a_\alpha(i) \triangleleft_T a_\beta(i).$$

Consider the type $q(x) = \{\psi(x, a_\alpha) \mid \alpha \in \gamma\}$. The set $q(x)$ satisfies $\star$ of Definition 4.5.4, since $a_\alpha(i) \in T$ is an element of the SOP_2-tree T of $\mathcal{M}$ for each $i \in I$ and $\alpha \in \gamma$. Now we show that $q(x)$ is a finitely consistent SOP_2-type. If $\alpha_1 \triangleleft \ldots \triangleleft \alpha_n$, then the set

$$d(\{\alpha_1\}) \cap \ldots \cap d(\{\alpha_n\}) \cap \{i \in I \mid c_{\alpha_1}(i) \triangleleft \ldots \triangleleft c_{\alpha_n}(i)\} \in \mathcal{U}$$

is a subset of

$$\{i \in I \mid a_{\alpha_1}(i) \triangleleft \ldots \triangleleft a_{\alpha_n}(i)\}$$

and this is contained in

$$\{i \in I \mid \exists x \bigwedge_{j=1}^{n} \psi(x, a_{\alpha_j}(i))\},$$

thus $q(x)$ is finitely consistent. By (i) there exists a distribution d_q of q with the property $*$. For each $i \in I$ the set $\Lambda(i) = \{c_\alpha(i) \mid i \in d_q(\{\alpha\})\}$ is finite and linearly ordered by $\trianglelefteq_T$ in $(T, \trianglelefteq)$. Let $c(i)$ be the maximum of the set $\Lambda(i)$ and put $c = [(c(i))_{i\in I}] \in (T, \trianglelefteq)^I/\mathcal{U}$. Now for each $\alpha \in \gamma$ we have

$$\mathcal{U} \ni d_q(\{\alpha\}) \subseteq \{i \in I \mid c_\alpha(i) \leq c(i)\},$$

hence c is an upper bound of the sequence $(c_\alpha)_{\alpha\in\gamma}$.

□

Theorem 4.6.2. *Let $\mathcal{U}$ be a regular ultrafilter on I, where $|I| = \lambda$. The following are equivalent:*

(i) $\mathcal{U}$ has the λ^+-treetops property.

(ii) $\mathcal{U}$ realizes all SOP_2-types of cardinality less or equal to λ.

Proof. Immediate by lemmas 4.5.6 and 4.6.1. □

Theorem 4.6.3 (Lemma 11.11 [5])**.** *If T is a theory with the SOP_2 property, then T is maximal in Keisler's order.*

Proof. By Theorem 4.2.4, it is sufficient to show that $\mathcal{U}$ is λ^+-good ultrafilter on λ, when $\mathcal{M}^\lambda/\mathcal{U}$ is a λ^+-saturated ultrapower and $\mathcal{M}$ is a model of a theory with the SOP_2-property. By Theorem 4.6.2, $\mathcal{U}$ has the λ^+-treetops property, when $\mathcal{M}^\lambda/\mathcal{U}$ is a λ^+-saturated ultrapower. We conclude the proof by Theorem 4.4.11. □

Chapter 5

Random graphs are not maximal in Keisler order

The purposes of this chapter is to prove that the statement *The theory of random graphs is not maximal in Keisler's order* is consistent with ZFC.

In Section 5.1, we introduced the technique of two-step iterated ultrapower and prove that this construction is equivalent to an ultrapower modulo a *tensor ultrafilter*. We study how the combinatorial properties of the tensor ultrafilter $\mathcal{U} \otimes \mathcal{V}$ are linked with the properties of $\mathcal{U}$ and $\mathcal{V}$.

In Section 5.2, we remark some basic facts on the theory of random graphs. Using Martin's axiom, we construct an ultrafilter $\mathcal{U}$ on $\aleph_1$ such that $\mathcal{U}$ is not $\aleph_2$-good, but each ultrapower $\mathcal{M}^{\aleph_1}/\mathcal{U}$ is $\aleph_2$-saturated for every random graph $\mathcal{M}$. In this way we deduce that the theory ZFC does not prove the maximality of the theory of random graphs in Keisler's order.

5.1 Iterated ultrapower

Definition 5.1.1. Let $\mathcal{U}, \mathcal{V}$ be filters on I, J, respectively. Consider the family $\mathcal{U} \otimes \mathcal{V}$ of subsets of $I \times J$ such that $S \in \mathcal{U}_1 \times \mathcal{U}_2$ if and only if

$$\{j \in J \mid \{i \in I \mid (i,j) \in S\} \in \mathcal{U}\} \in \mathcal{V}.$$

Remark 5.1.2. When $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$, we have $U_1 \times U_2 \in \mathcal{U}_1 \otimes \mathcal{U}_2$.

Proposition 5.1.3. *(i) Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on I and J, respectively. Then $\mathcal{U} \otimes \mathcal{V}$ is a ultrafilter on $I \times J$, called* tensor ultrafilter.

(ii) $\prod_{(i,j)\in I\times J} \mathcal{M}_{ij}/(\mathcal{U} \otimes \mathcal{V}) \cong \prod_{j\in J}(\prod_{i\in I} \mathcal{M}_{ij}/\mathcal{U})/\mathcal{V}.$

(iii) $\mathcal{M}^{I\times J}/(\mathcal{U} \otimes \mathcal{V}) \cong (\mathcal{M}^I/\mathcal{U})^J/\mathcal{V}.$

(iv) If $\mathcal{U}$ or $\mathcal{V}$ is λ-regular, then $\mathcal{U} \otimes \mathcal{V}$ is λ-regular.

(v) Assume $\mathcal{V}$ countably incomplete, then $\mathcal{U} \otimes \mathcal{V}$ is λ-good if and only if $\mathcal{V}$ is λ-good.

Proof. (i) Assume that S_1, S_2 belong to $\mathcal{U} \otimes \mathcal{V}$ and $S_1 \subseteq S$. Then the sets

$$\{j \in J \mid \{i \in I \mid (i,j) \in S_1\} \in \mathcal{U}\} \cap \{j \in J \mid \{i \in I \mid (i,j) \in S_2\} \in \mathcal{U}\} \in \mathcal{V}$$

$$\{j \in J \mid \{i \in I \mid (i,j) \in S_1\} \in \mathcal{U}\} \in \mathcal{V}$$

are subsets of

$$\{j \in J \mid \{i \in I \mid (i,j) \in S_1 \cap S_2\} \in \mathcal{U}\},$$

$$\{j \in J \mid \{i \in I \mid (i,j) \in S\} \in \mathcal{U}\},$$

respectively, hence $S_1 \cap S_2$ and S belong to $\mathcal{U} \otimes \mathcal{V}$. Now assume that $S \notin \mathcal{U} \otimes \mathcal{V}$, that is

$$\{j \in J \mid \{i \in I \mid (i,j) \in S\} \in \mathcal{U}\} \notin \mathcal{V}.$$

Hence

$$\mathcal{V} \ni \{j \in J \mid \{i \in I \mid (i,j) \in S\} \notin \mathcal{U}\} = \{j \in J \mid \{i \in I \mid (i,j) \notin S\} \in \mathcal{U}\}.$$

Then we conclude that $I \times J \setminus S \in \mathcal{U} \otimes \mathcal{V}$.

(ii) Define the map

$$f \colon \prod_{(i,j) \in I \times J} \mathcal{M}_{ij}/\mathcal{U} \otimes \mathcal{V} \to \prod_{j \in J} (\prod_{i \in I} \mathcal{M}_{ij}/\mathcal{U})/\mathcal{V}$$

such that $f([a]) = [b]$, where $b(j) = [a(\ ,j)]$. It easy to see that f is a well defined map and it is an isomorphism.

(iii) Follows by (ii).

(iv) Assume that $\mathcal{U}$ is λ-regular and the family $\{I_\alpha \mid \alpha \in \lambda\}$ regularizes $\mathcal{U}$. Consider the family $\{I_\alpha \times J \mid \alpha \in \lambda\}$. For each $\alpha \in \lambda$, we have $I_\alpha \times J \in \mathcal{U} \otimes \mathcal{V}$, since

$$\{j \in J \mid \{i \in I \mid (i,j) \in I_\alpha \times J\} \in \mathcal{U}\} = J \in \mathcal{V}.$$

If $(i,j) \in \bigcap_{\alpha \in u} I_\alpha \times J$ for some infinite $u \subseteq \lambda$, then $i \in \bigcap_{\alpha \in u} I_\alpha$, absurd.

(v) Assume that $\mathcal{V}$ is λ-good. Let $\mathcal{M}$ be a λ-saturated elementary extension of $(S_\omega(\gamma), \subseteq, P)$, for some $\gamma < \lambda$. By $iii)$ we have $\mathcal{M}^{I \times J}/(\mathcal{U} \otimes \mathcal{V}) \cong (\mathcal{M}^I/\mathcal{U})^J/\mathcal{V}$. We conclude that $\mathcal{M}^{I_1 \times I_2}/(\mathcal{U} \otimes \mathcal{V})$ is λ-saturated, hence $\mathcal{U} \otimes \mathcal{V}$ is λ-good, by Lemma 4.2.2. Now assume that $\mathcal{U} \otimes \mathcal{V}$ is λ-good. For $\gamma \in \lambda$, let $f \colon S_\omega(\gamma) \to \mathcal{V}$ be a monotone function. Consider $\bar{f} \colon S_\omega(\gamma) \to \mathcal{U} \otimes \mathcal{V}$ such that $\bar{f}(u) = I \times f(u)$ and note that $\bar{f}$ is monotone. Then there exists an additive map $\bar{g}$ such that $\bar{g} \leq \bar{f}$. Put $g \colon S_\omega(\gamma) \to \mathcal{V}$ such that

$$g(u) = \{j \in J \mid \{i \in I \mid (i,j) \in \bar{g}(u)\} \in \mathcal{U}\}.$$

Note that g is well defined since $\bar{g}(u) \in \mathcal{U} \otimes \mathcal{V}$. If $j \in g(u)$, then

$$\mathcal{U} \ni \{i \in I \mid (i,j) \in \bar{g}(u)\} \subseteq \{i \in I \mid (i,j) \in \bar{f}(u)\}.$$

Hence $g \leq f$. Finally we prove that g is additive. For $u, v \in S_\omega(\gamma)$, we have

$$\begin{aligned}
g(u \cup u') =& \{j \in J \mid \{i \in I \mid (i,j) \in \bar{g}(u \cup u')\} \in \mathcal{U}\} \\
=& \{j \in J \mid \{i \in I \mid (i,j) \in \bar{g}(u) \cap \bar{g}(u')\} \in \mathcal{U}\} \\
=& \{j \in J \mid \{i \in I \mid (i,j) \in \bar{g}(u)\} \in \mathcal{U},\ \{i \in I \mid (i,j) \in \bar{g}(u')\} \in \mathcal{U}\} \\
=& \{j \in J \mid \{i \in I \mid (i,j) \in \bar{g}(u)\} \in \mathcal{U}\} \cap \{j \in J \mid \{i \in I \mid (i,j) \in \bar{g}(u')\} \in \mathcal{U}\} \\
=& g(u) \cap g(u').
\end{aligned}$$

□

5.2 Random graphs and Keisler's order

Recall some facts on the theory of random graphs.

Definition 5.2.1. The theory of random graphs, abbreviated as T_{rg}, is the theory in the language $\mathcal{L}_{rg} = \{R\}$ with the following axioms:

(i) $\forall x\ \neg xRx$.

(ii) $\forall x, y\, [xRy \longleftrightarrow yRx]$.

(iii) For each $n, m \in \omega$ there are the following axioms:

$$\forall y_1, \ldots, y_n, z_1, \ldots, z_m \left(\bigwedge_{i,j} y_i \neq z_j \to \exists x \left[\bigwedge_{1 \leq j \leq n} xRy_j \wedge \bigwedge_{1 \leq j \leq m} \neg xRz_j \right] \right).$$

(iv) For each $n \in \omega$ there are the following axioms:

$$\exists x_1, \ldots, x_n \bigwedge_{i \neq j} \neg(x_i = x_j).$$

Definition 5.2.2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A formula $\psi(x)$ possibly with parameters is *algebraic*, if $\psi(x)$ has a finite number of realizations in $\mathcal{M}$, that is for some $n \in \omega$

$$\mathcal{M} \models \exists^{=n} x \psi(x).$$

Lemma 5.2.3. *The theory T_{rg} has the elimination of quantifiers. Moreover, for every model $\mathcal{M}$ of T and parameters $a_1, \ldots, a_n \in \mathcal{M}$, a formula $\psi(x, a_1, \ldots, a_n)$ is is non-algebraic, unless $x = a_i$ is not a subformula of $\psi(x, a_1, \ldots, a_n)$.*

Proof. To prove the first part we use Theorem B.2.3. Assume that $\mathcal{M} \models T_{rg}$, $A \subseteq \mathcal{M}$, $\mathcal{N} \models T_{rg}$ is $|\mathcal{M}|^+$-saturated and $f\colon A \to \mathcal{N}$ is a partial embedding. Let $\mathcal{M} = \{a_\alpha \mid \alpha \in \kappa\}$ be an enumeration of $\mathcal{M}$. We contruct a family of partial embeddings $\{f_\alpha \mid \alpha \leq \kappa\}$ such that

(i) the map $f_\alpha\colon \mathcal{M} \to \mathcal{N}$ extends f.

(ii) $a_\alpha \in \mathrm{dom}(f_{\alpha+1})$.

(iii) $f_\beta \subseteq f_\alpha$ for all $\beta \leq \alpha \leq \kappa$.

When the construction is complete, the map f_κ has the required properties. Put $f_0 = f$ and $f_\alpha = \bigcup_{\beta\in\alpha} f_\alpha$ when α is limit ordinal. Now assume that f_α is defined and $a_\alpha \notin \mathrm{dom}(f_\alpha)$. The type

$$\begin{aligned} p(x) = &\{xRf_\alpha(a_\gamma) \mid \mathcal{M} \models a_\alpha R a_\gamma,\ \ a_\gamma \in \mathrm{dom}(f_\alpha)\} \cup \\ &\{\neg xRf_\alpha(a_\gamma) \mid \mathcal{M} \models \neg a_\alpha R a_\gamma,\ \ a_\gamma \in \mathrm{dom}(f_\alpha)\} \end{aligned}$$

is finitely satisfiable in $\mathcal{N}$, since clause (iii) of Definition 5.2.1 holds in $\mathcal{N}$. Hence there exists a realization $b \in \mathcal{N}$ of $p(x)$, since $\mathcal{N}$ is $|\mathcal{M}|^+$-saturated. Conclude setting $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, b)\}$.

In order to prove the last part, fix $\mathcal{M} \models T_{rg}$ and $a_1, \dots, a_n \in \mathcal{M}$. By quantifier elimination of T_{rg}, we can assume that

$$\psi(x, a_1, \dots, a_n) = \bigwedge_{1\leq j\leq m} xRa_j \wedge \bigwedge_{m+1\leq j\leq n} \neg xRa_j.$$

By the axioms of T, the model $\mathcal{M}$ satisfies the sentence $\psi(b_1, a_1, \dots, a_n)$ for some $b_1 \in \mathcal{M}$. Now let $b_2 \in \mathcal{M}$ be a realization of the formula $\psi(x, a_1, \dots, a_n) \wedge xRb_1$. By axiom 1), we have $b_1 \neq b_2$. Repeating this argument, we conclude that the formula $\psi(x, a_1, \dots, a_n)$ has infinite realizations. □

From now on, we work under the assumption that Martin's axiom holds, see Section A.2 of the appendix to a summary of all facts that we use.

Notation. *Let I, J be two sets. We denote by $Fn(I, J)$ the set of all finite partial functions p with domain in I and range in J. On $Fn(I, J)$, we consider the following partial order $\leq$:*

$$p \leq q \iff p \supseteq q.$$

Theorem 5.2.4 (Theorem 3.10 (i) Chapter VI [9])**.** *Assume that Martin's axiom holds and $2^{\aleph_0} = \aleph_2$. There exists a regular ultrafilter $\mathcal{U}$ on ω, such that if $\mathcal{M}$ is a $2^{\aleph_0}$-saturated model of T_{rg}, then $\mathcal{M}^\omega/\mathcal{U}$ is also $2^{\aleph_0}$-saturated.*

The proof of the theorem reposes on the following two Lemmas:

Lemma 5.2.5. *Assume Martin's axiom holds. Let $\mathcal{M}$ be a $2^{\aleph_0}$-saturated model of T_{rg} and A be a subset of $\mathcal{M}$ of size less than $2^{\aleph_0}$. Then there exists a countable subset B of $\mathcal{M}$ such that every non-algebraic finite type over A is realized by some element of B.*

Lemma 5.2.6 (Lemma 3.9 Chapter VI [9]). *Assume that Martin's axiom holds. Let $\mathcal{L} = \mathcal{L}_{rg} \cup \{P\}$ be an expansion of the language of random graphs obtained adding an unary relational symbol P. There exists a regular ultrafilter $\mathcal{U}$ on ω with the following property: Assume that for each $i \in \omega$ $\mathcal{M}_i$ is a structure for $\mathcal{L}$ (of arbitrary size) such that:*

- *$\mathcal{M}_i$ restricted to the language $\mathcal{L}_{rg}$ models the theory of the random graph*
- *$P^{\mathcal{M}_i}$ is countable.*

Then the ultraproduct $\mathcal{M} = \prod_{i\in\omega} \mathcal{M}_i/\mathcal{U}$ realizes any type $p(x)$ with parameters in $\mathcal{M}$ which has cardinality less than $2^{\aleph_0}$ and is such that $P(x) \in p(x)$.

Assume that both Lemmas have been proved. Then the proof of the theorem is immediately obtained as follows:

Proof. Let $\mathcal{U}$ be the ultrafilter given by the Lemma 5.2.6 and $\mathcal{M}$ be a $2^{\aleph_0}$-saturated model of the theory of the random graph. We show that $\mathcal{M}^\omega/\mathcal{U}$ is still $2^{\aleph_0}$-saturated. Let $p(x) = \{\phi(x, \bar{a}_\alpha) \mid \alpha \in \mu\}$ be a type over $\mathcal{M}^\omega/\mathcal{U}$ such that $\mu < \lambda$ and put $A_i = \bigcup\{\bar{a}_\alpha(i) \mid \alpha \in \mu\}$. Now use Lemma 5.2.5 to expand for each i $\mathcal{M}$ to $\mathcal{M}_i = (\mathcal{M}, P^{\mathcal{M}_i})$ where $P^{\mathcal{M}_i}$ is a countable subset of $\mathcal{M}$ such that every non-algebraic finite type in the theory of the random graph over the parameters in A_i is realized by some element of $P^{\mathcal{M}_i}$. Now observe that the ultraproduct $\prod_{i\in\omega} \mathcal{M}_i/\mathcal{U}$ realizes the type $p(x) \cup \{P(x)\}$ by Lemma 5.2.6, in particular $\mathcal{M}^\omega/\mathcal{U}$ realizes $p(x)$. □

So we start with the proof of Lemma 5.2.5

Proof. Let $|A| = \nu$. Since the theory T_{rg} has quantifier elimination, we can assume that every non-algebraic formulas $\psi(x)$ with parameters in A has the form

$$\psi(x) = \bigwedge_{1\leq j\leq m} xRa_{\alpha_j} \wedge \bigwedge_{m+1\leq j\leq n} \neg xRa_{\alpha_j},$$

for some $a_{\alpha_1}, \ldots, a_{\alpha_n} \in A$. We can associate to every non-algebraic formula $\psi(x)$ with parameters $a_{\alpha_1}, \ldots, a_{\alpha_n} \in A$ a partial function $q\colon \mu \to 2$ such that

(i) $\mathrm{dom}(q) = \{\gamma \in \mu \mid a_\gamma \text{ is a parameter of } \psi(x)\}$.

(ii) $q(\gamma) = 1$, if xRa_γ is a subformula of $\psi(x)$, and $q(\gamma) = 0$, if $\neg xRa_\gamma$ is a subformula of $\psi(x)$.

Let P be the set of all these functions and

$$Q = \{(q_0, \ldots, q_n) \mid n \in \omega \text{ and } q_i \in P \text{ for all } i \leq n\}.$$

Define a partial order $\leq$ on Q such that:

$$(q_0, \ldots, q_n) \leq (q'_0, \ldots, q'_m) \iff m \leq n \text{ and } q'_i \subseteq q_i \text{ for each } i \leq m.$$

Now we show that $(Q, \leq)$ has the countable chain condition. Every $q = (q_0, \dots, q_n) \in Q$ can be seen as a finite function

$$\tilde{q} \colon \omega \times \mu \to 2$$

such that $\tilde{q}(h, \gamma) = q_h(\gamma)$. The order on $(Q, <)$ is a dense sub-order of

$$\mathrm{Fn}(\omega \times \mu, 2), \leq_{\mathrm{Fn}(\omega\times\mu,2)}),$$

in fact every functions $p \in \mathrm{Fn}(\omega \times \mu, 2)$ can be extended to a function of Q and, for each $p, q \in Q$, we clearly have

$$p \leq_Q q \iff p \leq_{\mathrm{Fn}(\omega\times\mu,2)} q.$$

Then, by Lemma A.2.9, we conclude that $(Q, <)$ has the countable chain condition. For every $q \in P$, the set

$$D_q = \{(q_0, \dots, q_n) \mid \exists i \leq n \; q \subseteq q_i\}$$

is clearly dense. Put $\mathcal{D} = \{D_q \mid q \in P\}$ and note that $\mathcal{D}$ has cardinality $\mu < 2^{\aleph_0}$, since P has cardinality μ. By Martin's axiom, there exists a $\mathcal{D}$-generic filter G such that $G \cap D_q \neq \emptyset$ for each $q \in P$. For $h \in \omega$, put

$$C_h = \bigcup \{q \mid \text{there exists } (q_1, \dots, q_n) \in G \text{ with } q_h = q\}.$$

If $(q_1, \dots, q_n), (q'_1, \dots, q'_m) \in G$, then q_h and q'_h are compatible, hence C_h is a partial function from μ to 2. For every $h \in \omega$ the type

$$p_h(x) = \{xRa_\xi \mid \xi < \mu \text{ such that } C_h(\xi) = 1\} \cup \{\neg xRa_\xi \mid \xi < \mu \text{ such that } C_h(\xi) = 0\}$$

is finitely consistent in $\mathcal{M}$, since $\mathcal{M}$ satisfies

$$\forall y_1, \dots, y_n, z_1, \dots, z_m \left(\bigwedge_{i,j} y_i \neq z_j \to \exists x \left[\bigwedge_{1\leq j\leq n} xRy_j \wedge \bigwedge_{1\leq j\leq m} \neg xRz_j \right] \right),$$

for each $n, m \in \omega$. Hence we can choose a realization b_h of $p_h(x)$. Finally, we prove that the set $\{b_h \mid h \in \omega\}$ has the required property. Let $\psi(x)$ be a non-algebraic formula with parameters in A. Let $q \in Q$ be the partial function associated to $\psi(x)$. Since $G \cap D_q \neq \emptyset$, for some $h \in \omega$ there exists a finite sequence $(q_0, \dots, q_n) \in G$ such that $q_h \supseteq q$. It follows that $q \subseteq C_h$ and hence b_h realizes $\psi(x)$. □

We now prove Lemma 5.2.6:

Proof. Without loss of generality it is enough to prove the conclusion of the lemma for the structures $\prod_{i\in\omega} M_i/\mathcal{U}$ such that each $\mathcal{M}_i$ has cardinality less than $2^{\aleph_0}$: Assume that we have proved the lemma for this type of ultraproducts, let $\prod \mathcal{N}_i/\mathcal{U}$ be an ultraproduct such that some $\mathcal{N}_i$ has size at least continuum and $p(x)$ be a type in $\prod \mathcal{N}_i/\mathcal{U}$ of size less than continuum. Let $A \in \prod \mathcal{N}_i$ be a set of size $|p(x)|$ such that

$$\{[a]_{\mathcal{U}} : a \in A\}$$

is the set of parameters appearing in some formula of $p(x)$. Let

$$A_i = \{a(i) : a \in A\} \subseteq \mathcal{N}_i.$$

Let $\mathcal{M}_i$ be the skolem hull of A_i inside $\mathcal{N}_i$, if $\mathcal{N}_i$ has size at least continuum, and $\mathcal{M}_i$ be $\mathcal{N}_i$ otherwise. Then $p(x)$ is a type on $\prod_{i\in\omega}\mathcal{M}_i/\mathcal{U}$ and the lemma applies to $\prod_{i\in\omega} M_i/\mathcal{U}$ which is an elementary substructure of $\prod_{i\in\omega}\mathcal{N}_i/\mathcal{U}$.

So we are left with the proof of the Lemma for ultraproducts of structures of size less than continuum. Let $\{S_\alpha \subseteq \omega \mid \alpha \in 2^{\aleph_0} \text{ is odd}\}$ be an enumeration of all subsets of ω. Consider the enumeration $\{(p_\alpha, (\mathcal{M}_n^\alpha \mid i \in \omega)) \mid \alpha \in 2^{\aleph_0} \text{ is even}\}$ of all couples $(p, (\mathcal{M}_i \mid n \in \omega))$ where $\mathcal{M}_i$ is an $\mathcal{L}$-structure, $\mathcal{M}_i, P^{\mathcal{M}_i}$ have cardinality less than $2^{\aleph_0}$ and $p(x)$ is a type with parameters in $\prod_{i\in\omega}\mathcal{M}_i$ of cardinality less than $2^{\aleph_0}$; moreover we fix the enumeration so that every couple appears $2^{\aleph_0}$-times in the enumeration. Now define by induction a set $\{U_\alpha \subseteq \mathcal{P}(\omega) \mid \alpha \in 2^{\aleph_0}\}$ such that

(i) The family U_α generates a filter on ω, which we denote by $[U_\alpha]$.

(ii) If $\beta \in \alpha$, then $U_\beta \subseteq U_\alpha$ and, if α is limit ordinal, then $U_\alpha = \bigcup_{\beta\in\alpha} U_\beta$ and $|U_\alpha| < 2^{\aleph_0}$.

(iii) For every α odd, we have $S_\alpha \in U_{\alpha+1}$ or $\omega \setminus S_\alpha \in U_{\alpha+1}$.

(iv) For α even, if for every $\phi_1(x, \bar{a}_1), \dots, \phi_n(x, \bar{a}_n) \in p_\alpha$ we have

$$\{i \in \omega \mid \mathcal{M}_i^\alpha \models \exists x P(x) \wedge \bigwedge_{j=1}^{n} \phi_j(x, \bar{a}_j(i))\} \in [U_\alpha],$$

then for some $\bar{b} \in \prod_{i\in\omega}\mathcal{M}_i^\alpha$ we have

$$\{i \in \omega \mid \mathcal{M}_i^\alpha \models P(b(i)) \wedge \phi(b(i), \bar{a}(i))\} \in [U_{\alpha+1}]$$

for every $\phi(x, \bar{a}) \in p_\alpha$.

When we complete the construction, the ultrafilter $\mathcal{U} = [\bigcup_{\alpha\in 2^{\aleph_0}} U_\alpha]$ has the properties required. For $\alpha = 0$, put $U_0 = \{I_n \mid n \in \omega\}$, where $I_n = \omega \setminus \{1, \dots, n\}$. The construction is clear when α is limit or α is odd. Hence we can assume that α is even and $(p_\alpha, (\mathcal{M}_i^\alpha \mid i \in \omega))$ satisfies the hypothesis of (iv). Define a partial order $(P, \supseteq)$, where $\supseteq$ is reverse inclusion. An element of P is a finite set p of equations $x(i) = a$, where $i \in \omega$ and $a \in P^{\mathcal{M}_i}$, such that $x(i) = a, x(i) = a' \in p$ imply $a = a'$. Since $P^{\mathcal{M}_i}$ is countable, the partial order P is countable, hence it has the countable chain condition. Put $p'_\alpha = \{\bigwedge q \mid q \subseteq p_\alpha \text{ is finite }\}$. For every $\phi(x, \bar{a}) \in p'_\alpha$ and finite intersection S of members of U_α, define the set $D(S, \phi(x, \bar{a}))$

$$\{p \in P \mid \mathcal{M}_i^\alpha \models \phi(b, \bar{a}(i)) \text{ for some } i \in S \text{ and } b \in P^{\mathcal{M}_i} \text{ such that } \{x(i) = b\} \subseteq p\}.$$

It can be easily shown that each $D(S, \phi(x, \bar{a}))$ is open dense in P, since the premise in condition (iv) holds for the type p_α. The cardinality of this family of dense open subsets of P is less than $2^{\aleph_0}$, hence there exists a generic filter $G \subseteq P$ such that $G \cap D(S, \phi(x, \bar{a})) \neq \emptyset$, for every $\phi(x, \bar{a}) \in p'_\alpha$ and finite intersection S of members of U_α. Put $b(i) = a \in \mathcal{M}_i$ if and only if $x(i) = a \in \bigcup\{p \mid p \in G\}$. Note that $(b(i))_{i\in\omega}$ is an element of $\prod_{i\in\omega} M_i^\alpha$, in fact if $x(i) = a \in v \in G$ and $x(i) = a' \in w \in G$, then v, w are compatible, hence $a = a'$. Now we conclude putting

$$U_{\alpha+1} = U_\alpha \bigcup \{\{i \in \omega \mid \mathcal{M}_i^\alpha \models \phi(b(i), \bar{a}(i))\} \mid \phi(x, \bar{a}) \in p_\alpha\}.$$

We must show that $U_{\alpha+1}$ has the finite intersection property: for any $T \in [U_\alpha]$ and $\phi(x, \bar{a}) \in p'_\alpha$, we can first find S finite subset of U_α contained in T and then a $p \in G \cap D(S, \phi(x, \bar{a}))$, such that for some $i \in S$ $x_i = b(i) \in p$ and $\mathcal{M}_i \models \phi(b(i), \bar{a}(i))$. This gives that for all $\phi_1(x, \bar{a}_1), \ldots, \phi_k(x, \bar{a}_k) \in p_\alpha$,

$$\{i \in \omega \mid \mathcal{M}_i^\alpha \models \phi(b(i), \bar{a}_1(i)) \wedge \cdots \wedge \phi_k(x, \bar{a}_k(i)))\}$$

has non empty intersection with all members of $[U_\alpha]$, thus $U_{\alpha+1}$ still has the finite intersection property, as it is required to carry on the induction anc conclude the proof of the Lemma. □

Theorem 5.2.7 (Theorem 3.10 ii) Chapter VI [9]). *Assume that Martin's axiom holds and $2^{\aleph_0} = \aleph_2$. There exists a regular ultrafilter $\mathcal{F}$ on $\aleph_1$, such that for any model $\mathcal{M}$ of T_{rg} the model $\mathcal{M}^{\omega_1}/\mathcal{F}$ is $\aleph_2$-saturated, but $\mathcal{F}$ is not $\aleph_2$-good. Hence it is consistent with ZFC, that the theory T_{rg} is not maximal in Keisler's order.*

Proof. Let $\mathcal{V}$ be a $\aleph_2$-good countably incomplete ultrafilter on $\aleph_1$. By Lemma 3.1.8, $\mathcal{V}$ is regular. Let $\mathcal{U}$ be the regular ultrafilter on ω given by Theorem 5.2.4. The ultrafilter $\mathcal{V} \otimes \mathcal{U}$ is regular on $\aleph_1 \times \omega$ by Proposition 5.1.3. Let $\mathcal{M}$ be a model of T_{rg}. Since $\mathcal{M}^{\aleph_1}/\mathcal{V}$ is $\aleph_2$-saturated by Theorem 3.1.26, then $\mathcal{M}^{\aleph_1 \times \omega}/\mathcal{V} \otimes \mathcal{U} \cong (\mathcal{M}^{\aleph_1}/\mathcal{V})^\omega/\mathcal{U}$ is $\aleph_2$-saturated, by Theorem 5.2.4. We prove that $\mathcal{V} \otimes \mathcal{U}$ is not $\aleph_2$-good. Assume for a contradiction that this holds, then $\mathcal{U}$ is a regular $\aleph_2$-good ultrafilter on ω, by Proposition 5.1.3(v). By Lemma 3.1.8, the ultrafilter $\mathcal{U}$ is $\aleph_1$-regular. We obtain a contradiction by Lemma 3.1.5. Noting that $|\aleph_1 \times \omega| = \aleph_1$ we conclude the proof. □

Appendix A

Set theory

In this Appendix we prove some results of set theory, that we use in Chapter 2 and 5. Since many results of this appendix are without proof, we refer the reader to Kunen's book [4] for a complete treatment.

In Section A.1, we give a short introduction to forcing. Under the assumptions of the existence of a transitive countable model $\mathcal{M} \in V$ of ZFC, we define the class of $\mathbb{P}$-*names*, the *relation of forcing* and remark some classical theorems. Then we define the λ-*closed* notion of forcing and we prove that this combinatorial property is sufficient for the model $V[G]$ to preserve each cardinal $\kappa \leq \lambda$, whenever G is V-generic for some $\mathbb{P}$ which is λ-closed. We conclude the section explaining how all proven results of Chapter 2, can be interpreted without the hypothesis of the existence of a transitive countable model $\mathcal{M} \in V$ of ZFC.

In Section A.2, we work under the assumptions that Martin's axiom holds and we show how this statement influences cardinal exponentiation. In particular, we prove that $2^\kappa = 2^{\aleph_0}$ for every $\kappa < 2^{\aleph_0}$.

A.1 Forcing

We assume that $\mathcal{M} \in V$ is a transitive countable model of ZFC.

Definition A.1.1. A set $(\mathbb{P}, \leq, \mathbb{1}_\mathbb{P})$ of $\mathcal{M}$ is a *notion of forcing*, if $(\mathbb{P}, \leq)$ is a preorder of $\mathcal{M}$ and $\mathbb{1}_\mathbb{P}$ is the maximum element of $\mathbb{P}$. A subset D of $\mathbb{P}$ is *dense*, if for each $p \in \mathbb{P}$, there is $q \in D$ such that $q \leq p$. A subset D of $\mathbb{P}$ is *dense below* $p \in \mathbb{P}$, if D is dense in the notion of forcing $\{q \in \mathbb{P} \mid q \leq p\}$. Two element p, q of P are *compatible*, if there exists $r \in P$ such that $r \leq p, q$, they are incompatible otherwise. A subset A of P is an *antichain*, if their elements are pairwise incompatible.

Definition A.1.2. Let $(\mathbb{P}, \leq, \mathbb{1}_\mathbb{P})$ be a notion of forcing in $\mathcal{M}$. A set $G \subseteq \mathbb{P}$ is a *filter* if the following hold:

(i) if $p, q \in G$, then there exists $r \in G$ such that $r \leq p, q$.

(ii) If $p \in G$ and $p \leq q$, then $q \in G$.

A filter G is $\mathcal{M}$-*generic* over $\mathbb{P}$, if G is a filter that meets every dense subset of $\mathbb{P}$, that is if $D \in \mathcal{M}$ is dense in $\mathbb{P}$, then $G \cap D \neq \emptyset$.

The hypothesis that $\mathcal{M}$ is countable ensures that for each notion of forcing $\mathbb{P}$ there exists a generic filter over $\mathbb{P}$, in fact the following holds:

Lemma A.1.3. *Assume that $\mathcal{M}$ is a transitive countable model of ZFC and $(\mathbb{P}, \leq, \mathbb{1}_{\mathbb{P}})$ is a notion of forcing in $\mathcal{M}$. If $p \in \mathbb{P}$, then there exists an $\mathcal{M}$-generic filter G over $\mathbb{P}$ such that $p \in G$.*

Proof. Working in V, let $\{D_n \mid n \in \omega\}$ be an enumeration of all dense subsets of $\mathbb{P}$. It easy to construct a set $\bar{G} = \{p_n \mid n \in \omega\}$ such that $p_0 = p$, $p_{n+1} \in D_n$ and $p_{n+1} \leq p_n$ for all $n \in \omega$. Let

$$G = \{q \mid \exists n \in \omega \; p_n \leq q\},$$

then G is a filter and meets every D_n. □

Lemma A.1.4. *Assume that $\mathcal{M}$ is a transitive countable model of ZFC with $\mathbb{P} \in \mathcal{M}$. Let G be an $\mathcal{M}$-generic filter over $\mathbb{P}$. Assume that the set $D \in \mathcal{M}$ is dense below some $p \in G$, then $G \cap D \neq \emptyset$.*

Proof. Consider the set

$$\tilde{D} = \{q \in \mathbb{P} \mid \exists r \in D(q \leq r)\} \cup \{q \in \mathbb{P} \mid \forall r \in D(r, q \text{ are incompatible})\}.$$

Notice that $\tilde{D} \in \mathcal{M}$. We prove that $\tilde{D}$ is dense. Assume that $q \notin \tilde{D}$, then there exists $r \in D$ such that q, r are compatible, that is there is $s \leq q, r$. We conclude $s \in \tilde{D}$. Assume for a contradiction that $G \cap D = \emptyset$, hence

$$G \cap \{q \in \mathbb{P} \mid \exists r \in D(q \leq r)\} = \emptyset.$$

Since G is $\mathcal{M}$-generic, there exists $q \in G$ such that q, r are incompatible for all $r \in D$. Since $p, q \in G$, there is $s \in G$ with $s \leq p, q$, then we can find $t \in D$ with $t \leq s$, since D is dense below p. We obtain that t, q are compatible, contradiction. □

The next Lemma give a sufficient condition so that every generic filter G is not in $\mathcal{M}$.

Definition A.1.5. A partial order $\mathbb{P}$ is *separative*, if for for every $p \in \mathbb{P}$ there exist two incompatible elements $q, r \in \mathbb{P}$ such that $q, r \leq p$

Lemma A.1.6. *Assume that $\mathcal{M}$ is a transitive countable model of ZFC with $\mathbb{P} \in \mathcal{M}$. Let $G \in V$ be an $\mathcal{M}$-generic filter for $\mathbb{P}$. If $\mathbb{P}$ is separative, then $G \notin \mathcal{M}$.*

Proof. Assume for a contradiction that $G \in \mathcal{M}$, then the set $\mathcal{M} \setminus G \in \mathcal{M}$ is dense in $\mathbb{P}$. In fact, if $p \in \mathbb{P}$, then there exists $q, r \leq p$ such that q, r are incompatible. If $q, r \in G$, we obtain a contradiction since G is a filter, hence $q \notin G$ or $r \notin G$. We conclude that $\mathcal{M} \setminus G$ is dense in $\mathbb{P}$, hence $G \cap (\mathcal{M} \setminus G) \neq \emptyset$, contradiction. □

From now on, we shall study only separative notion of forcing, hence the $\mathcal{M}$-generic filters do not belong to $\mathcal{M}$.

Definition A.1.7 (Definition 2.5 Chapter VII [4])**.** In a model $\mathcal{M}$ of ZFC such that $\mathbb{P} \in \mathcal{M}$, we can define with parameter $\mathbb{P}$ the characteristic function $H(\mathbb{P}, \tau)$ of a $\mathbb{P}$-name τ in such a way that

$$H(\mathbb{P}, \tau) = 1 \iff \tau \text{ is a binary relation and } \forall (\sigma, p) \in \tau \, [H(\mathbb{P}, \sigma) = 1 \wedge p \in \mathbb{P}]$$

and $H(\mathbb{P}, \tau) = 0$ otherwise. In $\mathcal{M}$, the proper class

$$\mathcal{M}^{\mathbb{P}} = \{\tau \in \mathcal{M} \mid \mathcal{M} \models H(\mathbb{P}, \tau) = 1\}$$

is called the *class of* $\mathbb{P}$*-names.*

Remark A.1.8. The class $\mathcal{M}^{\mathbb{P}}$ is Δ_1-definable in the parameter $\mathbb{P}$, hence it is absolute in each transitive models of ZFC, that is in our initial assumptions we have $\mathcal{M}^{\mathbb{P}} = V^{\mathbb{P}} \cap \mathcal{M}$.

Definition A.1.9. Let $\mathcal{M} \in V$ be transitive models of ZFC to which $\mathbb{P}$ belongs and $G \in V$ be an $\mathcal{M}$-generic filter on $\mathbb{P}$. In V we can define an absolute class function

$$F \colon \mathcal{M}^{\mathbb{P}} \to \mathcal{M}[G]$$

such that

$$F(\tau) = \{F(\pi) \mid (\pi, p) \in \tau \text{ and } p \in G\}.$$

Note that F is definable with parameters $\mathcal{M}$, $\mathbb{P}$ and G, hence this function is not defined in $\mathcal{M}$, if G does not belong to $\mathcal{M}$.

Definition A.1.10. Given $a \in \mathcal{M}[G]$, there exists a $\mathbb{P}$-names τ such that $F(\tau) = a$ and we denote by $\dot{a}$ this $\mathbb{P}$-names. Given $a \in \mathcal{M}$, we call $\check{a} = \{(\check{b}, \mathbb{1}_{\mathbb{P}}) \mid b \in a\}$ the *canonical name* of a. For an $\mathcal{M}$-generic filter G on $\mathbb{P}$, the $\mathbb{P}$-name $\Gamma = \{(\check{p}, p) \mid p \in \mathbb{P}\}$ is the *canonical name* of G in $\mathcal{M}[G]$.

Remark A.1.11. By definition of Γ, follows that $F(\Gamma) = G \in \mathcal{M}[G]$, if G is a $\mathcal{M}$-generic filter on $\mathbb{P}$.

The next result guarantees that $\mathcal{M}[G]$ is a model of ZFC, that extends $\mathcal{M}$.

Theorem A.1.12 (Theorem 4.2 Chapter VII [4])**.** *Assume that $\mathcal{M}$ is a transitive model of ZFC and $(\mathbb{P}, <, \mathbb{1}_{\mathbb{P}})$ is a notion forcing in $\mathcal{M}$. If G is an $\mathcal{M}$-generic filter over $\mathbb{P}$, then*

1. *$\mathcal{M} \subseteq \mathcal{M}[G]$ and $G \in \mathcal{M}[G]$.*

2. *$\mathcal{M}[G]$ is a transitive model of ZFC.*

Definition A.1.13. Let $(\mathbb{P}, <, \mathbb{1}_{\mathbb{P}})$ be a notion of forcing in $\mathcal{M}$. The condition $p \in P$ *forces* the sentence $\psi(\dot{a}_1, \ldots, \dot{a}_n)$ if and only if

$$\mathcal{M}[G] \models \psi(a_1, \ldots, a_n)$$

for every $\mathcal{M}$-generic filter G over $\mathbb{P}$ such that $p \in G$. In this case we write

$$p \Vdash_{\mathbb{P}} \psi(\dot{a}_1, \ldots, \dot{a}_n).$$

Note that this definition is external to $\mathcal{M}$, since G is not in $\mathcal{M}$, if G is $\mathcal{M}$-generic. When the notion of forcing $\mathbb{P}$ is clear from the context, we write $p \Vdash \psi(\dot{a}_1, \ldots, \dot{a}_n)$ in place of $p \Vdash_{\mathbb{P}} \psi(\dot{a}_1, \ldots, \dot{a}_n)$.

Now let $\mathcal{M}$ be a model of ZFC that contains a notion of forcing $(\mathbb{P}, \leq, \mathbb{1})$. For $p \in \mathbb{P}$ and $\dot{a}_1, \ldots, \dot{a}_n \in \mathcal{M}^{\mathbb{P}}$ we can define in $\mathcal{M}$ an *internal relation of forcing* $p \Vdash^*_{\mathbb{P}} \psi(\dot{a}_1, \ldots, \dot{a}_n)$, see Definition 3.3 of Chapter VII [4]. Also in this case we write $p \Vdash^* \psi(\dot{a}_1, \ldots, \dot{a}_n)$, when $(\mathbb{P}, \leq, \mathbb{1})$ is clear from the context. The following theorems are classical results, that outline the link between the semantics of $\mathcal{M}[G]$ and the internal relation of forcing $\Vdash^*$.

Theorem A.1.14 (Forcing Theorem, Theorem 3.6 Chapter VII [4])**.** *Let $(\mathbb{P}, <, \mathbb{1}_{\mathbb{P}})$ be a notion of forcing in $\mathcal{M}$ and G be an $\mathcal{M}$-generic filter over $\mathbb{P}$. If $\mathcal{M}[G] \models \psi(a_1, \ldots, a_n)$, then there exists some $p \in G$, such that $\mathcal{M} \models p \Vdash^* \psi(\dot{a}_1, \ldots, \dot{a}_n)$. Vice versa, if $p \in G$ and $\mathcal{M} \models p \Vdash^* \psi(\dot{a}_1, \ldots, \dot{a}_n)$, then $\mathcal{M}[G] \models \psi(a_1, \ldots, a_n)$.*

Theorem A.1.15 (Theorem 3.6 Chapter VII [4])**.** *Let $\mathcal{M} \in V$ be a countable transitive model of ZFC and $\mathbb{P} \in \mathcal{M}$ be a notion of forcing.*

(i) For every $p \in \mathbb{P}$ we have

$$p \Vdash \psi(\dot{a}_1, \ldots, \dot{a}_n) \Longleftrightarrow \mathcal{M} \models p \Vdash^* \psi(\dot{a}_1, \ldots, \dot{a}_n).$$

(ii) For every $\mathcal{M}$-generic filter G over $\mathbb{P}$ we have

$$\mathcal{M}[G] \models \psi(a_1, \ldots, a_n) \Longleftrightarrow \exists p \in G \; p \Vdash \psi(\dot{a}_1, \ldots, \dot{a}_n).$$

Definition A.1.16. Let λ be an infinite cardinal. A notion of forcing $(\mathbb{P}, <, \mathbb{1}_{\mathbb{P}})$ is λ-*closed*, if for every $\kappa < \lambda$ and every decreasing sequence $(p_\gamma)_{\gamma \in \kappa}$ in P, there exists $p \in P$ such that $p < p_\gamma$ for every $\gamma \in \kappa$.

Theorem A.1.17. *Assume that $(\mathbb{P}, <, \mathbb{1}_{\mathbb{P}})$ is a λ-closed notion of forcing in $\mathcal{M}$ and G is an $\mathcal{M}$-generic filter over $(P, <)$. Let α, β be ordinals such that $|\alpha| < \lambda$. If we have*

$$\mathcal{M}[G] \models f \colon \alpha \to \beta \textit{ is a function},$$

then $f \in M$.

Proof. Assume for a contradiction that

$$\mathcal{M}[G] \models f \notin \beta^\alpha \cap \mathcal{M}.$$

Let $\dot{f}$ and τ be $\mathbb{P}$-names for f and $\beta^\alpha \cap \mathcal{M}$, respectively. By Forcing Theorem A.1.14, there exists $p \in G$ such that

$$p \Vdash \dot{f} \text{ is a function from } \check{\alpha} \text{ to } \check{\beta} \text{ and } \dot{f} \notin \tau.$$

We construct inductively a sequence $(p_\gamma)_{\gamma \leq \alpha}$ in $\mathcal{M}$ such that for all $\eta \leq \gamma \in \alpha$ we have $p_\gamma \leq p_\eta$ and

$$p_{\gamma+1} \Vdash \dot{f}(\check{\gamma}) = \check{\beta}_\gamma,$$

for some $\beta_\gamma \in \beta$. Put $p_0 = p$. In the successor step, we have $p_\gamma \leq p$, hence

$$p_\gamma \Vdash \dot{f} \text{ is a function from } \check{\alpha} \text{ to } \check{\beta}.$$

Now there exists $p_{\gamma+1} \leq p_\gamma$ such that

$$p_{\gamma+1} \Vdash \dot{f}(\check{\gamma}) = \check{\beta}_\gamma.$$

In the limit step, the sequence $(p_\eta)_{\eta \in \gamma}$ is defined in $\mathcal{M}$. By hypothesis

$$\mathcal{M} \models \mathbb{P} \text{ is } \lambda\text{-closed}$$

hence there exists p_γ such that $p_\gamma \leq p_\eta$ for all $\eta \in \gamma$. When the construction is complete, we can define a map $g \colon \alpha \to \beta$ in $\mathcal{M}$ such that $g(\gamma) = \beta_\gamma$. Now let H be an $\mathcal{M}$-generic filter over $\mathbb{P}$ such that $p_\alpha \in H$. Note that $p_\gamma \in H$ for all $\gamma \in \alpha$. We have

$$\mathcal{M}[G] \models f \text{ is a function from } \alpha \text{ to } \beta,\ f \notin \beta^\alpha \cap \mathcal{M}\ ,$$

$g \in \mathcal{M}$ and $\mathcal{M}[G] \models f = g$, contradiction. □

Corollary A.1.18. *Assume that $(\mathbb{P}, <, \mathbb{1}_\mathbb{P})$ is a notion of forcing λ-closed in $\mathcal{M}$ and G is an $\mathcal{M}$-generic filter over $(\mathbb{P}, <)$.*

(i) If $\beta \leq \lambda$ is a regular cardinal in $\mathcal{M}$, then

$$\mathcal{M}[G] \models \beta \textit{ is regular.}$$

(ii) $\mathcal{M}[G]$ preserves cardinals $\alpha \leq \lambda$, that is

$$\mathcal{M} \models \alpha \textit{ is a cardinal} \Longleftrightarrow \mathcal{M}[G] \models \alpha \textit{ is a cardinal.}$$

Proof. (i) Assume for a contradiction that

$$\mathcal{M}[G] \models \beta \text{ is not regular,}$$

hence

$$\mathcal{M}[G] \models \text{for some } \alpha < \beta \text{ there exists a cofinal map } f\colon \alpha \to \beta.$$

By Theorem A.1.17, we conclude that $f \in \mathcal{M}$, hence

$$\mathcal{M} \models \beta \text{ is not regular,}$$

contradiction.

(ii) Note that the only non-trivial direction is from left to right. In fact, the formula

$$\psi(x) = \text{"}x \text{ is a cardinal"}$$

is Π_1 and $\mathcal{M} \subseteq \mathcal{M}[G]$, hence

$$\mathcal{M}[G] \models \psi(x) \Longrightarrow \mathcal{M} \models \psi(x).$$

For the other direction, assume that

$$\mathcal{M} \models \alpha \text{ is a regular cardinal.}$$

By (i), we have

$$\mathcal{M}[G] \models \alpha = cf(\alpha),$$

hence we conclude

$$\mathcal{M}[G] \models \alpha \text{ is a regular cardinal.}$$

If

$$\mathcal{M} \models \alpha \text{ is a singular cardinal,}$$

then α is limit of regular cardinals in $\mathcal{M}$ and these cardinals remain regular in $\mathcal{M}[G]$. Hence we conclude that

$$\mathcal{M}[G] \models \alpha \text{ is a cardinal.}$$

□

We have given a short introduction to forcing, but in Chapter 2 we use another approach to the method of forcing. In particular, we work in a transitive model V of ZFC, that contains a notion of forcing $(\mathbb{P}, \leq, \mathbb{1})$, and we prove that the generic extension $V[G]$ satisfies a sentence ψ for each V-generic filter G over $\mathbb{P}$. This approach is more simple compared to what we used here, in fact it avoids to relativize every sentence in a transitive countable model $\mathcal{M} \in V$, but it is not clear the meaning of $V[G]$, because we need a larger model $\mathcal{N}$ than V, in such a way that $G \in \mathcal{N}$. Now we explain the idea behind the proofs of Chapter 2 and the notation $V[G]$.

Without loss of generality we can assume that V has an inaccessible cardinal θ, such that the transitive set

$$H_\theta = \{x \in V \mid |\mathrm{TC}(x)| < \theta\}$$

contains $\mathbb{P}$. Since θ is inaccessible, H_θ is a transitive model of ZFC, see Theorem 6.6 Chapter IV [4] for a proof. Working in V we can use Löwenheim-Skolem's Theorem to find a countable set $\mathcal{M}$ such that $\mathcal{M} \preccurlyeq H_\theta$ and $\mathbb{P} \in \mathcal{M}$. But this set may not be transitive, hence we need the following classical result of set theory.

Theorem A.1.19 (Mostowski collapse, Theorem 5.13 [4])**.** *Let $R \subseteq A \times A$ be an extensional well-founded relation on the set A, that is*

$$\forall x, y \in A\,(\forall z\,[zRx \leftrightarrow zRy] \to x = y)\,,$$

$$\forall x \subseteq A\,(x \neq \emptyset \to \exists y \in x\,[\neg\exists z \in x(zRy)])\,.$$

Then there exists a unique function π such that:

(i) $dom(\pi) \subseteq A$ and $\pi[A]$ is a transitive set.

(ii) π is a isomorphism between the structures (A, R) and $(\pi(A), \in)$.

Moreover, the function π is defined recursively as follows:

$$\pi(x) = \{\pi(y) \mid y \in A,\ yRx\}$$

and the structure $(\pi[A], \in)$ is called the Mostowski collapse *of (A, R).*

Since $(\mathcal{M}, \in)$ is a models of ZFC, the Mostowski's map π is an isomorphism, hence the Mostowski collapse $(\pi[\mathcal{M}], \in)$ is a transitive model of ZFC and $\pi[\mathcal{M}] \in V$. Now, repeating every proof of Chapter 2 in $\pi[M]$, we obtain that

$$\pi[\mathcal{M}] \models \mathbb{1}_{\pi[\mathbb{P}]} \Vdash^*_{\pi[\mathbb{P}]} \psi(\pi(\dot{a}_1), \ldots, \pi(\dot{a}_n)).$$

Since π is an isomorphism, we have

$$\mathcal{M} \models \mathbb{1}_{\mathbb{P}} \Vdash^*_{\mathbb{P}} \psi(\dot{a}_1, \ldots, \dot{a}_n),$$

whence

$$H_\theta \models \mathbb{1}_{\mathbb{P}} \Vdash^*_{\mathbb{P}} \psi(\dot{a}_1, \ldots, \dot{a}_n).$$

A.2 Martin's axiom

Definition A.2.1. A partial order $(P, <)$ has the *countable chain condition*, abbreviated as *c.c.c.*, if every every antichain has cardinality at most countable.

Definition A.2.2. *Martin's axiom* is the statement:
for every c.c.c. partial order $(P, \leq)$, if $\mathcal{D}$ is family of dense subsets of P of cardinality less than $2^{\aleph_0}$, then there exists a filter G that meets every $D \in \mathcal{D}$.

Remark A.2.3. Clearly Martin's axiom is a theorem of $ZFC + GCH$. In fact, if $\mathcal{D}$ is a countable family of dense sets of $\mathbb{P}$, as in the proof of Lemma A.1.3, we obtain the desired $\mathcal{D}$-generic filter.

Using iterated forcing we can obtain the consistency of Martin's axiom + ¬CH, see [12] or [7].

Theorem A.2.4 (Solovay and Tennenbaum, Theorem 16.13 [12])**.** *Let $\kappa > \aleph_1$ be a regular cardinal in V. There exists a generic extension $V[G]$ that satisfies MA and $2^{\aleph_0} = \kappa$.*

Martin's axiom influences cardinal exponentiation:

Theorem A.2.5 (Martin and Solovay, Theorem 16.20 [12])**.** *If Martin's axiom holds, then $2^\kappa = 2^{\aleph_0}$ for every $\kappa < 2^{\aleph_0}$.*

Proof. Fix $\kappa < 2^{\aleph_0}$. Since $\aleph_0 \leq \kappa$, we have $2^{\aleph_0} \leq 2^\kappa$. Hence it is sufficient to find a surjective function from $\mathcal{P}(\omega)$ to $\mathcal{P}(\kappa)$. First of all, we need the following claim.

Claim A.2.6. *There exists a family $\{A_\alpha \subseteq \omega \mid \alpha \in \kappa\}$ such that $|A_\alpha| = \aleph_0$ and $|A_\alpha \cap A_\beta| \in \omega$ for all $\alpha < \beta \in \kappa$.*

Proof. We shall prove a stronger thesis, that is the existence of a family of cardinality $2^{\aleph_0}$ with the above properties. Let P be the set of all finite functions $p\colon \omega \to 2$ with domain an initial segment of ω. Since $|P| = \aleph_0$, it is sufficient to prove the lemma for P. For $f \in 2^{\aleph_0}$, consider $A_f = \{p \in S \mid p \subseteq f\}$. Note that A_f is always infinite and $|A_f \cap A_g| \in \omega$, for every $f \neq g$, hence the family $\{A_f \mid f \in 2^{\aleph_0}\}$ has the required properties. □

Choose a family $\{A_\alpha \subseteq \omega \mid \alpha \in \kappa\}$ like above. Put

$$G\colon \mathcal{P}(\omega) \to \mathcal{P}(\kappa)$$

such that $G(A) = \{\alpha \in \kappa \mid A \cap A_\alpha \text{ is infinite}\}$. In order to prove that f is surjective, fix $X \subseteq \kappa$. Consider the partial order $\mathbb{P}$ of all function $p\colon \omega \to 2$, ordered by reverse inclusion, with the following properties:

(i) $\mathrm{dom}(p) \cap A_\alpha$ is finite for all $\alpha \in X \subseteq \kappa$.

(ii) the set $\{n \in \mathrm{dom}(p) \mid p(n) = 1\}$ is finite.

We prove that $(\mathbb{P}, \supseteq)$ has the c.c.c. Assume for a contradiction that $\mathcal{A} = \{p_\alpha \mid i \in I\}$ is an antichain. For $i \neq j$, we have

$$\{n \in \mathrm{dom}(p_i) \mid p_i(n) = 1\} \neq \{n \in \mathrm{dom}(p_j) \mid p_j(n) = 1\},$$

otherwise $p_i \cup p_j$ is a function that extends p_i and p_j. Now note that the number of the above sets is at most countable, by condition (ii), hence I is countable. For $\alpha \in \kappa \setminus X$, the set

$$D_\alpha = \{p \in \mathbb{P} \mid A_\alpha \subseteq \mathrm{dom}(p)\}$$

is dense, in fact if p is such that $A_\alpha \not\subseteq \mathrm{dom}(p)$, then we can extend p to 0 on $\mathrm{dom}(p) \setminus A_\alpha$. For $\alpha \in X$ and $n \in \omega$, the set

$$D_{\alpha,n} = \{p \in \mathbb{P} \mid |\{n \in A_\alpha \mid p(n) = 1\}| \geq n \}$$

is dense. In fact, if $|\{n \in A_\alpha \mid p(n) = 1\}| < n$, we can find an extension $q \in D_{\alpha,n}$ of p, since A_α is infinite and $\mathrm{dom}(p) \cap A_\alpha$ is finite, by clause (i). The family

$$\mathcal{D} = \{D_\alpha \mid \alpha \in \kappa \setminus X\} \cup \{D_{\alpha,n} \mid \alpha \in X,\ n \in \omega\}$$

has size less than $2^{\aleph_0}$, hence there exists a filter G that meets every dense of $\mathcal{D}$. Since G is a filter, $f = \bigcup G$ is a function from ω to 2. Put

$$A = \{n \in \mathrm{dom}(f) \mid f(n) = 1\}.$$

We conclude the proof showing that $G(A) = X$, that is

$$\alpha \in X \iff |A_\alpha \cap A| = \aleph_0.$$

If $\alpha \in X$, then, for every $n \in \omega$, there exists $p \in G \cap D_{\alpha,n}$, hence $|A_\alpha \cap A| \geq n$. If $\alpha \notin X$, then there exists $p \in G \cap D_\alpha$. We conclude that $A_\alpha \subseteq \mathrm{dom}(p)$ and

$$\{n \in \mathrm{dom}(p) \mid p(n) = 1\}$$

is finite, hence $A_\alpha \cap A$ is finite. □

Corollary A.2.7. *If Martin's axiom holds, then $2^{\aleph_0}$ is regular.*

Proof. Assume for a contradiction that $cof(2^{\aleph_0}) = \kappa$ for some $\kappa \in 2^{\aleph_0}$. Since for every cardinal λ we have $\lambda^{cof(\lambda)} > \lambda$, we obtain

$$2^{\aleph_0} < (2^{\aleph_0})^\kappa = 2^{\aleph_0 \cdot \kappa} = 2^\kappa = 2^{\aleph_0},$$

contradiction. □

To use Martin's axiom, we need that certain partial orders of finite functions from an infinite cardinal to a finite one has the c.c.c. To prove this, we use the following combinatorial lemma, see [4] for a proof.

Lemma A.2.8 (Δ-system lemma, Theorem 1.6 Chapter II [4])**.** *Let λ be a regular uncountable cardinal and*

$$\mathcal{F} = \{a_\alpha \mid \alpha \in \lambda\}$$

be a subset of $[\lambda]^{<\aleph_0}$ of cardinality λ. There exist a family $\mathcal{F}' \subseteq \mathcal{F}$ and $r \in [\lambda]^{<\aleph_0}$ such that $\mathcal{F}'$ has cardinality λ and $a_\alpha \cap a_\beta = r$ for all $a_\alpha, a_\beta \in \mathcal{F}'$.

Notation. *Let I, J two sets. We denote by $Fn(I, J)$ the set of all finite partial functions p with domain in I and range in J. On $Fn(I, J)$, we consider the following partial order $\leq$:*

$$p \leq q \iff p \supseteq q.$$

Lemma A.2.9. *Fix an infinite cardinal λ. The partial order $(Fn(\omega \times \lambda, 2), \leq)$ has the countable chain condition.*

Proof. Assume for a contradiction that $\{p_\alpha \mid \alpha \in \aleph_1\}$ is an uncountable antichain in P. Since $\mathcal{A} = \bigcup_{n \in \omega} \{p \mid |p| = n\}$ and $\aleph_1$ is regular, we can assume that every element of $\mathcal{A}$ has cardinality n. Put $\mathcal{F} = \{a_\alpha \mid a_\alpha = \mathrm{dom}(p_\alpha)\}$. Assume that $\mathcal{F}$ is countable. We conclude that there are two compatible functions in $\mathcal{A}$, since the set $\mathrm{Fn}(\omega, 2)$ is countable. Now assume that $\mathcal{F}$ is uncountable. By Δ-system Lemma A.2.8, there exist $\mathcal{F}' \subseteq \mathcal{F}$ of cardinality $\aleph_1$ and a finite $r \subseteq \omega \times \lambda$, such that $a_\alpha \cap a_\beta = r$ for all $a_\alpha, a_\beta \in \mathcal{F}'$. Note that the function from r to 2 are exactly $2^{|r|}$, hence for some $a_\alpha, a_\beta \in \mathcal{F}'$, the function p_α, p_β are compatible, contradiction. $\square$

Appendix B

Model theory

In this Appendix, we give a brief introduction to model theory, recalling what we use in this thesis.

In Section B.1, we recall, without proofs, some basic results of model theory, such as the Compactness Theorem and Łoś's Theorem.

In Section B.2, we prove that the theory of discrete linear orders with minimum element and without maximum has quantifier elimination in the language $\mathcal{L} = \{0, s, <\}$.

B.1 Some basic results

In this Appendix, we assume that the reader has familiarity with some basic concepts of model theory, such as the notion of first order language, $\mathcal{L}$-theory and $\mathcal{L}$-structure. We refer the reader to Marker's book [6] for a complete treatment of these arguments.

We begin with a classical result of model theory.

Theorem B.1.1 (Compactness Theorem, Theorem 2.1.4 [6]). *A theory T is satisfiable if and only if every finite $T' \subseteq T$ is satisfiable.*

Definition B.1.2. Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{L}$-structures with universes M, N, respectively. We say that $\mathcal{N}$ is an *elementary substructure* of $\mathcal{M}$ and we write $\mathcal{N} \preccurlyeq \mathcal{M}$, if $N \subseteq M$ and for every formula $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in N$ we have

$$\mathcal{N} \models \phi(a_1, \ldots, a_n) \Longleftrightarrow \mathcal{M} \models \phi(a_1, \ldots, a_n).$$

Corollary B.1.3. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $p(x)$ be a finitely satisfiable type in $\mathcal{M}$. Then there exists an $\mathcal{L}$-structure $\mathcal{N}$ such that $\mathcal{M} \preccurlyeq \mathcal{N}$ and $\mathcal{N}$ realizes $p(x)$.*

Theorem B.1.4 (Löwenheim-Skolem, Theorem 2.3.7 [6]). *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and A be a subset of $\mathcal{M}$. There exists an $\mathcal{L}$-structure $\mathcal{N}$ such that $\mathcal{N} \preccurlyeq \mathcal{M}$, $A \subseteq \mathcal{N}$ and $|N| \leq |\mathcal{L}| + |A| + \aleph_0$.*

In Chapter 4, we study a preorder on a class of special theories, called *complete*.

Definition B.1.5. Let T be a $\mathcal{L}$-theory and ϕ be a sentence of $\mathcal{L}$. We write $T \models \phi$ to indicate that every model of T satisfies ϕ.

Definition B.1.6. A theory T is complete, if for every sentence ϕ either $T \models \phi$ or $T \models \neg\phi$.

Remark B.1.7. For every $\mathcal{L}$-structure $\mathcal{M}$, the theory $Th(\mathcal{M}) = \{\phi \mid \mathcal{M} \models \phi\}$ is complete.

Definition B.1.8. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* and we write $\mathcal{M} \equiv \mathcal{N}$, if for every sentence ϕ we have

$$\mathcal{M} \models \phi \Longleftrightarrow \mathcal{N} \models \phi.$$

Lemma B.1.9. *For an $\mathcal{L}$-theory T the following are equivalent:*

(i) T is complete.

(ii) Every two models of T are elementarily equivalent.

Proof.

$(i) \Rightarrow (ii)$ Let ϕ be a sentence in the language $\mathcal{L}$ and $\mathcal{M}$, $\mathcal{N}$ be two models of T. Assume that $\mathcal{M}$ satisfies ϕ. We obtain that $\mathcal{N} \models \phi$, since T is complete. In a similar way we conclude that $\mathcal{N} \models \neg\phi$, if $\mathcal{M}$ satisfies $\neg\phi$.

$(ii) \Rightarrow (i)$ Assume that T is not complete, hence there exist two models $\mathcal{M}, \mathcal{N}$ of T and a sentence ϕ such that $\mathcal{M} \models \phi$ and $\mathcal{N} \models \neg\phi$. We conclude that $\mathcal{M} \not\equiv \mathcal{N}$.

□

Definition B.1.10. Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{L}$-structures with universes M, N, respectively. A *morphism* $F\colon \mathcal{M} \to \mathcal{N}$ is a map $F\colon M \to N$ with the following properties:

(i) $F(c^{\mathcal{M}}) = c^{\mathcal{N}}$, for every constant symbol c of $\mathcal{L}$.

(ii) $F(g^{\mathcal{M}}(a_1, \ldots, a_n)) = g^{\mathcal{N}}(F(a_1), \ldots, F(a_n))$, for every n-ary function symbol g of $\mathcal{L}$ and $a_1, \ldots, a_n \in M$.

(iii) If R is a n-ary relation symbol and $a_1, \ldots, a_n \in M$, then

$$\mathcal{M} \models R(a_1, \ldots, a_n) \Longrightarrow \mathcal{N} \models R(F(a_1), \ldots, F(a_n)).$$

If in addition F is injective and

$$\mathcal{M} \models R(a_1, \ldots, a_n) \Longleftrightarrow \mathcal{N} \models R(F(a_1), \ldots, F(a_n))$$

for every n-ary relation symbol R and $a_1, \ldots, a_n \in M$, then the morphism F is called *embedding*. A bijective embedding is an *isomorphism*.

Lemma B.1.11 (Theorem 1.1.10 [6]). *Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{L}$-structures with universes M, N, respectively. If $F\colon \mathcal{M} \to \mathcal{N}$ is an isomorphism, then for every formula $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$ we have*

$$\mathcal{M} \models \phi(a_1, \ldots, a_n) \Longleftrightarrow \mathcal{N} \models \phi(F(a_1), \ldots, F(a_n)).$$

In particular $\mathcal{M} \equiv \mathcal{N}$.

Now we recall the construction of the ultraproduct of the structures $\mathcal{M}_i$ modulo an ultrafilter $\mathcal{U}$.

Definition B.1.12. Let I be a set of cardinality λ and $\mathcal{U}$ be a filter on I. Assume that $\{\mathcal{M}_i \mid i \in I\}$ is family of $\mathcal{L}$-structures such that every $\mathcal{M}_i$ has domain M_i. Consider the set

$$M = \prod_{i \in I} M_i / \sim,$$

where $\sim$ is an equivalence relation on $\prod_{i \in I} M_i$ defined as follows:

$$f \sim g \Longleftrightarrow \{i \in I \mid f(i) = g(i)\} \in \mathcal{U}.$$

Given an element $f \in \prod_{i \in I} M_i$, we indicate with $[f]$ its equivalence class in M. Interpreting the symbols of the language $\mathcal{L}$, we construct an $\mathcal{L}$-structure $\mathcal{M}$ with universe the set M: if c is a symbol of constant, then $c^{\mathcal{M}} = [(c^{\mathcal{M}_i})_{i \in I}]$. If $g(x_1, \ldots, x_n)$ is a symbol of function, then for all $[f_1], \ldots, [f_n], [f] \in M$ we have

$$\mathcal{M} \models g([f_1], \ldots, [f_n]) = [f] \Longleftrightarrow \{i \in I \mid \mathcal{M}_i \models g(f_1(i), \ldots, f_n(i)) = f(i)\} \in \mathcal{U}.$$

If $R(x_1, \ldots, x_n)$ is a symbol of relation, then for all $[f_1], \ldots, [f_n] \in M$ we have

$$\mathcal{M} \models R([f_1], \ldots, [f_n]) \Longleftrightarrow \{i \in I \mid \mathcal{M}_i \models R(f_1(i), \ldots, f_n(i))\} \in \mathcal{U}.$$

Since $\mathcal{U}$ is a filter, the interpretations of the $\mathcal{L}$-symbols are well defined. We call the new $\mathcal{L}$-structure $\mathcal{M}$ *reduced product* of the $\mathcal{L}$-structures $\{\mathcal{M}_i \mid i \in I\}$. When the filter $\mathcal{U}$ is an ultrafilter, the $\mathcal{L}$-structure $\mathcal{M}$ is the *ultraproduct* of the $\mathcal{L}$-structures $\{\mathcal{M}_i \mid i \in I\}$ modulo $\mathcal{U}$. In the special case that $\mathcal{U}$ is an ultrafilter and every $\mathcal{L}$-structure $\mathcal{M}_i$ is the same $\mathcal{L}$-structure $\mathcal{N}$, we say that $\mathcal{M}$ is the *ultrapower* of the $\mathcal{L}$-structure $\mathcal{N}$ modulo $\mathcal{U}$.

The ultraproduct of the $\mathcal{L}$-structures $\{\mathcal{M}_i \mid i \in I\}$ modulo an ultrafilter $\mathcal{U}$ is denoted by

$$\prod_{i \in I} \mathcal{M}_i / \mathcal{U}.$$

Now we remark a classical result on the ultraproducts.

Theorem B.1.13 (Loś, Theorem 4.1.9 [2]). *Let I be a set, $\mathcal{U}$ be an ultrafilter on I and $\{\mathcal{M}_i \mid i \in I\}$ be a family of $\mathcal{L}$-structures. If $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ is the ultrapower modulo $\mathcal{U}$ of the $\mathcal{L}$-structures $\mathcal{M}_i$, then*

(i) *for all $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$ and $[f_1], \ldots, [f_n] \in \mathcal{M}$ we have*

$$\mathcal{M} \models \psi([f_1], \ldots, [f_n]) \iff \{i \in I \mid \mathcal{M}_i \models \psi(f_1(i), \ldots, f_n(i))\} \in \mathcal{U}.$$

(ii) *For all $\mathcal{L}$-sentence ϕ we have*

$$\mathcal{M} \models \phi \iff \{i \in I \mid \mathcal{M}_i \models \phi\} \in \mathcal{U}.$$

Corollary B.1.14. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\mathcal{U}$ be an ultrafilter on I. The map $f \colon \mathcal{M} \to \mathcal{M}^I/\mathcal{U}$ such that $f(a) = [(a)_{i\in I}]$ is elementary, that is for every sentence $\psi(a_1, \ldots, a_n)$ with parameters $a_1, \ldots, a_n \in \mathcal{M}$ we have*

$$\mathcal{M} \models \psi(a_1, \ldots, a_n) \iff \mathcal{M}^I/\mathcal{U} \models \psi([a_1], \ldots, [a_n])$$

Corollary B.1.15. *When $\{\mathcal{M}_i \mid i \in I\}$ is a family of models of a theory T, every ultraproduct $\prod_{i\in I} \mathcal{M}_i/\mathcal{U}$ is a model of T.*

From now on, in order to simplify the notation, we shall confuse the structure $\mathcal{M}$ and its universe M, hence we write $a \in \mathcal{M}$ to indicate $a \in M$.

B.2 Quantifier elimination and discrete linear orders

Definition B.2.1. A theory T has *quantifier elimination*, if every formula $\phi(x_1, \ldots, x_n)$ is equivalent to a quantifier-free formula $\psi(x_1, \ldots, x_n)$ modulo T, that is

$$T \models \forall x_1, \ldots, x_n[\phi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)].$$

The next is a classical result.

Theorem B.2.2 (Theorem 3.1.3 [6])**.** *The theory of dense linear orders without end-points has quantifier elimination.*

The following is an equivalent condition for quantifier elimination.

Theorem B.2.3 (Proposition 4.3.28 [6])**.** *Assume that $\mathcal{L}$ is a language containing a constant symbol and T is an $\mathcal{L}$-theory. The theory T has quantifier elimination if and only if whenever $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$, $\mathcal{N} \models T$ is $|\mathcal{M}|^+$-saturated and $f \colon A \to \mathcal{N}$ is a partial embedding, f can be extended to an embedding of $\mathcal{M}$.*

Definition B.2.4. Fix $\mathcal{L} = \{0, s, <\}$ be a language where 0 is a constant symbol, s is a unary function symbol and $<$ is a binary relation symbol. The $\mathcal{L}$-theory of discrete linear orders with minimum element and without maximum has the following axioms:

(i) $\forall x \neg(x < x)$;

(ii) $\forall x, y, z \ (x < y \wedge y < z \to x < z)$;

(iii) $\forall x, y\ (x < y \vee y < x \vee x = y)$;

(iv) $\forall x\ [x < s(x) \wedge \neg\exists y\ (x < y \wedge y < s(x))]$;

(v) $\forall x\ \ [x \neq 0 \to (0 < x \wedge \exists y\ x < y)]$;

Theorem B.2.5. *Let $\mathcal{L} = \{0, s, <\}$ be the language of Definition B.2.4. The $\mathcal{L}$-theory T of discrete linear orders with minimum element and without maximum has quantifier elimination.*

Proof. It is sufficient to check the condition of Theorem B.2.3. For $n \in \omega$, we write $s^n(x)$ and $s^{-n}(x)$ to indicate the terms

$$\underbrace{s(s(\ldots(s(x))\ldots))}_{n \text{ times}} \text{ and } \underbrace{s^{-1}(s^{-1}(\ldots(s^{-1}(x))\ldots))}_{n \text{ times}},$$

respectively. Assume that $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$, $\mathcal{N} \models T$ is $|\mathcal{M}|^+$-saturated and $f\colon A \to \mathcal{N}$ is a partial embedding. Let $\mathcal{M} = \{a_\alpha \mid \alpha \in \kappa\}$ be an enumeration of $\mathcal{M}$. We construct a set of partial embeddings $\{f_\alpha \mid \alpha \leq \kappa\}$ such that the following properties hold:

(i) the map $f_\alpha\colon \mathcal{M} \to \mathcal{N}$ extends f.

(ii) $a_\alpha \in \mathrm{dom}(f_{\alpha+1})$.

(iii) $f_\beta \subseteq f_\alpha$ for all $\beta \leq \alpha \leq \kappa$.

Put $f_0 = f$. If α is limit ordinal, define $f_\alpha = \bigcup_{\beta\in\alpha} f_\beta$. Now assume that f_α is defined and $a_\alpha \notin \mathrm{dom}(f_\alpha)$. Set

$$\begin{aligned}
&A_1 = \{a \in \mathrm{dom}(f_\alpha) \mid a < a_\alpha\},\ \ A_2 = \{a \in \mathrm{dom}(f_\alpha) \mid a_\alpha < a\}\\
&B_1 = \{a \in A_1 \mid a_\alpha = s^n(a) \text{ for some } n \in \omega\},\\
&B_2 = \{a \in A_2 \mid s^n(a_\alpha) = a \text{ for some } n \in \omega\}.
\end{aligned}$$

There are two cases:

1. The sets B_1 and B_2 are empty. Since $\mathcal{N}$ is $|\mathcal{M}|^+$-saturated and f_α is a partial embedding, there exists $b \in \mathcal{N}$ such that for every $a_1 \in A_1$, $a_2 \in A_2$, $n \in \omega$ we have $f_\alpha(a_1) < b < f_\alpha(a_2)$, $s^n(b) \neq f_\alpha(a_1)$ and $s^n(f_\alpha(a_1)) \neq b$. Then the map $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, b)\}$ is a partial embedding.

2. At least one of them is not empty. Since the argument is similar, we can assume that B_1 is not empty, hence there exists a maximal element $a \in B_1$ such that for some $n \in \omega$ we have $s^n(a) = a_\alpha$. Put $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, s^n(f_\alpha(a)))\}$ and note that $f_{\alpha+1}$ is a partial embedding.

□

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