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Limits of Boolean algebras and Boolean valued models

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Contents

Abstract

The purpose of this research is to formalize forcing and iterated forcing in the Boolean algebras language. We present the argument principally trough the study of Boolean valued models, regular embeddings and projections between complete Boolean algebras, and iterated systems of complete Boolean algebras.

Introduction

This work in divided in four parts. For the first two chapters we refer to [1], [2] and [4], for the last two chapters we refer to [5] and [6].

In the first chapter, we introduce as preliminary notions: posets, lattices and Boolean algebras.

They constitute the matter of Sections 1.1 and 1.2. In particular, in Section 1.3 we present in detail the relation between posets and complete Boolean algebras.

We first prove that every poset P can be refined (Lemma 1.3.11). Consequently, we show in Theorem 1.3.12 that:
for every poset P there is a complete Boolean algebra $\mathbb{B}$, unique up to isomorphism, and a map $j : P \to \mathbb{B}$ such that:

- *$j[P]$ is dense in $\mathbb{B}$;*
- *j is order preserving;*
- $\forall p, q \in P(p \parallel q \leftrightarrow j(p) \wedge j(q) \neq 0_{\mathbb{B}})$.

To obtain this result we use a topological approach, defining on P the *order topology*, with base $O_p = \{q \in P : q \leq p\}$, and we then consider the *regular open algebra $RO(P)$*.

In the second chapter, we introduce Boolean valued-models. We make, by transfinite recursion, the construction of $V^{(2)}$ and thus of $V^{\mathbb{B}}$ (Section 2.1). We give then a Boolean truth value $[\![\cdot]\!]_{\mathbb{B}}$ to each $\mathbb{B}$-sentence (Section 2.2), and we expose basic results on $V^{\mathbb{B}}$ (Section 2.3).
In Section 2.4, we prove that if $\mathbb{B}'$ is a submodel of $\mathbb{B}$, then $V^{\mathbb{B}'}$ is a submodel of $V^{\mathbb{B}}$ (Theorem 2.4.1). This fact will be useful to prove that V is

canonically identified with a submodel of every $V^{\mathbb{B}}$, (Theorem 2.4.4) assigning a *standard* representative $\hat{x}$ to any element $x \in V$.
Finally, in Section 2.5 we prove the *Mixing Lemma* (Lemma 2.5.2), in order to get the *Maximum Principle* (Lemma 2.5.3) that states that $\bigvee_{u \in V^{\mathbb{B}}} [\![\phi(u)]\!]$ is actually attained at some element $u \in V^{\mathbb{B}}$.

In the third chapter, we define *regular embeddings* and *projections* between complete Boolean algebras.

We prove that for any regular embedding i there is an associated projection π_i (Proposition 3.1.5), and vice-versa (Proposition 3.1.7). Analyzing these maps we prove a variant of the Mixing Lemma (Lemma 3.2.2):
Assume $i : \mathbb{B} \to \mathbb{Q}$ is a regular embedding. Let $\dot{J}$ be a canonical name for the dual ideal of the filter generated by $i[\dot{G}_{\mathbb{B}}]$. For all $\dot{a} \in V^{\mathbb{B}}$ such that $[\![\dot{a} \in (\mathbb{Q}/\dot{J})^+]\!]_{\mathbb{B}} = 1_{\mathbb{B}}$, there is a unique $r_{\dot{a}} \in \mathbb{Q}$ such that

$$\pi(r_{\dot{a}}) = 1_{\mathbb{B}};$$
$$[\![\dot{a} = [r_{\dot{a}}]_{\dot{J}}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

Finally, in the last chapter, we present first of all *iteration systems* of complete Boolean algebras $\mathcal{F} = \{i_{\alpha,\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha < \beta \in \delta\}$ (Section 4.1). In particular, we study the set of all threads $T(\mathcal{F})$ and the set of constant threads $C(\mathcal{F})$. Then we summarize the general iterated forcing (Section 4.2) and we start a comparison of the two constructions (Section 4.3).

Chapter 1

Posets and Boolean algebras

In the current chapter we introduce posets, lattices and Boolean algebras. There are two main objectives. The first is to define the notion of complete Boolean algebra through the definition of lattices, and thus posets, and to present some basic properties. The second and more interesting purpose is to show how a partially ordered set can be uniquely embedded in a complete Boolean algebra, up to isomorphism.

1.1 Posets

In this section we summarize basic facts about partially ordered sets.

Definition 1.1.1. A *partially ordered set* or a *poset* $\langle P, \leq_P \rangle$ is a set P together with a binary relation $\leq_P$ on P which is transitive and reflexive.

Observe that $\leq_P$ may not be a total relation, that is, not every couple of elements of P are comparable. If no confusion arises we simply write $\leq$ instead of $\leq_P$.

Definition 1.1.2. Let $\langle P, \leq \rangle$ be a poset. Two elements $p, q \in P$ are said to be *compatible* , written $p \parallel q$, if there is a $r \in P$ such that $r \leq p$ and $r \leq q$. Two elements $p, q \in P$ are said *incompatible* $p \perp q$ if they are not compatible.

Definition 1.1.3. We say that a subset $A \subseteq P$ is an *antichain* if

$$\forall p, q \in A (p \perp q).$$

Definition 1.1.4. A *maximal antichain* is an antichain that is not a proper subset of any other antichain.

Definition 1.1.5. A subset $D \subseteq P$ is *dense* in P iff

$$\forall p \in P \, \exists d \in D \, (d \leq p).$$

Definition 1.1.6. A subset $D' \subseteq P$ is *predense* in P iff

$$\forall p \in P \, \exists d \in D' \, (d \parallel p).$$

Proposition 1.1.7. *Every dense set $D \subseteq P$ contains an antichain $A \subseteq D$, which is maximal in P, with respect to inclusion. Conversely, every maximal antichain $A \subseteq P$ is predense in P and the downward closure of A is dense.*

Proof. We begin with the second assertion. For every $p \in P$ we have two choices: $p \in A$ or $p \notin A$. For the last case, by A maximality, it can not happens that for all $a \in A$, $p \perp a$. So there is $a \in A$ such that $a \parallel p$. Thus A is predense. Remark that the downward closure of a predense set is dense, thus $A' = \{p \in P : \exists a \in A(p \leq a)\}$ is dense.

Consider now a dense subset $D \subseteq P$. By Zorn's lemma there is a maximal antichain $A \subseteq D$. We show that A is a maximal antichain for P too. That is, every $p \in P$ is compatible with each $a \in A$.
To this aim, fix an arbitrary $p \in P$, by D density, there is a $d \in D$ such that $d \leq p$. By A maximality in D, we have that for all $a \in A$, $a \parallel d$. Thus for every $a \in A$, there is a $r \in P$ such that $r \leq a$ and $r \leq d$. The last condition implies that $r \leq p$. Thus $a \parallel p$ for every $a \in A$, as we wanted. □

We present now the definitions of ideal and filter, that are dual to each other. Compare these definitions with the ones for Boolean algebras (Definitions 1.2.16 and 1.2.17).

Definition 1.1.8. A set $I \subseteq P$ is an *ideal* in P if for all $a, b \in I$ and for all $p \in P$:

(i) $\exists c \in I \, (a \leq c \wedge b \leq c)$;

(ii) $p \leq a \Rightarrow p \in I$.

Definition 1.1.9. A set $F \subseteq P$ is a *filter* if for all $f, g \in F$ and for all $h \in P$:

(i) $\exists r \in F \, (r \leq f \wedge r \leq g)$;

(ii) $f \leq h \Rightarrow h \in F$.

1.2 Lattices and Boolean Algebras

In the following, we describe lattices and we define Boolean algebras as complemented distributive lattices.

Definition 1.2.1. A *lattice* is a nonempty partially ordered set L with partial ordering $\leq_L$ in which each two-element subset $\{x, y\}$ has a supremum or *join* , $x \vee y$, and an infimum or *meet*, $x \wedge y$.

Definition 1.2.2. A *top* element of a lattice L is an element denoted by 1_L such that $x \leq_L 1_L$ for all $x \in L$. A *bottom* element is denoted by 0_L and is such that $0_L \leq_L x$ for all $x \in L$. A lattice with top and bottom elements is called *bounded.*

In a bounded lattice it is easy to see that the following hold:

$$\begin{aligned} x \vee 0_L = x, &\qquad x \wedge 1_L = x, \\ x \vee x = x, &\qquad x \wedge x = x, \\ x \vee y = y \vee x, &\qquad x \wedge y = y \wedge x, \\ x \vee (y \vee z) = (x \vee y) \vee z, &\qquad x \wedge (y \wedge z) = (x \wedge y) \wedge z, \\ (x \vee y) \wedge y = y, &\qquad (x \wedge y) \vee y = y. \end{aligned} \tag{1.1}$$

Conversely, we can create a lattice by the equations in (1.1), in the following way.

Remark 1.2.3. Suppose that $(L, \vee, \wedge, 0_L, 1_L)$ is an algebraic structure with $\vee$ and $\wedge$ binary operations that satisfy (1.1). Define the relation $\leq_L$ on L by

$$x \leq_L y \text{ iff } x \wedge y = y. \tag{1.2}$$

Then $(L, \leq_L)$ is a bounded lattice in which $\vee$ and $\wedge$ are respectively the join and meet operations, and 1_L and 0_L the top and bottom elements. This is the *equational characterization* of lattices.

Definition 1.2.4. A lattice is said to be *distributive* if the following identities are satisfied:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \tag{1.3}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{1.4}$$

Remark that (1.3) and (1.4) are equivalent. For example, assuming (1.3), we have:

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= [x \wedge (x \vee z)] \vee [y \wedge (x \vee z)] \\ &= x \vee [(y \wedge x) \vee (y \wedge z)] \\ &= [x \vee (y \wedge x)] \vee (y \wedge z) \\ &= x \vee (y \wedge z).\end{aligned}$$

Definition 1.2.5. Let L a bounded lattice. A *complement* for an element $a \in L$ is an element $b \in L$ satisfying:

$$\begin{aligned}a \vee b &= 1_L; \\ a \wedge b &= 0_L.\end{aligned}$$

Complement elements, if they exist, may not be unique. However, the following proposition proves that, if a lattice is distributive, complements are unique.

Proposition 1.2.6. *In a distributive lattice an element can have at most one complement.*

Proof. If b, c are complements of an element $a \in L$, then $a \vee b = a \vee c = 1_L$ and $a \wedge b = a \wedge c = 1_L$. We deduce:

$$\begin{aligned}b &= b \vee 0_L \\ &= b \vee (a \wedge c) \\ &= (b \vee a) \wedge (b \vee c) \\ &= 1_L \vee (b \vee c) \\ &= b \vee c.\end{aligned}$$

Similarly $c = c \vee b$ so that $b = c$. □

We will often use the sign $\neg a$ to design the complement of a.

Definition 1.2.7. A lattice is called *complemented* if it is bounded and each of its elements has a complement.

Definition 1.2.8. A lattice L is *complete* if every subset $X = \{x_i : i \in I\} \subseteq L$ has an infimum $\bigwedge_{i \in I} x_i$ and a supremum $\bigvee_{i \in I} x_i$. If $X = \emptyset$ and is L complete, then:

$$\bigwedge \emptyset = 1_L, \quad \bigvee \emptyset = 0_L.$$

Notice that for a lattice to be complete, it suffices that every subset have a supremum, or every subset an infimum. For the supremum (infimum), if it exists, of the set of lower (upper) bounds of a given subset X is the infimum (supremum) of X.

We turn finally to the definition of Boolean algebras collecting the arguments so far seen.

Definition 1.2.9. A *Boolean algebra* $\mathbb{B}$ is a complemented distributive lattice. If in addition $\mathbb{B}$ is also complete, we have a *complete Boolean algebra.*

In a complete Boolean algebra we have the following identities:

$$\neg(\bigvee_{i\in I} x_i) = \bigwedge_{i\in I} \neg x_i, \qquad \neg(\bigwedge_{i\in I} x_i) = \bigvee_{i\in I} \neg x_i,$$

$$x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i), \qquad x \vee \bigwedge_{i\in I} y_i = \bigwedge_{i\in I}(x \vee y_i).$$

Remark 1.2.10. Equivalently, it is possible to describe a Boolean algebra by *equational characterization*. In this case a Boolean algebra is a six-tuple $(\mathbb{B}, \wedge, \vee, \neg, 0_{\mathbb{B}}, 1_{\mathbb{B}})$ consisting of a set $\mathbb{B}$, equipped with two binary operations $\wedge$ and $\vee$, a unary operation $\neg$ and two elements $0_{\mathbb{B}}$ and $1_{\mathbb{B}}$, such that for all elements $a, b, c \in \mathbb{B}$, the following axioms hold:

$$a \vee (b \vee c) = (a \vee b) \vee c \qquad \text{associativity}$$
$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad \text{distributivity}$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee b = b \vee a \qquad \text{commutativity}$$
$$a \wedge b = b \wedge a$$
$$a \vee 0_{\mathbb{B}} = a \qquad \text{identity}$$
$$a \wedge 1_{\mathbb{B}} = a$$
$$a \vee \neg a = 1_{\mathbb{B}} \qquad \text{complements}$$
$$a \wedge \neg a = 0_{\mathbb{B}}$$

Definition 1.2.11. A complete Boolean algebra $\mathbb{B}'$ is said to be a *complete subalgebra* of $\mathbb{B}$ if $\mathbb{B}'$ is a subalgebra of $\mathbb{B}$ and for any $X \subseteq \mathbb{B}'$, $\bigvee X$ and $\bigwedge X$ formed in $\mathbb{B}'$ are the same as those formed in $\mathbb{B}$.

Definition 1.2.12. If $\mathbb{B}$ is a Boolean algebra, let $\mathbb{B}^+ = \mathbb{B} \setminus \{0\}$ denote the set of all nonzero elements of $\mathbb{B}$. If $a \in \mathbb{B}^+$, the set $\mathbb{B} \restriction a = \{u \in \mathbb{B} : u \leq a\}$ with the partial order inherited from $\mathbb{B}$, is a Boolean algebra.

Definition 1.2.13. An element $a \in \mathbb{B}$ is called an *atom* if it is a minimal element of $\mathbb{B}^+$; equivalently, if there is no x such that $0 < x < a$.

Definition 1.2.14. A Boolean algebra is *atomic* if for every $u \in \mathbb{B}^+$ there is an atom $a \leq u$; $\mathbb{B}$ is *atomless* if it has no atoms.

Definition 1.2.15. If $W \subseteq \mathbb{B}^+$ is an antichain and if $\bigvee W = u$ then we say that W is a *partition* of u. A partition of $1_{\mathbb{B}}$ is just a *partition* or a *maximal antichain.*

Let us now define ideals and filters for Boolean algebras. Remark that if we define a partial order as in (1.2) we turn to definitions (1.1.8) and (1.1.9).

Definition 1.2.16. A set $I \subseteq \mathbb{B}$ is an *ideal* in $\mathbb{B}$ iff

(i) $0_{\mathbb{B}} \in I$ and $1_{\mathbb{B}} \notin I$;

(ii) if $a, b \in I$, then $a \vee b \in I$;

(iii) if $a \in I$ and $b \in \mathbb{B}$, then $a \wedge b \in I$.

Definition 1.2.17. A set $F \subseteq \mathbb{B}$ is a *filter* in $\mathbb{B}$ iff

(i) $1_{\mathbb{B}} \in F$ and $0_{\mathbb{B}} \notin F$;

(ii) if $a, b \in F$, then $a \wedge b \in F$;

(iii) if $a \in F$ and $b \in \mathbb{B}$, then $a \vee b \in F$.

Definition 1.2.18. A set $U \subseteq \mathbb{B}$ is an *ultrafilter* in $\mathbb{B}$ if and only if U is a filter and

$$\forall p \in \mathbb{B} : p \in U \text{ or } \neg p \in U.$$

The dual notion is a *prime ideal* : for every $b \in \mathbb{B}$, $b \in I$ or $\neg b \in I$. One can see that an ideal is a prime ideal (and a filter is an ultrafilter) if and only if it is maximal. By AC, every ideal (filter) on $\mathbb{B}$ can be extended to a prime ideal (ultrafilter). Also, an ideal I is prime if and only if the quotient of $\mathbb{B}/I$ is the trivial algebra $\{0_{\mathbb{B}}, 1_{\mathbb{B}}\}$.

At this point let us say a few words about *quotient algebras.*

Definition 1.2.19. If I is an ideal of $\mathbb{B}$, the *quotient* $\mathbb{B}/I$ is the quotient of $\mathbb{B}$ with respect to an equivalence relation defined by:

$$[a] = [b] \text{ iff } a\Delta b \in I,$$

where $a\Delta b = (a \wedge \neg b) \vee (b \wedge \neg a)$. The operations in $\mathbb{B}/I$ are:

$$\begin{aligned} [a] \wedge_{\mathbb{B}/I} [b] &= [a \wedge_{\mathbb{B}} b], & 0_{\mathbb{B}/I} &= [0_{\mathbb{B}}], \\ [a] \vee_{\mathbb{B}/I} [b] &= [a \vee_{\mathbb{B}} b], & 1_{\mathbb{B}/I} &= [1_{\mathbb{B}}], \\ \neg_{\mathbb{B}/I}[a] &= [\neg_{\mathbb{B}} a]. \end{aligned}$$

We conclude the section with algebras homomorphisms.

Definition 1.2.20. A *homomorphism* between two Boolean algebras $\mathbb{B}$ and $\mathbb{Q}$ is a function $h : \mathbb{B} \to \mathbb{Q}$ such that for all $a, b \in \mathbb{B}$, h satisfies (by De Morgan's laws) either of the two equivalent conditions:

(i) $h(a \wedge b) = h(a) \wedge h(b)$ and $h(\neg a) = \neg h(a)$;

(ii) $h(a \vee b) = h(a) \vee h(b)$ and $h(\neg a) = \neg h(a)$.

It then follows that $h(0_{\mathbb{B}}) = 0_{\mathbb{Q}}$ and $h(1_{\mathbb{B}}) = 1_{\mathbb{Q}}$.
The *kernel* $h^{-1}[0_{\mathbb{Q}}] = \{x \in \mathbb{B} : h(x) = 0_{\mathbb{Q}}\}$ is an ideal, and the *hull* $h^{-1}[1_{\mathbb{Q}}] = \{x \in \mathbb{B} : h(x) = 1_{\mathbb{Q}}\}$ is a filter in $\mathbb{B}$.
Moreover, we have that h is *injective* iff $h^{-1}[0] = \{0\}$, or, equivalently, $h^{-1}[1] = \{1\}$. A bijective homomorphism is called *isomorphism.*
The map $f : \mathbb{B} \to \mathbb{B}/I$ given by $h(a) = [a]$ is a homomorphism onto $\mathbb{B}/I$ called the *canonical homomorphism.*

Definition 1.2.21. A homomorphism h of Boolean algebras is *complete* if for every $A \subseteq \mathbb{B}$ for which $\bigvee A$ exists,

$$\bigvee \{h(x) : x \in A\} \text{ exists and } h(\bigvee A) = \bigvee h[A].$$

1.3 Boolean completions of posets

The purpose of this section is to show that each poset P has a Boolean completion $\mathbb{B}(P)$ which is unique up to isomorphism. We will construct the Boolean completion $\mathbb{B}(P)$ of P that will be the regular open algebra $RO(P)$. Of course the approach we deal with is topological. We first work with refined posets and show that they have Boolean completions. Then, we show how to map a poset to a refined poset and thus prove the existence of the Boolean completion of any poset.

Definition 1.3.1. A poset P is *refined* if

$$\forall p, q \in P[q \not\leq p \to \exists p' \leq q : p \perp p']$$

Thus P is refined if, whenever q is not a refinement of p, q has a refinement which is incompatible with p.

We need a topology on P, the *order topology.*

Definition 1.3.2. For each $p \in P$, put

$$O_p = \{q \in P : q \leq p\}.$$

The O_p form a base for a topology on P called the *order topology.*

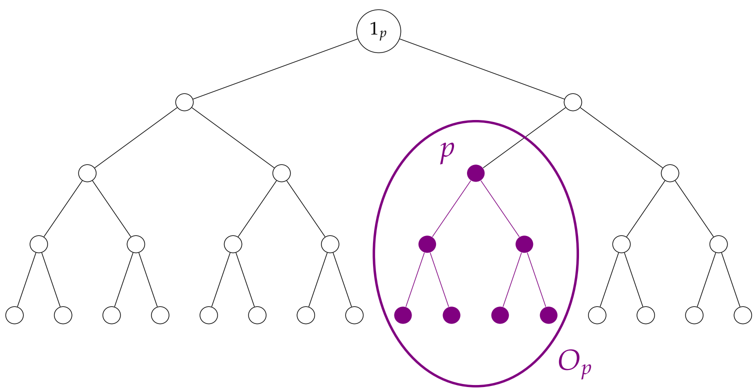


Figure 1.1: O_p in a binary tree

Definition 1.3.3. Consider P with the order topology, then the *regular open algebra* $RO(P)$ is the complete Boolean algebra of regular open sets of P, partially ordered by inclusion. Recall that a set $R \subset RO(P)$ is *regular open* if $\mathring{\overline{R}} = R$. The algebraic operations in $RO(P)$ are $\forall U, V \in RO(P), \forall \{U_i : i \in I\} \subseteq RO(P)$:

$$
\begin{aligned}
0_{RO(P)} &= \emptyset, & 1_{RO(P)} &= P, \\
U \vee V &= \mathring{\overline{U \cup V}}, & U \wedge V &= U \cap V, \\
\bigvee_{i \in I} U_i &= \mathring{\overline{\bigcup_{i \in I} U_i}}, & \bigwedge_{i \in I} U_i &= (\bigcap_{i \in I} U_i)^\circ, \\
\neg_{RO(P)} U &= (P \setminus U)^\circ.
\end{aligned}
$$

We prove that $RO(P)$ is indeed a complete Boolean algebra. We first verify that the operations defined return regular open sets. For Lemma 1.3.4 and Proposition 1.3.5 we refer to [3].

Lemma 1.3.4. *The right sides of the operations presented in Definition 1.3.3 are regular open sets.*

Proof. It is convenient to define, for every set $V \subseteq P$,

$$V^\perp = P \setminus \overline{V}$$

Thus $\mathring{\overline{V}} = V^{\perp\perp}$ since the interior of a set V can be regarded as $P \setminus \overline{P \setminus V}$.

So we must check that for U, V regular open the following are regular open:

(i) $\emptyset$;

(ii) P;

(iii) $U \cap V$;

(iv) $(U \cup V)^{\perp\perp}$;

(v) $U^\perp$.

For (i) and (ii) this is obvious.
To prove (iii)-(iv)-(v), let us first express some considerations.
First of all, remark that the closure preserves inclusions but complementation reverse them. So we have

$$U \subseteq V \to V^\perp \subseteq U^\perp \tag{1.5}$$

Subsequently, assuming U open, we observe that, since $U \subseteq \overline{U}$ then $U^{\perp} \subseteq P \setminus U$.
Since $P \setminus V$ is already closed, it follows that $\overline{U^{\perp}} \subseteq P \setminus U$.
By complementation we get:

$$U \subseteq U^{\perp\perp} \tag{1.6}$$

Now, if U is open, applying (1.5) to (1.6) we get $U^{\perp\perp\perp} \subseteq U^{\perp}$.
Conversely applying (1.6) to $U^{\perp}$ we get $U^{\perp} \subseteq U^{\perp\perp\perp}$.
Thus, for U open:

$$U^{\perp\perp\perp} = U^{\perp} \tag{1.7}$$

By (1.7) we conclude that (iv) and (v) are regular.

In order to prove (iii), we show that, for U, V open:

$$(U \cap V)^{\perp\perp} = U^{\perp\perp} \cap V^{\perp\perp} \tag{1.8}$$

Since $U \cap V \subseteq U$ and $U \cap V \subseteq V$, by (1.6), it follows that:

$$U \cap V \subseteq U^{\perp\perp} \quad \text{and} \quad U \cap V \subseteq V^{\perp\perp},$$

hence $U \cap V \subseteq U^{\perp\perp} \cap V^{\perp\perp}$.

Conversely, $U \cap \overline{V} \subseteq \overline{(U \cap V)}$. (If W is a neighborhood of a point of $U \cap \overline{V}$ then so is $W \cap U$ and this imply that $W \cap U$ meets V, or equivalently that W meets $U \cap V$).
Complementing we get $(U \cap V)^{\perp} \subseteq (P \setminus U) \cup V^{\perp}$.
Applying closure and complementation it follows

$$P \setminus (\overline{P \setminus U} \cup \overline{V^{\perp}}) \subseteq (U \cap V)^{\perp\perp}$$
$$(P \setminus \overline{P \setminus U}) \cap V^{\perp\perp} \subseteq (U \cap V)^{\perp\perp}$$

$$U \cap V^{\perp\perp} \subseteq (U \cap V)^{\perp\perp} \tag{1.9}$$

An application of (1.9) with $U^{\perp\perp}$ in place of U, followed by an application of (1.9) interchanging U and V yields:

$$U^{\perp\perp} \cap V^{\perp\perp} \subseteq (U^{\perp\perp} \cap V)^{\perp\perp} \subseteq (U \cap V)^{\perp\perp\perp\perp}$$

By (1.7):

$$U^{\perp\perp} \cap V^{\perp\perp} \subseteq (U \cap V)^{\perp\perp},$$

as was to be proved. □

Proposition 1.3.5. *$RO(P)$ is a complete Boolean algebra.*

Proof. $RO(P)$ **is a lattice:** We verify the laws on the operations.

- Idempotency of $\vee$ and $\wedge$: Clearly, $U = U \wedge U$ and $U \vee U = (U \cup U)^{\perp\perp} = U^{\perp\perp} = U$, since U is regular open.
- Commutativity of the binary operations are obvious.
- Associativity of $\wedge$ is clear. Associativity of $\vee$:

$$\begin{aligned}
U \vee (V \vee W) &= (U \cup (V \cup W)^{\perp\perp})^{\perp\perp} \\
&= (U^{\perp} \cap (V \cup W)^{\perp\perp\perp})^{\perp} \\
&= (U^{\perp} \cap (V \cup W)^{\perp})^{\perp} \\
&= (U^{\perp} \cap (V^{\perp} \cap W^{\perp}))^{\perp} \\
&= ((U^{\perp} \cap V^{\perp}) \cap W^{\perp})^{\perp} \\
&= ((U \cup V)^{\perp} \cap W^{\perp})^{\perp} \\
&= ((U \cup V)^{\perp\perp\perp} \cap W^{\perp})^{\perp} \\
&= ((U \cup V)^{\perp\perp} \cup W)^{\perp\perp} \\
&= (U \vee V) \vee W.
\end{aligned}$$

- Finally, we verify the absorption laws. First,

$$\begin{aligned}
U \wedge (U \vee V) &= U \cap (U \cup V)^{\perp\perp} \\
&= U^{\perp\perp} \cap (U \cup V)^{\perp\perp} \\
&= (U^{\perp} \cup (U \cup V)^{\perp})^{\perp} \\
&= (U^{\perp} \cup (U^{\perp} \cap V^{\perp}))^{\perp} \\
&= (U^{\perp})^{\perp} \\
&= U.
\end{aligned}$$

  Second, $U \vee (U \wedge V) = (U \cup (U \wedge W))^{\perp\perp} = U^{\perp\perp} = U$.

$RO(P)$ **is complemented:** First, it is easy to see that $\emptyset$ and P are the bottom and top elements of $RO(P)$. Furthermore, for any $U \in RO(P)$, $U \wedge U' = U \cap U^{\perp} = U \cap (P \setminus \overline{U}) \subseteq \overline{U} \cap (P \setminus \overline{U}) = \emptyset$. Finally, $U \vee U' = (U \cup U^{\perp})^{\perp\perp} = (U^{\perp} \cap U^{\perp\perp})^{\perp} = (U^{\perp} \cap U)^{\perp} = \emptyset^{\perp} = P$.

$RO(P)$ **is distributive:** We do this by direct computation:

$$\begin{aligned} U \wedge (V \vee W) &= U \cap (V \cup W)^{\perp\perp} \\ &= U^{\perp\perp} \cap (V \cup W)^{\perp\perp} \\ &= (U \cap (V \cup W))^{\perp\perp} \\ &= \big((U \cap V) \cup (U \cap W)\big)^{\perp\perp} \\ &= \big((U \wedge V) \cup (U \wedge W)\big)^{\perp\perp} \\ &= (U \wedge V) \vee (U \wedge W); \end{aligned}$$

$$\begin{aligned} U \vee (V \wedge W) &= (U \cup (V \cap W))^{\perp\perp} \\ &= ((U \cup V) \cap (U \cup W))^{\perp\perp} \\ &= (U \cup V)^{\perp\perp} \cap (U \cup W)^{\perp\perp} \\ &= (U \vee V) \wedge (U \vee W). \end{aligned}$$

Completness: Let $K = \{U_i : i \in I\} \subseteq RO(P)$, and $V = (\bigcup_{i\in I} U_i)^{\perp\perp}$. For any $i \in I$, $U_i \subseteq \bigcup_{i\in I} U_i$ so that $U_i = U_i^{\perp\perp} \subseteq (\bigcup_{i\in I} U_i)^{\perp\perp} = V$. This shows that V is an upper bound of elements of K. If W is another such upper bound, then $U_i \subseteq W$, so that $\bigcup_{i\in I} U_i \subseteq W$, whence $V = (\bigcup_{i\in I} U_i)^{\perp\perp} \subseteq W^{\perp\perp} = W$. The infimum is proved similarly.

Since a complete complemented distributive lattice is a complete Boolean algebra the proof is done. □

Definition 1.3.6. A subset X of a Boolean algebra $\mathbb{B}$ is *dense* if $0_{\mathbb{B}} \notin X$ and for each $0_{\mathbb{B}} \neq b \in \mathbb{B}$ there is a $x \in X$ such that $x \leq b$.

Lemma 1.3.7. *(i) P is refined iff $O_p \in RO(P)$ for all $p \in P$.*

(ii) If P is refined, the map $p \mapsto O_p$ is an order isomorphism of P onto a dense subset of $RO(P)$.

Proof. (i) First of all, it is useful to study how the interior of the closure of a subset $X \subset P$ is made. In the order topology, the least open containing $p \in P$ is O_p. The closure of X is constituted by all $p \in P$ such that, for all open $O \in RO(P)$ containing p, $O \cap X \neq \emptyset$. Remark that $O_p \subset O$ for all open O such that $p \in O$. We must then have

$$\overline{X} = \{p \in P : O_p \cap X \neq \emptyset\}.$$

Now, the interior of $\overline{X}$ is the set $\mathring{\overline{X}}$ of points $q \in P$ that have a neighborhood completely contained in $\overline{X}$. That is:

$$\mathring{\overline{X}} = \{q \in P : O_q \subseteq \overline{X}\} \tag{1.10}$$

Remark that:

$$\begin{aligned} O_q \subseteq \overline{X} &\text{ iff } \forall p' \in O_q \, (O_{p'} \cap X \neq \emptyset) \\ &\text{ iff } \forall p' \leq q \, \exists r \in X(r \leq p'). \end{aligned}$$

Thus,

$$\mathring{\overline{X}} = \{q \in P : \forall p' \leq q \exists r \in X(r \leq p')\} \tag{1.11}$$

If we choose $X = O_p$ in (1.11) we get that:

$$\begin{aligned} \mathring{\overline{O_p}} &= \{q \in P : \forall p' \leq q \exists r \leq p(r \leq p')\} \\ &= \{q \in P : \forall p' \leq q(p \parallel p')\}. \end{aligned} \tag{1.12}$$

Since O_p is open we have certainly $O_p \subseteq \mathring{\overline{O_p}}$. Suppose now that P is refined, if $q \notin O_p$, then $q \not\leq p$, so there is $p' \leq q$ such that $p \perp p'$ and by (1.12) $q \notin \mathring{\overline{O_p}}$. Therefore $O_p = \mathring{\overline{O_p}}$, that is, $O_p \in RO(P)$.

Conversely, if $O_p \in RO(P)$, then $O_p = \mathring{\overline{O_p}}$, so by (1.12):

$$q \not\leq p \to q \notin O_p \to q \notin \mathring{\overline{O_p}} \to \exists p \leq q(p \perp p'). \qquad \square$$

P is then refined.

(ii) Let $p, q \in P$ such that $p \leq q$, then $O_p \subseteq O_q$ and the map $p \mapsto O_p$ is order preserving. To show that the map is an isomorphism, let $O_p = O_q$. If, by contradiction, $p \neq q$ then suppose $q \not\leq p$. As P is refined, there is some $q' \leq q$ such that $q' \perp p$. Thus there is some $q' \in O_q$ such that $q' \notin O_p$, and hence $O_q \neq O_p$.

The density of P is easy to prove. We show that for every $\emptyset \neq R \in RO(P)$, there is a $p \in P$ such that $O_p \subseteq R$. That is immediate by (1.10): if $R \neq \emptyset$, then $\exists p \in R$ and $O_p \subseteq R$.

Corollary 1.3.8. *P is refined if and only if is order isomorphic to a dense subset of a complete Boolean algebra.*

Proof. If P is refined, then by Lemma 1.3.7 (ii), P is order isomorphic to a dense subset of the complete Boolean algebra $RO(P)$.

Conversely, suppose that P is order isomorphic to a dense subset D of a complete Boolean algebra $\mathbb{B}$. We may identify P with D. If $p, q \in P$ are such that $q \not\leq p$, then $q \wedge_{\mathbb{B}} \neg p \neq 0_{\mathbb{B}}$. Since P is dense, there is $p' \in P$ such that $p' \leq_{\mathbb{B}} q \wedge_{\mathbb{B}} \neg p$. We have then $p' \leq q$ and $p \perp p'$. Therefore P is refined. $\square$

We now prove that a Boolean completion of a partially ordered set is unique up to isomorphism.

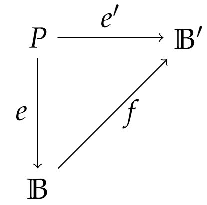
Definition 1.3.9. We say that a pair $\langle \mathbb{B}, e \rangle$ is a *Boolean completion* of P if the following conditions are met:

(i) $\mathbb{B}$ is a complete Boolean algebra;

(ii) e is an order isomorphism of P onto a dense subset of $\mathbb{B}$.

We will occasionally use the notation $\mathbb{B}(P)$ to denote the Boolean completion of P.

Lemma 1.3.10. *If $\langle \mathbb{B}, e \rangle$ and $\langle \mathbb{B}', e' \rangle$ are Boolean completions of P, then there is an isomorphism f between $\mathbb{B}$ and $\mathbb{B}'$ which interchanges $e[P]$ and $e'[P]$, that is, $f : \mathbb{B} \to \mathbb{B}'$ makes the following diagram commute:*



Proof. For each $x \in \mathbb{B}$ put

$$P_x = \{p \in P : e(p) \leq x\}.$$

Then the density of $e[P]$ in $\mathbb{B}$ implies that $\bigvee e[P_x] = x$, for each $x \in \mathbb{B}$. In fact, if $0_{\mathbb{B}} < \bigvee e[P_x] < x$, then $x \setminus \bigvee e[P_x] > 0_{\mathbb{B}}$ and, by the $e[P]$ density, there is some $p \in P$ such that $e(p) \leq x \setminus \bigvee e[P_x]$. Hence

$$e(p) \leq x; \tag{1.13}$$

$$e(p) \wedge \bigvee e[P_x] = 0_{\mathbb{B}}. \tag{1.14}$$

By (1.13), $p \in P_x$; by (1.14) $e(p) \wedge e(q) = 0_{\mathbb{B}}$ for all $q \in P_x$. In particular, for $q = p$ it follows that $e(p) = 0_{\mathbb{B}}$, which is not true.

Define now the following map:

$$\begin{aligned} f : \mathbb{B} &\to \mathbb{B}' \\ x &\mapsto \bigvee e'[P_x] \end{aligned}$$

We prove that f is an isomorphism of complete Boolean algebras.

First of all, remark that for any $q \in P$ and $x \neq 0_{\mathbb{B}}$,

$$e'(q) \leq f(x) \to e(q) \wedge x > 0_{\mathbb{B}}. \tag{1.15}$$

In fact, if that is not the case, assume $e(q) \wedge x = 0_{\mathbb{B}}$, then $e(q) \wedge e(p) = 0_{\mathbb{B}}$ for all $e(p) \leq x$, and thus $q \perp p$, for all $p \in P_x$. It follows that $e'(q) \wedge e'(p) = 0_{\mathbb{B}}$ for all $p \in P_x$, and hence the absurd: $0_{\mathbb{B}} = e'(q) \wedge \bigvee e'[P_x] = e'(q) \wedge f(x) = e'(q)$.

Commutativity:

$$\begin{aligned} f \circ e(p) &= \bigvee \{e'(q) : e(q) \leq e(p)\} \\ &= \bigvee \{e'(q) : e(q) = e(p)\} \\ &= e'(p) \end{aligned}$$

The last equality comes from the fact that e and e' are order isomorphisms, thus $e(q) = e(p)$, implies $q = p$ and then $e'(q) = e'(p)$.

Completeness: Let $A \subseteq \mathbb{B}$, then:

$$\begin{aligned} \bigvee f[A] &= \bigvee \{f(a) : a \in A\} \\ &= \bigvee_{a \in A} \bigvee \{e'(p) : e(p) \leq a\} \\ &\leq \bigvee \{e'(p) : e(p) \leq \bigvee A\} \\ &= f(\bigvee A). \end{aligned}$$

Assume, by contradiction, that $f(\bigvee A) \setminus \bigvee f[A] \neq 0_{\mathbb{B}'}$. By $e'[P]$ density, there is a $e'(r) \leq f(\bigvee A) \setminus f[A]$. Then $e(r) \wedge \bigvee A > 0_{\mathbb{B}}$ by (1.15).

We affirm furthermore that there is an $a \in A$ such that $e(r) \wedge a > 0_{\mathbb{B}}$. If not, then $e(r) \wedge a = 0_{\mathbb{B}}$ for all $a \in A$ would give $e(r) \wedge \bigvee A = 0_{\mathbb{B}}$.

Now, by $e[P]$ density, there is a $q \in P$ such that $e(q) \leq e(r) \wedge a$. Thus $e(q) \leq a$ and $e(q) \leq e(r)$ entail respectively that $e'(q) = f(e(q)) \leq f(a) \leq \bigvee f[A]$ and $e'(q) = f(e(q)) \leq f(e(r)) = e'(r)$. Finally,

$$\begin{aligned} e'(q) &\leq e'(r) \wedge \bigvee f[A] \\ &= 0_{\mathbb{B}'}. \end{aligned}$$

which is absurd.

Join preservation: It follows from completnes.

$1_{\mathbb{B}}$ **is mapped to** $1_{\mathbb{B}'}$ **:**

$$\begin{aligned} f(1_{\mathbb{B}}) &= \bigvee e'[P_1] \\ &= \bigvee \{e'(p) : e(p) \leq 1_{\mathbb{B}}\} \\ &= \bigvee \{e'(p) : p \in P\} \\ &= 1_{\mathbb{B}'}. \end{aligned}$$

The last equality is the result of $e'[P]$ density in $\mathbb{B}'$.

Complement preservation: We prove $f(\neg x) = \neg f(x)$, showing that:

$$f(\neg x) \vee f(x) = 1_{\mathbb{B}'} \text{ and } f(\neg x) \wedge f(x) = 0_{\mathbb{B}'}$$

The first equality is easily proved:

$$\begin{aligned} f(\neg x) \vee f(x) &= f(\neg x \vee x) \\ &= f(1_{\mathbb{B}}) \\ &= 1_{\mathbb{B}'}. \end{aligned}$$

As well as the second one:

$$\begin{aligned} f(\neg x) \wedge f(x) &= \bigvee \{e'(p) : e(p) \leq \neg x\} \wedge \bigvee \{e'(q) : e(q) \leq x\} \\ &= \bigvee \{e'(p) \wedge e'(q) : e(p) \leq \neg x, e(q) \leq x\} \\ &\leq \bigvee \{e'(p) \wedge e'(q) : p \perp q\} \\ &= \bigvee \{e'(p) \wedge e'(q) : e'(p) \wedge e'(q) = 0_{\mathbb{B}'}\} \\ &= 0_{\mathbb{B}'} \end{aligned}$$

Meet preservation: it comes out from join and complement preservations.

Injectivity:

$$\begin{aligned}\ker f &= \{x : f(x) = 0_{\mathbb{B}'}\}\\ &= \{x : \bigvee e'[P_x] = 0_{\mathbb{B}'}\}\\ &= \{0_{\mathbb{B}}\}\end{aligned}$$

The last equality comes from the fact that $\forall P' \subseteq P$, $\bigvee e[P'] = 0_{\mathbb{B}'}$ if and only if $P' = \emptyset$. In our case, $P' = P_x$. Then $P_x = \emptyset$ iff $x = 0_{\mathbb{B}}$.

Surjectivity: Let $y \in \mathbb{B}'$, then, by the density of $e'[P]$ in $\mathbb{B}'$, there is a $p \in P$ such that $e'(p) \leq y$. The preimage of y is thus

$$f^{-1}(y) = \bigvee\{e(p) : e'(p) \leq y\}.$$

In fact:

$$\begin{aligned}f(\bigvee\{e(p) : e'(p) \leq y\}) &= \bigvee\{f(e(p)) : e'(p) \leq y\}\\ &= \bigvee\{e'(p) : e'(p) \leq y\}\\ &= y.\end{aligned}$$ □

We consider now nonrefined posets and show the connection with refined posets.

Lemma 1.3.11. *Let $\langle P, \leq_P\rangle$ a partially ordered set, then there is a unique, up to isomorphism, refined poset $\langle Q, \leq_Q\rangle$ and an order preserving map j of P onto Q such that:*

$$\forall p, q \in P\ (p \parallel q \leftrightarrow j(p) \parallel j(q)). \tag{1.16}$$

Proof. **Existence:** Define the equivalence relation $\sim$ on P by

$$p \sim q \text{ iff } \forall x \in P(p \parallel x \leftrightarrow q \parallel x)$$

and let $Q = P/\sim$. Elements of Q are denoted by $[p]$. The partial order on Q is defined in the following way:

$$[p] \leq_Q [q] \text{ iff } \forall x \in P(x \parallel p \rightarrow x \parallel q)$$

Let j be the map of P onto Q :

$$\begin{aligned}j : P &\rightarrow Q\\ p &\mapsto [p].\end{aligned}$$

We have that j is order preserving; let $p \leq_P q$, then by definition of the equivalence relation, we get $[p] \leq_Q [q]$, that is $j(p) \leq_Q j(q)$.

Let us verify condition (1.16). If $p \parallel q$, then $\exists r \in P(r \leq_P p \wedge r \leq_P q)$, thus $j(r) \leq_Q j(p)$ and $j(r) \leq_Q j(q)$. That is $j(p) \parallel j(q)$.
Conversely, if $j(p) \parallel j(q)$, then

$$\exists j(r) \in Q\ (j(r) \leq_Q j(p) \wedge j(r) \leq_Q j(q)).$$

Now,

$$\begin{aligned} j(r) \leq_Q j(p) &\leftrightarrow [r] \leq_Q [p] \\ &\leftrightarrow \forall x \in P(x \parallel r \rightarrow x \parallel p) \end{aligned}$$

Choosing $x = r$, we get that $r \parallel p$, thus $\exists r' \in P(r' \leq_P r \wedge r' \leq_P p)$. We also have $j(r) \leq_Q j(q)$; thus $r' \parallel r$ implies $r' \parallel q$. Hence $\exists r'' \in P(r'' \leq_P r' \wedge r'' \leq_P q)$. We have obtained that $r'' \leq_P p \wedge r'' \leq_P q$, namely $p \parallel q$.

We check now that $\langle Q, \leq_Q \rangle$ is refined. To this aim, assume that $[q]$ is not a refinement of $[p]$, that is $[q] \not\leq_Q [p]$. Then certainly $\exists x \in P$ such that $x \parallel q$, but $x \perp p$. As a consequence, $\exists r \in P(r \leq_P x)$ such that $r \leq_P q$ and $r \perp p$, therefore $[r] \leq_Q [q]$ and $[r] \perp [p]$.

Uniqueness: We prove now that $Q = P/\sim$ is unique up to isomorphism. Let $\{S, \leq_S\}$ be another refined poset with a surjective order preserving map $k : P \rightarrow S$ satisfying (1.16). Consider the map:

$$\begin{aligned} f : Q &\rightarrow S \\ [q] &\mapsto k(q). \end{aligned}$$

The map f is well defined: let $[q] = [p]$, if $k(q) \neq k(p)$ then we may assume $k(q) \not\leq_S k(p)$. By S refinement $\exists k(r) \in S(k(r) \leq_S k(q) \wedge k(r) \perp k(p))$. By (1.16) applied to the map k, we get $r \parallel q$ and $r \perp p$, in contradiction with $[q] = [p]$.

As k is surjective, f is clearly surjective.

The map f is order preserving: let $[q] \leq_Q [p]$, if $k(q) \not\leq_S k(p)$, by S refinement $\exists k(r) \in S(k(r) \leq k(q) \wedge k(r) \perp k(p))$. Thus $r \parallel q \wedge r \perp p$, in contradiction with $[q] \leq_Q [p]$.

Finally, we check f injectivity: let $k(q) = k(p)$. If, by contradiction, $[q] \neq [p]$, then pick $[q] \not\leq_Q [p]$. Then, by Q refinement, $\exists [r] \in Q([r] \leq_Q$

$[q] \wedge [r] \perp [p])$. By f order preserving and by (1.16), we have $k(r) \leq_S k(q) \wedge k(r) \perp k(p)$, which is absurd. □

Putting all the Lemmas together we conclude that a poset P can always be carried in a complete Boolean algebra in the following way:

Theorem 1.3.12. *Let P a poset, then there is a complete Boolean algebra $\mathbb{B}$, unique up to isomorphism, and a map $j : P \to \mathbb{B}$ such that:*

- *$j[P]$ is dense in $\mathbb{B}$;*
- *j is order preserving, that is, $\forall p, q \in P(p \leq q \to j(p) \leq j(q))$;*
- *$\forall p, q \in P(p \parallel q \leftrightarrow j(p) \wedge j(q) \neq 0_{\mathbb{B}})$*

Proof. If P is refined, we already see the proof in Lemma 1.3.7. If P is non-refined, then use Lemma 1.3.11 and Lemma 1.3.7. The uniqueness comes from Lemma 1.3.10 □

Chapter 2

Boolean-valued models

In this chapter, we briefly expose the construction of Boolean-valued models, a generalization of first order models. By transfinite recursion we define $V^{(2)}$ and $V^{\mathbb{B}}$. We then give a Boolean truth value $[\![\cdot]\!]_{\mathbb{B}}$ to each $\mathbb{B}$-sentence and prove that all the axioms of the predicate calculus, equality and its rules of inference are true in $V^{\mathbb{B}}$. Furthermore, we study the relationship between complete Boolean subalgebras and submodels. At the end of the chapter, we state a useful method to construct elements of $V^{\mathbb{B}}$ and prove the Mixing Lemma and the Maximum Principle.

Boolean-valued models were introduced in the 1960's by Scott, Solovay and Vopěnka in order to help understand Cohen method of forcing, discovered in 1963. Our reference text for constructions and statements in the next few section is [2].

2.1 Construction of the model $V^{\mathbb{B}}$

We begin with the Boolean algebra $2 = \{0, 1\}$ and define the class $V^{(2)}$, called the *universe of two valued sets*.

Definition 2.1.1. By transfinite recursion on α we define

$$V_\alpha^{(2)} = \{x : \mathrm{Fun}(x) \wedge \mathrm{ran}(x) \subseteq 2 \wedge \exists \xi < \alpha[\mathrm{dom}(x) \subseteq V_\xi^{(2)}]\}$$

and then put

$$V^{(2)} = \{x : \exists \alpha [x \in V_\alpha^{(2)}]\}$$

It is easy to see that we have

$$x \in V^{(2)} \leftrightarrow \mathrm{Fun}(x) \wedge \mathrm{ran}(x) \subseteq 2 \wedge \mathrm{dom}(x) \subseteq V^{(2)}. \tag{2.1}$$

As we shall see in Theorem 2.4.4 we can identify $V^{(2)}$ with the standard universe V of sets.

We replace now the algebra $2 = \{0, 1\}$, with an arbitrary complete Boolean algebra $\mathbb{B}$, obtaining the *universe* $V^{\mathbb{B}}$ *of* $\mathbb{B}$*-valued sets.*

Definition 2.1.2. We define the *universe* $V^{\mathbb{B}}$ *of* $\mathbb{B}$*-valued sets* by analogy with Definition 2.1.1. Namely, we define, by recursion on α,

$$V_\alpha^{\mathbb{B}} = \{x : \mathrm{Fun}(x) \wedge \mathrm{ran}(x) \subseteq \mathbb{B} \wedge \exists \xi < \alpha[\mathrm{dom}(x) \subseteq V_\xi^{\mathbb{B}}]\}$$

and

$$V^{\mathbb{B}} = \{x : \exists \alpha[x \in V_\alpha^{\mathbb{B}}]\}$$

We see immediately that, as in (2.1), we have

$$x \in V^{\mathbb{B}} \leftrightarrow \mathrm{Fun}(x) \wedge \mathrm{ran}(x) \subseteq \mathbb{B} \wedge \mathrm{dom}(x) \subseteq V^{\mathbb{B}}.$$

That is a $\mathbb{B}$-valued set is a $\mathbb{B}$-valued function whose domain is a set of $\mathbb{B}$-valued sets. $V^{\mathbb{B}}$ is called a *Boolean extension of* V or more precisely the $\mathbb{B}$*-extension of* V.

We prove, by induction on rank argument, the following:

Definition 2.1.3. (Induction Principle for $V^{\mathbb{B}}$) For any formula $\phi(x)$,

$$\forall x \in V^{\mathbb{B}}[\forall y \in \mathrm{dom}(x)\phi(y) \to \phi(x)] \to \forall x \in V^{\mathbb{B}}\phi(x).$$

2.2 Boolean truth values $[\![\cdot]\!]_{\mathbb{B}}$

We now define the map $[\![\cdot]\!]_{\mathbb{B}}$ from the class of all $\mathbb{B}$ -sentences to $\mathbb{B}$, which assigns to each $\mathbb{B}$-sentence σ the *Boolean truth value* of σ in $V^{\mathbb{B}}$.

For the sake of argument, suppose that Boolean truth values have been assigned to all atomic sentences (those of the form $u = v$ and $u \in v$, for $u, v \in V^{\mathbb{B}}$). Then, in analogy with the classical two valued case, we extend the definition to all $\mathbb{B}$-sentences inductively as follows.

Definition 2.2.1. For $\mathbb{B}$-sentences σ, τ we put

$$[\![\sigma \wedge \tau]\!]_{\mathbb{B}} := [\![\sigma]\!]_{\mathbb{B}} \wedge_{\mathbb{B}} [\![\tau]\!]_{\mathbb{B}}; \tag{2.2}$$

$$[\![\neg\sigma]\!]_{\mathbb{B}} := \neg_{\mathbb{B}}[\![\sigma]\!]_{\mathbb{B}} \tag{2.3}$$

If $\phi(x)$ is a $\mathbb{B}$-formula with one free variable x, such that $[\![\phi(u)]\!]_{\mathbb{B}}$ has been defined for all $u \in V^{\mathbb{B}}$, we observe that the definable class $\{[\![\phi(u)]\!]_{\mathbb{B}} : u \in V^{\mathbb{B}}\}$ is a subset of $\mathbb{B}$ and define

$$[\![\exists x \phi(x)]\!]_{\mathbb{B}} := \bigvee_{u \in V^{\mathbb{B}}} [\![\phi(u)]\!]_{\mathbb{B}}. \tag{2.4}$$

From (2.2)-(2.4) it follow immediately that

$$[\![\sigma \vee \tau]\!]_{\mathbb{B}} = [\![\sigma]\!]_{\mathbb{B}} \vee_{\mathbb{B}} [\![\tau]\!]_{\mathbb{B}}; \tag{2.5}$$

$$[\![\sigma \rightarrow \tau]\!]_{\mathbb{B}} = [\![\sigma]\!]_{\mathbb{B}} \Rightarrow_{\mathbb{B}} [\![\tau]\!]_{\mathbb{B}}; \tag{2.6}$$

$$[\![\sigma \leftrightarrow \tau]\!]_{\mathbb{B}} = [\![\sigma]\!]_{\mathbb{B}} \Leftrightarrow_{\mathbb{B}} [\![\tau]\!]_{\mathbb{B}}; \tag{2.7}$$

$$[\![\forall x \phi(x)]\!]_{\mathbb{B}} = \bigwedge_{u \in V^{\mathbb{B}}} [\![\phi(u)]\!]_{\mathbb{B}}. \tag{2.8}$$

It remains to assign Boolean truth values to the atomic $\mathbb{B}$-sentences. We certainly want the axiom of extensionality to hold in $V^{\mathbb{B}}$, so we would have

$$[\![u = v]\!]_{\mathbb{B}} = [\![\forall x \in u[x \in v] \wedge \forall y \in v[y \in u]]\!]_{\mathbb{B}}. \tag{2.9}$$

Also, in accordance with the logical truth $u \in v \leftrightarrow \exists y \in v[u = y]$, which we certainly want to be true in $V^{\mathbb{B}}$, it should be the case that

$$[\![u \in v]\!]_{\mathbb{B}} = [\![\exists y \in v[u = y]]\!]_{\mathbb{B}}. \tag{2.10}$$

For restricted formulas like $\exists x \in u\ \phi(x)$ and $\forall x \in u\ \phi(x)$, we require that their Boolean truth values depend only on Boolean truth values of $\phi(x)$ for those x which are actually in $\mathrm{dom}(u)$. Moreover, we agree to be guided by the idea of "characteristic function," where, for $x \in \mathrm{dom}(u)$, the truth value of the formula $x \in u$ is $u(x)$.

It seems thus reasonable to require that

$$[\![\exists x \in u \phi(x)]\!]_{\mathbb{B}} = \bigvee_{x \in \mathrm{dom}(u)} [u(x) \wedge [\![\phi(x)]\!]_{\mathbb{B}}] \tag{2.11}$$

and

$$[\![\forall x \in u \phi(x)]\!]_{\mathbb{B}} = \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![\phi(x)]\!]_{\mathbb{B}}] \tag{2.12}$$

Definition 2.2.2. Putting (2.9)-(2.12) together, we see that we must have, for $u, v \in V^{\mathbb{B}}$,

$$[\![u \in v]\!]_{\mathbb{B}} = \bigvee_{y \in \mathrm{dom}(v)} [v(y) \wedge [\![u = y]\!]_{\mathbb{B}}]; \tag{2.13}$$

$$[\![u = v]\!]_{\mathbb{B}} = \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in v]\!]_{\mathbb{B}}] \wedge \bigwedge_{y \in \mathrm{dom}(v)} [v(y) \Rightarrow [\![y \in u]\!]_{\mathbb{B}}]. \tag{2.14}$$

Now (2.13) and (2.14) may and shall be regarded as a definition of $[\![u \in v]\!]_{\mathbb{B}}$ end $[\![u = v]\!]_{\mathbb{B}}$ by recursion on the following well-founded relation.

Definition 2.2.3. Define for $x, y, u, v \in V^{\mathbb{B}}$,

$\langle x, y \rangle < \langle u, v \rangle$ iff
either ($x \in \mathrm{dom}(u)$ and $y = v$) or ($x = u$ and $y \in \mathrm{dom}(v)$).

Then $<$ is easily seen to be a well-founded relation on the class $V^{\mathbb{B}} \times V^{\mathbb{B}} = \{\langle x, y \rangle : x \in V^{\mathbb{B}} \wedge y \in V^{\mathbb{B}}\}$. If we now put, for $u, v \in V^{\mathbb{B}}$,

$$G(\langle u, v \rangle) = \langle [\![u \in v]\!]_{\mathbb{B}}, [\![v \in u]\!]_{\mathbb{B}}, [\![u = v]\!]_{\mathbb{B}}, [\![v = u]\!]_{\mathbb{B}} \rangle,$$

then (2.13) and (2.14) may be written for some class function F

$$G(\langle u, v \rangle) = F(\langle u, v, G | \{\langle x, y \rangle : \langle x, y \rangle < \langle u, v \rangle\} \rangle).$$

This constitute a definition of G by recursion on $<$ and from G we obtain $[\![u \in v]\!]_{\mathbb{B}}, [\![u = v]\!]_{\mathbb{B}}$. Accordingly we take (2.13) and (2.14) as a definition of $[\![\sigma]\!]_{\mathbb{B}}$ for atomic $\mathbb{B}$-sentences σ, and then define $[\![\sigma]\!]_{\mathbb{B}}$ for all $\mathbb{B}$-sentences by induction on the complexity of σ in accordance with (2.2)-(2.4).

Definition 2.2.4. A $\mathbb{B}$-sentence σ is *true* or *holds with probability* 1 in $V^{\mathbb{B}}$, and write

$$V^{\mathbb{B}} \models \sigma$$

if $[\![\sigma]\!]_{\mathbb{B}} = 1_{\mathbb{B}}$. A $\mathbb{B}$-formula is *true* in $V^{\mathbb{B}}$ if its universal closure is true in $V^{\mathbb{B}}$. A rule of inference is *valid* in $V^{\mathbb{B}}$ if it preserves the truth of formulas in $V^{\mathbb{B}}$.

2.3 Basic results of $V^{\mathbb{B}}$

From now on we shall take the liberty of dropping the sub- or super-script from $[\![\sigma]\!]_{\mathbb{B}}, 0_{\mathbb{B}}, 1_{\mathbb{B}}$. Complete proofs of the following results can be found in [2].

Theorem 2.3.1. *All the axioms of the first-order predicate calculus with equality are true in $V^{\mathbb{B}}$, and all its rules of inference are valid in $V^{\mathbb{B}}$. In particular, we have:*

(i) $[\![u = u]\!] = 1_{\mathbb{B}}$;

(ii) $u(x) \leq [\![x \in u]\!]$ *for* $x \in \mathrm{dom}(u)$;

(iii) $[\![u = v]\!] = [\![v = u]\!]$;

(iv) $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$;

(v) $[\![u = v]\!] \wedge [\![u \in w]\!] \leq [\![v \in w]\!]$;

(vi) $[\![v = w]\!] \wedge [\![u \in v]\!] \leq [\![u \in w]\!]$;

(vii) $[\![u = v]\!] \wedge [\![\phi(u)]\!] \leq [\![\phi(v)]\!]$ *for any* $\mathbb{B}$*-formula* $\phi(x)$.

Proof. The points are proves using the induction principle for $V^{\mathbb{B}}$. A sketch of the proof can be founded in [2] □

It then follows that all the theorems of first order predicate calculus are true in $V^{\mathbb{B}}$. We can now prove the laws governing the assignment of Boolean truth values to formulas with restricted quantifiers.

Corollary 2.3.2. *For any* $\mathbb{B}$*-formula* $\phi(x)$ *with one free variable* x*, and all* $u \in V^{\mathbb{B}}$,

(i) $[\![\exists x \in u \phi(x)]\!] = \bigvee_{x \in \mathrm{dom}(u)} [u(x) \wedge [\![\phi(x)]\!]]$;

(ii) $[\![\forall x \in u \phi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![\phi(x)]\!]]$.

Proof. We prove the first equality, the second follows by duality.

$$\begin{aligned}
[\![\exists x \in u\phi(x)]\!] &= [\![\exists x[x \in u \wedge \phi(x)]]\!] \\
&= \bigvee_{y \in V^{\mathbb{B}}} [\![y \in u \wedge \phi(y)]\!] \\
&= \bigvee_{y \in V^{\mathbb{B}}} \bigvee_{x \in \mathrm{dom}(u)} [[\![x = y]\!] \wedge u(x) \wedge [\![\phi(y)]\!]] \\
&= \bigvee_{x \in \mathrm{dom}(u)} [u(x) \wedge \bigvee_{y \in V^{\mathbb{B}}} [\![x = y \wedge \phi(y)]\!]] \\
&= \bigvee_{x \in \mathrm{dom}(u)} [u(x) \wedge [\![\exists y[x = y \wedge \phi(y)]]\!]] \\
&= \bigvee_{x \in \mathrm{dom}(u)} [u(x) \wedge [\![\phi(x)]\!]].
\end{aligned}$$

□

The properties of $V^{\mathbb{B}}$ can be used to produce relative consistency proofs in set theory.

Theorem 2.3.3. *Let T, T' be extensions of* ZF *such that $Con(\mathsf{ZF}) \rightarrow Con(T')$, and suppose that in $\mathcal{L}$ we can define a constant term $\mathbb{B}$ such that:*

$T' \vdash \mathbb{B}$ is a complete Boolean algebra and, for each axiom τ of T, we have $T' \vdash [\![\tau]\!]_{\mathbb{B}} = 1_{\mathbb{B}}$.

Then $Con(\mathsf{ZF}) \rightarrow Con(T)$.

Proof. If T is inconsistent, then for some axioms $\tau_1, \tau_2, \ldots, \tau_n$ of T we would have, for any sentence σ,

$$T \vdash \tau_1 \wedge \ldots \wedge \tau_n \rightarrow \sigma \wedge \neg\sigma. \tag{2.15}$$

Now let $\mathbb{B}$ a complete Boolean algebra satisfying the hypothesis of our theorem. Then

$$T' \vdash [\![\tau_1 \wedge \ldots \wedge \tau_n]\!]_{\mathbb{B}} = 1_{\mathbb{B}}. \tag{2.16}$$

But by (2.15) we have:

$$T' \vdash [\![\tau_1 \wedge \ldots \wedge \tau_n]\!]_{\mathbb{B}} \leq [\![\sigma \wedge \neg\sigma]\!]_{\mathbb{B}} = 0_{\mathbb{B}},$$

so, by (2.16)

$$T' \vdash 1_{\mathbb{B}} \leq 0_{\mathbb{B}},$$

so T' and hence ZF would be inconsistent. □

2.4 Subalgebras and their models

We are going to show that, if $\mathbb{B}'$ is a complete subalgebra of $\mathbb{B}$, then $V^{\mathbb{B}'}$ is a submodel of $V^{\mathbb{B}}$. Furthermore, we will see that every element x of V has a natural representative $\hat{x}$ in $V^{\mathbb{B}}$. We will terminate the section with some properties abouts standard elements $\hat{x} \in V^{\mathbb{B}}$.

Theorem 2.4.1. *Let $\mathbb{B}'$ a complete subalgebra of $\mathbb{B}$. Then, for $u, v \in V^{\mathbb{B}'}$,*

(i) $V^{\mathbb{B}'} \subseteq V^{\mathbb{B}}$;

(ii) $[\![u \in v]\!]_{\mathbb{B}'} = [\![u \in v]\!]_{\mathbb{B}}$;

(iii) $[\![u = v]\!]_{\mathbb{B}'} = [\![u = v]\!]_{\mathbb{B}}$.

Proof. The first point comes out from definitions of $V^{\mathbb{B}}$ and $V^{\mathbb{B}'}$. The points (ii) and (iii) are proved simultaneously by induction on the well founded relation $y \in \mathrm{dom}(x)$, where the inductive hypothesis is, for all $y \in \mathrm{dom}(v)$ and all $u \in V^{\mathbb{B}}$:

$$
\begin{aligned}
[\![u \in y]\!]_{\mathbb{B}'} &= [\![u \in y]\!]_{\mathbb{B}};\\
[\![u = y]\!]_{\mathbb{B}'} &= [\![u = y]\!]_{\mathbb{B}};\\
[\![y \in u]\!]_{\mathbb{B}'} &= [\![y \in u]\!]_{\mathbb{B}}.
\end{aligned}
$$

□

Corollary 2.4.2. *If $\mathbb{B}'$ is a complete subalgebra of $\mathbb{B}$, then, for any restricted formula $\phi(v_1, \ldots, v_n)$ and any $u_1, \ldots, u_n \in V^{\mathbb{B}'}$,*

$$[\![\phi(u_1, \ldots, u_n)]\!]_{\mathbb{B}'} = [\![\phi(u_1, \ldots, u_n)]\!]_{\mathbb{B}}.$$

Proof. We prove the corollary by induction on the complexity of ϕ. If ϕ is atomic, the result follows by Theorem 2.4.1. The only non trivial induction step arises when ϕ is in the form $\exists x \in u\psi$. In this case, if $u, u_1, \ldots, u_n \in V^{\mathbb{B}'}$, then writing $\bigvee^{\mathbb{B}}$ and $\bigvee^{\mathbb{B}'}$ for joins in $\mathbb{B}$ and $\mathbb{B}'$, we have:

$$
\begin{aligned}
[\![\phi(u, u_1, \ldots, u_n)]\!]_{\mathbb{B}'} &= \bigvee_{x \in \mathrm{dom}(u)}^{\mathbb{B}'} [u(x) \wedge [\![\psi(x, u_1, \ldots, u_n)]\!]_{\mathbb{B}'}]\\
&= \bigvee_{x \in \mathrm{dom}(u)}^{\mathbb{B}} [u(x) \wedge [\![\psi(x, u_1, \ldots, u_n)]\!]_{\mathbb{B}}]\\
&= [\![\phi(u, u_1, \ldots, u_n)]\!]_{\mathbb{B}}.
\end{aligned}
$$

□

Remark that, as the two-element algebra $2 = \{0, 1\}$ is a complete subalgebra of every complete Boolean algebra $\mathbb{B}$, we have that $V^{(2)}$ is a submodel of every $V^{\mathbb{B}}$. We now want to prove that there is a bijection from V to $V^{(2)}$. To this end we need the following definition, constructed by recursion on the well-founded relation $y \in x$.

Definition 2.4.3. For each $x \in V$, define:

$$\hat{x} = \{\langle \hat{y}, 1\rangle : y \in x\}.$$

Observe that for each $x \in V$, then $\hat{x} \in V^{(2)} \subseteq V^{\mathbb{B}}$. By Theorem 2.4.1, for $x, y \in V$:

$$[\![\hat{x} \in \hat{y}]\!]_{\mathbb{B}} = [\![\hat{x} \in \hat{y}]\!]_2 \in 2;$$
$$[\![\hat{x} = \hat{y}]\!]_{\mathbb{B}} = [\![\hat{x} = \hat{y}]\!]_2 \in 2.$$

We can think of $\hat{x}$ being a natural representative in $V^{\mathbb{B}}$ for $x \in V$. Elements of the form $\hat{x}$ are called *standard.* The next and last theorem establishes some results about standard members of $V^{\mathbb{B}}$. It will follow from (v) that V and $V^{(2)}$ have the same true sentences.

Theorem 2.4.4. *(i) For $x \in V, u \in V^{\mathbb{B}}$,*

$$[\![u \in \hat{x}]\!] = \bigvee_{y \in x} [\![u = \hat{y}]\!].$$

(ii) For $x, y \in V$,

$$x \in y \leftrightarrow V^{\mathbb{B}} \models \hat{x} \in \hat{y};$$
$$x = y \leftrightarrow V^{\mathbb{B}} \models \hat{x} = \hat{y}.$$

(iii) The map $x \mapsto \hat{x}$ is one-one from V to $V^{(2)}$.

(iv) For each $u \in V^{(2)}$ there is a unique $x \in V$ such that $V^{\mathbb{B}} \models u = \hat{x}$.

(v) For any formula $\phi(v_1, \ldots, v_n)$ and any $x_1, \ldots, x_n \in V$,

$$\phi(x_1, \ldots, x_n) \leftrightarrow V^{(2)} \models \phi(\hat{x_1}, \ldots, \hat{x_n})$$

and if ϕ is restricted then

$$\phi(x_1, \ldots, x_n) \leftrightarrow V^{\mathbb{B}} \models \phi(\hat{x_1}, \ldots, \hat{x_n}.)$$

Proof. (i)

$$\llbracket u \in \hat{x} \rrbracket = \bigvee_{v \in \mathrm{dom}(\hat{x})} [\hat{x}(v) \wedge \llbracket u = v \rrbracket]$$
$$= \bigvee_{y \in x} [\hat{x}(\hat{y}) \wedge \llbracket u = \hat{y} \rrbracket]$$
$$= \bigvee_{y \in x} \llbracket u = \hat{y} \rrbracket.$$

(ii) By induction on $\mathrm{rank}(y)$. The induction hypothesis is for all z with $(\mathrm{rank}(z) < \mathrm{rank}(y))$:

$$\forall x[x \in z \leftrightarrow \llbracket \hat{x} \in \hat{z} \rrbracket = 1];$$
$$\forall x[x = z \leftrightarrow \llbracket \hat{x} = \hat{z} \rrbracket = 1];$$
$$\forall x[z \in x \leftrightarrow \llbracket \hat{z} \in \hat{x} \rrbracket = 1].$$

(iii) It follows from (ii).

(iv) The uniqueness follows from (ii). For the existence, use induction on the well founded relation $x \in \mathrm{dom}(u)$. Suppose then that $u \in V^{(2)}$ and

$$\forall x \in \mathrm{dom}(u) \exists y \in V(\llbracket x = \hat{y} \rrbracket = 1).$$

We want to show that for some $v \in V$, $\llbracket u = \hat{v} \rrbracket = 1$. By definition,

$$\llbracket u = \hat{v} \rrbracket = \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow \llbracket x \in \hat{v} \rrbracket] \wedge \bigwedge_{y \in v} \llbracket \hat{y} \in u \rrbracket.$$

We want that $\llbracket u = \hat{v} \rrbracket = 1$, so it must happens that:

$$x \in \mathrm{dom}(u) \to u(x) \leq \llbracket x \in \hat{v} \rrbracket = \bigvee_{y \in v} \llbracket x = \hat{y} \rrbracket; \tag{2.17}$$
$$y \in v \to 1 = \llbracket \hat{y} \in u \rrbracket = \bigvee_{x \in \mathrm{dom}(u)} [u(x) \wedge \llbracket x = \hat{y} \rrbracket]. \tag{2.18}$$

In order to satisfy (2.18), we create

$$v = \{y \in V : \exists x \in \mathrm{dom}(u)[u(x) = 1 \wedge \llbracket x = \hat{y} \rrbracket = 1]\}.$$

By (ii) and Replacement, $v \in V$. By inductive hypothesis v satisfies (2.17).

(v) For the first part, use induction on complexity of ϕ, (ii) and (iv). If ϕ is atomic the result holds by (ii).

If ϕ is in the form $\exists x\psi$, then suppose that $x_1, \ldots, x_n \in V$. If

$$[\![\phi(\hat{x}_1, \ldots, \hat{x}_n)]\!]_2 = 1,$$

then

$$\bigvee_{x \in V^{(2)}} [\![\psi(x, \hat{x}_1, \ldots, \hat{x}_n)]\!]_2 = 1.$$

Thus

$$[\![\psi(x, \hat{x}_1, \ldots, \hat{x}_n)]\!]_2 = 1$$

for some $x \in V^{(2)}$. By (iv), for some $y \in V$, we have $[\![x = \hat{y}]\!]_2 = 1$ so that

$$\begin{aligned} 1 &= [\![\psi(x, \hat{x}_1, \ldots, \hat{x}_n)]\!]_2 \wedge [\![x = \hat{y}]\!]_2 \\ &\le [\![\phi(\hat{y}, \hat{x}_1, \ldots, \hat{x}_n)]\!]_2. \end{aligned}$$

The inductive hypothesis gives $\psi(y, x_1, \ldots, x_n)$, which implies $\phi(x_1, \ldots, x_n)$. The converse is similar. The second part of (v) follows from the first part and Corollary 2.4.2. □

2.5 Mixtures and the Maximum Principle

We are going to formulate a useful general method for constructing elements of $V^{\mathbb{B}}$.

Definition 2.5.1. Given a subset $\{a_i : i \in I\} \subseteq \mathbb{B}$, and a subset $\{u_i : i \in I\} \subseteq V^{\mathbb{B}}$, we define the *mixture* $\sum_{i \in I} a_i \cdot u_i$ *of* $\{u_i : i \in I\}$ *with respect to* $\{a_i : i \in I\}$ to be that element $u \in V^{\mathbb{B}}$ such that

$$\mathrm{dom}(u) = \bigcup_{i \in I} \mathrm{dom}(u_i)$$

and, for $z \in \mathrm{dom}(u)$,

$$u(z) = \bigvee_{i \in I} [a_i \wedge [\![z \in u_i]\!]].$$

The following lemma, the *Mixing Lemma*, justifies the use of the term *mixture* by showing that under certain conditions (in particular, when $\{a_i : i \in I\}$ is an antichain) $\sum_{i\in I} a_i \cdot u_i$ behaves as if it were obtained by mixing the $\mathbb{B}$-valued sets $\{u_i : i \in I\}$ together in the proportions $\{a_i : i \in I\}$.

Lemma 2.5.2. *(**Mixing Lemma**) Let $\{a_i : i \in I\} \subseteq \mathbb{B}$, let $\{u_i : i \in I\} \subseteq V^{\mathbb{B}}$ and put $\sum_{i\in I} a_i \cdot u_i = u$. Suppose that, for all $i, j \in I$,*

$$a_i \wedge a_j \leq [\![u_i = u_j]\!]. \tag{2.19}$$

Then, for all $i \in I$,

$$a_i \leq [\![u = u_i]\!].$$

In particular, the result holds if $\{a_i : i \in I\}$ is an antichain.

Proof. We have $[\![u = u_i]\!] = a \wedge b$, where

$$a = \bigwedge_{z\in \mathrm{dom}(u)} [u(z) \Rightarrow [\![z \in u_i]\!]]$$
$$b = \bigwedge_{z\in \mathrm{dom}(u_i)} [u_i(z) \Rightarrow [\![z \in u]\!]].$$

If $z \in \mathrm{dom}(u)$, then

$$a_i \wedge u(z) = \bigvee_{j\in I} a_i \wedge a_j \wedge [\![z \in u_j]\!] \tag{2.20}$$
$$\leq \bigvee_{j\in I} [\![u_i = u_j]\!] \wedge [\![z \in u_j]\!] \quad \text{(by 2.19)} \tag{2.21}$$
$$\leq [\![z \in u_i]\!], \tag{2.22}$$

so that $a_i \leq [u(z) \Rightarrow [\![z \in u_i]\!]]$ for any $z \in \mathrm{dom}(u)$, whence $a_i \leq a$. On the other hand, if $z \in \mathrm{dom}(u_i)$, then

$$\begin{aligned} a_i \wedge u_i(z) &\leq a_i \wedge [\![z \in u_i]\!] \\ &\leq u(z) \\ &\leq [\![z \in u]\!], \end{aligned}$$

so that $a_i \leq [u_i(z) \Rightarrow [\![z \in u]\!]]$, whence $a_i \leq b$. Hence $a_i \leq a \wedge b$, and the result follows. □

Recall that we assigned a Boolean truth value to the formula $\exists x\phi(x)$ by putting

$$[\![\exists x\phi(x)]\!] = \bigvee_{u\in V^{\mathbb{B}}} [\![\phi(u)]\!].$$

We now show, using the Mixing Lemma, that $V^{\mathbb{B}}$ contains so many members that the supremum on the right side of the above equality is actually attained at some element $u \in V^{\mathbb{B}}$. Thus $V^{\mathbb{B}}$ is *full* . In fact, the next Lemma is also called *fullness Lemma.*

Lemma 2.5.3. ***(The Maximum Principle or Fullness Lemma)*** *If $\phi(x)$ is any $\mathbb{B}$-formula, then there is $u \in V^{\mathbb{B}}$ such that*

$$[\![\exists x\phi(x)]\!] = [\![\phi(u)]\!].$$

In particular, if $V^{\mathbb{B}} \models \exists x\phi(x)$, then $V^{\mathbb{B}} \models \phi(u)$ for some $u \in V^{\mathbb{B}}$.

Proof. By definition, we have

$$[\![\exists x\phi(x)]\!] = \bigvee_{u\in V^{\mathbb{B}}} [\![\phi(u)]\!].$$

Since $\mathbb{B}$ is a set, so is $\{[\![\phi(u)]\!] : u \in V^{\mathbb{B}}\} \in \mathcal{P}(\mathbb{B})$ and, by AC, there is an ordinal α and a set $\{u_\xi : \xi < \alpha\} \subseteq V^{\mathbb{B}}$ such that $\{[\![\phi(u)]\!] : u \in V^{\mathbb{B}}\} = \{[\![\phi(u_\xi)]\!] : \xi < \alpha\}$. Accordingly,

$$[\![\exists x\phi(x)]\!] = \bigvee_{\xi<\alpha} [\![\phi(u_\xi)]\!].$$

For each $\xi < \alpha$, put

$$a_\xi = [\![\phi(u_\xi)]\!] \wedge \neg[\bigvee_{\eta<\xi} [\![\phi(u_\eta)]\!]].$$

Then $\{a_\xi : \xi < \alpha\}$ is an antichain in $\mathbb{B}$ and $a_\xi \leq [\![\phi(u_\xi)]\!]$ for all $\xi < \alpha$. Put $u = \sum_{\xi<\alpha} a_\xi \cdot u_\xi$; then by the Mixing Lemma we have $a_\xi \leq [\![u = u_\xi]\!]$ for all $\xi < \alpha$. Also, clearly,

$$[\![\phi(u)]\!] \leq [\![\exists x\phi(x)]\!].$$

On the other hand,

$$[\![\phi(u)]\!] \geq [\![u = u_\xi]\!] \wedge [\![\phi(u_\xi)]\!] \geq a_\xi$$

so that

$$[\![\phi(u)]\!] \geq \bigvee_{\xi<\alpha} a_\xi = \bigvee_{\xi<\alpha} [\![\phi(u_\xi)]\!] = [\![\exists x\phi(x)]\!].$$

□

Corollary 2.5.4. *Let $\phi(x)$ be a $\mathbb{B}$-formula such that $V^{\mathbb{B}} \models \exists x \phi(x)$.*

(i) For any $v \in V^{\mathbb{B}}$ there is a $u \in V^{\mathbb{B}}$ such that $[\![\phi(u)]\!] = 1$ and $[\![\phi(v)]\!] = [\![u = v]\!]$.

(ii) If $\psi(x)$ is a $\mathbb{B}$-formula such that for any $u \in V^{\mathbb{B}}$, $V^{\mathbb{B}} \models \phi(u)$ implies $V^{\mathbb{B}} \models \psi(u)$, then $V^{\mathbb{B}} \models \forall x[\phi(x) \to \psi(x)]$.

Proof. (i) Apply the Maximum Principle to obtain $w \in V^{\mathbb{B}}$ such that $[\![\phi(w)]\!] = 1_{\mathbb{B}}$, put $b = [\![\phi(v)]\!]$ and $u = b \cdot v + (\neg b) \cdot w$. Then

$$\begin{aligned} [\![\phi(u)]\!] &\geq [\![u = v \wedge \phi(v)]\!] \vee [\![u = w \wedge \phi(w)]\!] \\ &\geq b \vee \neg b \\ &= 1_{\mathbb{B}}, \end{aligned}$$

and $[\![u = v]\!] = [\![u = v]\!] \wedge [\![\phi(u)]\!] \leq [\![\phi(v)]\!]$. Since $[\![u = v]\!] \geq b = [\![\phi(v)]\!]$, the result follows.

(ii) Assume the hypothesis, and let $v \in V^{\mathbb{B}}$. Using (i), choose $u \in V^{\mathbb{B}}$ such that $[\![\phi(u)]\!] = 1_{\mathbb{B}}$ and $[\![\phi(v)]\!] = [\![u = v]\!]$. Then $[\![\psi(u)]\!] = 1_{\mathbb{B}}$ and

$$[\![\phi(v)]\!] = [\![u = v]\!] = [\![u = v]\!] \wedge [\![\psi(u)]\!] \leq [\![\psi(v)]\!].$$

The result follows. □

Chapter 3

Regular embeddings and retractions

In this chapter, we present the construction and some properties about regular embeddings and retractions between complete Boolean algebras. Given a regular embedding i we show that the associated map π_i is in fact a retraction (Proposition 3.1.5). In Proposition 3.1.7 we prove the converse: to a retraction π, we can associate a regular embedding i_π.
In Section 3.2, we show how to embed a quotient algebra and we prove a variant of the Mixing Lemma (Lemma 3.2.2).

3.1 Regular embeddings and retractions of complete Boolean algebras

We are now interested in defining two kinds of maps between Boolean algebras: regular embeddings and retractions. In the following, we show how one can associate a retraction to a regular embedding, and vice-versa.

Definition 3.1.1. Let $\mathbb{B}$, $\mathbb{Q}$ be non-atomic Boolean algebras. $i : \mathbb{B} \to \mathbb{Q}$ is a *regular embedding* if it is a complete and injective homomorphism of Boolean algebras.

Definition 3.1.2. A map $\pi : \mathbb{Q} \to \mathbb{B}$ between complete Boolean algebras is a *retraction* if the following conditions are satisfied:

(i) π preserves joins and is surjective;

(ii) $\pi^{-1}(0_{\mathbb{B}}) = \{0_{\mathbb{Q}}\}$;

(iii) For all $q \in \mathbb{Q}, b \in \mathbb{B}$:

$$\pi(q) \wedge b = \bigvee \{\pi(s) : s \leq q, \pi(s) \leq b\}. \tag{3.1}$$

Remark 3.1.3. Analyzing the definition of retaction we can add some properties. Join preservation implies that π is also order preserving. Moreover, it follows from (ii), that $\pi(q) > 0_{\mathbb{B}}$ for all $q > 0_{\mathbb{Q}}$ and $\pi(0_{\mathbb{Q}}) = 0_{\mathbb{B}}$.

Equation (3.1) describes π behavior when restricted to a parameter b : π preserves $\bigvee$-operation in a homogeneous way. This is equivalent to a property to which we will often refer:

$$\pi(q) \wedge b = \pi(q \wedge \bigvee \{s : \pi(s) \leq b\}). \tag{3.2}$$

In fact,

$$\begin{aligned} \bigvee \{\pi(s) : s \leq q, \pi(s) \leq b\} &= \pi(\bigvee \{s : s \leq q, \pi(s) \leq b\}) \\ &= \pi(\bigvee \{s \wedge q : \pi(s) \leq b\}) \\ &= \pi(q \wedge \bigvee \{s : \pi(s) \leq b\}). \end{aligned}$$

We finally highlight that we do NOT assume that π is a homomorphism, thus π may not preserve meets and complements and may map different positive elements of $\mathbb{Q}$ to the same object of $\mathbb{B}^+$.

We want now to construct an associated map π_i to any regular embedding $i : \mathbb{B} \to \mathbb{Q}$. As we shall see, π_i will be a retraction.

Definition 3.1.4. The *retraction associated to i* is the map

$$\begin{aligned} \pi_i : \mathbb{Q} &\to \mathbb{B} \\ q &\mapsto \bigwedge_{\mathbb{B}} \{b \in \mathbb{B} : i(b) \geq q\} \end{aligned}$$

Let us verify some properties of the embedding i and its associated map π_i. The next proposition states, in particular, that π_i is in fact a retraction.

Proposition 3.1.5. *Assume that $i : \mathbb{B} \to \mathbb{Q}$ is a regular embedding. Then:*

(i) $i \circ \pi_i(q) \geq q$, for all $q \in \mathbb{Q}$ and π_i maps $\mathbb{Q}^+$ to $\mathbb{B}^+$;

(ii) $\pi_i \circ i(a) = a$, for all $a \in \mathbb{B}$ and thus π_i is surjective;

(iii) π_i *preserves joins of subsets of* $\mathbb{Q}$*, that is, for every* $X \subseteq \mathbb{Q}$*,*

$$\pi_i(\bigvee_{\mathbb{Q}} X) = \bigvee_{\mathbb{B}} \pi_i[X];$$

(iv) *If* i *is not surjective,* $\pi_i : \mathbb{Q} \to \mathbb{B}$ *does not preserve neither meets nor complements;*

(v) *If* $\pi_i(a) \wedge \pi_i(b) = 0_{\mathbb{B}}$*, then* $a \wedge b = 0_{\mathbb{Q}}$*;*

(vi) *For any* $q \in \mathbb{Q}$ *and* $b \in \mathbb{B}$*, then* $\pi_i(q) \wedge b = \pi_i(q \wedge i(b))$*.*
In particular, for all $b \leq \pi_i(q)$*,*

$$\pi_i(q \wedge i(b)) = b; \tag{3.3}$$

(vii) $i(b) = \bigvee\{s : \pi_i(s) \leq b\}$*, for all* $b \in \mathbb{B}$*;*

(viii) π_i *is a retraction.*

Proof. (i) Recall that

$$\pi_i(q) = \bigwedge_{\mathbb{B}}\{b \in \mathbb{B} : i(b) \geq q\}.$$

Thus

$$\begin{aligned} i \circ \pi_i(q) &= i(\bigwedge\{b \in \mathbb{B} : i(b) \geq q\}) \\ &= \bigwedge\{i(b) : b \in \mathbb{B}, \, i(b) \geq q\} \\ &\geq \bigwedge\{p \in \mathbb{Q} : p \geq q\} \\ &= q. \end{aligned}$$

This proves the first part of (i). Now if, by contradiction, for $q > 0_{\mathbb{Q}}$, $\pi_i(q) = 0_{\mathbb{B}}$, we would have $0_{\mathbb{Q}} = i \circ \pi(q) \geq q > 0_{\mathbb{Q}}$. Hence π_i must preserve positivity.

(ii) We prove that π_i is surjective:

$$\pi_i \circ i(a) = \bigwedge\{b \in \mathbb{B} : i(b) \geq i(a)\}) = a,$$

since i is injective.

(iii) Let $X = \{q_j : j \in J\} \subseteq \mathbb{Q}$. Thus, for all $j \in J$,

$$\begin{aligned}\pi_i(\bigvee\{q_j : j \in J\}) &= \bigwedge\{b \in \mathbb{B} : i(b) \geq \bigvee\{q_j : j \in J\}\} \\ &\geq \bigwedge\{b \in \mathbb{B} : i(b) \geq q_j\} \\ &= \pi_i(q_j).\end{aligned}$$

In that way, we obtain the first inequality:

$$\pi_i(\bigvee\{q_j : j \in J\}) \geq \bigvee\{\pi_i(q_j) : j \in J\},$$

that is $\pi_i(\bigvee X) \geq \bigvee \pi_i[X]$.

Now if $r = \bigvee \pi_i[X]$, we have that $r \geq \pi_i(q_j)$ for all $j \in J$. Thus, by (i) for all $j \in J$:

$$i(r) \geq i \circ \pi_i(q_j) \geq q_j.$$

In particular $i(r) \geq \bigvee\{q_j : j \in J\}$. From π_i definition we get that π_i is increasing, so:

$$r = \pi_i(i(r)) \geq \pi(\bigvee\{q_j : j \in J\}),$$

that is, the second inequality $\bigvee \pi_i[X] \geq \pi_i(\bigvee X)$ holds.

(iv) If $i : \mathbb{B} \to \mathbb{Q}$ is not surjective, then pick $q \in \mathbb{Q} \setminus i[\mathbb{B}]$. Then $i(\pi_i(q)) \neq q$ and we have, by (i), $i(\pi_i(q)) > q$. Thus $p = i(\pi_i(q)) \setminus q > 0_{\mathbb{Q}}$ and $\pi_i(p) > 0_{\mathbb{B}}$.

Now,

$$\begin{aligned}\pi_i(q) \vee \pi_i(p) &= \pi_i(q \vee p) \\ &= \pi_i(i(\pi_i(q))) \\ &= \pi_i(q).\end{aligned}$$

Thus $\pi_i(p) \wedge \pi_i(q) = \pi_i(p) > 0_{\mathbb{B}}$. But $\pi_i(p \wedge q) = \pi_i(0_{\mathbb{Q}}) = 0_{\mathbb{B}}$, so π_i does not preserve meets.

We check now that $\pi_i(\neg_{\mathbb{Q}} f) \neq \neg_{\mathbb{B}} \pi_i(f)$ for some $f \in \mathbb{Q}$.
Let:

$$q \in \mathbb{Q} \setminus i[\mathbb{B}],\ d = \neg_{\mathbb{B}} \pi_i(q) \text{ and } e = i(d).$$

We have

$$\begin{aligned} f &= q \vee e \\ &= q \vee i(d) \\ &= q \vee i(\neg \pi_i(q)) \\ &= q \vee \neg i(\pi_i(q)) \\ &< q \vee \neg q \\ &= 1_{\mathbb{Q}}. \end{aligned}$$

Thus $f < 1_{\mathbb{Q}}$ implies that $\neg f > 0_{\mathbb{Q}}$ and $\pi_i(\neg f) > 0_{\mathbb{Q}}$.

On the other hand,

$$\begin{aligned} \pi_i(f) &= \pi_i(q) \vee \pi_i(e) \\ &= \pi_i(q) \vee \pi_i(i(d)) \\ &= \pi_i(q) \vee d \\ &= \pi_i(q) \vee \neg \pi_i(q) \\ &= 1_{\mathbb{B}}. \end{aligned}$$

Thus $\neg \pi_i(f) = 0_{\mathbb{B}}$ and $\pi_i(\neg f) > 0_{\mathbb{B}}$, imply $\neg \pi_i(f) \neq \pi_i(\neg f)$.

(v) Assume $a \wedge b > 0_{\mathbb{Q}}$, then we would have a contradiction:

$$\begin{aligned} 0_{\mathbb{Q}} &= i(\pi_i(a) \wedge \pi_i(b)) \\ &= i(\pi_i(a)) \wedge i(\pi_i(b)) \\ &\geq a \wedge b \\ &> 0_{\mathbb{Q}}. \end{aligned}$$

(vi) For $b \in \mathbb{B}, q \in \mathbb{Q}$, the following three equations hold:

$$\pi_i(q \wedge i(b)) \vee \pi_i(q \wedge \neg i(b)) = \pi_i(q); \tag{3.4}$$

$$(\pi_i(q) \wedge b) \vee (\pi_i(q) \wedge \neg b) = \pi_i(q); \tag{3.5}$$

$$(\pi_i(q) \wedge b) \wedge (\pi_i(q) \wedge \neg b) = 0_{\mathbb{B}}. \tag{3.6}$$

Furthermore, by π_i definition, we have:

$$\pi_i(q \wedge i(b)) \leq \pi_i(q) \wedge b \tag{3.7}$$

and

$$\begin{aligned} \pi_i(q \wedge \neg i(b)) &= \pi_i(q \wedge i(\neg b)) \\ &\leq \pi_i(q) \wedge \neg b. \end{aligned} \tag{3.8}$$

Now, in order to abridge the notation, set:

$$a = \pi_i(q \wedge i(b)), \qquad c = \pi_i(q) \wedge b,$$
$$b = \pi_i(q \wedge \neg i(b)), \qquad d = \pi_i(q) \wedge \neg b.$$

By (3.7) and (3.8) we get

$$a \leq c;$$
$$b \leq d.$$

So, by (3.6),

$$a \wedge b \leq c \wedge d = 0_{\mathbb{B}},$$

that is, a and b are disjoint. Moreover, by (3.4) and (3.5), a and b have the same supremum as c and d, that is,

$$a \vee b = c \vee d.$$

All in all, we conclude that

$$a = c \text{ and } b = d.$$

In particular, translating $a = c$ we have:

$$\pi_i(q) \wedge b = \pi_i(q \wedge i(b)),$$

as was to be proved.

(vii) Let $s \in \mathbb{Q}$ such that $\pi_i(s) \leq b$. By (i), and i order preserving,

$$s \leq i(\pi_i(s)) \leq i(b).$$

Thus

$$\bigvee \{s : \pi_i(s) \leq b\} \leq i(b). \tag{3.9}$$

In order to prove the other inequality, recall that by (ii) we can write $b = \pi_i(i(b))$. So

$$\begin{aligned} i(b) &\leq \bigvee \{s : \pi_i(s) \leq \pi_i(i(b))\} \\ &= \bigvee \{s : \pi_i(s) \leq b\}. \end{aligned} \tag{3.10}$$

The result follow immediately putting (3.9) and (3.10) together.

(viii) Join preservation and surjectivity follow from points (iii) and (ii). In order to show that $\pi^{-1}(0_{\mathbb{B}}) = \{0_{\mathbb{Q}}\}$ put $a = 0_{\mathbb{B}}$ in (ii), then

$$\pi_i(0_{\mathbb{Q}}) = \pi_i(i(0_{\mathbb{B}})) = 0_{\mathbb{B}}.$$

Thus $0_{\mathbb{Q}}$ is certainly mapped in $0_{\mathbb{B}}$ and moreover by (i) it is the only element of $\mathbb{Q}$ mapped in $0_{\mathbb{B}}$. The last point of Definition 3.1.2 follows putting (vi) and (vii) together and using the equivalent equation (3.2). □

We just proved that we can associate a retraction to any regular embedding. We want now to construct the converse: we start from a retraction π and we associate a regular embedding i_π.

Definition 3.1.6. Let $\pi : \mathbb{Q} \to \mathbb{B}$ a retraction. The *regular embedding associated to* π is the map

$$\begin{aligned} i_\pi : \mathbb{B} &\to \mathbb{Q} \\ b &\mapsto \bigvee_{\mathbb{Q}} \{s \in \mathbb{Q}^+ : \pi(s) \leq b\} \end{aligned}$$

In the next Proposition we see some properties of i_π and show that i_π is actually a regular embedding and that the associated retraction of i_π is π again.

Proposition 3.1.7. *Assume that* $\pi : \mathbb{Q} \to \mathbb{B}$ *is a retraction. Then the map* i_π *satisfies the following properties:*

(i) $\pi \circ i_\pi(a) = a$ *for all* $a \in \mathbb{B}$;

(ii) $i_\pi \circ \pi(q) \geq q$ *for all* $q \in \mathbb{Q}$;

(iii) i_π *preserves joins;*

(iv) $i_\pi(1_{\mathbb{B}}) = 1_{\mathbb{Q}}$ *and* $i_\pi(0_{\mathbb{B}}) = 0_{\mathbb{Q}}$;

(v) i_π *preserves complements and meets;*

(vi) i_π *is injective;*

(vii) $\pi_{i_\pi} = \pi$;

(viii) $i_{\pi_i} = i$.

In particular, it follows from points (iii)-(vi) that i_π is a regular embedding.

Proof. (i)

$$\begin{aligned}\pi \circ i_\pi(a) &= \pi(\bigvee\{s \in \mathbb{Q} : 0_{\mathbb{B}} < \pi(s) \leq a\}) \\ &= \bigvee\{\pi(s) : 0_{\mathbb{B}} < \pi(s) \leq a\} \\ &= a.\end{aligned}$$

In the second equality, we used the fact that π preserves joins and in the last equality we used π surjectivity.

(ii)

$$\begin{aligned}i_\pi \circ \pi(q) &= \bigvee\{s : \pi(s) \leq \pi(q)\} \\ &\geq q\end{aligned}$$

(iii) Let $\{b_k : k \in K\} \subseteq \mathbb{B}$. Certainly, by i_π definition, $i_\pi(b_k) \leq i_\pi(\bigvee_{k\in K} b_k)$. Thus

$$\bigvee_{k\in K} i_\pi(b_k) \leq i_\pi(\bigvee_{k\in K} b_k).$$

Assume, by contradiction, that the last inequality is strict, then we define

$$q = i_\pi(\bigvee_{k\in K} b_k) \setminus \bigvee_{k\in K} i_\pi(b_k) > 0_{\mathbb{Q}}. \tag{3.11}$$

By (3.2), we have

$$\begin{aligned}\pi(q) \wedge b_k &= \pi(q \wedge i(b_k)) \\ &= \pi(0_{\mathbb{Q}}) \\ &= 0_{\mathbb{B}}.\end{aligned}$$

We obtain thus that

$$\forall k \in K\ \pi(q) \wedge b_k = 0_{\mathbb{B}}. \tag{3.12}$$

By (3.11), it follows that $q \leq i(\bigvee_{k\in K} b_k)$, and by π properties we have:

$$\begin{aligned}\pi(q) &\leq \pi(i(\bigvee_{k\in K} b_k)) \\ &= \bigvee_{k\in K} b_k.\end{aligned} \tag{3.13}$$

Putting (3.13) and (3.12) together we get:

$$\begin{aligned}\pi(q) &= \pi(q) \wedge \bigvee_{k\in K} b_k \\ &= \bigvee_{k\in K} (\pi(q) \wedge b_k) \\ &= 0_{\mathbb{B}}.\end{aligned}$$

Hence $q = 0_{\mathbb{Q}}$, in contradiction with (3.11).

(iv) $i_\pi(1_{\mathbb{B}}) = \bigvee\{s \in \mathbb{Q} : 0_{\mathbb{B}} < \pi(s)\} = 1_{\mathbb{Q}}$ and
$i_\pi(0_{\mathbb{B}}) = \bigvee\{s \in \mathbb{Q} : 0_{\mathbb{B}} < \pi(s) \leq 0_{\mathbb{B}}\} = \bigvee_{\mathbb{Q}} \emptyset = 0_{\mathbb{Q}}$.

(v) Let $a, b \in \mathbb{B}$ such that $a \wedge b = 0_{\mathbb{B}}$. Then

$$\begin{aligned}i_\pi(a) \wedge i_\pi(b) &= \bigvee\{s \in \mathbb{Q}^+ : \pi(s) \leq a\} \wedge \bigvee\{t \in \mathbb{Q}^+ : \pi(t) \leq b\} \\ &= \bigvee\{s \wedge t : 0_{\mathbb{B}} < \pi(s) \leq a, 0_{\mathbb{B}} < \pi(t) \leq b\} \\ &\leq \bigvee\{q : \pi(q) \leq a \wedge b = 0_{\mathbb{B}}\} \\ &= 0_{\mathbb{Q}} \\ &= i_\pi(a \wedge b).\end{aligned}$$

On the other hand since we already showed that i_π preserves joins we have that $i_\pi(a) \vee i_\pi(b) = i_\pi(a \vee b)$. By these equalities, we can easily get that $i_\pi(\neg b) = \neg i_\pi(b)$.

Meet preservation follows from complement and join preservation. By De Morgan laws we get:

$$\begin{aligned}i_\pi(a \wedge b) &= i_\pi(\neg(\neg a \vee \neg b)) \\ &= \neg i_\pi(\neg a \vee \neg b) \\ &= \neg(i_\pi(\neg a) \vee i_\pi(\neg b)) \\ &= \neg(\neg i_\pi(a) \vee \neg i_\pi(b)) \\ &= i_\pi(a) \wedge i_\pi(b).\end{aligned}$$

(vi) We have already shown that i_π is an homomorphism, thus it suffices to check that $ker(i_\pi) = \{0_{\mathbb{B}}\}$. Applying π to $i_\pi(a) = 0_{\mathbb{Q}}$, we get by (i) $a = \pi \circ i_\pi(a) = \pi(0_{\mathbb{Q}}) = 0_{\mathbb{B}}$.

(vii) Using Proposition 3.1.5 (i) and applying π_{i_π} to point (ii) of the current Proposition, we get for all $q \in \mathbb{Q}$:

$$\begin{aligned}\pi(q) &= [\pi_{i_\pi} \circ i_\pi](\pi(q)) \\ &= \pi_{i_\pi} \circ [i_\pi \circ \pi(q)] \\ &\geq \pi_{i_\pi}(q)\end{aligned}$$

On the other hand, by (i) and applying π to Proposition 3.1.5 (i) we have:

$$\begin{aligned}\pi_{i_\pi}(q) &= [\pi \circ i_\pi](\pi_{i_\pi}(q)) \\ &= \pi \circ [i_\pi \circ \pi_{i_\pi}(q)] \\ &\geq \pi(q)\end{aligned}$$

(viii) By Proposition 3.1.5(ii) and Proposition 3.1.7(ii) we have for all $b \in \mathbb{B}$:

$$\begin{aligned}i_{\pi_i}(b) &= i_{\pi_i}(\pi_i \circ i(b)) \\ &= [i_{\pi_i} \circ \pi_i](i(b)) \\ &\geq i(b).\end{aligned}$$

The second disequality comes from Proposition 3.1.7(i) and Proposition 3.1.5(i):

$$\begin{aligned}i(b) &= i(\pi_i \circ i_{\pi_i}(b)) \\ &= [i \circ \pi_i](i_{\pi_i}(b)) \\ &\geq i_{\pi_i}(b).\end{aligned}$$

□

3.2 Embeddings and quotients

With a slight abuse of terminology for any $b \in \mathbb{B}$ we let $i \restriction b : \mathbb{B} \restriction b \to \mathbb{Q} \restriction i(b)$ denote the natural restriction of i to $\mathbb{B} \restriction b$.

Given a regular embedding $i : \mathbb{B} \to \mathbb{Q}$ with associated retraction $\pi_i : \mathbb{Q} \to \mathbb{B}$, let $\dot{G}_{\mathbb{B}} \in V^{\mathbb{B}}$ be the canonical name for the V-generic filter and $\dot{I}_{\mathbb{B}}$ be the canonical name for its dual ideal:

$$\begin{aligned}\dot{G}_{\mathbb{B}} &= \{\langle \check{b}, b\rangle : b \in \mathbb{B}\}; \\ \dot{I}_{\mathbb{B}} &= \{\langle \check{b}, \neg b\rangle : b \in \mathbb{B}\}.\end{aligned}$$

We want that $V^{\mathbb{B}}$ models that $i[\dot{I}_{\mathbb{B}}]$ generates an ideal $\dot{J}$ on $\mathbb{Q}$ by the requirement that

$$[\![q \in \dot{J}]\!]_{\mathbb{B}} = [\![\pi_i(q) \in \dot{I}_{\mathbb{B}}]\!]_{\mathbb{B}}.$$

That is,

$$\dot{J} = \{\langle \check{q}, \neg\pi_i(q)\rangle : q \in \mathbb{Q}\}.$$

Thus we get that for all $q, r \in \mathbb{Q}$

$$\begin{aligned} [\![[q]_{\dot{J}} = [r]_{\dot{J}}]\!]_{\mathbb{B}} &= [\![q\Delta r \in \dot{J}]\!]_{\mathbb{B}} \\ &= [\![\pi_i(q\Delta r) \in \dot{I}_{\mathbb{B}}]\!]_{\mathbb{B}} \\ &= \neg(\pi_i(q\Delta r)) \end{aligned}$$

The quotient forcing $\mathbb{Q}/i[\dot{G}_{\mathbb{B}}]$ is a canonical $\mathbb{B}$-name for the Boolean algebra $\mathbb{Q}/\dot{J}$.

Lemma 3.2.1. *If $i : \mathbb{B} \to \mathbb{Q}$ is a regular embedding of complete Boolean algebras, then $\mathbb{B} * (\mathbb{Q}/i[\dot{G}_{\mathbb{B}}])$ (see Definition 4.2.3) is forcing equivalent to $\mathbb{Q}$.*

Proof. Let $\pi_i : \mathbb{Q} \to \mathbb{B}$ be the retraction map associated to i.

The map

$$i^* : \mathbb{Q} \to \mathbb{B} * (\mathbb{Q}/i[\dot{G}_{\mathbb{B}}]))$$

which maps $r \mapsto (\pi(r), [\check{r}]_{i[\dot{J}]})$ is a complete embedding such that $i^*[\mathbb{Q}]$ is dense in $\mathbb{B} * (\mathbb{Q}/i[\dot{G}_{\mathbb{B}}])$. The conclusion follows. □

We show the following variant of the mixing Lemma:

Lemma 3.2.2. *Assume $i : \mathbb{B} \to \mathbb{Q}$ is a regular embedding. Let $\dot{J}$ be a canonical name for the dual ideal of the filter generated by $i[\dot{G}_{\mathbb{B}}]$. For all $\dot{a} \in V^{\mathbb{B}}$ such that $[\![\dot{a} \in (\mathbb{Q}/\dot{J})^+]\!]_{\mathbb{B}} = 1_{\mathbb{B}}$, there is a unique $r_{\dot{a}} \in \mathbb{Q}$ such that*

$$\begin{aligned} \pi_i(r_{\dot{a}}) &= 1_{\mathbb{B}}; \\ [\![\dot{a} = [r_{\dot{a}}]_{\dot{J}}]\!]_{\mathbb{B}} &= 1_{\mathbb{B}}. \end{aligned}$$

Proof. Remember that for $r, t \in \mathbb{Q}$ we can compute the Boolean values:

$$\begin{aligned} [\![[r]_{\dot{J}} \in (\mathbb{Q}/\dot{J})^+]\!]_{\mathbb{B}} &= \pi_i(r); \\ [\![[r]_{\dot{J}} = [t]_{\dot{J}}]\!]_{\mathbb{B}} &= \neg\pi_i(r\Delta t). \end{aligned}$$

Uniqueness. By contradiction, assume $r \neq t$ are such that:

$$\pi_i(r) = \pi_i(t) = 1_{\mathbb{B}};$$
$$[\![[r]_{\dot{J}} = \dot{a}]\!]_{\mathbb{B}} = [\![[t]_{\dot{J}} = \dot{a}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

The second condition entails that $[\![[r]_{\dot{J}} = [t]_{\dot{J}}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}$.

Now, $r \neq t$ involves $r\Delta t > 0_{\mathbb{Q}}$ and thus:

$$\pi_i(r\Delta t) = [\![[r]_{\dot{J}} \neq [t]_{\dot{J}}]\!]_{\mathbb{B}} > 0_{\mathbb{B}}.$$

Finally,

$$0_{\mathbb{B}} < \pi_i(r\Delta t) = [\![[r]_{\dot{J}} \neq [t]_{\dot{J}}]\!]_{\mathbb{B}} = \neg[\![[r]_{\dot{J}} = [t]_{\dot{J}}]\!]_{\mathbb{B}} = 0_{\mathbb{B}},$$

which is impossible.

Existence. We now turn to the proof of the existence of the desired element $r_{\dot{a}}$. Observe that if $\dot{a}$ is a $\mathbb{B}$-name for an element of $(\mathbb{Q}/\dot{J})^+$, that is $[\![\dot{a} \in (\mathbb{Q}/\dot{J})^+]\!] = 1_{\mathbb{B}}$, then:

$$\begin{aligned}
[\![\dot{a} = [r]_{\dot{J}}]\!]_{\mathbb{B}} &= [\![\dot{a} = [r]_{\dot{J}}]\!]_{\mathbb{B}} \wedge [\![\dot{a} \in (\mathbb{Q}/\dot{J})^+]\!]_{\mathbb{B}} \\
&\leq [\![[r]_{\dot{J}} \in (\mathbb{Q}/\dot{J})^+]\!]_{\mathbb{B}} \\
&= \pi_i(r).
\end{aligned}$$

and

$$\begin{aligned}
&[\![\dot{a} = [r]_{\dot{J}}]\!]_{\mathbb{B}} \wedge [\![\dot{a} = [t]_{\dot{J}}]\!]_{\mathbb{B}} \\
&\leq [\![[r]_{\dot{J}} \in (\mathbb{Q}/\dot{J})^+]\!]_{\mathbb{B}} \wedge [\![[t]_{\dot{J}} \in (\mathbb{Q}/\dot{J})^+]\!]_{\mathbb{B}} \wedge [\![[r]_{\dot{J}} = [t]_{\dot{J}}]\!]_{\mathbb{B}} \\
&= \pi(r) \wedge \pi(t) \wedge \neg\pi(r\Delta t). \qquad (3.14)
\end{aligned}$$

Consider the set $A_{\dot{a}} = \{b \in \mathbb{B}^+ : \exists r \in \mathbb{Q}, b \leq [\![\dot{a} = [r]_{\dot{J}}]\!]_{\mathbb{B}}\}$. We want to construct a maximal antichain from $A_{\dot{a}}$, for this purpose we first prove that $A_{\dot{a}}$ is dense and then apply Proposition 1.1.7. By the fundamental forcing theorem, we have that for any formula ϕ and element $q \in \mathbb{B}$:

$$q \leq [\![\phi]\!]_{\mathbb{B}}$$
$$\text{iff}$$
$$\forall G \text{ filter } V\text{-generic for } \mathbb{B} \text{ such that } q \in G, \text{then} V[G] \models \phi.$$

We have to show that $\forall q \in \mathbb{B}^+ \exists b \in A_{\dot{a}}$ such that $b \leq q$. Fix $q \in \mathbb{B}^+$ and let G a V-generic filter for $\mathbb{B}$ such that $q \in G$. Thus

$$\sigma_G(\dot{a}) \in \mathbb{Q}^+/\sigma_G(\dot{J})$$
$$\text{says that}$$
$$\exists r \in \mathbb{Q}^+ : V[G] \models \sigma_G(\dot{a}) = [r]_{\sigma_G(\dot{J})}$$

Then, for the forcing theorem,

$$\exists s \in G : s \leq [\![\dot{a} = [r]_j]\!]_{\mathbb{B}}.$$

Let $b = s \wedge q$, then $b \in G$ and

$$b = s \wedge q \leq [\![\dot{a} = [r]_j]\!]_{\mathbb{B}};$$

so $b \in A_{\dot{a}}$ and $b \leq q$, as we were looking for. Let now fix a maximal antichain $\{b_i : i \in I\} \subseteq A_{\dot{a}}$. For every $i \in I$ define:

$$A_i = \{r \in \mathbb{Q} : b_i \leq [\![\dot{a} = [r]_j]\!]_{\mathbb{B}}\}$$

and let

$$a_i = \bigwedge_{\mathbb{B}} \{\pi(r) : r \in A_i\} \wedge_{\mathbb{B}} \bigwedge_{\mathbb{B}} \{\neg_{\mathbb{B}} \pi(s \Delta_{\mathbb{Q}} t) : s, t \in A_i\}. \tag{3.15}$$

Then for all $i \in I$, $b_i \leq a_i$, by (3.14).
For every $i \in I$, pick some $t_i \in A_i$ and let $r_i = i(b_i) \wedge t_i$.
We get that

$$b_i \leq [\![[r_i] = [t_i]]\!]_{\mathbb{B}},$$

in fact:

$$\begin{aligned}
[\![[r_i]_j = [t_i]_j]\!]_{\mathbb{B}} &= [\![[i(b_i) \wedge t_i]_j = [t_i]_j]\!]_{\mathbb{B}} \\
&= \neg\pi((i(b_i) \wedge t_i)\Delta t_i) \\
&= \neg\pi(r \setminus i(b_i)) \\
&\geq \neg\pi(1_{\mathbb{Q}} \setminus i(b_i)) \\
&= \neg\pi(\neg i(b_i)) \\
&= \neg\pi(i(\neg b_i)) \\
&= \neg\neg b_i \\
&= b_i.
\end{aligned}$$

From $b_i \leq [\![[r_i]_j = [t_i]_j]\!]_{\mathbb{B}}$ and $b_i \leq [\![\dot{a} = [t_i]_j]\!]_{\mathbb{B}}$ we get that $\{r_i : i \in I\}$ is a subset of $\mathbb{Q}$ such that, for all $i \in I$,

$$b_i \leq [\![[r_i]_j = \dot{a}]\!]_{\mathbb{B}}.$$

Now, define:

$$r_{\dot{a}} = \bigvee_{\mathbb{Q}} \{r_i : i \in I\}.$$

We prove that $[\![[r_{\dot{a}}]_{\dot{I}} = \dot{a}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}$. First of all, remark that:

$$
\begin{aligned}
\neg i(b_i) &\geq r_{\dot{a}} \setminus i(b_i) \\
&= (\bigvee_{j \neq i} r_j \vee r_i) \setminus i(b_i) \qquad (3.16) \\
&= \bigvee_{j \neq i} (r_j \setminus i(b_i)) \vee (r_i \setminus i(b_i)) \\
&= \bigvee_{j \neq i} r_j .
\end{aligned}
$$

The last equality comes from two facts. The first is $r_i \setminus i(b_i) = 0_{\mathbb{Q}}$, by r_i definition. The second is that $\forall j \neq i$, $(b_j \wedge b_i = 0_{\mathbb{B}})$ implies $i(b_j) \wedge i(b_i) = 0_{\mathbb{Q}}$ and thus $r_j \wedge i(b_i) = 0_{\mathbb{Q}}$. Therefore,

$$
\begin{aligned}
[\![[r_{\dot{a}}]_{\dot{J}} = [r_i]_{\dot{J}}]\!]_{\mathbb{B}} &= \neg \pi (r_{\dot{a}} \Delta r_i) \\
&= \neg \pi (r_{\dot{a}} \setminus r_i) \\
&= \neg \pi (\bigvee_{j \neq i} r_j) \\
&\geq \neg(\neg i(b_i)) \text{ by 3.16} \\
&= \neg \pi (i(\neg b_i)) \\
&= \neg\neg b_i \\
&= b_i .
\end{aligned}
$$

All in all, since $\{b_i : i \in I\}$ is a maximal antichain in $\mathbb{B}^+$, we get that $[\![[r_{\dot{a}}]_{\dot{J}} = [r_i]_{\dot{J}}]\!]_{\mathbb{B}} \geq \bigvee \{b_i : i \in I\} = 1_{\mathbb{B}}$, as we were looking for.

This gives a canonical representation of any $\mathbb{B}$-name $\dot{a}$ for an element of $(\mathbb{Q}/\dot{I})^+$ by an element $r_{\dot{a}} \in \mathbb{Q}^+$.

Moreover, applying the last point of Proposition 3.1.5 to $\pi(t_i) \geq b_i$ we get:

$$
\begin{aligned}
\pi(r_i) &= \pi(i(b_i) \wedge t_i) \\
&= b_i .
\end{aligned}
$$

Thus, because $\pi(r_{\dot{a}}) \geq \pi(r_i) = b_i$, we finally have

$$
\pi(r_{\dot{a}}) \geq \bigvee_{\mathbb{B}} \{b_i : i \in I\} = 1_{\mathbb{B}} .
$$

The proof of the lemma is completed. □

Chapter 4

Limits of Boolean algebras and iterated forcing

In this chapter, we study how to apply embeddings and retractions in order to get iteration systems of complete Boolean algebras. We then summarize the construction of iterated forcing with posets and start in the last section a comparison between iterated forcing and iteration systems of complete Boolean algebras.

4.1 Iteration systems of complete Boolean algebras

Definition 4.1.1.

$$\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \delta\}$$

is a *complete iteration system* of complete Boolean algebras if for all $\alpha, \beta, \gamma \in \delta$:

(i) each $\mathbb{B}_\alpha$ is a Boolean algebra;

(ii) each $i_{\alpha\beta}$ is a regular embedding with associated retraction $\pi_{\alpha\beta}$ and $i_{\alpha\alpha}$ is the identity;

(iii) for $\alpha \leq \beta \leq \gamma$ we have $i_{\beta\gamma} \circ i_{\alpha\beta} = i_{\alpha\gamma}$.

Lemma 4.1.2. *It follows from point (iii) of Definition 4.1.1 that the composition of the associated retractions is an associated retraction, i.e.*

$$\forall \alpha \leq \beta \leq \gamma : \quad \pi_{\alpha\beta} \circ \pi_{\beta\gamma} = \pi_{\alpha\gamma}.$$

Proof. Let $g \in \mathbb{B}_\gamma$, then:

$$\begin{aligned}
\pi_{\alpha\beta}(\pi_{\beta\gamma}(g)) &= \bigwedge\{a \in \mathbb{B}_\alpha : i_{\alpha\beta}(a) \geq \pi_{\beta\gamma}(g)\} \\
&\geq \bigwedge\{a \in \mathbb{B}_\alpha : i_{\beta\gamma} \circ i_{\alpha\beta}(a) \geq i_{\beta\gamma} \circ \pi_{\beta\gamma}(g)\} \\
&\geq \bigwedge\{a \in \mathbb{B}_\alpha : i_{\alpha\gamma}(a) \geq g\} \\
&= \pi_{\alpha\gamma}(g)
\end{aligned}$$

and

$$\begin{aligned}
\pi_{\alpha\gamma}(g) &= \bigwedge\{a \in \mathbb{B}_\alpha : i_{\alpha\gamma}(a) \geq g\} \\
&= \bigwedge\{a \in \mathbb{B}_\alpha : i_{\beta\gamma} \circ i_{\alpha\beta}(a) \geq g\} \\
&\geq \bigwedge\{a \in \mathbb{B}_\alpha : \pi_{\beta\gamma} \circ i_{\beta\gamma} \circ i_{\alpha\beta}(a) \geq \pi_{\beta\gamma}(g)\} \\
&= \bigwedge\{a \in \mathbb{B}_\alpha : i_{\alpha\beta}(a) \geq \pi_{\beta\gamma}(g)\} \\
&= \pi_{\alpha\beta} \circ \pi_{\beta\gamma}(g).
\end{aligned}$$

□

Definition 4.1.3. We say that $f \in \Pi_{\alpha\in\delta}\mathbb{B}_\alpha$ is a *thread* (Figure 4.1) for $\mathcal{F}$ if for all $\alpha \leq \beta < \delta$:

$$\pi_{\alpha\beta}(f(\beta)) = f(\alpha).$$

Definition 4.1.4. We say that $c \in \Pi_{\alpha\in\delta}\mathbb{B}_\alpha$ is a *constant thread* (Figure 4.2) if c is a thread and there is some $\alpha < \delta$ such that, for all $\alpha \leq \beta < \delta$:

$$i_{\alpha\beta} \circ \pi_{\alpha\beta}(c(\beta)) = c(\beta).$$

Definition 4.1.5. The *support* S of a constant thread c is the least α for which the above holds, that is:

$$S(c) = \min\{\alpha < \delta : i_{\alpha\beta} \circ \pi_{\alpha\beta}(c(\alpha)) = c(\alpha)\}$$

Remark that generally we only have $i_{\alpha\beta} \circ \pi_{\alpha\beta}(f(\beta)) \leq f(\beta)$.

Definition 4.1.6. The set of all threads is denoted by $T(\mathcal{F})$. The set of all constant threads is denoted by $C(\mathcal{F})$.

$$f = \langle\, f(0), f(1), \ldots, f(\alpha), f(\beta), f(\gamma), \ldots \,\rangle$$
$$\pi_{\alpha\gamma}$$

Figure 4.1: A thread $f \in T(\mathcal{F})$.

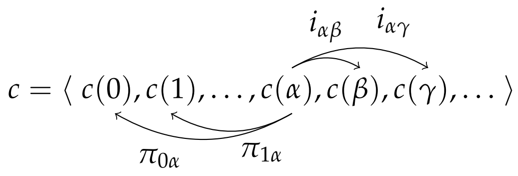


Figure 4.2: A constant thread $c \in C(\mathcal{F})$ with support α.

It is easy to check that a thread is uniquely determined by its restriction to D for any D cofinal subset of δ and a constant thread by its value on any α greater or equal than its support.

Definition 4.1.7. Given $p \in \mathbb{B}_\alpha$ we let $c^\alpha_p \in C(\mathcal{F})$ be the unique constant thread with support α such that $c^\alpha_p(\alpha) = p$.

Proposition 4.1.8. *There is a natural partial order on $T(\mathcal{F})$ and $C(\mathcal{F})$ given by $f \leq_{T(\mathcal{F})} g$ ($f \leq_{C(\mathcal{F})} g$) iff $f(\alpha) \leq_{\mathbb{B}_\alpha} g(\alpha)$ for all $\alpha \in \delta$.*

Proposition 4.1.9. *For every complete iteration system $\mathcal{F}$ of complete Boolean algebras there is a natural Boolean algebra structure on $C(\mathcal{F})$, where the operations are given component wise i.e. if $f = \langle f_\alpha : \alpha \in \delta\rangle$ and $g = \langle g_\alpha : \alpha \in \delta\rangle$ are in $C(\mathcal{F})$ we have that $f \wedge_{C(\mathcal{F})} g = h$ if h is the unique thread such that for eventually all $\alpha < \delta$ $g(\alpha) \wedge_{\mathbb{B}_\alpha} f(\alpha) = h(\alpha)$. All the other Boolean operations are defined similarly.*

Notice however that $C(\mathcal{F})$ most likely is not a complete Boolean algebra.

Remark 4.1.10. $C(\mathcal{F})$ may not be a complete Boolean algebra and most likely it is not a complete Boolean subalgebra of $\mathbb{B}(T(\mathcal{F}))$. In general for a linear system indexed by ω, let $p = \langle p_n : n < \omega\rangle$ be a thread in $T(\mathcal{F}) \setminus C(\mathcal{F})$. Thus $\forall n < \omega$:

$$\begin{aligned} i_{n,n+1}(p_n) &= i_{n,n+1}(\pi_{n,n+1}(p(n+1))) \\ &> p_{n+1} \end{aligned}$$

then

$$\{c^n_{p(n)} : n < \omega\} \subset C(\mathcal{F})$$

is a strictly decreasing sequence in $C(\mathcal{F})$:

$$\begin{aligned}\langle p_0, i_{01}(p_0), i_{02}(p_0), \ldots, i_{0n}(p_0), \ldots\rangle &>_{C(\mathcal{F})} \langle p_0, p_1, i_{12}(p_1), \ldots, i_{1n}(p_1), \ldots\rangle \\ &\vdots \\ &>_{C(\mathcal{F})} \langle p_0, p_1, p_2, \ldots, p_n, i_{n,n+1}(p_n), \ldots\rangle\end{aligned}$$

which may have no exact lower bound or could have a lower bound different from its exact lower bound in $T(\mathcal{F})$ which is p.

For example let

$$\mathcal{F} = \{i_{nk} : \mathbb{B}(\prod_{i<n} 2^{<\omega}) \to \mathbb{B}(\prod_{i<k} 2^{<\omega}) : n \leq k < \omega\}$$

with each i_{nk} the natural injection. Observe that $\mathcal{F}$ is a complete iteration system. Now $\mathbb{B}(C(\mathcal{F}))$ is the Boolean completion of the finite support product of $\prod_{i<\omega} 2^{<\omega}$ and is forcing equivalent to Cohen forcing. If $s \in 2^{<\omega}$, we get that $\{c(\langle s : i < n\rangle) : n < \omega\}$ is a strictly decreasing sequence in $C(\mathcal{F})^+$ with no lower bound in $C(\mathcal{F})^+$, thus in $C(\mathcal{F})$ its exact lower bound is $0_{C(\mathcal{F})}$. On the other hand in $T(\mathcal{F})$ its exact lower bound will be $\{\langle s : i < n\rangle : n < \omega\}$ which is a non 0-element of $T(\mathcal{F})$. This shows that the inclusion map of $C(\mathcal{F})$ into $\mathbb{B}(T(\mathcal{F}))$ is an injective homomorphism of Boolean algebras which is not complete.

Remark 4.1.11. We can not identify $T(\mathcal{F})$ with its Boolean completion. In fact, let $\{f_i : i \in I\} \subseteq T(\mathcal{F})$, then $\langle \bigvee_{i\in I} f_i(0), \bigvee_{i\in I} f_i(1), \ldots\rangle$ is a thread that can not be the supremum in $\mathbb{B}(T(\mathcal{F}))$ of $\{f_i : i \in I\}$.

As an example, we can find a thread $b = \langle b_n : n < \omega\rangle$ such that $\pi_{n,n+1}(i_{n,n+1}(b_n) \wedge \neg b_{n+1}) = b_n$, that is: $\pi_{n,n+1}(\neg b_{n+1}) = 1_{\mathbb{B}_n}$. Let $K = \{c^n_{\neg b_n} : n \in \omega\}$, that is K contains elements of the form:

$$\begin{aligned}c^0_{\neg b_0} &= \langle \neg b_0, i_{01}(\neg b_0), \ldots, i_{0n}(\neg b_0), \ldots\rangle, \\ c^1_{\neg b_1} &= \langle 1_{\mathbb{B}_0}, \neg b_1, i_{12}(\neg b_0), \ldots, i_{1n}(\neg b_1), \ldots\rangle, \\ &\ldots \\ c^n_{\neg b_n} &= \langle 1_{\mathbb{B}_0}, 1_{\mathbb{B}_1}, \ldots, 1_{\mathbb{B}_{n-1}}, \neg b_n, i_{n,n+1}(\neg b_n), \ldots\rangle.\end{aligned}$$

Each element of K is incompatible with b, but the pointwise supremum of K which is $\langle 1_{\mathbb{B}_0}, 1_{\mathbb{B}_1}, \ldots, 1_{\mathbb{B}_n}, \ldots\rangle$ is compatible with b, so $1_{T(\mathcal{F})}$ can not be $\bigvee_{\mathbb{B}(T(\mathcal{F}))} K$.

4.2 General iterated forcing

We summarize definitions and properties related to general iterated forcing. For further informations and proofs we refer to [5][VII-VIII]. Here M is a transitive model of ZFC. A *p.o.* is a triple $\langle P, \leq_P, 1_P\rangle$ such that $\leq_P$ partially orders P and 1_P is a largest element of P.

Definition 4.2.1. We mean for *complete embedding* of two p.o. $\langle P, \leq_P, 1_P\rangle$ and $\langle Q, \leq_Q, 1_Q\rangle$ a map $i : P \to Q$ such that:

(i) $\forall p, p' \in P[p \leq_P p' \to i(p) \leq_Q i(q)]$;

(ii) $\forall p_1, p_2 \in P[p_1 \perp p_2 \leftrightarrow i(p_1) \perp i(p_2)]$;

(iii) $\forall q \in Q \exists p \in P \forall p' \in P[p' \leq_P p \to i(p') \parallel q]$. We call p a *reduction* of q to P.

Definition 4.2.2. If P is a p.o. in M, a P-name for a p.o. is a triple of P-names, $\langle \nu, \nu', \nu''\rangle \in M$, such that $\nu'' \in \text{dom}(\nu)$ and

$$1_P \Vdash_P [(\nu'' \in \nu) \wedge (\nu' \text{ is a partial order of } \nu \text{ with largest element } \nu'')]$$

We often write ν for $\langle \nu, \nu', \nu''\rangle$, $\leq_\nu$ for ν' and 1_ν for ν''.

Definition 4.2.3. If P is a p.o. in M and $\langle \nu, \leq_\nu, 1_\nu\rangle$ is a P-name for a p.o., define

$$X_\nu = \{\tau \in M^P : 1_P \Vdash \tau \in \nu, \forall \rho \in M^P[1_P \Vdash \rho = \nu \to \text{rank}(\rho) \geq \text{rank}(\nu)]\}$$

then $P * \nu$ is the p.o. whose base set is

$$\{\langle p, \tau\rangle : p \in P \wedge \tau \in X_\nu\}$$

In $P * \nu$, we define $\langle p, \tau\rangle \leq_{P*\nu} \langle q, \sigma\rangle$ iff

$$p \leq_P q \wedge p \Vdash \tau \leq_\nu \sigma,$$

and we set

$$1_{P*\tau} = \langle 1_P, 1_\nu\rangle.$$

Define $i : P \to P * \nu$ by $i(p) = \langle p, 1_\nu\rangle$.

Lemma 4.2.4. *In the notation of Definition 4.2.3:*

(i) $\forall p, p' \in P(p \leq_P p' \leftrightarrow \langle p, 1_\nu \rangle \leq_{P*\nu} \langle p', 1_\nu \rangle)$;

(ii) $i(1_P) = 1_{P*\nu}$;

(iii) $\forall \langle p, \tau \rangle, \langle p', t' \rangle \in P * \nu (p \perp p' \rightarrow \langle p, \tau \rangle \perp \langle p', t' \rangle)$;

(iv) $\forall \langle p, \tau \rangle \in P * \nu \, \forall p' \in P(p \perp p' \leftrightarrow \langle p, \tau \rangle \perp \langle p', 1_\nu \rangle)$;

(v) $\forall p, p' \in P(p \perp p' \leftrightarrow i(p) \perp i(p'))$;

(vi) $i : P \rightarrow P * \nu$ *is a complete embedding.*

Proof. Proofs of (i)-(iii) come from the definitions. To prove (iv) from right to left, if $q \leq_P p$ and $q \leq_P p'$ then $\langle q, \tau \rangle \leq_{P*\nu} \langle p, \tau \rangle$ and $\langle q, \tau \rangle \leq_{P*\nu} \langle p', 1_\nu \rangle$. (v) is a special case of (iv). (vi) follows from (i), (v) and (vi), where p is a reduction of $\langle p, \tau \rangle$ to P. ☐

Definition 4.2.5. In the notation of Definition 4.2.3, if G is P-generic over M and $H \subseteq \nu_G$, then

$$G * H = \{\langle p, \tau \rangle \in P * \nu : p \in G \wedge \tau_G \in H\}.$$

Theorem 4.2.6. *Assume that P is a p.o. in M and ν is a P-name for a p.o. Let K be $P * \nu$-generic over M, and let*

$$H = \{\tau_G : \tau \in \mathrm{dom}(\tau) \wedge \exists q(\langle q, \tau \rangle \in K)\}.$$

Then G is P-generic over M, H is ν_G-generic over $M[G]$, $K = G * H$, and $M[K] = M[G][H]$.

We now turn to consider *α-stage iterated forcing* for α any ordinal. Remark that for $\alpha = 1$ we have ordinary forcing and for $\alpha = 2$ we have iterations of the form $P * \nu$ just discussed.

Definition 4.2.7. In M, suppose α is any ordinal, $\mathcal{I} \subseteq \mathcal{P}(\alpha)$, $\mathcal{I}$ is an ideal on $\alpha + 1$, and $\mathcal{I}$ contains all finite subsets of α. An *α-stage iterated forcing construction with supports in $\mathcal{I}$* is an object in M of the form,

$$\langle \langle \langle P_\xi, \leq_{P_\xi}, 1_{P_\xi} \rangle : \xi \leq \alpha \rangle, \ \langle \langle \nu_\xi, \leq_{\nu_\xi}, 1_{\nu_\xi} \rangle : \xi < \alpha \rangle \rangle,$$

which satisfies the following conditions:

(i) Each $\langle P_\xi, \leq_\xi, 1_{P_\xi} \rangle$ is a p.o.;

(ii) Elements of P_ξ are all sequences of length ξ;

(iii) If $p \in P_\eta$ and $\xi < \eta$, then $p \restriction \xi \in P_\xi$;

(iv) Each $\langle \nu_\xi, \leq_{P_\xi}, 1_{P_\xi} \rangle$ is a P_ξ-name for a p.o.;

(v) If $\langle \rho_\mu : \mu < \xi \rangle \in P_\xi$, then each $\rho_\mu \in X_{\nu_\mu}$;

(vi) 1_{P_ξ} is the sequence $\langle \rho_\mu : \mu < \xi \rangle$ such that each $\rho_\mu = 1_{\nu_\mu}$.

Define $\mathrm{supt}(\langle \rho_\mu : \mu < \xi \rangle) = \{\mu < \xi : \rho_\mu \neq 1_{\nu_\mu}\}$. We demand further that the construction satisfy:

(a) *Basis.* $P_0 = \{0\}$.

(b) *Successors.* If $p = \langle \rho_\mu : \mu \leq \xi \rangle$, then $p \in P_{\xi+1}$ iff $p \restriction \xi \in P_\xi$, $\rho_\xi \in X_{\nu_\xi}$, and $p \restriction \xi \Vdash_{P_\xi} (\rho_\xi \in \nu_\xi)$. If also $p' = \langle \rho'_\mu : \mu \leq \xi \rangle$, then $p \leq p'$ iff $p \restriction \xi \leq p' \restriction \xi$ and $p \restriction \xi \Vdash (\rho_\xi \leq \rho'_\xi)$.

(c) *Limits.* If η is a limit ordinal and $p = \langle \rho_\mu : \mu < \eta \rangle$, then $p \in P_\eta$ iff

$$\forall \xi < \eta (p \restriction \xi \in P_\xi \wedge \mathrm{supt}(p) \in \mathcal{I}).$$

If $p, p' \in P_\eta$, then $p \leq p'$ iff $\forall \xi < \eta (p \restriction \xi \leq p' \restriction \xi)$.

Definition 4.2.8. In Definition 4.2.7, we say the iteration is of *finite support* iff $\mathcal{I} = \{X \subseteq \alpha : |X| < \omega\}$, and of *countable support* iff $\mathcal{I} = \{X \subseteq \alpha : (|X| \leq \omega)^M\}$. Iterations with *full limits* means $\mathcal{I} = (\mathcal{P}(\alpha))^M$.

Generalizing the $i : P \to P * \nu$ from two stage iterations, we have the following.

Definition 4.2.9. In the notation of Definition 4.2.7, if $\xi \leq \eta \leq \alpha$, define $i_{\xi\eta} : P_\xi \to P_\eta$ so that $i_{\xi\eta}(p)$ is the $p' \in P_\eta$, such that $p' \restriction \xi = p$ and $p'(\mu) = 1_{\nu_\mu}$ for $\xi \leq \mu < \eta$.

We summarize now some abstract order-theoretic properties of iterated forcing constructions.

Lemma 4.2.10. *In the notation of Definitions 4.2.7 and 4.2.9, assume that $\xi \leq \eta \leq \zeta \leq \alpha$. Then:*

(a) $i_{\xi\zeta} = i_{\eta\zeta} \circ i_{\xi\eta}$;

(b) $i_{\xi\eta}(1_{P_\xi}) = 1_{P\eta}$;

(c) $\forall p, p' \in P_\eta (p \leq p' \to p \restriction \xi \leq p' \restriction \xi)$;

(d) $\forall p, p' \in P_\xi (p \leq p' \leftrightarrow i_{\xi\eta}(p) \leq i_{\xi\eta}(p'))$;

(e) $\forall p, p' \in P_\eta (p \restriction \xi \perp p' \restriction \xi \to p \perp p')$;

(f) $\forall p, p' \in P_\eta [\mathrm{supt}(p) \cap \mathrm{supt}(p') \subseteq \xi \to (p \restriction \xi \perp p' \restriction \xi \leftrightarrow p \perp p')]$;

(g) $\forall p, p' \in P_\xi (p \perp p' \leftrightarrow i_{\xi\eta}(p) \perp i_{\xi\eta}(p'))$;

(h) $i_{\xi\eta}$ *is a complete embedding.*

Lemma 4.2.11. *Assume that in M,*

$$\langle \langle P_\xi : \xi \leq \alpha \rangle, \langle \nu_\xi : \xi < \alpha \rangle \rangle$$

is an α-stage iterated forcing construction. Let G be $P_\alpha-$ generic over M. For each $\xi \leq \alpha$, let $G_\xi = i_{\xi\alpha}^{-1}(G)$. Then G_ξ is $P_\xi-$ generic over M and

$$\xi \leq \eta \to M[G_\xi] \subseteq M[G_\eta].$$

4.3 Iterated forcing and Boolean algebras iteration systems

In this last section we start to relate α-stage iterated forcing and linear iteration systems of complete Boolean algebras.

We begin by recalling the definitions of p.o. complete embedding, Boolean algebras regular embedding and Boolean completion of p.o.

Complete embedding: We mean for *complete embedding* of two p.o. $\langle P, \leq_P, 1_P \rangle$ and $\langle Q, \leq_Q, 1_Q \rangle$ a map $i : P \to Q$ such that:

- $\forall p, p' \in P(p \leq_P p' \to i(p) \leq_Q i(q))$;
- $\forall p_1, p_2 \in P(p_1 \perp p_2 \leftrightarrow i(p_1) \perp i(p_2))$;
- $\forall q \in Q \exists p \in P \forall p' \in P(p' \leq_P p \to i(p') \parallel q)$. We call p a *reduction* of q to P.

Regular embedding: Let $\mathbb{B}$, $\mathbb{Q}$ be complete Boolean algebras. $\hat{\imath} : \mathbb{B} \to \mathbb{Q}$ is a *regular embedding* if it is a complete and injective homomorphism of Boolean algebras.

Boolean completion: Let P a poset, then there is a complete Boolean algebra $\mathbb{B}$, unique up to isomorphism, and a map $j : P \to \mathbb{B}$ such that:

- $j[P]$ is dense in $\mathbb{B}$;
- j is order preserving, that is, $\forall p, q \in P(p \leq q \to j(p) \leq j(q))$;
- $\forall p, q \in P(p \parallel q \leftrightarrow j(p) \wedge j(q) \neq 0_{\mathbb{B}})$

We want to study how the previous definitions are related. The next lemma states the equivalence between regular embeddings and complete embeddings of Boolean algebras.

Lemma 4.3.1. *$\hat{\imath} : \mathbb{B} \to \mathbb{Q}$ is a regular embedding if and only if $\hat{\imath} : \mathbb{B}^+ \to \mathbb{Q}^+$ is a complete embedding.*

Proof. We begin the proof with the implication from left to right.

Predensity preservation: Let $D \subseteq \mathbb{B}$ a predense subset of $\mathbb{B}$. By hypothesis, we know that $\forall q \in \mathbb{Q} \exists p \in \mathbb{B}$ such that $\forall p' \leq p \, (\hat{\imath}(p') \parallel q)$. By D predensity, $\forall p' \leq p \exists d \in D(d \parallel p')$, that is $\exists r \in P$ such that $r \leq d$ and $r \leq p'$. It follows that $\hat{\imath}(r) \leq \hat{\imath}(d)$ and $\hat{\imath}(r) \parallel q$. Hence $\hat{\imath}(d) \parallel q$ and $\hat{\imath}[D]$ is predense.

$1_{\mathbb{B}}$ is mapped in $1_{\mathbb{Q}}$: By the predensity preservation, for $D \subseteq P$ predense, it follows:

$$1_{\mathbb{Q}} = \bigvee \hat{\imath}[D] \leq \hat{\imath}\,(\bigvee D) = \hat{\imath}\,(1_{\mathbb{B}})$$

Complement preservation: We already know that for $a \in \mathbb{B}$ we have $\hat{\imath}\,(a) \vee \hat{\imath}\,(\neg a) = 1_{\mathbb{Q}}$ since the subset $\{a, \neg a\}$ is predense.

We check instead that $\hat{\imath}(a) \wedge \hat{\imath}\,(\neg a) = 0_{\mathbb{Q}}$. By hypothesis, $\hat{\imath}$ preserves incompatibility so $a \wedge \neg a = 0_{\mathbb{B}}$ must implies $\hat{\imath}(a) \wedge \hat{\imath}\,(\neg a) = 0_{\mathbb{Q}}$

Completeness: Let $A \subseteq \mathbb{B}$. If A is not predense we complete it with $A \cup \{\neg \bigvee A\} = A'$.

A' is predense. If not, for all $b \in \mathbb{B}$ and $a \in A'$, we wold have $a \wedge b = 0_{\mathbb{B}}$, that imply

$$\begin{aligned} \bigvee_{a \in A'} (a \wedge b) = 0_{\mathbb{B}} &\rightarrow (\bigvee A \vee \neg \bigvee A) \wedge p = 0_{\mathbb{B}} \\ &\rightarrow 1_{\mathbb{B}} \wedge p = 0_{\mathbb{B}} \\ &\rightarrow p = 0_{\mathbb{B}}. \end{aligned}$$

Now, applying $\hat{\imath}$ to elements of A', we get that $\hat{\imath}[A']$ is predense, and

$$\bigvee \hat{\imath}[A] \vee \hat{\imath}(\neg \bigvee A) = 1_{\mathbb{B}} = \hat{\imath}(\bigvee A) \vee \hat{\imath}(\neg \bigvee A). \tag{4.1}$$

We also know that

$$\bigvee \hat{\imath}[A] \leq \hat{\imath}(\bigvee A) \tag{4.2}$$

and

$$\begin{aligned} \bigvee \hat{\imath}[A] \wedge \hat{\imath}(\neg A) &= \bigvee \hat{\imath}[A] \wedge \neg \hat{\imath}(A) \\ &\leq \hat{\imath}(\bigvee A) \wedge \neg \hat{\imath}(\bigvee A) \\ &= 0_{\mathbb{Q}}. \end{aligned} \tag{4.3}$$

By (4.1)- (4.2)- (4.3), we conclude that

$$\bigvee \hat{\imath}[A] = \hat{\imath}(\bigvee A).$$

Injectivity: It is obvious from the fact that, by hypothesis, $\hat{\imath}$ goes from positive elements of $\mathbb{B}$ to positive elements of $\mathbb{Q}$.

To the other implication, assume $\hat{\imath}$ is a complete and injective homomorphism of Boolean algebras.

Order preservation ("→"): Let $a \leq b$, then $\hat{\imath}(a) \leq \hat{\imath}(a) \vee \hat{\imath}(b) = \hat{\imath}(a \vee b) = \hat{\imath}(b)$.

Incompatibility preservation ("↔"): Let $a \wedge b = 0_{\mathbb{B}}$ then $0_{\mathbb{Q}} = \hat{\imath}(a \wedge b) = \hat{\imath}(a) \wedge \hat{\imath}(b)$. Conversely, let $\hat{\imath}(a) \wedge \hat{\imath}(b) = 0_{\mathbb{Q}}$, then $\hat{\imath}(a) \wedge \hat{\imath}(b) = \hat{\imath}(a \wedge b) = 0_{\mathbb{Q}}$. By $\hat{\imath}$ injectivity, $a \wedge b = 0_{\mathbb{B}}$.

Reduction: Let $q \in \mathbb{Q}^+$. We claim that $D = \{b \in \mathbb{B} : \hat{\imath}(b) \perp q\}$ is not predense. By contradiction, if $\bigvee D = 1_{\mathbb{B}}$ then $\exists d \in D : \hat{\imath}(d) \wedge q > 0_{\mathbb{Q}}$. Now letting $r = 1_{\mathbb{B}} \setminus \bigvee D$, the reduction of q to $\mathbb{B}$ is r. In fact, $\forall s \leq r$, $\hat{\imath}(s) \parallel q$. If not, we would have a contradiction: $s \in D$ but, by definition, $s \wedge \bigvee D = 0_{\mathbb{B}}$. □

We now show how a complete embedding of two posets defines a regular embedding of their Boolean completions.

Lemma 4.3.2. *Consider two refined p.o.* $\langle P, \leq_P, 1_P \rangle$ *and* $\langle Q, \leq_q, 1_Q \rangle$ *and let* $i : P \to Q$ *be a complete embedding. Let* $\langle \mathbb{B}(P), e \rangle$ *and* $\langle \mathbb{B}(Q), f \rangle$ *be the Boolean completions of P and Q.*
Then i defines a regular embedding $\hat{\imath} : \mathbb{B}(P) \to \mathbb{B}(Q)$ *that makes commutative the following diagram:*

$$\begin{array}{ccc} P & \xrightarrow{\ i\ } & Q \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{B}(P) & \xrightarrow[\ \hat{\imath}\ ]{} & \mathbb{B}(Q) \end{array}$$

Proof. We define the map between $\mathbb{B}(P)$ and $\mathbb{B}(Q)$ as:

$$\begin{aligned} \hat{\imath} : \mathbb{B}(P) &\to \mathbb{B}(Q) \\ a &\mapsto \bigvee \{f \circ i(p) : e(p) \leq a\}. \end{aligned}$$

We check that $\hat{\imath}$ makes commutative the diagram and that it is a regular embedding by means of Lemma 4.3.1.

Commutativity: For all $p \in P$:

$$\begin{aligned} \hat{\imath} \circ e(p) &= \{f \circ i(p') : e(p') \leq e(p)\} \\ &= f \circ i(p). \end{aligned}$$

Double incompatibility preservation ("$\leftrightarrow$") : We shall control that for all $a, b \in P$:

$$a \perp b \;\leftrightarrow\; \hat{\imath}(a) \wedge \hat{\imath}(b) = 0_{\mathbb{B}(Q)}.$$

For the implication from left to right, define:

$$A_a = \{p \in P : e(p) \leq a\};$$
$$A_b = \{q \in P : e(q) \leq b\}.$$

For all $p \in A_a$ and $q \in A_b$, we have that $e(p) \perp e(q)$ and, in particular, the following implications hold:

$$\begin{aligned} e(p) \perp e(q) &\rightarrow p \perp q \\ &\rightarrow i(p) \perp i(q) \\ &\rightarrow f \circ i(p) \wedge f \circ i(q) = 0_{\mathbb{B}(Q)} \\ &\rightarrow \bigvee\{f \circ i(p) : e(p) \leq a\} \wedge \bigvee\{f \circ i(q) : e(q) \leq b\} = 0_{\mathbb{B}(Q)} \\ &= \hat{\imath}(a) \wedge \hat{\imath}(b) = 0_{\mathbb{B}(Q)}. \end{aligned}$$

To prove the implication from right to left, assume $a \wedge b > 0_{\mathbb{B}(P)}$. By $e[P]$ density in $\mathbb{B}(P)$, there is a $p \in P$ such that $e(p) \leq a \wedge b$. In particular $e(p) \leq a$ and $e(p) \leq b$. Thus $f \circ i(p) \leq \bigvee\{f \circ i(p') : e(p') \leq a\} = \hat{\imath}(a)$ and $f \circ i(p) \leq \bigvee\{f \circ i(p') : e(p') \leq b\} = \hat{\imath}(b)$. Hence the conclusion: $0_{\mathbb{B}} < f \circ i(p) \leq \hat{\imath}(a) \wedge \hat{\imath}(b)$.

Order preservation ("$\rightarrow$"): By $\hat{\imath}$ definition, it follows that for all $a, b \in \mathbb{B}(P)$:

$$a \leq b \rightarrow \hat{\imath}(a) \leq \hat{\imath}(b).$$

Reduction existence: Let $q \in \mathbb{B}(Q)$. By $f[Q]$ density in $\mathbb{B}(Q)$, there is a $q' \in Q$ such that $f(q') \leq q$. Now, we can pick a reduction $p \in P$ of q' to P, so we have:

$$\forall p' \leq p : \; i(p') \parallel q'.$$

It follows that $\forall p' \leq p$:

$$\begin{aligned} i(p') \parallel q' &\leftrightarrow f(i(p')) \wedge f(q') > 0_{\mathbb{B}(Q)} \\ &\leftrightarrow \hat{\imath}(e(p')) \wedge f(q') > 0_{\mathbb{B}(Q)}. \end{aligned} \tag{4.4}$$

We affirm that the reduction of q to $\mathbb{B}(P)$ is $e(p)$.
In fact, let $0_{\mathbb{B}(P)} < r \leq e(p)$. By $e[P]$ density in $\mathbb{B}(P)$, there is a $p'' \in P$

such that $e(p'') \leq r \leq e(p)$. Then in particular, $p'' \parallel p$, that is, there exists a $t \in P$ such that $t \leq p''$ and $t \leq p$. Hence we get the following chain:

$$e(t) \leq e(p'') \leq r \leq e(p).$$

Applying $\hat{\imath}$ we get in particular that

$$\hat{\imath}(e(t)) \leq \hat{\imath}(r),$$

and, by (4.4), we finally conclude that

$$0_{\mathbb{B}(Q)} < \hat{\imath}(e(t)) \wedge q \leq \hat{\imath}(r) \wedge q. \qquad \square$$

Lemma 4.3.3. *Let $P * \nu$ a two step iteration and consider the Boolean completion maps $j_0 : P \to \mathbb{B}(P)$, $j_1 : P * \nu \to \mathbb{B}(P * \nu)$. Let $i : P \to P * \nu$ the complete embedding such that $i(p) = \langle p, 1_\nu \rangle$, $\hat{\imath} : \mathbb{B}(P) \to \mathbb{B}(P * \nu)$ the regular embedding generated by i and $\hat{\pi}$ the retraction associated to $\hat{\imath}$. If $b_1 \in \mathbb{B}(P * \nu)$ and $b_0 = \hat{\pi}(b_1)$, by $j_0[P]$ density in $\mathbb{B}(P)$ pick $r \in P$ such that $j_0(r) \leq b$. Then there is a $\tau \in X_\nu$ such that $j_1(\langle r, \tau \rangle) \leq b_1$.*

Finally, by means of the previous lemmas, we can connect iterated forcing and iterations systems of Boolean algebras.

Theorem 4.3.4. *In the notation of Definition 4.2.7, consider an α-stage iterated forcing construction with support $\mathcal{I} \subseteq \mathcal{P}(\alpha)$:*

$$\langle \langle \langle P_\xi, \leq_{P_\xi}, 1_{P_\xi} \rangle : \xi \leq \alpha, \langle \langle \nu_\xi, \leq_{\nu_\xi}, 1_{\nu_\xi} \rangle : \xi < \alpha \rangle \rangle \rangle.$$

Then

$$\mathcal{F} = \{ \hat{\imath}_{\xi\eta} : \mathbb{B}_\xi \to \mathbb{B}_\eta : \xi < \eta \leq \alpha \}$$

is such that:

(a) $\forall \xi \leq \alpha : \langle \mathbb{B}_\xi, j_\xi \rangle$ *is a Boolean completion of* P_ξ.

(b) $\forall \xi < \eta \leq \alpha$, *the map* $\hat{\imath}_{\xi\eta}$ *makes commutative the following diagram:*

$$\begin{array}{ccc} P_\xi & \xrightarrow{\;i_{\xi\eta}\;} & P_\eta \\ {\scriptstyle j_\xi}\downarrow & & \downarrow{\scriptstyle j_\eta} \\ \mathbb{B}_\xi & \xrightarrow[\;\hat{\imath}_{\xi\eta}\;]{} & \mathbb{B}_\eta \end{array}$$

where $i_{\xi\eta}(p) = p^\frown\langle 1_{\pi_\mu} : \xi \leq \mu < \eta\rangle$

(c) *$\forall\xi < \eta \leq \alpha$, the retraction $\hat{\pi}_{\xi\eta}$ associated to $\hat{\imath}_{\xi\eta}$ makes commutative the following diagram:*

$$\begin{array}{ccc} P_\xi & \xleftarrow{\pi_{\xi\eta}} & P_\eta \\ {\scriptstyle j_\xi}\downarrow & & \downarrow{\scriptstyle j_\eta} \\ \mathbb{B}_\xi & \xleftarrow[\hat{\pi}_{\xi\eta}]{} & \mathbb{B}_\eta \end{array}$$

where $\pi_{\eta\xi}(q) = q \restriction \eta$

(d) *$\mathcal{F}$ is an iteration system of complete Boolean algebras;*

(e) *if $I = P(\alpha)$, P_α is dense in $T(\mathcal{F})$ and thus $\mathbb{B}_\alpha \cong \mathbb{B}(T(\mathcal{F}))$; if $\forall X \in I(\cup X < \alpha)$ then $\mathbb{B}_\alpha \cong C(\mathcal{F})$.*

Proof. (a) We choose $\mathbb{B}_\xi \cong \mathbb{B}(P_\xi)$.

(b) Use Lemma 4.3.2.

(c) Remember that $\hat{\pi}_{\xi\eta}(j_\eta(p)) = \bigwedge_{\mathbb{B}_\xi}\{b : \hat{\imath}_{\xi\eta}(b) \geq j_\eta(p)\}$. We certainly have that $j_\xi(p \restriction \xi) \geq \hat{\pi}_{\xi\eta} \circ j_\eta(p)$ because

$$\begin{aligned} \hat{\imath}_{\xi\eta}(j_\xi(p \restriction \xi)) &= j_\eta \circ i_{\xi\eta}(p \restriction \xi) \\ &\geq j_\eta(p) \end{aligned}$$

and thus $j_\xi(p \restriction \xi) \in \{b : \hat{\imath}_{\xi\eta}(b) \geq j_\eta(p)\}$.

Conversely, if $j_\xi(p \restriction \xi) > \hat{\pi}_{\xi\eta} \circ j_\eta(p)$, then there is a $s \in P_\xi$ such that $j_\xi(s) \leq j_\xi(p \restriction \xi)$ and $j_\xi(s) \wedge \hat{\pi}_{\xi\eta} \circ j_\xi(p) = 0_{\mathbb{B}_\xi}$, we can choose $s \leq p \restriction \xi$. Applying $\hat{\imath}_{\xi\eta}$ we get:

$$\begin{aligned} 0_{\mathbb{B}_\eta} &= \hat{\imath}_{\xi\eta}(0_{\mathbb{B}_\xi}) \\ &= \hat{\imath}_{\xi\eta}(j_\xi(s) \wedge \hat{\pi}_{\xi\eta}(j_\xi(p))) \\ &= \hat{\imath}_{\xi\eta}(j_\xi(s)) \wedge \hat{\imath}_{\xi\eta}(\hat{\pi}_{\xi\eta}(j_\xi(p)))) \\ &\geq j_\eta(i_{\xi\eta}(s)) \wedge j_\xi(p) \\ &= j_\eta(s^\frown\langle 1_{\pi_\mu} : \xi \leq \mu < \eta\rangle) \wedge j_\xi(p) \end{aligned}$$

thus $s^\frown\langle 1_{\pi_\mu} : \xi \leq \mu < \eta\rangle \perp p$.

But we also have that $s \leq p \restriction \xi$ so $s^\frown\langle p(\mu) : \xi \leq \mu < \eta\rangle \leq i_{\xi\eta}(s)$ and $s^\frown\langle p(\mu) : \xi \leq \mu < \eta\rangle \leq p$, a contradiction.

(d) We already know that: $\mathbb{B}_\xi$ for $\xi \leq \alpha$ are complete Boolean algebras, $\hat{\imath}_{\xi\eta}$ are regular embeddings with associated retractions $\hat{\pi}_{\xi\eta}$; it remains to show that for all $\xi < \eta < \nu \leq \alpha$ the composition works: $\hat{\imath}_{\eta\nu} \circ \hat{\imath}_{\xi\eta} = \hat{\imath}_{\xi\nu}$. In fact, for all $a \in \mathbb{B}_\xi$

$$\begin{aligned}
\hat{\imath}_{\xi\eta}(a) &= \bigvee\{j_\nu \circ i_{\xi\nu}(p) : j_\xi(p) \leq a\} \\
&= \bigvee\{j_\nu \circ i_{\eta\nu} \circ i_{\xi\eta}(p) : j_\xi(p) \leq a\} \\
&= \bigvee\{\hat{\imath}_{\eta\nu} \circ j_\eta \circ i_{\xi\eta}(p) : j_\xi(p) \leq a\} \\
&= \bigvee\{\hat{\imath}_{\eta\nu} \circ \hat{\imath}_{\xi\eta} \circ j_\xi(p) : j_\xi(p) \leq a\} \\
&= \hat{\imath}_{\xi\nu} \circ \hat{\imath}_{\xi\eta}(a),
\end{aligned}$$

the last equality comes from $j_\xi[P_\xi]$ density in $\mathbb{B}_\xi$.

(e) Define the map

$$\begin{aligned}
f : P_\alpha &\to T(\mathcal{F}) \\
p = \langle \rho_\mu : \mu < \alpha\rangle &\mapsto \langle j_\mu(p \restriction \mu) : \mu < \alpha\rangle
\end{aligned}$$

f is a well-defined map, that is $f(p) \in T(\mathcal{F})$, in fact for all $\mu < \xi < \alpha$

$$\begin{aligned}
\hat{\pi}_{\mu\xi}(f(p)_\xi) &= \hat{\pi}_{\mu\xi}(j_\xi(p \restriction \xi)) \\
&= j_\mu \circ \pi_{\mu\xi}(p \restriction \xi) \\
&= j_\mu(p \restriction \mu) \\
&= f(p)_\mu
\end{aligned}$$

If $\cup supp(p) < \alpha$ then $f(p) \in C(\mathcal{F})$.
f is order preserving. In fact if $p \leq_{P_\alpha} q$ then $p \restriction \mu \leq_{P_\mu} q \restriction \mu$ for all $\mu < \alpha$, thus $j_\mu(p \restriction \mu) \leq_{\mathbb{B}_\mu} j_\mu(q \restriction \mu)$ and hence $f(p) \leq_{T(\mathcal{F})} f(q)$.

$f[P_\alpha]$ is dense in $T(\mathcal{F})$. Let $b = \langle b_\xi : \xi < \alpha\rangle \in T(\mathcal{F})$, we can build a $p \in P_\alpha$ such that $j_\xi(p \restriction \xi) \leq_{\mathbb{B}_\xi} b_\xi$ for all $\xi < \alpha$ by means of Lemma 4.3.3.

f preserves incompatibility (i.e. $p \perp q \to f(p) \perp f(q)$)
Let $p \perp q \in P_\alpha$, if by absurd $f(p) \parallel f(q)$, then there is a $t \in T(\mathcal{F})$ such that $t \leq f(p)$ and $t \leq f(q)$. Now, by $f[P_\alpha]$ density, we can choose a $r \in P_\alpha$ so that $f(r) \leq t$. It follows that $f(r) \leq f(p)$ and $f(r) \leq f(q)$,

i.e. $\langle j_\xi(r \restriction \xi) : \xi < \alpha\rangle \leq_{T(\mathcal{F})} \langle j_\xi(p \restriction \xi) : \xi < \alpha\rangle$ and $\langle j_\xi(r \restriction \xi) : \xi < \alpha\rangle \leq_{T(\mathcal{F})} \langle j_\xi(q \restriction \xi) : \xi < \alpha\rangle$
In particular, for all $\xi < \alpha$ we have $j_\xi(r \restriction \xi) \leq_{P_\xi} j_\xi(p \restriction \xi)$ and $j_\xi(r \restriction \xi) \leq_{P_\xi} j_\xi(q \restriction \xi)$.
By j_ξ property, it follows that $r \restriction \xi \parallel p \restriction \xi$ and $r \restriction \xi \parallel q \restriction \xi$ for all $\xi < \alpha$. □

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