A binary modal logic for the intersection types of lambda-calculus

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Intersection types discipline allows to define a wide variety of models for the type free lambda-calculus, but the Curry-Howard isomorphism breaks down for this kind of type systems. In this paper we show that the correspondence between types and suitable logical formulas can still be recovered appealing to the fact that there is a strict connection between the semantics for lambda-calculus induced by the intersection types and a Kripke style semantics for modal and relevant logics. Indeed, we present a modal logic hinted by the analysis of the sub-typing relation for intersection types, and we show that the deduction relation for such a modal system is a conservative extension of the relation of sub-typing. Then, we define a Kripke-style semantics for the formulas of such a system, present suitable sequential calculi, prove a completeness theorem and give a syntactical proof of the cut elimination property. Finally, we define a decision procedure for theorem-hood and we show that it yields the finite model property and cut-redundancy.

1. INTERSECTION TYPES

Pure lambda-calculus Λ formalizes the notion of computable function with no reference to the concepts of domain and co-domain, contrary to what happens in the set theoretic or the categorical approach (see [3]). Indeed, a lambda term is built inductively, starting from variables, by means of lambda abstraction and an unrestricted form of application. Thus, we have the following term formation rules:

$$\mathsf{Term} := \mathsf{Var} \mid (\lambda\, \mathsf{Var}.\mathsf{Term}) \mid \mathsf{Term}(\mathsf{Term})$$

where Var is a countable set whose elements are called variables.

Not only the syntax of the objects of Λ is simple, but also the notion of computation for this very abstract notion of function becomes the simple notion of *β-reduction* (notation $\rightsquigarrow_\beta$). This is the relation between lambda terms obtained by closing under the term construction operations the relation of *β-contraction*, that is, $(\lambda x.c)(a) \rightsquigarrow c[x := a]$. The computation of the value of a lambda term is then defined as a *reduction process*, i.e. successive steps of β-reduction, until a *normal form* of the term is possibly reached, that is, a form where no β-contraction can be applied. Given a lambda term c, there are in general many different reduction processes, according to the choice of the β-contraction to be expanded within c; hence, it is well possible that only some of the reduction processes eventually terminate into a normal form. Moreover, since it is possible to have a code within Λ for any recursive function, there is no possibility to know if a reduction process for c will eventually terminate, because of the halting problem.

On the other hand, in the usual mathematical practice - both in the set theoretic and in the categorical approach - and in many concrete algorithms, functions are intended to operate over objects of a certain type in order to produce objects of some other type. Following this idea, the rule of application should be no longer completely free; in fact a function should be applicable only to arguments of the correct type. Thus, it will be no longer possible to build all the terms of Λ. However, a main advantage of this approach is the possibility to prove more properties on the terms which can be built because of the greater quantity of information. For instance, one of the main problems on the terms of Λ is to determine whether all the reduction processes for a certain term will eventually terminate, that is, the *strong normalization* problem, which reflects in the λ-formalism one of the key problem in Computer science, that is, the problem of finding a suitable methods to deal with total correctness of programs. In the case of lambda-calculi where functions and their arguments have a type there are suitable tools to deal with this problem. For instance, a possibility is to use the *simply typed* lambda-calculus $\Lambda_\rightarrow$; its rules of type formation are the following:

$$\mathsf{Type} := \mathsf{BasTypes} \mid \mathsf{Type} \rightarrow \mathsf{Type}$$

where $\mathsf{BasTypes}$ is a set whose elements are called *basic types*.

The intended meaning is that a type $\sigma \rightarrow \tau$ denotes a set of functions from elements of the set denoted by the type σ into elements of the set denoted by the type τ. Thus, in order to build the elements of these types, we use the following rules:

$$\begin{array}{ll} \text{(variable)} & \Gamma, x : \sigma \vdash x : \sigma \\ \text{(lambda abstraction)} & \dfrac{\Gamma, x : \sigma \vdash c : \tau}{\Gamma \vdash \lambda x.c : \sigma \rightarrow \tau} \\ \text{(application)} & \dfrac{\Gamma \vdash c : \tau \rightarrow \sigma \qquad \Gamma \vdash a : \tau}{\Gamma \vdash c(a) : \sigma} \end{array} \tag{1}$$

where Γ is a commutative list of assumptions of the form $x : \sigma$, for some type σ, such that no variable appears more than once.

A striking aspect of this typing system is that after a close inspection of the rules of $\Lambda_\rightarrow$ it is easily shown that when we strip away variables and terms from the typing system, we obtain a complete sequent calculus for the implicational fragment of intuitionistic logic; in fact, this is the *Curry-Howard Isomorphism* (see [6]).

$\Lambda_\rightarrow$ has many other desirable features; for example it is well known (see for instance [11]) that all the terms of $\Lambda_\rightarrow$ are strongly normalizing. Hence, the terms of $\Lambda_\rightarrow$ form a subset of the set of strongly normalizing terms of Λ. But, not all of the strongly normalizing terms of Λ have a type in $\Lambda_\rightarrow$; for instance, consider the term $\lambda x.x(x)$: it is in normal form, and hence it is trivially strongly normalizing, but it cannot have a type within $\Lambda_\rightarrow$ because of the instance of *self-application*. From a computational point of view this is a great loss, since it is clear that a complete solution of the strong normalization problem would be a typing system which allows to assign a type to all of the strongly normalizing terms of Λ, and only to them. Surprisingly, this typing system exists and can be obtained from $\Lambda_\rightarrow$ by adding just one type (see [19] or [25] for a recent new proof). The abstract syntax of the types of this calculus $\Lambda_\wedge$ of *intersection types* is the following:

$$\mathsf{Type} := \mathsf{BasTypes} \mid \mathsf{Type} \rightarrow \mathsf{Type} \mid \mathsf{Type} \wedge \mathsf{Type}$$

The intended meaning of the new type $\sigma \wedge \tau$ of $\Lambda_\wedge$ is that $\sigma \wedge \tau$ denotes the intersection of the two sets denoted by the type σ and τ respectively. Thus, in order to build the elements for these new types, we add the following rules to the previous ones:

$$\begin{array}{ll} \text{(intersection introduction)} & \dfrac{\Gamma \vdash c : \sigma \qquad \Gamma \vdash c : \tau}{\Gamma \vdash c : \sigma \wedge \tau} \\ \text{(intersection elimination)} & \dfrac{\Gamma \vdash c : \sigma \wedge \tau}{\Gamma \vdash c : \sigma} \quad \dfrac{\Gamma \vdash c : \sigma \wedge \tau}{\Gamma \vdash c : \tau} \end{array} \tag{2}$$

The starting question of our search is: "Can the *Curry-Howard Isomorphism* be somehow recovered also for this extended typing system?"

A first inspection shows that all the types which can be assigned to a closed λ-term are theorems of the fragment of the intuitionistic propositional logic containing only implication and conjunction; but it is possible to find theorems of the intuitionistic logic of conjunction and implication which are inhabited by no closed λ-term in pure intersection type system, for example $(\alpha \rightarrow \alpha) \wedge (\alpha \rightarrow (\beta \rightarrow \alpha))$ (see [12]). This example should be sufficient to show that it is not a straightforward task to recover the *Curry-Howard isomorphism* for this type system: a deeper analysis of the properties of intersection types is needed.

A key step towards a better comprehension of $\Lambda_\wedge$ can be found in [4]. We will briefly sum up the content of that paper since it has been the starting point of our research. In [4] the authors show that the interesting computational properties enjoyed by the intersection type system are a consequence of the fact that intersection types allow to define a natural, expressive and flexible semantic for the lambda-calculus. The starting point of their work is to define a *sub-typing* relation $\leq_\wedge$ between types of $\Lambda_\wedge$ whose intended meaning is that $\alpha \leq_\wedge \beta$ holds if α is more informative about the term to which it is assigned than β. A new type constant ω is added to the set of basic types; its intended meaning is coding the vacuous information. Finally, the following axioms and rules are proposed to characterize the sub-typing relation.

Axioms

$$\begin{array}{lll} \alpha \leq_\wedge \omega & \omega \leq_\wedge \omega \to \omega & \\ \alpha \leq_\wedge \alpha \wedge \alpha & \alpha \wedge \beta \leq_\wedge \alpha & \alpha \wedge \beta \leq_\wedge \beta \\ (\alpha \to \beta) \wedge (\alpha \to \gamma) \leq_\wedge (\alpha \to (\beta \wedge \gamma)) & & \end{array}$$

Rules

$$\frac{\alpha \leq_\wedge \beta \qquad \beta \leq_\wedge \gamma}{\alpha \leq_\wedge \gamma} \quad \frac{\alpha_1 \leq_\wedge \beta_1 \qquad \alpha_2 \leq_\wedge \beta_2}{\alpha_1 \wedge \alpha_2 \leq_\wedge \beta_1 \wedge \beta_2} \quad \frac{\alpha_1 \leq_\wedge \alpha_2 \qquad \beta_2 \leq_\wedge \beta_1}{\alpha_2 \to \beta_2 \leq_\wedge \alpha_1 \to \beta_1}$$

To support the intuition about the relation $\leq_\wedge$ let us analyze one of the axioms above:

$$(\alpha \to \beta) \wedge (\alpha \to \gamma) \leq_\wedge (\alpha \to (\beta \wedge \gamma))$$

It states that all the lambda terms to which it can be assigned both type $\alpha \to \beta$ and $\alpha \to \gamma$ can also be typed by $\alpha \to (\beta \wedge \gamma)$. And indeed in the pure intersection type system, if we are able to prove $\Gamma \vdash \lambda x.M : (\alpha \to \beta) \wedge (\alpha \to \gamma)$, then also $\Gamma \vdash \lambda x.M : \alpha \to (\beta \wedge \gamma)$ can be proved.

One of the reason for introducing the sub-typing relation is to extend such kind of property of the type assignment system to any term and not only to terms of a particular shape. Of course, in order to obtain this result, it is necessary to add to the type assignment system defined by rules 1 and 2 not only the sub-typing relation but also the following assignment rule which allows to use it:

$$\frac{\Gamma \vdash M : \alpha \quad \alpha \leq_\wedge \beta}{\Gamma \vdash M : \beta} \tag{3}$$

The sub-typing relation suggests a natural way to define a semantical counterpart to the notion of type assignment. To illustrate this fact let us recall the following definitions and results of [4]. We feel free to present them in a setting more suitable for our aims.

DEFINITION 1.1. Let $\mathcal{A} = (A, \cdot, \leq)$ be an *ordered weakly extensional λ-algebra*, namely, $(A, \cdot)$ is a weakly extensional λ-algebra[1] and $\leq$ is an order

[1] A complete development of the theory of weakly extensional λ-algebras can be found in [3], chapter 5.

relation on elements of A such that if $x \leq y$ and $z \leq w$ then $x \cdot z \leq y \cdot w$. Then a map $\nu(-)$ is a *valuation* of the types of $\Lambda_\wedge$ into subsets of A if:

- $\nu(\omega) = A$
- $\nu(\alpha \wedge \beta) = \nu(\alpha) \cap \nu(\beta)$
- $\nu(\alpha \rightarrow \beta) = \{x \in A | \ (\forall y \in \nu(\alpha))\ x \cdot y \in \nu(\beta)\}$

It is worth noting the following lemma whose proof is immediate.

LEMMA 1.1. *Let $\nu(-)$ be any valuation of the types of $\Lambda_\wedge$ into $\mathcal{A}$. Then $\nu(-)$ is upward closed, that is, for any type α, if $w \in \nu(\alpha)$ and $w \leq z$ then $z \in \nu(\alpha)$.*

After definition 1.1, it is immediatly possible to state a completeness theorem for the logic defined by the axioms and the rules of the sub-typing relation with respect to the class of the ordered weakly extensional λ-algebras:

THEOREM 1.1. *$\alpha \leq_\wedge \beta$ holds if and only if, for all ordered weakly extensional λ-algebras $\mathcal{A}$ and all type valuations ν into subsets of A, $\nu(\alpha) \subseteq \nu(\beta)$ holds.*

This theorem states the main property of the sub-typing relation $\leq_\wedge$ introduced in [4]. But, while that paper aims to show that the axioms and the rules which characterize $\leq_\wedge$ are strong enough to prove that the set $\mathcal{F}$ of filters of types defines a weakly extensional λ-algebra, here we want to point out that this happens also because $\mathcal{F}$ is a sort of *canonical Kripke frame* of the sub-typing relation $\leq_\wedge$. So, let us recall that a filter of $\Lambda_\wedge$ is a non-empty subset F of the set of types of $\Lambda_\wedge$ which is closed under $\wedge$, that is, if $\alpha, \beta \in F$ then $\alpha \wedge \beta \in F$, and up-ward closed, that is, if $\alpha \in F$ and $\alpha \leq_\wedge \beta$ then $\beta \in F$. We will need the following filter construction lemma.

LEMMA 1.2. *Let α be any type. Then*

$$\uparrow\alpha \equiv \{\beta \in \Lambda_\wedge |\ \alpha \leq_\wedge \beta\}$$

is a filter that will be called the filter generated *by α.*

Consider now the set

$$\mathcal{F} \equiv \{F |\ F \text{ is a filter of } \Lambda_\wedge\}$$

and, provided F and G are two elements of $\mathcal{F}$, define the following operation on filters:

$$F \cdot G \equiv \{\gamma \in \Lambda_\wedge |\ (\exists \delta \in \Lambda_\wedge)\ (\delta \rightarrow \gamma \in F)\ \&\ (\delta \in G)\}$$

Note that if $F_1 \subseteq F_2$ and $G_1 \subseteq G_2$ then $F_1 \cdot G_1 \subseteq F_2 \cdot G_2$. Then, the following theorem can be proved (see [4]).

THEOREM 1.2. *Let $F, G \in \mathcal{F}$. Then $F \cdot G$ is a filter, i.e., $\mathcal{F}$ is closed under $\cdot$, and $(\mathcal{F}, \cdot, \subseteq)$ is an ordered weakly extensional λ-algebra.*

As we already said, we will ignore the difficult part in the proof of this theorem, that is, to show that $(\mathcal{F}, \cdot)$ is a weakly extensional λ-algebra, and we will just show that it is the canonical Kripke frame of the logic of $\leq_\wedge$. To this aim, consider the map $\phi(-)$ of types of $\Lambda_\wedge$ into subsets of $\mathcal{F}$ defined by setting

$$\phi(\alpha) = \{F \in \mathcal{F} |\ \alpha \in F\}$$

It is immediate to check the following lemma.

LEMMA 1.3. *The map $\phi(-)$ defined as above is a valuation.*

After one has proved that $(\mathcal{F}, \cdot)$ is a weakly extensional λ-algebra, this lemma immediately yields the completeness theorem 1.1. Indeed, it is easy to provide a direct check of the left to right implication. On the other hand, let us suppose that for any valuation $\nu(-)$ of the types of $\Lambda_\wedge$ into subset of an ordered weakly extensional λ-algebra $\mathcal{A}$, $\nu(\alpha) \subseteq \nu(\beta)$ holds; then, if we specialize this assumption to the ordered weakly extensional λ-algebra $(\mathcal{F}, \cdot, \subseteq)$ and to the valuation $\phi(-)$ that we defined above, then we obtain that $\phi(\alpha) \subseteq \phi(\beta)$; hence, for any filter $F \in \mathcal{F}$, if $F \in \phi(\alpha)$ then $F \in \phi(\beta)$. But this means that if $\alpha \in F$ then $\beta \in F$. Let us consider now the filter $\uparrow\alpha$; it clearly contains α and hence $\beta \in \uparrow\alpha$, that is, $\alpha \leq_\wedge \beta$, follows.

We can present all the previous considerations in a slightly different, but deeply related, setting if we use a "relational" model instead of a ordered weakly extensional λ-algebra. Indeed a map $\nu(-)$ from the set of the types of $\Lambda_\wedge$ into the set of the subsets of a set A can be presented also like a standard modal forcing relation $\Vdash^\nu$ between elements of A and types provided that we adopt the following position:

$$x \Vdash^\nu \alpha \text{ if and only if } x \in \nu(\alpha)$$

Then the requirements on the map $\nu(-)$ in definition 1.1 force immediately the following inductive conditions on $\Vdash_\nu$:

$$\begin{array}{lll} x \Vdash^\nu \alpha & \text{iff} & x \in \nu(\alpha), \text{ for any basic type } \alpha \\ x \Vdash^\nu \omega & \text{iff} & \mathsf{true} \\ x \Vdash^\nu \alpha \wedge \beta & \text{iff} & x \Vdash^\nu \alpha \text{ and } x \Vdash^\nu \beta \end{array}$$

More complex is to state the condition on the forcing relation when the type $\alpha \to \beta$ is considered. We can solve this problem if we introduce a three places relation R

over A whose intended meaning is to state that $R(x, y, z)$ holds when $x \cdot y \leq z$. Then, since $w \leq z$ and $w \Vdash^\nu \beta$ yields $z \Vdash^\nu \beta$ as a consequence of upward closure of any valuation, it is not difficult to check that the correct condition on the forcing relation becomes

$$x \Vdash^\nu \alpha \to \beta \text{ iff } (\forall z \in M)\ ((\exists y \in M)\ R(x, y, z)\ \&\ y \Vdash^\nu \alpha) \Rightarrow (z \Vdash^\nu \beta)$$

This forcing relation can be used to define an interpretation of the relation $\leq_\wedge$ in a model (A, R, ν). In fact, we can set

$$(A, R, \nu) \models \alpha \leq_\wedge \beta \text{ iff } (\forall x \in A)\ (x \Vdash^\nu \alpha) \Rightarrow (x \Vdash^\nu \beta)$$

which, recalling the position above, means that $(A, R, \nu) \models \alpha \leq_\wedge \beta$ if and only if $\nu(\alpha) \subseteq \nu(\beta)$.

This interpretation can be generalized to any structure (A, R) by setting

$$(A, R) \models (\alpha \leq_\wedge \beta) \text{ iff } (A, R, \nu) \models (\alpha \leq_\wedge \beta), \text{ for any map } \nu : \mathsf{BasTypes} \longrightarrow \mathcal{P}(M)$$

Thus we arrived at a relational semantics for the subtype relation, that is,

$$\alpha \models \beta \text{ iff } (A, R) \models \alpha \leq_\wedge \beta, \text{ for any structure } (A, R)$$

It is now possible to state the following theorem of validity and completeness for relational structures which is the analogous of theorem 1.1.

THEOREM 1.3. $\alpha \models \beta$ *if and only if* $\alpha \leq_\wedge \beta$.

Even if the proof of this theorem is just a rewriting of the proof of theorem 1.1, let us show the relevant steps since they will be useful in the next section. The proof of validity is straightforward while in order to prove completeness let us consider again the set $\mathcal{F}$ of the filters of $\Lambda_\wedge$ and define a three place relation R on its elements by setting

$$R(F, G, H) \equiv (\forall \beta)\ ((\exists \alpha \in G)\ \alpha \to \beta \in F) \Rightarrow (\beta \in H)$$

that is, $R(F, G, H)$ holds if and only if $F \cdot G \subseteq H$.

Then, consider the interpretation map ϕ defined by setting, for any basic type α,

$$\phi(\alpha) = \{F \mid \alpha \in F\}$$

and extend it by induction to a forcing relation $\Vdash^\phi$. Then, it is not difficult to prove that $(\mathcal{F}, R, \phi)$ is a model for $\leq_\wedge$. Moreover, it is possible to prove by induction on type complexity the following lemma.

LEMMA 1.4. *Let α be any type and F be any filter of $\Lambda_\wedge$. Then $F \Vdash^\phi \alpha$ if and only if $\alpha \in F$.*

Now, this lemma immediately yields the completeness theorem 1.3 since supposing $\alpha \models \beta$ we obtain $(\mathcal{F}, R, \phi) \models (\alpha \leq_\wedge \beta)$ and hence, for any filter $F \in \mathcal{F}$, if $F \Vdash^\phi \alpha$ then $F \Vdash^\phi \beta$. But, after lemma 1.4, this means that if $\alpha \in F$ then $\beta \in F$. Let us consider now the filter $\uparrow\alpha$; it clearly contains α and hence $\beta \in\uparrow\alpha$, that is, $\alpha \leq_\wedge \beta$.

The semantics we considered here is clearly recalling a sort of non-standard Kripke semantics for a modal logic: the idea to define a modal interpretation for the connective $\rightarrow$ started here. The intuitive explanations is that lambda-terms are thought of as worlds in which their types are true formulas.

Now, the sub-typing axioms and rules are quite similar to a logical axiom system in which intersection behaves like the classical connective $\wedge$, while $\rightarrow$ axioms and rules are sound for intuitionistic implication but are surely not complete; in fact, it can be shown (see [26]) that this sub-typing system is the restriction to $\wedge$ and $\rightarrow$ of the logic B of relevant implication introduced in [16]. So, the sub-typing relation suggests a different approach to the problem of setting a logic that reflects the properties of the intersection types assignment system, an approach which is alternative to the *Curry-Howard isomorphism* paradigma: one does not try to define a proof system whose logic reflects the rules of type assignment, but defines a modal logic whose Kripke style semantics is as close as possible to the natural semantics of the intersection types system. Thus, we will develop a modal logical system in which the type constructor $\rightarrow$ is interpreted as a suitable modal operator and whose semantics is a natural extension of the semantics for the sub-typing relation. To this aim, in section 2 we generalize the semantics for relevant logics that was introduced in [16] and that was shown in [26] to interpret faithfully the sub-typing relation. Then we define a complete sequent calculus for the logical system so obtained and study its main properties. In particular, we establish the cut elimination property, the decidability property and the finite model property. In section 3 we show that our logic can be characterized as the logic over partial applicative structures and that under this interpretation it is well possible that our logic is a first step towards the the definition of a type system for Λ which extends the intersection types and introduce a disjunction and a negation type constructor.

2. THE TWO-PLACE MODAL LOGIC BK

In this section we present the modal logic BK for which we state and prove a completeness theorem. To this aim consider the propositional modal language whose formulas are inductively defined as follows

- Any propositional variable is a formula;
- $\bot$ and $\top$ are formulas;

- If α and β are formulas then also $\alpha \wedge \beta$, $\alpha \vee \beta$, $\neg\alpha$, $\alpha \supset \beta$ are formulas;
- If α and β are formulas then $\Box(\alpha, \beta)$ is a formula.

We can define a *kripke-like* semantics for the formulas of this language as follows. Let A be a set and R be a ternary relation over A and suppose that v is a map of the propositional variables into subsets of A. Then, supposing $x \in A$ and p is a propositional variable, set

$x \Vdash^{\mathsf{v}} p$	iff	$x \in \mathsf{v}(p)$
$x \Vdash^{\mathsf{v}} \bot$	iff	falsum
$x \Vdash^{\mathsf{v}} \top$	iff	true
$x \Vdash^{\mathsf{v}} \alpha \wedge \beta$	iff	$x \Vdash^{\mathsf{v}} \alpha$ and $x \Vdash^{\mathsf{v}} \beta$
$x \Vdash^{\mathsf{v}} \alpha \vee \beta$	iff	$x \Vdash^{\mathsf{v}} \alpha$ or $x \Vdash^{\mathsf{v}} \beta$
$x \Vdash^{\mathsf{v}} \neg\alpha$	iff	$x \nVdash^{\mathsf{v}} \alpha$
$x \Vdash^{\mathsf{v}} \alpha \supset \beta$	iff	$x \Vdash^{\mathsf{v}} \alpha$ yields $x \Vdash^{\mathsf{v}} \beta$
$x \Vdash^{\mathsf{v}} \Box(\alpha, \beta)$	iff	for all y and z such that $R(x, y, z)$, if $y \Vdash_{\mathsf{v}} \alpha$ then $z \Vdash_{\mathsf{v}} \beta$

To understand the intended meaning of the modal operator it can be useful to consider the following explanation. Let A be the set of the non-deterministic programs; then a formula α *is true for the program* x (notation $x \Vdash \alpha$) if and only if the type α can be assigned to x. Moreover, $R(x, y, z)$ holds if and only if y is an input accepted by the program x and z is a possible output of x when applied to y; so, provided $\cdot$ means the application operation, $R(x, y, z)$ holds if and only if $x \cdot y$ may give z as an output. Then, $\Box(\alpha, \beta)$ holds for x if and only if, for every input y of type α which is accepted by the program x, every possible output z of x applied to y has type β.

Let us recall now the standard conditions for validity of a formula in a Kripke-style semantics: a formula α is true in the model (A, R, v) if, for every element $x \in A$, $x \Vdash^{\mathsf{v}} \alpha$; moreover, a formula is true in the frame (A, R) if, for every valuation v, it is true in the model (A, R, v); finally, a formula is valid if it is true in every frame.

It is interesting to note that what we defined is a generalization of the usual modal situation. In fact, we can define a standard modality by setting $\Box(\beta) \equiv \Box(\top, \beta)$ and then we obtain the usual definition for a forcing relation by setting $\overline{R}(x, z) \equiv (\exists y \in A)\ R(x, y, z)$. Since no extra condition is required on the relation R, the models that we defined directly generalize the situation for the modal logic K. This is the reason why we called BK this *binary* modal logic.

Consider now any complete sequent calculus for the classical propositional logic such that sequents are couples of finite sets of formulas[2]. To such a calculus

[2]This last requirement is just a simplification which allows to consider cut and weakening as the only structural rules.

add the following modal rule:

$$\Box\text{-rule} \quad \frac{\alpha \vdash \alpha_1, \ldots, \alpha_n \qquad \beta_1, \ldots, \beta_m \vdash \beta}{\{\Box(\alpha_i, \beta_j) |\ i = 1 \ldots n, j = 1 \ldots m\} \vdash \Box(\alpha, \beta)} \qquad n \geq 0, m \geq 0$$

In the following we will call the sequent calculus obtained in this way BKS.

We will adopt for BKS the standard terminology for a sequent calculus, that is, we will say that a sequent *is provable* if it can be obtained from the axioms by a finite number of applications of the deduction rules, a sequent $\alpha_1, \ldots, \alpha_n \vdash \beta_1, \ldots, \beta_m$ is *valid* if and only if the formula $\alpha_1 \wedge \ldots \wedge \alpha_n \supset \beta_1 \vee \ldots \vee \beta_m$ of BK is valid, a *formula* α *is provable* if and only if the sequent $\vdash \alpha$ is a provable, and, if Γ is a set of formulas, then Γ is consistent if and only if, for any finite $\Gamma' \subseteq \Gamma$, the sequent $\Gamma' \vdash \emptyset$ is not a provable.

THEOREM 2.1. *The sequent calculus* BKS *is correct for* BK.

Proof. Only correctness of the $\Box$-rule deserves a proof, since all the other rules of BKS are shown to be correct by standard arguments. So, let us show that the $\Box$-*rule* is valid in any frame. To this aim, let us suppose that its conclusion is not valid in some frame (A, R), that is, let us suppose that there exists a point $x \in A$ and a valuation v such that $x \Vdash^{\mathsf{v}} \neg\Box(\alpha, \beta)$ whereas for all $i = 1 \ldots n$ and $j = 1 \ldots m$, $x \Vdash^{\mathsf{v}} \Box(\alpha_i, \beta_j)$. Then there must exist two points $y, z \in A$ such that $R(x, y, z)$ holds and $y \Vdash^{\mathsf{v}} \alpha$ and $z \Vdash^{\mathsf{v}} \neg\beta$. Hence, by the left premise, we obtain that there must be some index i such that $y \Vdash^{\mathsf{v}} \alpha_i$ and thus, for any $j = 1 \ldots m$, $z \Vdash^{\mathsf{v}} \beta_j$, since $x \Vdash^{\mathsf{v}} \Box(\alpha_i, \beta_j)$. But then the right premise forces $z \Vdash^{\mathsf{v}} \beta$, contradiction. ■

In the sequel we will show that BKS is also complete for BK. To this aim it is convenient to consider two instances of the $\Box$-*rule*, which are indeed sufficient to obtain the result. The first one is obtained for $n = 1$ and $\alpha_1 \equiv \alpha$ and the second one for $m = 1$ and $\beta_1 \equiv \beta$.

$$\Box\text{-monotonicity} \qquad \frac{\beta_1, \ldots, \beta_m \vdash \beta}{\Box(\alpha, \beta_1), \ldots, \Box(\alpha, \beta_m) \vdash \Box(\alpha, \beta)} \quad m \geq 0$$

$$\Box\text{-anti-monotonicity} \qquad \frac{\alpha \vdash \alpha_1, \ldots, \alpha_n}{\Box(\alpha_1, \beta), \ldots, \Box(\alpha_n, \beta) \vdash \Box(\alpha, \beta)} \quad n \geq 0$$

Note that setting $n = 0$ and $\alpha \equiv \bot$ in $\Box$-*anti-monotonicity* we obtain that $\Box(\bot, \beta)$ is provable and setting $m = 0$ and $\beta \equiv \top$ in $\Box$-*monotonicity* we obtain that $\Box(\alpha, \top)$ is provable.

Moreover, the $\Box$-*rule* is sufficient to prove that the binary modal operator is an operation in the Lindenbaum algebra $\mathcal{L}_{BK}$ of BK[3]. In fact, the following theorem holds.

[3]By $\mathcal{L}_{BK}$ we mean the set of equivalence classes over the formulas of BK induced by the equivalence relation defined by setting $\alpha \equiv \beta$ if and only if $\vdash (\alpha \supset \beta) \wedge (\beta \supset \alpha)$, endowed with the

THEOREM 2.2. *Let* $\vdash \alpha_1 \leftrightarrow \alpha_2$ *and* $\vdash \beta_1 \leftrightarrow \beta_2$. *Then* $\vdash \Box(\alpha_1, \beta_1) \leftrightarrow \Box(\alpha_2, \beta_2)$.

Proof: It is sufficient to show that if $\alpha_2 \vdash \alpha_1$ and $\beta_1 \vdash \beta_2$ hold then also $\Box(\alpha_1, \beta_1) \vdash \Box(\alpha_2, \beta_2)$ holds, which is immediate by $\Box$-*rule*. ■

It is worth noting that the proof of this theorem shows that the modality that we are considering enjoys some of the features of an implication, even if one should be aware that the usual rule of implication introduction is not valid for such a modality, that is, $\alpha \vdash \beta$ does not yield $\vdash \Box(\alpha, \beta)$.

We can now prove the completeness theorem.

THEOREM 2.3. *The sequent* $\alpha_1, \ldots, \alpha_n \vdash \beta_1, \ldots, \beta_m$ *is provable in* BKS *if and only if it is valid in any frame.*

We already proved that all the rules of BKS are valid. To prove that they are also sufficient we will adapt to BKS the standard approach to prove completeness for modal logical systems, that is, we start from the frame induced by the Lindenbaum Algebra of BKS and define a canonical model which is shown to yield the desired completeness result[4]. To this aim, let us consider the set $\mathcal{U}$ of ultrafilters of $\mathcal{L}_{BK}$[5] and define a ternary relation R over $\mathcal{U}$ by setting

$$R(F, G, H) \equiv F \cdot G \subseteq H$$

where $F \cdot G \equiv \{\delta \mid \text{there is } \gamma \in G \text{ such that } \Box(\gamma, \delta) \in F\}$.

It is interesting to note that $F \cdot G$ is a filter, as we noted in the previous section, but, in general, it is not an ultrafilter on $\mathcal{L}_{BK}$[6]. This is the reason why we cannot simply adapt the completeness proof of the previous section to the case of BKS, where also a negation connective is considered, and a new proof must be provided.

The last step in our completeness proof is to define a canonical valuation V of the propositional variables into the set of the subsets of $\mathcal{U}$:

$$\mathsf{V}(p) = \{F \in \mathcal{U} \mid p \in F\}$$

---

boolean structure given by the operations induced by the classical connectives: $[\alpha]^c = [\neg\alpha]$ and $[\alpha] \cap [\beta] = [\alpha \wedge \beta]$. On any boolean algebra the operations induce a natural order relation which in the case of a Lindenbaum Algebras can also be defined as: $[\alpha] \leq [\beta]$ if and only if $\alpha \vdash \beta$ is provable. In the sequel of the paper, following standard use, we will often identify the equivalence class $[\alpha]$ with any of its representative (for example α), in order to simplify the notation.

[4] A detailed account on the techniques used to construct such canonical models can be found in any introductory text on modal logic (see for instance [13]).

[5] An ultrafilter F on a boolean algebra B is just a filter such that for any $x \in B$, $x \in F$ or $x^c \in F$ but not both.

[6] Consider, for instance, the case for some $\beta \in G$, $\Box(\beta, \bot) \in F$; in this case $F \cdot G$ is the trivial filter, that is, it coincides with the whole algebra $\mathcal{L}_{BK}$. Then, for no H we have $R(F, G, H)$, that is, G is not an acceptable input for F.

The completeness theorem will then be achieved if the following key lemma holds.

LEMMA 2.1. *For any formula α, $F \Vdash^{\mathsf{V}} \alpha$ if and only if $\alpha \in F$.*

In fact, if $\alpha_1, \ldots, \alpha_n \vdash \beta_1, \ldots, \beta_m$ is not provable, then, by obvious properties of the calculus the formula $(\alpha_1 \wedge \ldots \wedge \alpha_n) \supset (\beta_1 \vee \ldots \vee \beta_m)$ is not provable. Now, let α be any formula; then, if α is not provable then $[\alpha] \neq 1_{\mathcal{L}_{BK}}$ and hence $[\alpha]^c \neq 0_{\mathcal{L}_{BK}}$. But a fundamental property of boolean algebra is that any non-zero element is contained in some utrafiter (see [2]); hence, there exists an ultrafilter F such that $\neg((\alpha_1 \wedge \ldots \wedge \alpha_n) \supset (\beta_1 \vee \ldots \vee \beta_m)) \in F$. By consistency of ultrafilters, this yields that $(\alpha_1 \wedge \ldots \wedge \alpha_n) \supset (\beta_1 \vee \ldots \vee \beta_m) \notin F$ and hence, by lemma 2.1, $F \nVdash^{\mathsf{V}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \supset (\beta_1 \vee \ldots \vee \beta_m)$ and this means that the sequent $\alpha_1, \ldots, \alpha_n \vdash \beta_1, \ldots, \beta_m$ is not valid in the model $(\mathcal{U}, R, \mathsf{V})$.

The rest of this section will be dedicated to the proof of lemma 2.1. Let us argue according to the complexity of the formula α. The cases of the propositional connectives are immediate:

- if α is the propositional variable p then by definition $F \Vdash^{\mathsf{V}} p$ if and only if $F \in \mathsf{V}(p)$ if and only if $p \in F$;
- if $\alpha \equiv \bot$ then the result is immediate since F is a proper filter;
- if $\alpha \equiv \top$ or $\alpha \equiv \alpha_1 \wedge \alpha_2$ the result follows by induction from the fact that F is a filter of a boolean algebra;
- if $\alpha \equiv \alpha_1 \vee \alpha_2$, $\alpha \equiv \neg\alpha_1$ or $\alpha \equiv \alpha_1 \supset \alpha_2$ then the result follows by induction from the fact that F is an ultrafilter.

The proof for the modal case $\alpha \equiv \Box(\alpha_1, \alpha_2)$ is more elaborate, and will go through the rest of this section. We can immediately prove that $\Box(\alpha_1, \alpha_2) \in F$ yields $F \Vdash^{\mathsf{V}} \Box(\alpha_1, \alpha_2)$. In fact, let us suppose that $G, H \in \mathcal{U}$ and $R(F, G, H)$ and $G \Vdash^{\mathsf{V}} \alpha_1$ hold. Then $\alpha_1 \in G$ by inductive hypothesis and hence $\Box(\alpha_1, \alpha_2) \in F$ and $R(F, G, H)$ yields $\alpha_2 \in H$. Then $H \Vdash^{\mathsf{V}} \alpha_2$ by inductive hypothesis and hence $F \Vdash^{\mathsf{V}} \Box(\alpha_1, \alpha_2)$ by definition.

The hard part is the proof that $F \Vdash^{\mathsf{V}} \Box(\alpha_1, \alpha_2)$ yields $\Box(\alpha_1, \alpha_2) \in F$. In fact, we will prove the converse, that is, we will assume that $\Box(\alpha_1, \alpha_2) \notin F$ and we will show that it is possible to build two ultrafilters G and H such that $R(F, G, H)$ holds, $G \Vdash^{\mathsf{V}} \alpha_1$ and $H \nVdash^{\mathsf{V}} \alpha_2$, that is, $F \nVdash^{\mathsf{V}} \Box(\alpha_1, \alpha_2)$. The idea is to build the ultrafilter G with a continuous attention for the possibility to build H. To this aim let us consider the following inductive definition of a sequence $(Y_i)_{i \in \omega}$ of filters. Let $(\phi_i)_{i \in \omega}$ be any surjective numbering of the elements of $\mathcal{L}_{BK}$ and set

$$\begin{array}{lcl} Y_0 & = & \uparrow \{\alpha_1\} \\ Y_{i+1} & = & \begin{cases} \uparrow (Y_i \cup \{\phi_i\}) & \text{if } \uparrow (Y_i \cup \{\phi_i\}) \text{ is } \langle F, \neg\alpha_2 \rangle\text{-consistent} \\ \uparrow (Y_i \cup \{\neg\phi_i\}) & \text{otherwise} \end{cases} \end{array}$$

where we write $\uparrow A$ to mean the minimal filter of $\mathcal{L}_{BK}$ which contains the subset A, that is, $\uparrow A \equiv \{\gamma \in \mathcal{L}_{BK} | \, (\exists \alpha_1, \ldots, \alpha_n \in A) \; \alpha_1 \wedge \ldots \wedge \alpha_n \vdash \gamma\}$, and we say that a set of formulas A is $\langle F, \neg\alpha_2 \rangle$-consistent to mean that the set $(F \cdot A) \cup \{\neg\alpha_2\}$ is consistent.

LEMMA 2.2. *For any $i \geq 0$, the filter Y_i is generated by one formula, that is, there exists a formula ψ_i such that $Y_i = \uparrow \{\psi_i\}$.*

Proof: By induction. By definition, Y_0 is generated by α_1 and, supposing that Y_i is generated by ψ_i, then $Y_{i+1} = \uparrow \{\psi_i \wedge \phi_i\}$ or $Y_{i+1} = \uparrow \{\psi_i \wedge \neg\phi_i\}$ according to the clause which applies in the definition of Y_{i+1}. In fact, it is immediate to verify that $\uparrow (Y_i \cup \{\gamma\}) = \uparrow \{\psi_i \wedge \gamma\}$ because $\delta \in \uparrow (Y_i \cup \{\gamma\})$ means that there exist $\gamma_1, \ldots, \gamma_n \in Y_i$ such that $\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \gamma \vdash \delta$ and hence, by using the cut-rule, $\psi_i \wedge \gamma \vdash \delta$ because, for each $1 \leq k \leq n$, $\psi_i \vdash \gamma_k$; in the other direction the result is an immediate consequence of the fact that ψ_i is an element of $Y_i = \uparrow \{\psi_i\}$. ∎

LEMMA 2.3. *For any $i \geq 0$, the filter Y_i is $\langle F, \neg\alpha_2 \rangle$-consistent.*

Proof. By induction on i.

• Case $i = 0$. Let us suppose that Y_0, which is equivalent to $\uparrow \{\alpha_1\}$, is not $\langle F, \neg\alpha_2 \rangle$-consistent; then there exist $\gamma_1, \delta_1, \ldots, \gamma_n, \delta_n$ such that

$$\delta_1, \ldots, \delta_n, \neg\alpha_2 \vdash \emptyset \tag{4}$$

and, for any $1 \leq k \leq n$,

$$\alpha_1 \vdash \gamma_k \tag{5}$$

and $\Box(\gamma_k, \delta_k) \in F$. By cut and negation rules, from equation 4, we obtain

$$\delta_1, \ldots, \delta_n \vdash \alpha_2 \tag{6}$$

and hence

$$\{\Box(\alpha_1, \delta_j) | \; j = 1 \ldots n\} \vdash \Box(\alpha_1, \alpha_2) \tag{7}$$

follows by $\Box$-*monotonicity* applied to the sequents $\alpha_1 \vdash \alpha_1$ and 6. But, for each $k \leq n$, by hypothesis 5, $\alpha_1 \vdash \gamma_k$ and hence we can use $\Box$-*anti-monotonicity* to obtain:

$$\Box(\gamma_k, \delta_k) \vdash \Box(\alpha_1, \delta_k)$$

Hence, for each $k \leq n$, $\Box(\alpha_1, \delta_k) \in F$ since F is upward closed. But then, by 7 above, we would obtain that $\Box(\alpha_1, \alpha_2) \in F$ which is contrary to our assumption.

• Induction Step. Suppose now, by inductive hypothesis, that Y_i is $\langle F, \neg\alpha_2 \rangle$-consistent and let us assume that both $\uparrow (Y_i \cup \phi_i)$ and $\uparrow (Y_i \cup \neg\phi_i)$ are not $\langle F, \neg\alpha_2 \rangle$-consistent. Then there exist $\gamma_1, \delta_1, \ldots, \gamma_n, \delta_n$ and $\gamma'_1, \delta'_1, \ldots, \gamma'_m, \delta'_m$ such that, for any $k \leq n$ and any $h \leq m$ the following conditions are satisfied:

$$
\begin{array}{lll}
(1.) & \gamma_k \in\uparrow (Y_i \cup \phi_i) & \Box(\gamma_k, \delta_k) \in F \\
(2.) & \gamma'_h \in\uparrow (Y_i \cup \neg\phi_i) & \Box(\gamma'_h, \delta'_h) \in F \\
(3.) & \delta_1 \wedge \ldots \wedge \delta_n \vdash \alpha_2 & \delta'_1 \wedge \ldots \wedge \delta'_m \vdash \alpha_2
\end{array}
$$

By lemma 2.2, we know that $Y_i \equiv\uparrow \{\psi_i\}$ for some formula ψ_i. Hence, for each $k \leq n$, $\psi_i \wedge \phi_i \vdash \gamma_k$ and, for each $h \leq m$, $\psi_i \wedge \neg\phi_i \vdash \gamma'_k$. Then, by $\Box$-*anti-monotonicity*, for each $k \leq n$, $\Box(\gamma_k, \delta_k) \vdash \Box(\psi_i \wedge \phi_i, \delta_k)$ and hence, by the condition (1.) above,

$$\Box(\psi_i \wedge \phi_i, \delta_k) \in F$$

For the same reason for each $h \leq m$, $\Box(\gamma'_h, \delta'_h) \vdash \Box(\psi_i \wedge \neg\phi_i, \delta'_h)$ and hence, by the condition (2.) above,

$$\Box(\psi_i \wedge \neg\phi_i, \delta'_h) \in F$$

Now, we use the latter sequents together with the conditions (3.) to apply $\Box$-*monotonicity* in the following ways:

$$\frac{\psi_i \wedge \phi_i \vdash \psi_i \wedge \phi_i \qquad \delta_1, \ldots, \delta_n \vdash \alpha_2}{\Box(\psi_i \wedge \phi_i, \delta_1), \ldots, \Box(\psi_i \wedge \phi_i, \delta_n) \vdash \Box(\psi_i \wedge \phi_i, \alpha_2)}$$

and

$$\frac{\psi_i \wedge \neg\phi_i \vdash \psi_i \wedge \neg\phi_i \qquad \delta'_1, \ldots, \delta'_m \vdash \alpha_2}{\Box(\psi_i \wedge \neg\phi_i, \delta'_1) \wedge \ldots \wedge \Box(\psi_i \wedge \neg\phi_i, \delta'_m) \vdash \Box(\psi_i \wedge \neg\phi_i, \alpha_2)}$$

Hence both $\Box(\psi_i \wedge \phi_i, \alpha_2) \in F$ and $\Box(\psi_i \wedge \neg\phi_i, \alpha_2) \in F$. We can now conclude immediately if we observe that $\psi_i \vdash (\psi_i \wedge \phi_i) \vee (\psi_i \wedge \neg\phi_i)$ is a tautology and then, by using again $\Box$-*anti-monotonicity*, we can infer that

$$\Box(\psi_i \wedge \phi_i, \alpha_2) \wedge \Box(\psi_i \wedge \neg\phi_i, \alpha_2) \vdash \Box(\psi_i, \alpha_2)$$

and hence $\Box(\psi_i, \alpha_2) \in F$ which means that Y_i is not $\langle F, \neg\alpha_2 \rangle$-consistent against the inductive hypothesis. ■

We are now almost arrived to the end of the proof of lemma 2.1. In fact, lemma 2.3 suggests how to build the desired ultrafilter G. Let us set:

$$G \equiv \bigcup_{i \in \omega} Y_i$$

Then, we can prove the following lemma.

LEMMA 2.4. *G is a $\langle F, \neg\alpha_2\rangle$-consistent ultrafilter.*

Proof. G is a filter because $\top \in G$ since $\top \in Y_0 \equiv\uparrow \{\alpha_1\}$ and, supposing $\gamma_1, \gamma_2 \in G$, there is an index i such that $\gamma_1, \gamma_2 \in Y_i$, i.e. $\psi_i \vdash \gamma_1$ and $\psi_i \vdash \gamma_2$, because for any i, $Y_i \subseteq Y_{i+1}$ obviously holds; hence $\psi_i \vdash \gamma_1 \wedge \gamma_2$, i.e. $\gamma_1 \wedge \gamma_2 \in Y_i$, and hence $\gamma_1 \wedge \gamma_2 \in G$; finally, if $\gamma_1 \in G$ and $\gamma_1 \vdash \gamma_2$ then there is an index i such that $\gamma_1 \in Y_i$, i.e. $\psi_i \vdash \gamma_1$, and hence $\psi_i \vdash \gamma_2$ by cut-rule, i.e. $\gamma_2 \in Y_i$, so that $\gamma_2 \in G$. Moreover, if G was not $\langle F, \neg\alpha_2\rangle$-consistent then there would be γ_1, $\delta_1, \ldots, \gamma_n, \delta_n$ such that $\gamma_1, \ldots, \gamma_n \in G$, $\Box(\gamma_1, \delta_1) \in F, \ldots, \Box(\gamma_n, \delta_n) \in F$ and $\delta_1 \wedge \ldots \wedge \delta_n \vdash \alpha_2$; but then there would exist an index i such that $\gamma_1, \ldots, \gamma_n \in Y_i$, that is Y_i would not be $\langle F, \neg\alpha_2\rangle$-consistent, contrary to lemma 2.3.

To prove that G is an ultrafilter we have only to prove it is a complete consistent filter. Since any formula γ appears in the sequence $(\phi_i)_{i\in\omega}$, i.e. $\gamma \equiv \phi_i$ for some $i \in \omega$, we obtain that $\gamma \in Y_{i+1}$ or $\neg\gamma \in Y_{i+1}$, and thus $\gamma \in G$ or $\neg\gamma \in G$, that is, G is complete. Finally consistency is a consequence of the fact that G is $\langle F, \neg\alpha_2\rangle$-consistent. In fact, if G was not consistent then $\bot \in G$ and hence $\bot \in F \cdot G$ because $\Box(\bot, \bot)$ is provable and hence it belongs to every filter. ■

In order to build the ultrafilter H, let us consider the set $Z \equiv (F \cdot G) \cup \{\neg\alpha_2\}$. The set Z is consistent by definition since G is $\langle F, \neg\alpha_2\rangle$-consistent; then Z can be extended to a proper ultrafilter H in the usual way (see [2]). Moreover, $R(F, G, H)$, that is, $F \cdot G \subseteq H$, holds by construction. Finally $\alpha_1 \in G$ by definition and $\neg\alpha_2 \in H$ because $\neg\alpha_2 \in Z$. We have thus completed the proof of lemma 2.1 and hence also that one of theorem 2.3.

Some comments on the previous proof are in order. What we did is just a refinement of the proof used in [16] to show completeness of various positive relevant logics. In fact, BK can be considered as the boolean completion of the minimal relevant logic B; and our completeness proof shows that adding classical negation to B yields to a conservative extension. The same result was also obtained in [17] by using a different proof. We will show later that there are other connections between the logic BK and the system B.

2.1. Cut-elimination

In the previous section we proved that $\Box$-*rule* is valid with respect to the Kripke models that we proposed and sufficient to obtain a completeness proof. However to obtain such a completeness proof it is essential to use also the cut-rule which should be explicitly consider among the structural rules of BKS. Indeed, it is possible to show that the rules we introduced are not sufficient to obtain a cut elimination theorem; for example, the following sequent is valid, but it cannot be proved without using instances of the cut-rule[7]:

$$\Box(\alpha, \alpha), \Box(\beta, \beta) \vdash \Box(\alpha \vee \beta, \alpha \vee \beta)$$

[7] This example was suggested to us by R.K.Meyer.

However, it is possible to prove the cut-elimination theorem for a version of the sequent calculus for BK obtained by a slight modification of the modal rule. To this aim, let us consider the following rule:

$$(\Box\text{-gen-rule})\quad \frac{\alpha \vdash \bigwedge_{i=1\dots n} \bigvee_{j=1\dots m_i} \gamma_{ij} \qquad \bigwedge_{i=1\dots n} \bigvee_{j=1\dots m_i} \delta_{ij} \vdash \beta}{\Box(\gamma_{11}, \delta_{11}), \dots, \Box(\gamma_{nm_n}, \delta_{nm_n}) \vdash \Box(\alpha, \beta)}$$

with the obvious meaning of the generalized connectives. We will call BKS* the sequent calculus obtained from BKS by substituting $\Box$-*rule* with the rule above.

Note that distributivity of $\wedge$ over $\vee$ allows to present $\Box$-*gen-rule* like a more standard rule, provided we use rules with a non-fixed number of premises instead of generalized quantifiers, that is,

$$(\Box\text{-gen-rule})\quad \frac{\begin{array}{c} \alpha \vdash \gamma_{11}, \dots, \gamma_{1m_1} \\ \vdots \\ \alpha \vdash \gamma_{n1}, \dots, \gamma_{nm_n} \end{array} \qquad \begin{array}{c} \delta_{11}, \delta_{21}, \dots, \delta_{n1} \vdash \beta \\ \delta_{12}, \delta_{21}, \dots, \delta_{n1} \vdash \beta \\ \vdots \\ \delta_{1m_1}, \delta_{21}, \dots, \delta_{n1} \vdash \beta \\ \vdots \\ \delta_{1m_1}, \delta_{2m_2}, \dots, \delta_{nm_n} \vdash \beta \end{array}}{\Box(\gamma_{11}, \delta_{11}), \dots, \Box(\gamma_{nm_n}, \delta_{nm_n}) \vdash \Box(\alpha, \beta)}$$

It is easy to check that $\Box$-*gen-rule* is valid in any of the considered Kripke model.

THEOREM 2.4. $\Box$-gen-rule *is valid with respect to the models for* BK.

Proof. Let us suppose that there is a point x in a model such that $x \Vdash \neg\Box(\alpha, \beta)$ and $x \Vdash \Box(\gamma_{ij}, \delta_{ij})$ for any $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Hence there must exist in the model two points y and z, in relation with x, such that $y \Vdash \alpha$ and $z \Vdash \neg\beta$. Then $y \Vdash \bigwedge_{i=1\dots n} \bigvee_{j=1\dots m_i} \gamma_{ij}$, and hence for all $i = 1 \dots n$, there is at least one $1 \leq j \leq m_i$ such that $y \Vdash \gamma_{ij}$ holds. Hence $z \Vdash \delta_{ij}$, because $x \Vdash \Box(\gamma_{ij}, \delta_{ij})$, and so $z \Vdash \bigwedge_{i=1\dots n} \bigvee_{j=1\dots m_i} \delta_{ij}$ which yields $z \Vdash \beta$. Contradiction. ■

It is worth noting that $\Box$-*monotonicity* and $\Box$-*anti-monotonicity* are special instances of $\Box$-*gen-rule*. In fact, let us put $m_i = 1$ for each $1 \leq i \leq n$ and $\gamma_{ij} \equiv \alpha$ in the $\Box$-*gen-rule* rule, then we obtain

$$\frac{\alpha \vdash \alpha \wedge \dots \wedge \alpha \quad \delta_1 \wedge \dots \wedge \delta_n \vdash \beta}{\Box(\alpha, \delta_1), \dots, \Box(\alpha, \delta_n) \vdash \Box(\alpha, \beta)}$$

which is equivalent to $\Box$-*monotonicity*. And if we put $n = 1$, $m_1 = m$ and $\delta_{1j} \equiv \beta$, then we obtain

$$\frac{\alpha \vdash \gamma_1 \vee \dots \vee \gamma_m \quad \beta \vee \dots \vee \beta \vdash \beta}{\Box(\gamma_1, \beta), \dots, \Box(\gamma_m, \beta) \vdash \Box(\alpha, \beta)}$$

which is equivalent to $\Box$-*anti-monotonicity*. Thus, after theorem 2.3 of validity and completeness, a calculus in which the unique modal rule is $\Box$-*gen-rule* is sufficient to prove all the valid sequents. The main reason we did not introduce $\Box$-*gen-rule* directly in the previous section is that, in our opinion, it is much harder to grasp what this inference figure does. On the other hand, under a proof theoretical standpoint this rule is much stronger; in fact, in this section we will show that $\Box$-*gen-rule* makes possible to devise a syntactical proof of the cut elimination property and in the next one a decision procedure for theorem-hood for BK and, as a by product of such a decision procedure, we will get the finite model property.

THEOREM 2.5 (Syntactic cut-elimination theorem). *Any sequent provable in* BKS^* *admits a derivation in which no cut appear.*

The proof of cut-eliminability is almost standard, that is, supposing Π is a proof of the sequent S and

$$\frac{\Gamma \vdash \Delta, \gamma \qquad \Gamma, \gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

is one of the top-most occurrence of the cut-rule within Π, we will prove that it can be eliminated by principal induction on the structural complexity $\delta(\gamma)$ of the cut-formula γ, which is extended here to the modal case in the obvious way by putting $\delta(\Box(\alpha, \beta)) = \delta(\alpha) + \delta(\beta) + 1$, and secondary induction on the length of the thread of γ (see [21]). The reductions to lower the length of the threads and those for lowering the complexity of the cut-formula in the non-modal cases are standard. Thus, we consider here only the case the cut-formula is $\Box(\alpha, \beta)$ and a modal rule is applied both on the right and the left thread, namely, the following case:

$$\frac{\dfrac{\alpha \vdash \bigwedge_i \bigvee_{j_i} \gamma_{ij_i} \quad \bigwedge_i \bigvee_{j_i} \delta_{ij_i} \vdash \beta}{\{\Box(\gamma_{ij_i}, \delta_{ij_i})\}_{i,j_i} \vdash \Box(\alpha, \beta)} \quad \dfrac{\phi \vdash \bigwedge_h \bigvee_{k_h} \phi_{hk_h} \quad \bigwedge_h \bigvee_{k_h} \psi_{hk_h} \vdash \psi}{\{\Box(\phi_{hk_h}, \psi_{hk_h})\}_{h,k_h} \vdash \Box(\phi, \psi)}}{\{\Box(\gamma_{ij_i}, \delta_{ij_i})\}_{i,j_i} \cup (\{\Box(\phi_{hk_h}, \psi_{hk_h})\}_{h,k_h} \setminus \Box(\alpha, \beta)) \vdash \Box(\phi, \psi)}$$

where one of the formulas in the set $\{\Box(\phi_{hk_h}, \psi_{hk_h})\}_{h,k_h}$ is $\Box(\alpha, \beta)$.

In this case,

$$\alpha \vdash A_1 \wedge \ldots \wedge A_n$$

and

$$\phi \vdash B_1 \wedge \ldots \wedge (B_h \vee \alpha) \wedge \ldots \wedge B_m$$

where $A_1 \equiv \bigvee_{j_1} \gamma_{1j_1}, \ldots, A_n \equiv \bigvee_{j_n} \gamma_{nj_n}$ and $B_1 \equiv \bigvee_{k_1} \phi_{1k_1}, \ldots, B_m \equiv \bigvee_{k_m} \phi_{mk_m}$. Hence

$$\phi \vdash B_1 \qquad \ldots \qquad \phi \vdash B_h, \alpha \qquad \ldots \qquad \phi \vdash B_m$$

follows since the property of permutability of the propositional rules holds for BKS (see [14]). Then, by using a cut on the formula α, whose structural complexity is lower than that of $\Box(\alpha, \beta)$, we obtain that

$$\phi \vdash B_h, A_1 \wedge \ldots \wedge A_n$$

and hence we can construct, by using no cut, a proof of

$$\phi \vdash B_1 \wedge \ldots \wedge (B_h \vee A_1) \wedge \ldots \wedge (B_h \vee A_n) \wedge \ldots \wedge B_m$$

In a similar way, from

$$A'_1 \wedge \ldots \wedge A'_n \vdash \beta$$

and

$$B'_1 \wedge \ldots \wedge (B'_h \vee \beta) \wedge \ldots \wedge B'_m \vdash \psi$$

where $A'_1 \equiv \bigvee_{j_1} \delta_{1j_1}, \ldots, A'_n \equiv \bigvee_{j_n} \delta_{nj_n}$ and $B'_1 \equiv \bigvee_{k_1} \psi_{1k_1}, \ldots, B'_m \equiv \bigvee_{k_m} \psi_{mk_m}$, we obtain both that

$$B'_1 \wedge \ldots \wedge B'_h \wedge \ldots \wedge B'_m \vdash \psi$$

and that

$$B'_1 \wedge \ldots \wedge \beta \wedge \ldots \wedge B'_m \vdash \psi$$

Hence, by using a cut on β, whose structural complexity is lower than that of $\Box(\alpha, \beta)$, we obtain

$$B'_1 \wedge \ldots \wedge A'_1 \wedge \ldots \wedge A'_n \wedge \ldots \wedge B'_m \vdash \psi$$

Thus, by using no cut, we can construct also a proof of

$$B'_1 \wedge \ldots \wedge (B'_h \vee A'_1) \wedge \ldots \wedge (B'_h \vee A'_n) \wedge \ldots \wedge B'_m \vdash \psi$$

Then we can conclude; in fact, by using an instance of $\Box$-*gen-rule* we obtain the sequent in the conclusion of the application of the cut-rule, except for the non essential repetition of some of the boxed assumptions.

2.2. Decidability and the finite models property

Nice consequences of the theorem of cut-elimination that we proved in the previous section 2.1 are decidability of BK and the finite model property.

In order to obtain these results, in this section, instead of using a generic sequent calculus for classical propositional logic, as we did till now, we will consider a sequent calculus in which the rules for the classical connectives are double sound, that is, a sequent calculus such that a sequent in the conclusion of a non-structural rule is valid if and only if all sequents in the premises of that rule are valid[8].

Decidability and finite model property for BK are an immediate consequence of the fact that we can provide an always terminating procedure for looking for the derivability of any sequent which does not use the cut-rule; and such a procedure is correct, that is, when it fails we can use the proof tentative to build a finite counter-model for the non provable sequent.

The proof of this statement follows the general ideas of a cut-redundancy proof (see for instance [21], or [23] for an application in a modal case); we have only to add a special treatment for the modal case. To deal with this case we need to introduce a new notion of complexity of a sequent. It will be used in the sequel to prove that our decision procedure is always terminating. In fact, in the case of the $\Box$-*gen-rule* we cannot state that the premises of the rule are simpler that the conclusion by simply counting the number of the connectives in the formulas in the sequents that appear in the premises. Nevertheless, we can recognize that the premises are simpler if we introduce a suitable notion of complexity which allows to compute the number of nested boxes within a sequent. Here is the definition of complexity of a sequent that we will use: we first define the maps $C_1(-)$ and $C_2(-)$ from formulas to natural numbers and then we use them to define a map $C(-)$ from sequents to couple of natural numbers.

- $C_1(\alpha) = 0$ if α is a propositional variable
- $C_1(\alpha \wedge \beta) = \mathsf{max}\{C_1(\alpha), C_1(\beta)\}$
- $C_1(\alpha \vee \beta) = \mathsf{max}\{C_1(\alpha), C_1(\beta)\}$
- $C_1(\neg\alpha) = C_1(\alpha)$
- $C_1(\alpha \supset \beta) = \mathsf{max}\{C_1(\alpha), C_1(\beta)\}$
- $C_1(\Box(\alpha, \beta)) = C_1(\alpha) + C_1(\beta) + 1$

- $C_2(\alpha) = 0$ if α is a propositional variable or $\alpha \equiv \Box(\alpha_1, \alpha_2)$
- $C_2(\alpha \wedge \beta) = C_2(\alpha) + C_2(\beta) + 1$
- $C_2(\neg\alpha) = C_2(\alpha) + 1$

- $C(\alpha) = \langle C_1(\alpha), C_2(\alpha)\rangle$
- $C(\Gamma \vdash \Phi) = C(\bigwedge_{\gamma\in\Gamma} \gamma \supset \bigvee_{\beta\in\Phi} \beta)$

Next, we order the pairs according to the lexicographical order. It is easy to check that with this definition of complexity the formula $\Box(p, q)$ is more complex than any formula of classical propositional logic. Note that, according to this notion of complexity of a sequent, the complexity of the sequent in the conclusion of any $\Box$-*gen-rule* is higher than the complexity of any sequent in its premises.

[8]A calculus of this kind can be easily defined; for example, see [27] or just consider Gentzen's original sequent calculus for classical propositional logic (see [10]) and consider sequents as couples of finite sets instead of couples of finite lists.

The decision strategy for the non-modal case is simply to apply any applicable propositional rule. Since the premise(s) of each propositional rule is (are) strictly simpler than the conclusion, this search procedure is going to arrive in a finite number of steps at an axiom or at a sequent of the following shape:

$$p_1, \ldots, p_r, \Box(\alpha_1, \beta_1), \ldots, \Box(\alpha_n, \beta_n) \vdash \Box(\phi_1, \psi_1), \ldots, \Box(\phi_m, \psi_m), q_1, \ldots, q_s \tag{8}$$

where $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ are propositional variables.

If all of the leaves of the search tree we arrived at in this way are axioms then our search procedure stops with a success. On the other hand, let us suppose that we did not arrive at an axiom. To begin with a simple case let us first consider the case that the sequent 8 that we are examining is

$$p_1, \ldots, p_r, \Box(\alpha_1, \beta_1), \ldots, \Box(\alpha_n, \beta_n) \vdash q_1, \ldots, q_s$$

that is, the case $m = 0$ and $\{p_1, \ldots, p_r\} \cap \{q_1, \ldots, q_s\} = \emptyset$. In this case the sequent can easily be falsified in the finite model $(\{*\}, R, \nu)$ defined on the one element set $\{*\}$ by setting $R = \emptyset$ and $\nu(p) = \{*\}$ if and only if $p \in \{p_1, \ldots, p_r\}$.

On the other hand, that is, when we consider the case $m \geq 1$, the cut-elimination theorem suggests that the sequent 8, provided it is not already an axiom, can only be obtained by weakening from:

$$\Box(\alpha_1, \beta_1), \cdots, \Box(\alpha_n, \beta_n) \vdash \Box(\phi_h, \psi_h) \tag{9}$$

for some $1 \leq h \leq m$. Indeed, if we will be able to find a suitable index h and prove the corresponding sequent 9, then we will eventually obtain a proof of the sequent 8 by using some instances of weakening. Of course, the problem will be in proving that if, for no index h, a proof tentative is successful then the sequent 8 is not valid and it can be falsified by using some finite counter-model.

In general, a cut free proof of the sequent 9 should be obtained by an application of the $\Box$-*gen-rule* possibly followed by an instance of weakening. Thus, our proof search algorithm is supposed to find a suitable subset W of the set $\{1, \ldots, n\}$ such that the sequent

$$\{\Box(\alpha_i, \beta_i)\}_{i \in W} \vdash \Box(\phi_h, \psi_h) \tag{10}$$

is provable by an application of the $\Box$-*gen-rule*. And the left premise of such a rule should have the following shape:

$$\phi_h \vdash \bigwedge_{A \in \mathcal{G}} \bigvee_{j \in A} \alpha_j \tag{11}$$

for some collection $\mathcal{G}$ of subsets of the set $\{1, \ldots, n\}$[9].

We remark that the sequent 11 is provable if and only if, for any $A \in \mathcal{G}$, $\phi_h \vdash \bigvee_{j \in A} \alpha_j$. So, in the search for the left premise of the required $\Box$-*gen-rule* we can consider only the following collection of set of indexes:

$$\mathcal{H} = \{A \subseteq \{1, \ldots, n\} |\ \phi_h \vdash \bigvee_{j \in A} \alpha_j\}$$

By the definition of complexity given above, it is easily shown that all of the sequents $\phi_h \vdash \bigvee_{j \in A} \alpha_j$ are simpler than the sequent 9 and thus we can assume to be able to decide on membership to $\mathcal{H}$.

Let us note that supposing $\mathcal{H}$ is empty, that is, supposing there is no subset $A \subseteq \{1, \ldots, n\}$ such that $\phi_h \vdash \bigvee_{j \in A} \alpha_j$, yields in particular that $\phi_h \not\vdash \alpha_1, \ldots, \alpha_n$ and hence, by inductive hypothesis, a finite model (M'_h, R'_h, ν'_h) can be built which contains a point y_h such that $y_h \Vdash \phi_h$ and, for all $1 \leq j \leq n$, $y_h \Vdash \neg\alpha_j$.

Let us now observe that $\not\vdash \psi_h$, otherwise the sequent 9 is obviously provable by an instance of $\Box$-*gen-rule* with premises $\phi_h \vdash \bigwedge_{i=1\ldots0, j=1\ldots m_i} \alpha_{ij}$ and $\bigwedge_{i=1\ldots0, j=1\ldots m_i} \beta_{ij} \vdash \psi_h$. So a finite model (M''_h, R''_h, ν''_h) can be built which contains a point z_h such that $z_h \Vdash \neg\psi_h$.

Then, a finite model (M_h, R_h, ν_h) which falsifies the sequent 9 can be built by adding a new point x_h to M'_h and M''_h, in order to obtain $M_h = \{x\} \cup M'_h \cup M''_h$, and setting $R_h = \{\langle x, y, z\rangle\} \cup R'_h \cup R''_h$ and $\nu_h = \nu'_h \cup \nu''_h$.

So, let us continue under the assumption that $\mathcal{H} \neq \emptyset$. We will use in the sequel the fact that in this case $\{1, \ldots, n\} \in \mathcal{H}$.

If $\mathcal{H} \neq \emptyset$, then if we would be able to find a subset $\mathcal{G}$ of $\mathcal{H}$ such that:

$$\bigwedge_{A \in \mathcal{G}} \bigvee_{j \in A} \beta_j \vdash \psi_h \tag{12}$$

we would have found the required instance of $\Box$-*gen-rule*[10].

To this aim, we need some preliminary lemmas. Let us consider the set $\mathcal{F}$ of all the functions $\phi : \mathcal{H} \longrightarrow \{1, \ldots, n\}$ such that $\phi(A) \in A$[11].

LEMMA 2.5. *Suppose no subset $\mathcal{G}$ of $\mathcal{H}$ can be found such that the sequent 12 holds. Then it is constructively given a (choice) function $\phi^* \in \mathcal{F}$ such that*

$$\bigwedge_{A \in \mathcal{H}} \beta_{\phi^*(A)} \not\vdash \psi_h \tag{13}$$

[9] For a better comprehension of the sequel, it can be useful to note that this condition doesn't mean that the elements of the set $\mathcal{G}$ are a partition of the set $\{1, \ldots, n\}$.

[10] Notice again that all of the sequents 12 can be assumed to be decidable since, according to the definition of complexity that we gave in the beginning of this section, they also are simpler than the sequent 9.

[11] This is the set of choice functions on $P(\{1, \ldots, n\}) \setminus \emptyset$.

Proof. If no subset $\mathcal{G}$ of $\mathcal{H}$ can be found which satisfies the condition in the hypothesis, then in particular, namely, for $\mathcal{G} = \mathcal{H}$, we have that

$$\bigwedge_{A\in\mathcal{H}} \bigvee_{j\in A} \beta_j \not\vdash \psi_h$$

Then, by distributivity, we obtain:

$$\bigvee_{\phi\in\mathcal{F}} \bigwedge_{A\in\mathcal{H}} \beta_{\phi(A)} \not\vdash \psi_h$$

Hence the result is immediate. ■

The function ϕ^* that we pointed out in the previous lemma is useful for finding a suitable subset of indexes $B = \{\phi^*(A) | A \in \mathcal{H}\}$ of the set $\{1, \dots, n\}$ such that, by induction on the complexity of the considered sequent, a finite model (M'_h, R'_h, ν'_h) can be built which contains a point z_h such that, for any β_i with $i \in B$, $z_h \Vdash \beta_i$ whereas $z_h \Vdash \neg\psi_h$.

Note that to build a finite counter-model for the sequent 9 when $B = \{1, \dots, n\}$ we need only to build a finite model (M''_h, R''_h, ν''_h) which contains a point y_h such that $y_h \Vdash \phi_h$. Since the sequent 9 is clearly provable if $\vdash \neg\phi_h$, and hence our proof search would have stopped with a proof in this case, we can suppose, by inductive hypothesis, to know how to build such a model.

The next lemma will show how to proceed in building the finite counter-model for the sequent 9 when the set of indexes B is not $\{1, \dots, n\}$.

LEMMA 2.6. *Suppose* $B \neq \{1, \dots, n\}$ *and set* $C = \{1, \cdots, n\} \setminus \phi^*(\mathcal{H})$. *Then*

$$\phi_h \not\vdash \bigvee_{j\in C} \alpha_j \tag{14}$$

Proof. Suppose the sequent $\phi_h \vdash \bigvee_{j\in C} \alpha_j$ is provable. Then $C \in \mathcal{H}$. Consider now the function ϕ^* that we pointed out in the previous lemma 2.5. Then, we get that $\phi^*(C) \in C$ since $\phi^* \in \mathcal{F}$ whereas the very definition of C yields that $\phi^*(C) \notin C$. Contradiction. ■

Thus, by inductive hypothesis, we can build a finite model (M''_h, R''_h, ν''_h) such that there is a point y_h such that for any α_j, with $j \in \{1, \dots, n\} \setminus \phi^*(\mathcal{H})$, $y_h \Vdash \neg\alpha_j$ and $y_h \Vdash \phi_h$.

In order to build a finite counter-model (M_h, R_h, ν_h) for the sequent 9 we can now put together the two models we built and add them a new point x_h, that is,

$$M_h = \{x_h\} \cup M'_h \cup M''_h$$

and define the relation R_h by setting

$$R_h \equiv \{< x_h, y_h, z_h >\} \cup R'_h \cup R''_h$$

and the interpretation ν_h by setting, for any $w \in M_h$ and any propositional variable p,

$$w \in \nu_h(p) \text{ if and only if } w \in \nu'_h(p) \text{ or } w \in \nu''_h(p)$$

Let us go back now to the problem of the proof of the sequent 8 and let us suppose that for no $1 \leq h \leq m$, the corresponding sequent 9 is provable, otherwise we would have the required proof of the sequent 8. Then, for each $1 \leq h \leq m$, we can construct as above the finite models (M'_h, R'_h, ν'_h) and (M''_h, R''_h, ν''_h) with suitable points y_h and z_h. Then in order to build a finite counter-model (M, R, ν) for the sequent 8, it is sufficient to put all of these models together, that is, we have to add a new point x and connect it with all the couple (y_h, z_h). So,

$$\begin{aligned}
M &\equiv \{x\} \cup M'_1 \cup M''_1 \cup \ldots \cup M'_m \cup M''_m \\
R &\equiv \{\langle x, y_1, z_1\rangle, \ldots, \langle x, y_m, z_m\rangle\} \cup R'_1 \cup R''_1 \cup \ldots \cup R'_m \cup R''_m \\
\nu(p) &= \begin{cases} \nu'_1(p) \cup \nu''_1(p) \cup \ldots \nu'_m(p) \cup \nu''_m(p) \cup \{x\} & \text{if } p \in \{p_1, \ldots, p_r\} \\ \nu'_1(p) \cup \nu''_1(p) \cup \ldots \nu'_m(p) \cup \nu''_m(p) & \text{otherwise} \end{cases}
\end{aligned}$$

It is now obvious that the point x falsifies the sequent 8. In fact, for each $p \in \{p_1, \ldots, p_r\}$, $x \Vdash p$ holds by definition of the valuation ν and, for $1 \leq i \leq n$, $x \Vdash \Box(\alpha_i, \beta_i)$ since, for each $1 \leq h \leq m$ and for each y_h and z_h, if $y_h \Vdash \alpha_i$ then $z_h \Vdash \beta_i$. Finally, for no $q \in \{q_1, \ldots, q_s\}$, $x \Vdash q$, again by definition of the valuation ν, and, for each $1 \leq h \leq m$, there are suitable points y_h and z_h in M such that $R(x, y_h, z_h)$ holds and $y_h \Vdash \phi_h$ and $z_h \Vdash \neg\psi_h$ and hence $x \Vdash \neg\Box(\phi_h, \psi_h)$.

3. RELATIONS BETWEEN THE LOGIC BK, THE INTERSECTION TYPES SYSTEM AND RELEVANTS LOGICS

In this section we point out the possible relations between BK, the intersection types systems and the relevant logics introduced in [16].

First we refine the notion of model for BK. Let us define the following semantics which clearly recalls and extends the semantics proposed in [4] for the sub-typing relation $\leq_\wedge$ (see section 1).

DEFINITION 3.1. Let $\mathcal{A} = (A, \cdot)$ be any applicative structure. Then ν is a *valuation* of the formulas of BK into subsets of A if the following conditions are satisfied:

$$
\begin{array}{lcl}
\nu(\top) & = & A \\
\nu(\bot) & = & \emptyset \\
\nu(\alpha \wedge \beta) & = & \nu(\alpha) \cap \nu(\beta) \\
\nu(\alpha \vee \beta) & = & \nu(\alpha) \cup \nu(\beta) \\
\nu(\neg \alpha) & = & \nu(\alpha)^c \\
\nu(\Box(\alpha, \beta)) & = & \{x \in A \mid (\forall y \in \nu(\alpha)) \text{ if } x \cdot y \text{ is defined then } x \cdot y \in \nu(\beta)\}
\end{array}
$$

Let α be any formula of BK. Then it will be said *valid* if and only if, for every applicative structure $\mathcal{A}$ and every valuation ν into $\mathcal{A}$, $\nu(\alpha) = \mathcal{A}$.

By using the results in the previous section we can obtain the following completeness result.

THEOREM 3.1. *Let $\Gamma \vdash \Phi$ be any sequent. Then $\Gamma \vdash \Phi$ is derivable if and only if for every applicative structure $\mathcal{A}$ and every valuation ν, $\bigcap_{\alpha \in \Gamma} \nu(\alpha) \subseteq \bigcup_{\beta \in \Phi} \nu(\beta)$.*

Proof. For the proof of validity only the correctness of the $\Box$-*gen-rule* deserves some comments; but it is not difficult to show that this rule is correct by mimicking the correctness proof we exhibited in the past sections.

For what concerns completeness, let us observe that the finite models (M, R, ν) of the previous section 2.2 can be constructed in such a way that for every $x, y \in M$ there exists at most one element $z \in M$ such that $R(x, y, z)$ holds. Thus, provided $\Gamma \not\vdash \Phi$, let (M, R, ν) be such a finite counter-model for $\Gamma \vdash \Phi$. Then we can define the applicative structure $\mathcal{A} = (M, \cdot)$ such that $x \cdot y = z$ if and only if $R(x, y, z)$ and the valuation ν such that, for every α, $\nu(\alpha) = \{x \mid x \Vdash \alpha\}$. It is easy to check that these definitions are correct and that they yield $\bigcap_{\alpha \in \Gamma} \nu(\alpha) \not\subseteq \bigcup_{\beta \in \Phi} \nu(\beta)$. ■

It is now possible to establish a conservativity result for the sub-typing relation:

THEOREM 3.2. *Let I be the interpretation of the types of $\Lambda_\wedge$ into formulas of* BK *defined by setting:*

$$
\begin{array}{l}
I(\alpha) = \alpha \ \ \textit{for every type variable } \alpha \\
I(\omega) = \top \\
I(\alpha \wedge \beta) = I(\alpha) \wedge I(\beta) \\
I(\alpha \to \beta) = \Box(I(\alpha), I(\beta))
\end{array}
$$

Then $\alpha \leq_\wedge \beta$ if and only if $I(\alpha) \vdash I(\beta)$.

Proof. If $\alpha \leq_\wedge \beta$ then $I(\alpha) \vdash I(\beta)$ follows by the fact that all the axioms on the sub-typing relation are translated into valid sequents and BKS^* is closed under the translation of all the rules for the sub-typing relation.

To establish the converse, observe that if $\alpha \not\leq_\wedge \beta$ then there exists a weakly extensional λ-algebra $\mathcal{A}$ and a valuation of the types ν such that $\nu(\alpha) \not\subseteq \nu(\beta)$; it is easy to check that the same $\mathcal{A}$ and the same ν are such that $\nu(I(\alpha)) \not\subseteq \nu(I(\beta))$, so the completeness theorem 3.1 yields $I(\alpha) \not\vdash I(\beta)$, since any λ-model is clearly a partial applicative structure. ■

Thus, it seems that the new semantics we proposed for BK naturally extends the sub-typing relation $\leq_\wedge$. Unfortunately, our completeness result holds if we let $\mathcal{A}$ vary over all kind of applicative structures; and the ones that we used to show the completeness theorem are far from being weakly extensional λ-algebras or even combinatorially complete applicative structures.

Nevertheless, BK shows that an alternative approach can be followed in the search of interesting models for computation, that is, one can select a suitable sub-logic L of BK such that the completeness theorem for this logic holds with respect to the class of λ-models. If such a task will be achieved then the sub-logic L immediately suggests how to define a complete typing system for the lambda-calculus which extends the intersection type system introduced in section 1. In fact a similar approach has been pursued in [9] in order to obtain an intersection types style semantic for the language XML.

Another example of the same idea has been noticed by R.K. Meyer; in fact he observed that if we drop negation from BK the completeness result for this fragment of BK holds with respect to the class of structures with a total binary operation.

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