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BOOLEAN VALUED MODELS FOR SET THEORY AND GROTHENDIECK TOPOI

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Introduction

This dissertation presents some aspects of the connection between boolean valued models for Set Theory (a subject pertaining to logic and Set Theory) with sheaves and toposes (which are mostly studied by category theorists and algebraic geometers).

Boolean valued models for Set Theory are a standard method to present Forcing. The forcing technique was introduced by Cohen in 1963 (see [2]) in order to prove the independence of the Continuum Hypotesis from the ZFC axioms for Set Theory. Since then it has been applied to prove the undecidability of many problems arsing in various branches of mathematics, among others: group theory, topology, functional analysis.

Category Theory arose from a 1945 article written by Mac Lane and Eilenberg ([9]) on algebraic topology. It provides a foundation of mathematics alternative to the one given by Set Theory. Its high degree of abstraction allows to find applications of category theoretic ideas and methods everywhere in mathematics. Category theory is also used in quantum physics and computer science.

Even if the idea of dealing with forcing from a categorial point of view has been well developed, as far as we know the interpretation of boolean valued models for Set Theory as categories of sheaves on a boolean topological space has not been exposed in full details yet. This dissertation does exactly that.

The thesis is divided in four chapters.

The first chapter introduces the tools needed to develop forcing by means of boolean valued models. The first two sections outline the basics about boolean algebras, Stone spaces, and boolean valued models respectively, and the third and fourth sections use these results to present the standard approach to forcing via boolean valued models for Set Theory of the form V^{B}.

Chapter 2 is a short introduction to Category Theory and to the notion of a sheaf on a topological space. The first section presents the basic definitions, while the second introduces some important categorial constructions. In the third section the notions of a sheaf and of a Grothendieck topos are defined.

In chapter 3 a step towards the sheaf interpretation of B-names in V^{B} is made: every $\tau \in V^{\mathsf{B}}$ can be canonically identified with a continuous function with domain $\mathrm{St}(\mathsf{B})$ (the Stone space of B) yelding a new boolean valued model for Set Theory $V^{\mathrm{St}(\mathsf{B})}$, which we will prove to be isomorphic to V^{B}.

Chapter 4 combines all the results of the previous chapters. We prove that the cateogory associated to $V^{\mathrm{St}(\mathsf{B})}$ can be faithfully embedded in the category $\mathsf{Sh}(\mathrm{St}(\mathsf{B}))$ given by all sheaves on a topological space $\mathrm{St}(\mathsf{B})$. Particular attention is given to the class of *full*

B-names. On the one hand these are relevant in order to analyze the boolean semantic on V^{B}. On the other hand it will be shown that their key properties give rise to a natural associated sheaf. Conversely we show how to attach to any sheaf a corresponding *full* B-name. The association is natural in the categorial sense. Finally, we prove that $V^{\mathrm{St}(\mathsf{B})}$ is a *Grothendieck topos*, this follows by an application of Giraud's theorem. Nonetheless we are not able to establish that this Topos is the category of *all* sheaves on $\mathrm{St}(\mathsf{B})$. We expect this to be the case and we leave the write-up of the answer to a future investigation.

Chapter 1

Boolean valued models for Set Theory and forcing

This chapter is a short review on the theory of boolean valued models and forcing. We assume the reader is acquainted with the basics on first order logic and Set Theory as axiomatized by ZFC. For a complete introduction to these subjects see for example [11, Chapters 1-5 and 9], while [5] is the reference for results involving boolean algebras and [4], [7] for boolean valued models for Set Theory and forcing.

1.1 Partial orders, boolean algebras and Stone's duality

In this section we recall some properties of boolean algebras and their associated Stone spaces.

Definition 1.1.1. A *boolean algebra* is a poset $(\mathsf{B}, \leq)$ satisfying the following conditions:

- every couple $a, b \in \mathsf{B}$ has a unique least upper bound, denoted $a \vee b$;
- every couple $a, b \in \mathsf{B}$ has a unique greatest lower bound, denoted $a \wedge b$;
- it is limited, that is there are two elements 1_{B} and 0_{B} such that $0_{\mathsf{B}} \leq a \leq 1_{\mathsf{B}}$ for every $a \in \mathsf{B}$;
- (distributivity) for every $a, b, c \in \mathsf{B}$

  $$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

  and

  $$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

- every $a \in \mathsf{B}$ admits a unique complement $\neg a$ with the properties $a \vee \neg a = 1_{\mathsf{B}}$ and $a \wedge \neg a = 0_{\mathsf{B}}$.

A boolean algebra B is *complete* if every subset $X \subseteq \mathsf{B}$ has a supremum (denoted $\bigvee X$) and an infimum (denoted $\bigwedge X$).

Remark 1.1.2. A boolean algebra can equivalently be defined as a tuple $(\mathsf{B}, \wedge, \vee, \neg, 1_{\mathsf{B}}, 0_{\mathsf{B}})$ where B is a set, $\vee$ and $\wedge$ are two commutative and associative binary operations on B, $\neg$ is a unary operation on B, the distributive laws hold, 1_{B} and 0_{B} are the identity elements of $\wedge$ and $\vee$ respectively and the complement law holds.

Given such an algebraic structure, one can recover the partial order on B setting

$$a \leq b \iff a \wedge b = a$$

for every $a, b \in \mathsf{B}$.

We give some examples of boolean algebras:

- $2 = \{0, 1\}$ with $0 < 1$. It is the simplest boolean algebra one can construct.
- Given a set X, its powerset $\mathcal{P}(X)$ with the operations of intersection, union and complement yelds a complete boolean algebra. Stone's representation theorem states that every boolean algebra is isomorphic to a subalgebra of the powerset of some X.
- let X be a topological space. Then the collection $\mathrm{CLOP}(X)$ of its clopen subsets is a boolean algebra with the same operations of $\mathcal{P}(X)$. It may not be complete.
- Recall that a subset A of the space X is *regular* if $A = \mathring{\overline{A}}$. The collection $\mathrm{RO}(X)$ of the regular open subsets of X is a complete boolean algebra with operations

$$\begin{aligned} A \wedge B &= A \cap B \\ A \vee B &= \mathring{\overline{A \cup B}} \\ \neg A &= \mathring{\overline{X \setminus A}} \\ A \leq B &\iff A \subseteq B. \end{aligned}$$

  Since every clopen subset A is trivially regular, $\mathrm{CLOP}(X)$ is always a subalgebra of $\mathrm{RO}(X)$.

Definition 1.1.3. Given two boolean algebras B and C, a function $i\colon \mathsf{B} \to \mathsf{C}$ is a *morphism* of boolean algebras if

- $i(a \vee_{\mathsf{B}} b) = i(a) \vee_{\mathsf{C}} i(b)$,
- $i(a \wedge_{\mathsf{B}} b) = i(a) \wedge_{\mathsf{C}} i(b)$,
- $i(\neg_{\mathsf{B}} a) = \neg_{\mathsf{C}} i(a)$

for every $a, b \in \mathsf{B}$, $i(1_{\mathsf{B}}) = 1_{\mathsf{C}}$ and $i(0_{\mathsf{B}}) = 0_{\mathsf{C}}$.

i is an *isomorphism* if it is a bijective morphism.

Before stating the properties of boolean algebras needed in the following, we outline some facts about general posets.

Definition 1.1.4. Let $(P, \leq)$ a partial order. A subset $X \subseteq P$ is said to be

- an *antichain* if every couple of elements in $p, q \in X$ are incompatible, i.e. there is no $r \in X$ such that $r \leq a$ and $r \leq b$;
- *open* if it is downward closed, that means if $p \in X$ and $q \leq p$ then $q \in X$;
- *dense* if for every $p \in P$ there is some $q \in X$ such that $q \leq p$. More generally, X is *dense below* $r \in P$ if for every $p \leq r$ there is some $q \in X$ with $q \leq p$;
- *predense* (below r) if

$$\downarrow X = \{p \in P : \exists q \in X (p \leq q)\}$$

  is dense (below r);
- a *maximal antichain* if it is a predense antichain;
- a *filter* if it is upward closed, and for every $p, q \in X$ there is $r \in X$ such that $r \leq p$ and $r \leq q$.

Remark 1.1.5. Every partial order $(P, \leq)$ has an associated topology, which we call the *forcing topology*. It is defined to be the topology generated by the collection $\{\downarrow \{p\} : p \in P\}$. It can be checked that $X \subseteq P$ is open (resp. dense) if and only if it is open (resp. dense) with respect to the forcing topology.

Remark 1.1.6. For a boolean algebra B seen as a partial order, $\{0_{\mathsf{B}}\}$ is a dense open subset. For our purposes it is much more interesting to consider the partial order $\mathsf{B}^+ = \mathsf{B} \setminus \{0_{\mathsf{B}}\}$ when considering B as a partial order, and we will do so hereafter.

Theorem 1.1.7. *Let* B *be a complete boolean algebra (sometimes abbreviated with cba). Then:*

- *If* $X \subseteq \mathsf{B}^+$ *is dense below* b*, then* X *contains a maximal antichain in itself;*
- X *is predense below* b *if and only if* $\bigvee X \geq b$*.*

For a proof, see [5, Thm 4.9]
This definition plays a key role in the develpment of the forcing method.

Definition 1.1.8. Let $(P, \leq)$ is a partial order and $D \subseteq \mathcal{P}(P)$. A filter F on P is said to be D-*generic* if it intersects all the subset belonging to D.

Theorem 1.1.9. *Let* $(P, \leq)$ *be a partial order,* $D = \{D_i : i < \omega\}$ *a countable family of predense subsets of* P*, and* $p \in P$*. Then there exists a filter* F *on* P *which is* D*-generic and contains* p*.*

Proof. Define by recursion the set $\{p_i : i < \omega\}$ setting $p_0 = p$, p_{i+1} requiring that $p_{i+1} \leq p_i$ and $p_{i+1} \in \downarrow D_i$ (this is possible since each D_i is predense). Then

$$F = \uparrow\{p_i : i < \omega\} = \{q \in P : \exists i \in \omega(p_i \leq q)\}$$

is the desired filter. It is upward closed by definition, and if $q, r \in F$ by construction $p_j \leq q$ and $p_j \leq r$ for some $j \in \omega$. □

Remark 1.1.10. If P has an atom p, i.e. there is no $q \in P$ with $q < p$, then $\uparrow \{p\}$ is D-generic if D is the collection of the predense subsets of P. The notion of genericity is of interest only when applied to atomless partial orders, and in what follows we will consider this kind of orders and boolean algebras.

Remark 1.1.11. The thesis of theorem 1.1.9 is no more true if D is uncountable. For example, consider the partial order $(2^{<\omega}, \supseteq)$, and the dense sets

- $D_f = \{s \in 2^{<\omega} : s \nsubseteq f\}$ for $f \in 2^\omega$, and
- $E_n = \{s \in 2^{<\omega} : |s| > n\}$ for $n \in \omega$.

If G is a generic filter for $\{D_f : f \in 2^\omega\} \cup \{E_n : n \in \omega\}$, then $g = \cup G \in 2^\omega$. But $G \cap D_g \neq \emptyset$, hence there is some $t \in 2^{<\omega}$ such that $t \nsubseteq g$ and $t \subseteq g$, a contradiction.

Definition 1.1.12. Let B be a boolean algebra. An *ultrafilter* on B is a filter G on B such that for every element $a \in \mathsf{B}$ either $a \in G$ or $\neg a \in G$.

Remark 1.1.13. It follows immediately from the definition that if G is an ultrafilter on B and $a \in G$, then exactly one among $a \in G$ and $\neg a \in G$ holds. In fact, if $a \in G$ and $\neg a \in G$, then $a \wedge \neg a = 0_{\mathsf{B}} \in G$, and this situation is excluded by 1.1.6 and 1.1.10.

Remark 1.1.14. Assuming the Axiom of Choice, every filter F on B can be extended to an ultrafilter (see [5, Thm 2.16]).

Definition 1.1.15. Let B be a boolean algebra. $\mathrm{St}(\mathsf{B})$ is the set of all the ultrafilters on B. On $\mathrm{St}(\mathsf{B})$ we consider the topology generated by the sets

$$N_b = \{G \in \mathrm{St}(\mathsf{B}) \mid b \in G\}.$$

for every $b \in \mathsf{B}$.

The following theorem summarizes the Stone duality, giving a link between the algebraic properties of B and the topological properties of $\mathrm{St}(\mathsf{B})$.

Theorem 1.1.16. *Given a boolean algebra* B*,* $\mathrm{St}(\mathsf{B})$ *is a compact,* 0*-dimensional Hausdorff space. Moreover*

- *every boolean algebra* B *is isomorphic to* $\mathrm{CLOP}(\mathrm{St}(\mathsf{B}))$*, the isomorphism given by the map*
$$b \mapsto N_b;$$

- *every compact* 0*-dimensional Hausdorff topological space* X *is homeomorphic to* $\mathrm{St}(\mathrm{CLOP}(X))$, *the homeomorphism given by the map*

$$x \mapsto \{A \in \mathrm{CLOP}(X) : x \in A\}.$$

For a proof, see [5, Thms 7.8 and 7.10]

We conclude this section with the following result:

Proposition 1.1.17. *Given a boolean algebra* B, $G \in \mathrm{St}(\mathsf{B})$ *is* principal *if* $G = \uparrow\{a\}$ *for some* $a \in \mathsf{B}$. *Then*

- G *is principal if and only it is an isolated point in* $\mathrm{St}(\mathsf{B})$, *therefore* $\mathrm{St}(\mathsf{B})$ *has no isolated points whenever* B *is atomless;*
- *consequently if* B *is atomless* $\mathrm{St}(\mathsf{B}) \setminus \{G\}$ *is open and dense in* $\mathrm{St}(\mathsf{B})$ *for every* $G \in \mathrm{St}(\mathsf{B})$*;*
- *it follows that the intersection of all dense subsets of* $\mathrm{St}(\mathsf{B})$ *for* B *atomless is empty.*

The proof can be found for example in [5, Thm 7.18]

1.2 Boolean valued models

In this section, we brifely discuss an approach to the method of forcing via boolen valued models. Boolean valued logic generalize the Tarski semantics for first order logic, letting the truth values of a given statement (expressed by a first order formula ϕ) range over a boolean algebra B. In other words, ϕ can be neither true nor false, but it can be assigned an intermediate value between "completely true" and "completely false".

Since we are interested in boolean valued models for Set Theory (which is axiomatized in a first order relational language), and to simplify slightly the definition of the boolean semantics, we will assume that all the languages we will deal with are *relational*, i.e. they contain no function symbols.

Definition 1.2.1. Given a complete boolean algebra B and a first order relational language L, a B-valued model $\mathcal{M}$ of signature L consists of

- a nonempty set M called the *domain* of $\mathcal{M}$, whose elements are called B-names;
- a function giving the boolean value of the equality symbol $=$

$$\begin{aligned} =^{\mathcal{M}} \colon M^2 &\to \mathsf{B} \\ (\tau, \sigma) &\mapsto [\![\tau = \sigma]\!]^{\mathcal{M}}_{\mathsf{B}}\,; \end{aligned}$$

- for every n-ary relation symbol r in L, a function giving the boolean value of r

$$\begin{aligned} r^{\mathcal{M}} \colon M^n &\to \mathsf{B} \\ (\tau_1, \ldots, \tau_n) &\mapsto [\![r(\tau_1, \ldots, \tau_n)]\!]^{\mathcal{M}}_{\mathsf{B}}\,; \end{aligned}$$

- for every constant symbol c in L, an element $c^{\mathcal{M}} \in M$.

The apex $\mathcal{M}$ and the pedix B will be omitted if this does not generate confusion. We also require the following conditions to hold:

- $[\![\tau = \tau]\!] = 1_{\mathsf{B}}$;
- $[\![\tau = \sigma]\!] = [\![\sigma = \tau]\!]$;
- $[\![\tau = \sigma]\!] \wedge_{\mathsf{B}} [\![\sigma = \eta]\!] \leq [\![\tau = \eta]\!]$;

for every $\tau, \sigma, \eta \in M$, and

$$\left(\bigwedge_{i=1}^{n} [\![\tau_i = \sigma_i]\!]\right) \wedge_{\mathsf{B}} [\![r(\tau_1, \ldots, \tau_n)]\!] \leq [\![r(\sigma_1, \ldots, \sigma_n)]\!]$$

for every n-ary relation symbol r and $(\tau_1, \ldots, \tau_n), (\sigma_1, \ldots, \sigma_n) \in M^n$.

Remark 1.2.2. If $\mathsf{B} = 2$, then $\mathcal{M}$ is a classical first order model of signature L.

Given a B-valued model $\mathcal{M}$, the boolean semantic for $\mathcal{M}$ is a function attaching an element $[\![\phi]\!] \in \mathsf{B}$ to every L-statement ϕ. To define this semantic on a boolean valued model $\mathcal{M}$ (with domain M) we expand the language L so that it contains a constant symbol c_a for every element $a \in M$, c_a will be often denoted by a.

Definition 1.2.3. Let L be a relational first order signature and $\mathcal{M}$ a boolean valued L-model. Let $L_{\mathcal{M}} = L \cup \{c_a : a \in \mathcal{M}\}$. Given an $L_{\mathcal{M}}$-statement ϕ (i.e. a formula with no free variables), the boolean truth value $[\![\phi]\!]$ in $\mathcal{M}$ is given recursively as follows:

- if ϕ is $c_1 = c_2$
$$[\![c_1 = c_2]\!] = =^{\mathcal{M}}(c_1^{\mathcal{M}}, c_2^{\mathcal{M}});$$
- if ϕ is $r(c_1, \ldots, c_n)$
$$[\![r(c_1, \ldots, c_n)]\!] = r^{\mathcal{M}}(c_1^{\mathcal{M}}, \ldots, c_n^{\mathcal{M}});$$
- $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \wedge_{\mathsf{B}} [\![\psi]\!]$;
- $[\![\phi \vee \psi]\!] = [\![\phi]\!] \vee_{\mathsf{B}} [\![\psi]\!]$;
- $[\![\neg\phi]\!] = \neg_{\mathsf{B}} [\![\phi]\!]$;
- $[\![\phi \rightarrow \psi]\!] = \neg_{\mathsf{B}} [\![\phi]\!] \vee_{\mathsf{B}} [\![\psi]\!]$;
- $[\![\exists x \phi(x)]\!] = \bigvee_{\tau \in M} [\![\phi(\tau)]\!]$;
- $[\![\forall x \phi(x)]\!] = \bigwedge_{\tau \in M} [\![\phi(\tau)]\!]$.

In what follows we will omit the pedix B in $\wedge_{\mathsf{B}}, \vee_{\mathsf{B}}$ and $\neg_{\mathsf{B}}$ when there is no risk of ambiguity.

Remark 1.2.4. We used the completeness of B in the last two clauses of the definition to be able to compute the supremum and the infimum of certain subsets of B. The requirement that B must be complete is stronger than what is really needed to interpret quantifier in this semantics. Nonetheless, we will use complete boolean algebras in our semantics, also to simplify a number of calculations.

Recall that every boolean algebra B has a unique boolean completion up to isomorphism (see [5, Thms 4.13 and 4.14]).

The following can be proved by induction on the complexity of ϕ (or using 1.2.9).

Theorem 1.2.5. *Let L be a relational first order signature and $\mathcal{M}$ a boolean valued L-model with domain M. Let $L_{\mathcal{M}} = L \cup \{c_a : a \in \mathcal{M}\}$. Given an $L_{\mathcal{M}}$-formula $\phi(x)$ (i.e. a formula with just x as a free variable), and $\tau, \sigma \in M$*

$$\llbracket \tau = \sigma \rrbracket \wedge \llbracket \phi(\tau) \rrbracket \leq \llbracket \phi(\sigma) \rrbracket .$$

Given two first order models of signature L, a morphism between them is a function respecting the interpretation of every symbol in L. In the same way it can be defined what a morphism between boolean valued models is.

Definition 1.2.6. Given $\mathcal{M}$ a B-valued model and $\mathcal{N}$ a C-valued model, both of signature L, the couple (Ψ, i) is a *morphism* between $\mathcal{M}$ and $\mathcal{N}$ if

- $\Psi \subseteq M \times N$ is a relation, and $\operatorname{dom} \Psi = M$;
- $i \colon \mathsf{B} \to \mathsf{C}$ is a morphism of boolean algebras;
- for every (τ_1, σ_1) and (τ_2, σ_2) belonging to Ψ
$$i(\llbracket \tau_1 = \tau_2 \rrbracket^{\mathcal{M}}_{\mathsf{B}}) \leq \llbracket \sigma_1 = \sigma_2 \rrbracket^{\mathcal{N}}_{\mathsf{C}} ;$$
- for all n-ary relation symbols r and $(\tau_1, \sigma_1), \ldots, (\tau_n, \sigma_n) \in \Psi$
$$i(\llbracket r(\tau_1, \ldots, \tau_n) \rrbracket^{\mathcal{M}}_{\mathsf{B}}) \leq \llbracket r(\sigma_1, \ldots, \sigma_n) \rrbracket^{\mathcal{N}}_{\mathsf{C}} ;$$
- for every constant symbol c and $(\tau, \sigma) \in \Psi$
$$i(\llbracket \tau = c \rrbracket^{\mathcal{M}}_{\mathsf{B}}) \leq \llbracket \sigma = c \rrbracket^{\mathcal{N}}_{\mathsf{C}} .$$

We say that (Ψ, i) is *injective* when it holds

$$i(\llbracket \tau_1 = \tau_2 \rrbracket^{\mathcal{M}})_{\mathsf{B}} \leq \llbracket \sigma_1 = \sigma_2 \rrbracket^{\mathcal{N}}_{\mathsf{C}}$$

for all $(\tau_1, \sigma_1), (\tau_2, \sigma_2) \in \Psi$.

If in addition the equality sign instead of the $\leq$ holds in all the clauses above, we speak of *embedding*.

Finally, an *isomorphism* is an embedding such that

- for all $\sigma \in N$ there exists $(\tau, \eta) \in \Psi$ with $[\![\eta = \sigma]\!]^{\mathcal{N}}_{\mathsf{C}} = 1_{\mathsf{B}}$,
- i is an isomorphism of boolean algebras.

In such case $\mathcal{M}$ and $\mathcal{N}$ are said to be isomorphic.

We will assume that $i = \mathrm{Id}_{\mathsf{B}}$ when $\mathsf{C} = \mathsf{B}$.

The following can be proved by induction on the complexity of ϕ:

Theorem 1.2.7. *If $\mathcal{M}$ and $\mathcal{N}$ are isomorphic boolean valued models, then for all formulas $\phi(x_1, \dots, x_n)$ and $(\tau_1, \sigma_1), \dots, (\tau_n, \sigma_n) \in \Psi$*

$$i([\![\phi(\tau_1, \dots, \tau_n)]\!]^{\mathcal{M}}) = [\![\phi(\sigma_1, \dots, \sigma_n)]\!]^{\mathcal{N}}.$$

We now state the soundness and completeness theorem for this boolean semantics.

Definition 1.2.8. Let $\mathcal{M}$ be a B-valued model of signature L and ϕ an L-sentence. ϕ is *valid* in $\mathcal{M}$ if $[\![\phi]\!]^{\mathcal{M}} = 1_{\mathsf{B}}$.

A first order theory T consisting of L-sentences is valid in $\mathcal{M}$ if every $\phi \in T$ is valid in $\mathcal{M}$.

Recall that a formal proof of ϕ from a first order theory T is a sequence $(\phi_1, \dots, \phi_n)$ of formulas where $\phi_n = \phi$, and every ϕ_i can be deduced from $(\phi_1, \dots, \phi_{i-1})$ through a rule of inference by a deductive system for first order logic. In this case ϕ is said to be *syntactically provable* from T.

There are many types of deductive systems, here we assume throught that the deductive system we may emply is that one presented in [11, Section 2.6].

The analogous of the Soundness and Completeness theorem holds for boolean valued models:

Theorem 1.2.9. *A sentence ϕ is syntactically provable from a theory T consisting of sentences if and only if ϕ is valid in every boolean valued model $\mathcal{M}$ such that T is valid in $\mathcal{M}$.*

The Completeness direction of the proof is trivial since, as already remarked, the collection of Tarski models is a subcollection of the boolean valued models so the Completeness theorem for first order logic applies. For the Soundness direction, the proof depends on the deductive system for first order logic chosen, and is done by induction on the length of the proof with respect to its applications of the rules of inference of the deductive system.

We now define the Tarski quotient of a boolean valued model.

Definition 1.2.10. Let $\mathcal{M}$ be a B-valued model of signature L, and G an ultrafilter on B. The model $\mathcal{M}/G$ is defined as follows:

- The domain of $\mathcal{M}/G$ is $\{[\tau]_G : \tau \in M\}$, where $\tau \sim_G \sigma \iff [\![\tau \in \sigma]\!] \in G$;

- for every n-ary relation symbol r and $(\tau_1, \ldots, \tau_n) \in M^n$

$$([\tau_1]_G, \ldots, [\tau_n]_G) \in r^{\mathcal{M}/G} \iff [\![r(\tau_1, \ldots, \tau_n)]\!] \in G;$$

- for every constant symbol $c \in L$ and $\tau \in M$

$$c^{\mathcal{M}/G} = [\tau]_G \iff [\![\tau = c]\!] \in G.$$

1.1.13 entails that $\sim_G$ is an equivalence relation on M, and that the above definition yelds a 2-valued model, i.e. a Tarski model.

One would expect that for every formula $\phi(x_1 \ldots, x_n)$, $[\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$ if and only if $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$. This however is not the case for all B-models $\mathcal{M}$, hence we give a definition which characterizes this property of $\mathcal{M}$.

Definition 1.2.11. A B-valued model $\mathcal{M}$ is *full* when for every existential formula $\exists x \phi(x)$ there is some $\tau \in M$ such that

$$[\![\exists x \phi(x)]\!] = [\![\phi(\tau)]\!] .$$

Theorem 1.2.12 (Łos Theorem)**.** *Let $\mathcal{M}$ be a full B-valued model. Then for every $G \in$ $\mathsf{St}(\mathsf{B})$, every formula $\phi(x_1, \ldots, x_n)$ and all $(\tau_1, \ldots, \tau_n) \in M$*

$$\mathcal{M}/G \models \phi(\tau_1, \ldots, \tau_n) \iff [\![\phi([\tau_1]_G, \ldots, [\tau_n]_G)]\!] \in G.$$

Proof. By induction on the complexity of ϕ.

- If $\phi(x_1, \ldots, x_n) = r(x_1, \ldots, x_n)$, the tesis follows immediately from the definition of $\mathcal{M}/G$;

- If $\phi(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_n) \wedge \chi(x_1, \ldots, x_n)$

$$\begin{aligned} &\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n)]_G) \\ &\iff \mathcal{M}/G \models \psi([\tau_1]_G, \ldots, [\tau_n]_G) \quad \text{and} \quad \mathcal{M}/G \models \chi([\tau_1]_G, \ldots, [\tau_n)]_G) \\ &\iff [\![\psi(\tau_1, \ldots, \tau_n)]\!] \in G \quad \text{and} \quad [\![\chi(\tau_1, \ldots, \tau_n)]\!] \in G \\ &\iff [\![\psi(\tau_1, \ldots, \tau_n) \wedge \chi(\tau_1, \ldots, \tau_n)]\!] \in G \end{aligned}$$

- If $\phi(x_1, \ldots, x_n) = \neg\psi(x_1, \ldots, x_n)$

$$\begin{aligned} &\mathcal{M}/G \models \neg\psi([\tau_1]_G, \ldots, [\tau_n)]_G) \\ &\iff \mathcal{M}/G \quad \text{does not model} \quad \psi([\tau_1]_G, \ldots, [\tau_n)]_G) \\ &\iff [\![\psi(\tau_1, \ldots, \tau_n)]\!] \notin G \\ &\iff [\![\neg\psi(\tau_1, \ldots, \tau_n)]\!] \in G \end{aligned}$$

- If $\phi(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_n) \vee \chi(x_1, \ldots, x_n)$ the tesis follows from the previous items and the fact that $\psi \vee \chi$ is logically equivalent to $\neg(\neg\psi \wedge \neg\chi)$;

- If $\phi(x_1, \dots, x_n) = \psi(x_1, \dots, x_n) \to \chi(x_1, \dots, x_n)$ the tesis follows from the previous items and the fact that $\psi \to \chi$ is logically equivalent to $\neg\psi \vee \chi$;

- If $\phi(x_1, \dots, x_n) = \exists x \psi(x, x_1, \dots, x_n)$

$$\begin{aligned}
&\mathcal{M}/G \models \exists x \psi(x, [\tau_1]_G, \dots, [\tau_n]_G) \\
&\iff \mathcal{M}/G \models \psi([\sigma]_G, [\tau_1]_G, \dots, [\tau_n]_G) \quad \text{for some } \sigma \in M \\
&\iff [\![\psi(\sigma, \tau_1, \dots, \tau_n)]\!] \in G \quad \text{for some } \sigma \in M \\
&\implies [\![\exists x \psi(\tau_1, \dots, \tau_n)]\!] \in G.
\end{aligned}$$

  Conversely, if $[\![\exists x \psi(\tau_1, \dots, \tau_n)]\!] \in G$, since $\mathcal{M}$ is full there is some $\sigma \in \mathcal{M}$ such that
$$[\![\psi(\sigma, \tau_1, \dots, \tau_n)]\!] = [\![\exists x \psi(\tau_1, \dots, \tau_n)]\!] \in G$$
  yelding $\mathcal{M}/G \models \psi([\sigma]_G, [\tau_1]_G, \dots, [\tau_n]_G)$, hence $\mathcal{M}/G \models \exists x \psi(x, [\tau_1]_G, \dots, [\tau_n]_G)$.

- If $\phi(x_1, \dots, x_n) = \forall x \psi(x, x_1, \dots, x_n)$ the tesis follows from the previous items and the fact that $\forall x \psi(x)$ is logically equivalent to $\neg(\exists x \neg\psi(x))$. □

1.3 Boolean valued models for Set Theory

In this section we introduce the boolean valued model for Set Theory V^{B}. For these models we require some extra conditions to hold:

Definition 1.3.1. Let $\mathcal{L} = \{\in, \subseteq, =\}$ be the language for Set Theory with three binary relation symbols (including equality), and B a cba. A B-valued model $\mathcal{M}$ of signature $\mathcal{L}$ is a boolean valued model for Set Theory if for any $\sigma, \tau, \eta \in M$

- $[\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!] = [\![\tau = \sigma]\!]$;

- $[\![\tau \in \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!] \leq [\![\tau \in \sigma]\!]$.

We proceed to construct explicitly a boolean valued model for Set Theory. Recall that the universe V of all the sets can be construed as

$$\begin{aligned}
V_0 &= \emptyset \\
V_{\alpha+1} &= \mathscr{P}(V_\alpha) \\
V_\lambda &= \bigcup_{\beta < \lambda} V_\beta \quad \text{if } \lambda \text{ is limit}
\end{aligned}$$

and
$$V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha.$$

Notice that $\mathscr{P}(V_\alpha)$ is isomorphic to the set 2^{V_α} of all functions from V_α to 2: every subset $X \subseteq V_\alpha$ corresponds naturally to its characteristic function $\chi_X \colon V_\alpha \to 2 = \{0,1\}$ with $\chi_X(y) = 1 \iff y \in X$. So, if we wish to get some B-valued model for Set Theory, we try to replace 2 with B in the definition of the universe V.

Definition 1.3.2. Given a complete boolean algebra B, the boolean universe V^{B} is defined recursively as follows:

$$V_0^{\mathsf{B}} = \emptyset;$$
$$V_{\alpha+1}^{\mathsf{B}} = \left\{f \colon V_\alpha^{\mathsf{B}} \to \mathsf{B} : f \text{ is a partial function}\right\};$$
$$V_\lambda^{\mathsf{B}} = \bigcup_{\beta<\lambda} V_\beta^{\mathsf{B}} \quad \text{for } \lambda \text{ limit ordinal.}$$

Finally
$$V^{\mathsf{B}} = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha^{\mathsf{B}}.$$

Remark 1.3.3. V^{B} contains an isomorphic copy of V: every set x can be represented in V^{B} with the B-name $\check{x}$ defined recursively as

$$\check{x} = \{(\check{y}, 1_{\mathsf{B}}) : y \in x\};$$

and $\check{V} = \{\check{x} : x \in V\} \simeq V$.

Remark 1.3.4. Assume $M \in V$ is a transitive model of ZFC, and let $\mathsf{B} \in M$ be such that M models B to be complete. Then one can construct inside M as a definable subclass the boolean valued model M^{B} in the same way as V^{B} is construed in V.

Definition 1.3.5. For every $\tau \in V^{\mathsf{B}}$, we define by recursion

$$\mathrm{rank}(\tau) = \sup\{\mathrm{rank}(\sigma) + 1 : \sigma \in \mathrm{dom}(\tau)\}.$$

The boolean semantic for V^{B} is defined as follows:

Definition 1.3.6. For every $\tau, \sigma \in V^{\mathsf{B}}$ we define by recursion on the pair $(\mathrm{rank}(\tau), \mathrm{rank}(\sigma))$ with respect to Godel's well ordering of Ord^2

$$[\![\tau \in \sigma]\!] = \bigvee_{\tau_0 \in \mathrm{dom}(\sigma)} ([\![\tau = \tau_0]\!] \wedge \sigma(\tau_0));$$
$$[\![\tau \subseteq \sigma]\!] = \bigwedge_{\sigma_0 \in \mathrm{dom}(\tau)} (\neg\tau(\sigma_0) \vee [\![\sigma_0 \in \sigma]\!]);$$
$$[\![\tau = \sigma]\!] = [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!].$$

Theorem 1.3.7. *V^{B} with the boolean predicates defined above is a boolean valued model for Set Theory, and* ZFC *is valid in V^{B}.*

The proof is done by induction on the semantic of V^{B} and the rank of the B-names. For the details, see for example [4, Lemma 14.16 and Thm 14.24].

Moreover, it can be shown that V^{B} has nice properties, in particular it is a full B-valued model.

Theorem 1.3.8. *For every existential formula $\exists x \phi(x)$ there is some $\tau \in V^{\mathsf{B}}$ such that*

$$[\![\exists x \phi(x)]\!] = [\![\phi(\tau)]\!] .$$

Therefore V^{B} is a full boolean valued model for Set Theory.

This is a consequence of the following:

Lemma 1.3.9 (Mixing lemma)**.** *Let B be a complete boolean algebra in V, and A an antichain in B. Assume that for every $a \in A$, $\tau_a \in V^{\mathsf{B}}$ is a B-name in V^{B}. Then there exists $\tau \in V^{\mathsf{B}}$ such that $a \leq [\![\tau = \tau_a]\!]$ for each $a \in A$.*

For a proof, see [4, Lemma 14.18]

Proof. (of 1.3.8) Let

$$D = \left\{ b \in \mathsf{B}^+ : \exists \sigma_b \in V^{\mathsf{B}} (b \leq [\![\phi(\sigma_b)]\!]) \right\} .$$

D is open and dense below $[\![\exists x \phi(x)]\!]$ by definition, so it contains a maximal antichain A below $[\![\exists x \phi(x)]\!]$. Therefore it holds that

$$\bigvee A = [\![\exists x \phi(x)]\!] .$$

By the Mixing lemma, there is $\tau \in V^{\mathsf{B}}$ such that $[\![\tau = \tau_b]\!] \geq b$ for every $b \in A$. This entails that

$$b \leq [\![\tau_b = \tau]\!] \wedge [\![\phi(\tau_b)]\!] \leq [\![\phi(\tau)]\!]$$

for every $b \in A$, hence

$$[\![\exists x \phi(x)]\!] = \bigvee A \leq [\![\phi(\tau)]\!] .$$

Since the inequality $[\![\phi(\tau)]\!] \leq [\![\exists x \phi(x)]\!]$ holds by befinition, we have $[\![\phi(\tau)]\!] = [\![\exists x \phi(x)]\!]$. $\square$

1.4 Cohen's forcing theorem

In this section we briefly outline how the forcing technique works. Assume we want to prove that a certain statement ϕ (for example the Continuum Hypotesis or its negation) is consistent with ZFC. This can be done building a model of ZFC in which ϕ is true. The strategy adopted is to start from a countable transitive model M of ZFC, choose a boolean algebra $\mathsf{B} \in M$ such that M models B to be complete, and then extend M to a larger model $M[G]$ of ZFC with the property that $M[G] \models \phi$.

Remark 1.4.1. Working in ZFC, by Godel's second incompleteness theorem one cannot produce a countable transitive model of ZFC (unless ZFC is inconsistent). This is roughly how this issue is overcomen: when saying that ϕ is consistent with ZFC, what we really mean that if ZFC + ϕ is contradictory, then so is ZFC. Hence, resorting to the compactness theorem, it can be shown it suffices for M to model only a finite subset of the ZFC axioms, and the existence of such a countable transitive model M is a consequence of the Reflection principles. The forcing method will grant that the generic extension $M[G]$ will be a model of this same set of axioms. For the details, see for example [6, Section II.5].

Definition 1.4.2. Let M be a transitive countable model of ZFC, $\mathsf{B} \in M$ be such that M models B to be a cba, and $G \in \mathrm{St}(\mathsf{B})$ be *M-generic*. For every $\tau \in M^{\mathsf{B}}$, define by recursion

$$\tau_G = \{\sigma_G : \tau(\sigma) \in G\}.$$

$M[G] = \{\tau_G : \tau \in M\}$ is the *generic extension* of M with G.

Remark 1.4.3. The transitivity of M is not strictly necessary, but it simplifies the work, since we can appeal to standard absoluteness results (see [6, SectionsI.16 and II.4]) to grant that computations in V^{B} yeld the same result when applied to M^{B}.

Remark 1.4.4. It can easily be proved by induction that for all $x \in M$ it holds $\check{x}_G = x$. Furthermore if $\bar{G} = \left\{(\check{b}, b) : b \in \mathsf{B}\right\}$, then $\bar{G}_G = G$.

Remark 1.4.5. We have seen that there exists no V-generic ultrafilter on B when B is atomless, nonetheless, by theorem 1.1.9, there exist M-generic ultrafilters G if M is countable, moreover the set of such M-generic ultrafilters on B is dense in $\mathrm{St}(\mathsf{B})$. Furthermore if G is M-generic, then $G \notin M$; since $G \in M[G]$ (by the previous remark), we see that $M[G]$ strictly extends M.

Now we analyze the relation between $M[G]$ and the usual Tarski quotient M^{B}/G.

Theorem 1.4.6. *Let M be a countable transitive model of* ZFC*, B a cba in M and G an M-generic ultrafilter on B. Then for all $\tau, \sigma \in M^{\mathsf{B}}$*

- $[\![\tau \in \sigma]\!] \in G \iff \tau_G \in \sigma_G$*;*
- $[\![\tau = \sigma]\!] \in G \iff \tau_G = \sigma_G$*.*

Proof. By induction on $(\mathrm{rank}(\tau), \mathrm{rank}(\sigma))$. For the details, see [4, Lemma 14.28] □

Combining this result with Łos theorem for full boolean valued models, we obtain

Theorem 1.4.7. *Let M be a countable transitive model of* ZFC*, B a cba in M and G an M-generic ultrafilter on B. Then the map*

$$\begin{aligned} \pi_G^M : M^{\mathsf{B}}/G &\to M[G] \\ [\tau]_G &\mapsto \tau_G \end{aligned}$$

is an isomorphism between M^B/G and $M[G]$.

The last result we presents outlines how the theory of $M[G]$ is related to the topology of $\mathrm{St}(\mathsf{B})$:

Theorem 1.4.8 (Cohen's forcing theorem)**.** *Let M be a ctm for set theory. Then*

$$M[G] \models \phi \iff [\![\phi]\!] \in G.$$

Furthermore, for every $b \in \mathsf{B}$ and every $\mathcal{L}$-statement ϕ, the following are equivalent:

- *$M[G] \models \phi$ for every M-generic ultrafilters G such that $b \in G$;*
- *the set $\{G \in \mathrm{St}(\mathsf{B}) : G$ is M-generic and $M[G] \models \phi\}$ is dense in N_b;*
- *$b \leq [\![\phi]\!]$.*

For a proof, see [13, Lemma 4.2.6].

Chapter 2

Category theory and sheaves

In this chapter we introduce some notions of Category theory, and we define the notion of sheaf on a topological space. For a complete introduction to the subject, see [8] and [1].

2.1 Categories and functors

Definition 2.1.1. A *category* $\mathcal{C}$ consists of:

- a class $C = \mathrm{Ob}_{\mathcal{C}}$ whose elements are called *objects*;
- a class $\mathrm{Arw}_{\mathcal{C}}$ whose elemens are called *arrows* or *morphisms*;
- a function $\mathrm{dom}_{\mathcal{C}}\colon \mathrm{Arw}_{\mathcal{C}} \to C$ assigning to every arrow its *domain*;
- a function $\mathrm{cod}_{\mathcal{C}}\colon \mathrm{Arw}_{\mathcal{C}} \to C$ assigning to every arrow its *codomain*;
- a function $\mathrm{Id}_{\mathcal{C}}\colon C \to \mathrm{Arw}$ attaching to every object c its *identity arrow* Id_c;
- a function $\circ_{\mathcal{C}}\colon E \to \mathrm{Arw}_{\mathcal{C}}$ where $E = \big\{(f,g) \in {\mathrm{Arw}_{\mathcal{C}}}^2 : \mathrm{cod}_{\mathcal{C}}(g) = \mathrm{dom}_{\mathcal{C}}(f)\big\}$.

The pedix $\mathcal{C}$ is often omitted if the category $\mathcal{C}$ is clear form the context, and $f \circ g$ is abbreviated fg. If $f \in \mathrm{Arw}$, $\mathrm{dom}(f) = c$ and $\mathrm{cod}(f) = d$ we will write $f\colon c \to d$.

We require the following conditions to hold:

- $\mathrm{dom}(\mathrm{Id}_c) = \mathrm{cod}(\mathrm{Id}_c) = c$ for every $c \in C$;
- if $\mathrm{dom}(f) = c$ and $\mathrm{cod}(g) = c$, then $f \circ \mathrm{Id}_c = f$ and $\mathrm{Id}_c \circ g = g$;
- $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$, and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$;
- $\circ$ is associative, $(f \circ g) \circ h = f \circ (g \circ h)$ for every $f, g, h \in \mathrm{Arw}$ such that this expression makes sense.

Here are some examples of categories

- Set is the category of all the sets. The collection of the objects is V, an arrow from X to Y is a function $f\colon X \to Y$;
- Grp is the category where the objects are the groups, and the arrows are the group homomorphisms. Likewise Ab, Ring, Field, $\mathsf{Vect}_{\mathbb{K}}$ are respectively the categories of abelian groups, rings, fields and vector spaces over a field $\mathbb{K}$ with group, ring, field homomorphisms and linear maps between vector spaces;
- more generally, given a class C of sets equipped with a certain type of structure, there is a category $\mathcal{C}$ with C as the collection of the objects and the arrows are the structure-preserving functions;
- as a special case of the previos point, Top is the category where topological spaces are the objects and continuous functions between them are the arrows;
- every partial order $(P, \leq)$ can be seen as a category where P is the collection of objects, and for every $p, q \in P$ there is an arrow from p to q if and only if $p \leq q$;
- Poset has partially ordered sets as objects, and order-preserving maps between them as the morphisms.

Definition 2.1.2. Given a category $\mathcal{C}$ and two objects $x, y \in C$, the collection of all the arrows from x to y is denoted by $\hom_{\mathcal{C}}(x, y)$.

Definition 2.1.3. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we say that $\mathcal{D}$ is a *subcategory* of $\mathcal{C}$ if $\mathsf{Ob}_{\mathcal{D}} \subseteq \mathsf{Ob}_{\mathcal{C}}$ and $\mathrm{Arw}_{\mathcal{D}} \subseteq \mathrm{Arw}_{\mathcal{C}}$.

$\mathcal{D}$ is a *full* subcategory of $\mathcal{C}$ if it is a subcategory of $\mathcal{C}$ and $\hom_{\mathcal{D}}(x, y) = \hom_{\mathcal{C}}(x, y)$ for every $x, y \in \mathsf{Ob}_{\mathcal{D}}$.

For example the category Ab of abelian groups is a (full) subcategory of the category Grp of all the groups.

As stated above, a category may consist of a proper class of objects and of a proper class of arrows. Hence it is important to recognise when the objects (or arrows) of a category form a set (for example this is the case for a preorder $(P, \leq)$ if P is a set) or a proper class as in Set. This motivates the following

Definition 2.1.4. A category $\mathcal{C}$ is said to be

- *small* if $\mathrm{Arw}_{\mathcal{C}}$ is a set;
- *locally small* if $\hom_{\mathcal{C}}(x, y)$ is a set for every $x, y \in C$.

Remark 2.1.5. Notice that the conditions posed on $\mathrm{Id}_{\mathcal{C}}\colon C \to \mathrm{Arw}_{\mathcal{C}}$ makes it an injective function. So, if $\mathrm{Arw}_{\mathcal{C}}$ is a set, then so is C.

For example, except for $(P, \leq)$, all the categories mentioned above are not small, but they are locally small.

Among the arrows, we can distinguish those having certain properties.

Definition 2.1.6. An arrow $f\colon c \to d$ is

- a *monomorphism* or *mono* when for every couple g, h of arrows with $\mathrm{cod}(h) = \mathrm{cod}(g) = c$, if $fg = fh$ then $g = h$;
- an *epimorphism* or *epi* when for every couple g, h of arrows with $\mathrm{dom}(h) = \mathrm{dom}(g) = d$, if $gf = hf$ then $g = h$;
- an *isomorphism* or *iso* if there is an arrow $g\colon d \to c$ such that $fg = \mathrm{Id}_g$ and $gf = \mathrm{Id}_c$. In this case it is easily proved that g is unique, and it is denoted by f^{-1} and called the *inverse* of f.

For example, in Set an arrow f is mono if and only if it is an injective function, epi if and only if it is a surjective function an iso if and only if it is a bijection.

Likewise in a category we can distinguish some objects:

Definition 2.1.7. An object $c \in C$ in $\mathcal{C}$ is

- *initial* if for every $x \in C$ there is a unique arrow $f\colon c \to x$;
- *terminal* if for every $x \in C$ there is a unique arrow $f\colon x \to c$.

For example, in Set the object $\{\emptyset\}$ is terminal; the trivial group is both initial and terminal in Grp.

A major tool in category theory is the concept of *duality*. This can be abstractly formulate introducing the notion of the opposite of a category, which is obtained by "reversing all the arrow".

Definition 2.1.8. Given a category $\mathcal{C}$, its *opposite category* $\mathcal{C}^{\mathrm{op}}$ is constructed as follows:

- $\mathrm{Ob}_{\mathcal{C}^{\mathrm{op}}} = \mathrm{Ob}_{\mathcal{C}}$;
- $\mathrm{Arw}_{\mathcal{C}^{\mathrm{op}}} = \mathrm{Arw}_{\mathcal{C}}$;
- $\mathrm{dom}_{\mathcal{C}^{\mathrm{op}}} = \mathrm{cod}_{\mathcal{C}}$, and $\mathrm{cod}_{\mathcal{C}^{\mathrm{op}}} = \mathrm{dom}_{\mathcal{C}}$;
- $\circ_{\mathcal{C}^{\mathrm{op}}}\colon E' \to \mathrm{Arw}_{\mathcal{C}}$ where $E' = \big\{(f,g) \in {\mathrm{Arw}_{\mathcal{C}}}^2 : \mathrm{cod}_{\mathcal{C}}(f) = \mathrm{dom}_{\mathcal{C}}(g)\big\}$;
- $f \circ_{\mathcal{C}^{\mathrm{op}}} g = g \circ_{\mathcal{C}} f$ for every $f, g \in \mathrm{Arw}_{\mathcal{C}}$ such that this expression makes sense.

It can be checked that an arrow f is mono (epi) in $\mathcal{C}$ if and only if it is epi (mono) in $\mathcal{C}^{\mathrm{op}}$, and an object c is initial (terminal) in $\mathcal{C}$ if and only if it is terminal (initial) in $\mathcal{C}^{\mathrm{op}}$. In this vein, duality allows to produce new interesting statements about categories from a known theorem.

We now introduce the notion of a morphism between categories, i.e. a functor.

Definition 2.1.9. Given two categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* from $\mathcal{C}$ to $\mathcal{D}$ is a function $F\colon C \cup \mathrm{Arw}_{\mathcal{C}} \to D \cup \mathrm{Arw}_{\mathcal{D}}$ such that

- $F(c) \in D$ for every object $c \in C$;
- $F(f) \in \mathrm{Arw}_{\mathcal{D}}$, $\mathrm{dom}_{\mathcal{D}} F(f) = F(\mathrm{dom}_{\mathcal{C}}(f))$ and $\mathrm{cod}_{\mathcal{D}} F(f) = F(\mathrm{cod}_{\mathcal{C}}(f))$ for every arrow f of $\mathcal{C}$;
- $F(\mathrm{Id}_c) = \mathrm{Id}_{F(c)}$ for every $c \in C$;
- $F(g \circ_{\mathcal{C}} f) = F(g) \circ_{\mathcal{D}} F(f)$ for every pair of composable arrows $f, g \in \mathrm{Arw}_{\mathcal{C}}$.

Here are some example of functors:

- $U\colon \mathsf{Grp} \to \mathsf{Set}$ assigning to every group G its underlying set and the underlying function to every group homomorphism is trivially a functor. The same argument applies to every category of algebraic or topological structures, and such a functor is called *forgetful functor*;
- the fundamental group $\pi\colon \mathsf{Top} \to \mathsf{Grp}$ assigning to every topological space its fundamantal group, and to every continuous function the induced homomorphism of fundamental group.

Definition 2.1.10. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is

- *faithful* if for every $c, d \in C$ the map

$$\hom_{\mathcal{C}}(c, d) \to \hom_{\mathcal{D}}(F(c), F(d)), \qquad f \mapsto F(f)$$

  is injective;
- *full* if for every $c, d \in C$ the above map is surjective;
- *injective on objects* if the map $c \mapsto F(c)$ is injective;
- *surjective on objects* if the map $c \mapsto F(c)$ is surjective;
- *essentially surjective on objects* if for every $d \in D$ there exists $c \in C$ such that $F(c)$ is isomorphic to d;
- an *isomorphism of categories* if it is full, faithful, injective and surjective on objects;
- an *equivalence of categories* if it is full, faithful, injective and essentially surjective on objects;

Functors behaving like in 2.1.9 are called *covariant*. There is another type of functor:

Definition 2.1.11. A *contravariant* functor from the category $\mathcal{C}$ to the category $\mathcal{D}$ is a covariant functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. More explicitly, F is a contravariant functor when

- $\mathrm{dom}_{\mathcal{D}} F(f) = F(\mathrm{cod}_{\mathcal{C}}(f))$, $\mathrm{cod}_{\mathcal{D}} F(f) = F(\mathrm{dom}_{\mathcal{C}}(f))$ for every arrow f of $\mathcal{C}$;
- $F(g \circ_{\mathcal{C}} f) = F(f) \circ_{\mathcal{D}} F(g)$ for every pair of composable arrows $f, g \in \mathrm{Arw}_{\mathcal{C}}$.

An example of a contravariant functor is

$$\mathcal{O}\colon \mathsf{Top} \to \mathsf{Poset}, \qquad X \mapsto \mathcal{O}(X)$$

attachig to every X the poset $\mathcal{O}(X)$ of its open subsets ordered by inclusion, and for every continuous function $f\colon X \to Y$

$$\mathcal{O}(f)\colon \mathcal{O}(Y) \to \mathcal{O}(X), \qquad \mathcal{O}(f)(U) = f^{-1}(U)$$

for every open subset $U \subseteq Y$.

If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and $G\colon \mathcal{D} \to \mathcal{E}$ is another functor, then the composition $G \circ F$ is a functor from $\mathcal{C}$ to $\mathcal{E}$. Moreover the identity function on $\mathcal{C}$ is a functor. This means that there is a category Cat where the objects are the small categories, and an arrow from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$.

Making a step further, we will see in the next section that the collection of all the functors between two fixed categories is a category as well.

2.2 Natural transformations, limits and colimits

In category theory it is common to use *commutative diagrams*. A commutative diagram consists of a picture where objects are points, and morphisms between objects are represented by graphic arrows connecting them. Moreover it is assumed that if there are more paths connecting two objects, then the corresponding composition of arrow commute.

As an example, the following commutative diagram

$$\begin{tikzcd} x \arrow[d, "g"'] \arrow[dr, "h"] & \\ y \arrow[r, "f"'] & z \end{tikzcd}$$

says that g is an arrow between x and y, f is an arrow between y and z, h is an arrow between x and z, and $h = fg$.

Our aim here is to define what a category of functors is. To this aim, the first step is to characterise a morphism between functors.

Definition 2.2.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ two functors with the same domain and codomain. A *natural transformation* from F to G consists of a collection of arrows $\alpha = \{\alpha_c : c \in C\}$ such that

- $\alpha_c\colon F(c) \to G(c)$ for every object $c \in C$;
- for every arrow $f\colon c \to d$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\ G(c) \arrow[r, "\alpha_d"'] & G(d) \end{tikzcd}$$

is commutative.

Theorem 2.2.2. *Let α and β be natural transformations from the two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$. Then $\beta \circ \alpha$ defined by $(\beta \circ \alpha)_c = \beta_c \circ \alpha_c$ for every $c \in C$ is also a natural transformation.*

Definition 2.2.3. Given two locally small categories $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{C}^{\mathcal{D}}$ is the category of all functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations between them. It is actually a category due to the previous theorem.

We now introduce the notions of a limit and colimit of a given diagram, and see how these concepts generalize many common set-theoretic constructions.

Definition 2.2.4. Given a small category J and a category $\mathcal{C}$, a *diagram of shape* J in $\mathcal{C}$ is an object in $\mathcal{C}^{\mathsf{J}}$, that is a functor from J to $\mathcal{C}$.

If D is a diagram of shape J in $\mathcal{C}$, a *cone over D with summit $c \in C$* is a natural transformation from the constant functor $\Delta_{\mathsf{J}}(c)$ (that one sending every object in J to c and every arrow in J to Id_c) to D.

For example, if D is the the diagram displayed as

$$x \xrightarrow{f} y$$

with $x, y \in C$, a cone over D with summit c is represented by the diagram

$$\begin{tikzcd} & c \arrow[dl, "\lambda_x"'] \arrow[dr, "\lambda_y"] & \\ x \arrow[rr, "f"] & & y \end{tikzcd}$$

$\lambda = \{\lambda_x, \lambda_y\}$ being the natural transformation in question.

Definition 2.2.5. Given a diagram D, $\lim D$ is a terminal object in the category of the cones over D.

For example, the cone in the previos example is a limit cone if, for every cone over D with summit c' there is a unique arrow $\alpha \colon c' \to c$ making the diagram

$$\begin{tikzcd} & c' \arrow[d, "\alpha"] \arrow[ddl, bend right, "\mu_x"'] \arrow[ddr, bend left, "\mu_y"] & \\ & c \arrow[dl, "\lambda_x"] \arrow[dr, "\lambda_y"'] & \\ x \arrow[rr, "f"'] & & y \end{tikzcd}$$

commute.

It can be shown tht if a limit exists, then it is unique up to isomorphism. We now list the major examples of limits.

- Given two objects $a, b \in C$, a *product* of a and b is a limit of the diagram

$$a \qquad b$$

that is an object $a \times b$ with two arrows $\pi_a\colon a \times b \to a$ and $\pi_b\colon a \times b \to b$ such that for every object $x \in C$ with two arrows $f\colon x \to a$ and $g\colon x \to b$ there is a unique arrow $(f, g)\colon x \to a \times b$ making the diagram

$$\begin{tikzcd} & x \arrow[dl, bend right, "f"'] \arrow[d, "(f,g)"] \arrow[dr, bend left, "g"] & \\ a & a \times b \arrow[l, "\pi_a"] \arrow[r, "\pi_b"'] & b \end{tikzcd}$$

commute.

For example, in Set a product of a and b is the usual cartesian product $a \times b$.

- A *pullback* is a limit of a diagram of type

$$a \xrightarrow{f} c \xleftarrow{g} y$$

that is an object $a \times_c b$ with two arrows $p_a\colon a \times_c b \to a$ and $p_b\colon a \times_c b \to b$ such that for every object $x \in C$ with two arrows $r\colon x \to a$ and $s\colon x \to b$ there is a unique arrow $(r, s)\colon x \to a \times b$ making the diagram

$$\begin{tikzcd} x \arrow[dr, "(r,s)"] \arrow[drr, bend left, "s"] \arrow[ddr, bend right, "r"'] & & \\ & a \times_c b \arrow[r, "p_a"] \arrow[d, "p_b"] & a \arrow[d, "f"] \\ & b \arrow[r, "g"'] & c \end{tikzcd}$$

commute.

For example, in Set a pullback of $f\colon a \to b$ and $g\colon b \to c$ is the fibered product $a \times_c b = \{(y, w) \in a \times b : f(y) = g(w)\}$.

- an *equalizer* is a limit of a diagram of type

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & b \end{tikzcd}$$

explicitly an object e with an arrow $i\colon e \to a$ such that for every object x with an arrow $h\colon x \to a$ satisfying $fh = gh$ there is a unique arrow u making the following diagram

$$\begin{tikzcd} x \arrow[d, "u"] \arrow[dr, "h"] & & \\ e \arrow[r, "i"] & a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & b \end{tikzcd}$$

commute.

In Set, the equalizer of f and g is the set $e = \{y \in a : f(x) = g(x)\}$, with $i : e \to a$ the inclusion.

The dual notion of the limit is the *colimit* of a diagram of shape J.

Definition 2.2.6. If D is a diagram of shape J in a category $\mathcal{C}$, a *cone under D with nadir* $c \in C$ is a natural transformation from D to the constant functor $\Delta_{\mathsf{J}}(c)$.

Definition 2.2.7. Given a diagram D, the *colimit* of $\lim D$ is an initial object in the category of the cones under D.

The dual notions of product, pullback and equalizer are respectively the *coproduct*, the *pushout* and the *coequalizer*, obtained by reversing all the arrows in the definitions. For example, in Set, the coproduct of a and b is the disjoint union of a and b.

2.3 Sheaves on a topological space

We can now introduce the main concept in category theory we are interested in.

We start with an example: let X be a topological space, and let $\mathbf{C}(X)$ be the set of continuous functions from X to $\mathbb{R}$. If U is an open subset of X, then for every $f \in \mathbf{C}(X)$ we have $f \restriction_U$ is a continuous function from U to $\mathbb{R}$. Moreover, if $\{U_i : i \in I\}$ is an open covering of X, and f_i is a continuous function from U_i to $\mathbb{R}$ for every $i \in I$, then there exists $f : X \to \mathbb{R}$ continuous with $f \restriction_{U_i} = f_i$ if and only if for every $i, j \in I$ it holds $f_i \restriction_{U_i \cap U_j} = f_j \restriction_{U_i \cap U_j}$, and this f is uniquely determined. Notice that to every inclusion $V \subseteq U$ of open subsets of X it corresponds an operation of restriction of continuouus functions

$$\restriction_V : \mathbf{C}(U) \to \mathbf{C}(V), \qquad f \mapsto f \restriction_V$$

which is compatible with nested inclusions, that is $(f \restriction_V) \restriction_W = f \restriction_W$ when $W \subseteq V \subseteq U$. This means that the association

$$\mathbf{C} : \mathcal{O}(X) \to \mathsf{Set}, \qquad U \mapsto \mathbf{C}(U)$$

is in fact a contravariant functor.

Definition 2.3.1. Given a category $\mathcal{C}$, $\hat{\mathcal{C}}$ is the category of contravarant functors from $\mathcal{C}$ to Set and natural transformations between them. A *presheaf* on $\mathcal{C}$ is an element of $\hat{\mathcal{C}}$.

Consider for any topological space (X, τ)

$$B^\infty(X) = \{f : \ f : X \to \mathbb{R} \text{ is a continuous function with bounded range } \}.$$

The functor

$$B^\infty : \tau \to \mathsf{Set}, \qquad U \mapsto B^\infty(U)$$

is a presheaf mapping an inclusion of subsets to the restriction of bounded functions as in the previous example. However, there is a substantial difference between the two functors. Let $X = \mathbb{R}$ and $\{U_i = (i, i+2) : i \in \mathbb{Z}\}$; this is an open covering of $\mathbb{R}$, and let $f_i \in B^\infty(U_i)$ be the identity function on U_i for every $i \in \mathbb{Z}$. Clearly $f_i \restriction_{U_i \cap U_j} = f_j \restriction_{U_i \cap U_j}$ for every $i, j \in I$. The identity function $\mathrm{Id}\colon \mathbb{R} \to \mathbb{R}$ is the gluing of the family $\{f_i : i \in I\}$ is not bounded, therefore it is not an element of $B^\infty(\mathbb{R})$.

Definition 2.3.2. A presheaf F on a topological space X is a *sheaf* if for every open covering $\{U_i : i \in I\}$ of an open subset $U \subseteq X$ and every collection $\{f_i : i \in I\}$ with $f_i \in F(U_i)$ such that $F(U_i \cap U_j)(f_i) = F(U_j \cap U_j)(f_j)$ for every $i, j \in I$ there is a unique $f \in F(U)$ satisfying $F(U_i \subseteq U)(f_i) = f$ for every $i \in I$.

We have just seen that the collection $\mathbf{C}(X)$ determines a sheaf on X. Furthermore the above definition can be easily generalized for a presheaf on any category $\mathcal{C}$.

Definition 2.3.3. $\mathsf{Sh}(\mathcal{C})$ is the full subcategory of $\hat{C}$ spanned by all the sheaves on C.

Definition 2.3.4. A *Grothendieck Topos* is a category which is equivalent to the category $\mathsf{Sh}(\mathcal{O}(X))$ for some topological space X.

The aim of the following chapters is to show that B-valued models for Set Theory as categories of sheaves on $\mathrm{St}(\mathsf{B})$, we will find of some interest the next result:

Theorem 2.3.5. *Let $U = \{U_i : i \in I\}$ be a basis for the topology of some space X. Then* $\mathsf{Sh}(\mathcal{O}(X))$ *is equivalent to* $\mathsf{Sh}(U)$*.*

We now introduce Giraud's theorem.

Definition 2.3.6. Let X be the coproduct of $\{X_i : i \in I\}$ with universal arrows $\{f_i : i \in I\}$ in a category $\mathcal{C}$. X is *disjoint* if every f_i is mono, and the pullback $X_i \times_X X_j$ is an initial object for every $i \neq j$.

X is *stable* when the coproduct of $\{Y \times_X X_i : i \in I\}$ is isomorphic to Y for every object $Y \in \mathcal{C}$.

Definition 2.3.7. Let X be an object in $\mathcal{C}$. R is an *equivalence relation* on X if there is a mono arrow $(i_0, i_1)\colon R \to X \times X$ such that

- the diagonal arrow $\Delta\colon X \to X \times X$ factors through (i_0, i_1);
- (i_0, i_1) factors through (i_1, i_0);
- given the pullback diagram

$$\begin{tikzcd} R \times_X R \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & R \arrow[d, "i_0"] \\ R \arrow[r, "i_1"'] & X \end{tikzcd}$$

then $(i_0\pi_1, i_1\pi_0)$ factors through R.

The *quotient* of R is the coequalizer of (i_0, i_1).

Definition 2.3.8. Let f be any arrow in a category $\mathcal{C}$. The *kernel pair* of f is the pullback of f along itself.

A *fork* is a diagram like

$$\begin{tikzcd} R \arrow[r, bend left, "i_0"] \arrow[r, bend right, "i_1"'] & X \arrow[r, "f"] & Q \end{tikzcd}$$

and it is said to be *exact* if f is the coequalizer of i_0 and i_1 while the latter form the pullback of f along itself, *stably exact* if it reamins exact after composing f with any arrow g such that $\mathrm{dom}(g) = Q$.

Definition 2.3.9. A set of objects $\{x_i : i \in I\}$ is said to generate a category $\mathcal{C}$ if for any couple of parallel arrows $f, g \colon a \to b$ in $\mathcal{C}$

$$fh = gh \quad \text{for all } h \colon x_i \to a \text{ and } i \in I \implies f = g.$$

Theorem 2.3.10 (Giraud's theorem)**.** *A locally small category $\mathcal{C}$ is a Grothendieck topos if and only if it satisfies the following properties:*

- *$\mathcal{C}$ has all small coproducts, ad these are disjoint and stable under pullback;*
- *every epi arrow in $\mathcal{C}$ is a coequalizer;*
- *every equivalence relation in $\mathcal{C}$ is a kernel pair and has a quotient;*
- *every exaxt fork is stably so;*
- *there is a set of object in $\mathcal{C}$ generating $\mathcal{C}$.*

For a proof, see [10, Thm 1, p. 575]

It can be checked using the standard set-theoretical constructions corresponding to the categorical ones involved in Giraud's theorem that Set is a Grothendieck topos (see [10, Chapter IV]).

Chapter 3

The model $V^{\mathrm{St}(\mathsf{B})}$

In this chapter, we introduce the boolean valued model for Set Theory $V^{\mathrm{St}(\mathsf{B})}$ where B is a complete boolean algebra, and explore its connection with the standard boolean valued model V^{B} commonly used to introduce the technique of forcing. We first present its construction, which is similar to that of V^{B}, then we prove that $V^{\mathrm{St}(\mathsf{B})}$ and V^{B} are isomorphic boolean valued models.

3.1 Construction of $V^{\mathrm{St}(\mathsf{B})}$

Definition 3.1.1. Let B be a complete boolean algebra.

We define by recursion on the ordinals

$$V_0^{\mathrm{St}(\mathsf{B})} = \emptyset;$$
$$V_1^{\mathrm{St}(\mathsf{B})} = \{f\colon \mathrm{St}(\mathsf{B}) \to \{\emptyset\}\} = \{\{(G,\emptyset) \mid G \in \mathrm{St}(\mathsf{B})\}\};$$
$$V_{\alpha+1}^{\mathrm{St}(\mathsf{B})} = \left\{f\colon \mathrm{St}(\mathsf{B}) \to 2^{\bigcup_{\beta\leq\alpha} V_\alpha^{\mathrm{St}(\mathsf{B})}} \mid f \text{ is continuous}\right\};$$

where the product topology on $2^{\bigcup_{\beta\leq\alpha} V_\alpha^{\mathrm{St}(\mathsf{B})}}$ is considered;

$$V_\lambda^{\mathrm{St}(\mathsf{B})} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathrm{St}(\mathsf{B})}$$

for λ limit ordinal.

Finally we set

$$V^{\mathrm{St}(\mathsf{B})} = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha^{\mathrm{St}(\mathsf{B})}.$$

Notation 3.1.2. $V_{\leq\alpha}^{\mathrm{St}(\mathsf{B})}$ will stand for $\bigcup_{\beta\leq\alpha} V_\alpha^{\mathrm{St}(\mathsf{B})}$.

We define the boolean semantic in $V^{\mathrm{St}(\mathsf{B})}$ in way analogous to that used for V^{B}. In this case it's convenient to identify B with $\mathrm{RO}(\mathrm{St}(\mathsf{B})) = \mathrm{CLOP}(\mathrm{St}(\mathsf{B}))$ (by 1.1.16 these are isomorphic via the duality map $b \mapsto N_b$).

Definition 3.1.3. For $\emptyset \neq f \in V^{\mathrm{St}(\mathsf{B})}$ $\mathrm{rank}(f) = \alpha$ where α is the unique ordinal such that $f\colon \mathrm{St}(\mathsf{B}) \to 2^{V^{\mathrm{St}(\mathsf{B})}_{\leq\alpha}}$. (clearly $\mathrm{rank}(\emptyset) = 0$).

Definition 3.1.4. Let $\mathscr{L} = \{\in, \subseteq, =\}$ be the language of Set Theory, and $f, g \in V^{\mathrm{St}(\mathsf{B})}$. We define, by recursion on the pair $(\mathrm{rank}(f), \mathrm{rank}(g))$ with respect to Godel's well-ordering of Ord^2, the boolean interpretation of the symbols of $\mathscr{L}$:

$$[\![f \in g]\!]^{V^{\mathrm{St}(\mathsf{B})}} = \bigvee_{h \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(g)}} \left([\![h = f]\!]^{V^{\mathrm{St}(\mathsf{B})}} \wedge \{G \in \mathrm{St}(\mathsf{B}) \mid g(G)(h) = 1)\}\right);$$

$$[\![f \subseteq g]\!]^{V^{\mathrm{St}(\mathsf{B})}} = \bigwedge_{h \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(f)}} \left([\![h \in g]\!]^{V^{\mathrm{St}(\mathsf{B})}} \vee \{G \in \mathrm{St}(\mathsf{B}) \mid f(G)(h) = 0\}\right);$$

$$[\![f = g]\!]^{V^{\mathrm{St}(\mathsf{B})}} = [\![f \subseteq g]\!]^{V^{\mathrm{St}(\mathsf{B})}} \wedge [\![g \subseteq f]\!]^{V^{\mathrm{St}(\mathsf{B})}}.$$

Remark 3.1.5. The definition just given makes sense: the set $\{G \in \mathrm{St}(\mathsf{B}) \mid f(G)(h) = j\}$ is a clopen subset of $\mathrm{St}(\mathsf{B})$ for every $h \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(f)}$ since

- for all $h \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(f)}$

$$\{G \in \mathrm{St}(\mathsf{B}) \mid f(G)(h) = j\} = f^{-1}\left[\left\{s \in 2^{V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(f)}} \mid s(h) = j\right\}\right],$$

- the set $\left\{s \in 2^{V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(f)}} \mid s(h) = j\right\}$ is clopen in $2^{V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(f)}}$, and f is continuous.

Our goal is to prove the following

Theorem 1. *$V^{\mathrm{St}(\mathsf{B})}$ with the boolean interpretation of the symbols in $\mathscr{L}$ introduced above is a full boolean valued model for Set Theory isomorphic to V^{B}.*

In fact it suffices only to construct the isomorphism since we already know by Theorem 1.3.8 that V^{B} with B a cba is a full boolean valued model for Set Theory.

3.2 The isomorphism between V^{B} and $V^{\mathrm{St}(\mathsf{B})}$

We must find a boolean isomorphism between V^{B} and $V^{\mathrm{St}(\mathsf{B})}$. To this aim it is convenient to resort to a special class of B-names in V^{B}. More specifically, let

$$\widetilde{V^{\mathsf{B}}_0} = \emptyset;$$

$$\widetilde{V^{\mathsf{B}}_{\alpha+1}} = \left\{\tau \in \mathsf{B}^{\bigcup_{\beta\leq\alpha} \widetilde{V^{\mathsf{B}}_\beta}} : \tau \text{ is a total function}\right\};$$

$$\widetilde{V^{\mathsf{B}}_\lambda} = \bigcup_{\alpha<\lambda} \widetilde{V^{\mathsf{B}}_\alpha} \qquad \text{for } \lambda \text{ limit ordinal;}$$

and
$$\widetilde{V^{\mathsf{B}}} = \bigcup_{\alpha \in \mathrm{Ord}} \widetilde{V^{\mathsf{B}}_\alpha}.$$
It is convenient to denote $\bigcup_{\beta\leq\alpha}\widetilde{V^{\mathsf{B}}_\beta}$ by $\widetilde{V^{\mathsf{B}}_{\leq\alpha}}$.

Obviously $\widetilde{V^{\mathsf{B}}_\alpha} \cap \widetilde{V^{\mathsf{B}}_\beta} = \emptyset$ when $\alpha \neq \beta$ since the functions in the two sets have different domains and are total. Furthermore by induction it is easily proved that if $\tau \in \widetilde{V^{\mathsf{B}}_{\alpha+1}}$ then $\tau \in V^{\mathsf{B}}$ and $\mathrm{rank}(\tau) = \alpha$ in V^{B}. The boolean semantic for $\widetilde{V^{\mathsf{B}}}$ is the restriction of the semantic defined for V^{B}. The following holds:

Proposition 3.2.1. *For every $\tau \in V^{\mathsf{B}}_\alpha$ there is some $\tilde{\tau} \in \widetilde{V^{\mathsf{B}}_\alpha}$ such that $[\![\tau = \tilde{\tau}]\!]^{V^{\mathsf{B}}} = 1_{\mathsf{B}}$. So V^{B} and $\widetilde{V^{\mathsf{B}}}$ are isomorphic boolean valued models.*

Proof. Suppose by induction that the thesis is true for all $\beta \leq \alpha$. Let for all $\eta \in V^{\mathsf{B}}_\alpha$, $\tilde{\eta} \in \widetilde{V^{\mathsf{B}}_{\leq\alpha}}$ be such that $[\![\eta = \tilde{\eta}]\!]^{V^{\mathsf{B}}} = 1_{\mathsf{B}}$. For $\tau \colon X \to \mathsf{B}$ with $X \subseteq V^{\mathsf{B}}_\alpha$, define
$$\begin{aligned}\tilde{\tau} \colon \widetilde{V^{\mathsf{B}}_{\leq\alpha}} &\to \mathsf{B} \\ \tilde{\sigma} &\mapsto \bigvee_{\eta \in X} (\tau(\eta) \wedge [\![\eta = \tilde{\sigma}]\!]).\end{aligned}$$
We compute the boolean value $[\![\tau = \tilde{\tau}]\!]^{V^{\mathsf{B}}}$:
$$[\![\tau \subseteq \tilde{\tau}]\!] = \bigwedge_{\sigma \in X} ([\![\sigma \in \tilde{\tau}]\!] \vee \neg\tau(\sigma)) = \bigwedge_{\sigma\in X}\Big(\Big(\bigvee_{\tilde{\eta}\in\widetilde{V^{\mathsf{B}}_{\leq\alpha}}} ([\![\tilde{\eta} = \sigma]\!] \wedge \tilde{\tau}(\tilde{\eta}))\Big) \vee \neg\tau(\sigma)\Big) \geq$$
$$\geq \bigwedge_{\sigma\in X} (([\![\sigma = \tilde{\sigma}]\!] \wedge \tau(\sigma)) \vee \neg\tau(\sigma)) \geq \bigwedge_{\sigma\in X} (\tau(\sigma) \vee \neg\tau(\sigma)) = 1_{\mathsf{B}}.$$
On the other hand
$$[\![\tilde{\tau} \subseteq \tau]\!] = \bigwedge_{\tilde{\sigma}\in\widetilde{V^{\mathsf{B}}_{\leq\alpha}}} ([\![\tilde{\sigma} \in \tau]\!] \vee \neg\tilde{\tau}(\tilde{\sigma})) = \bigwedge_{\tilde{\sigma}\in\widetilde{V^{\mathsf{B}}_{\leq\alpha}}}\Big(\Big(\bigvee_{\eta\in X}([\![\eta = \tilde{\sigma}]\!] \wedge \tau(\eta))\Big) \vee \neg\tilde{\tau}(\tilde{\sigma})\Big) =$$
$$= \bigwedge_{\tilde{\sigma}\in\widetilde{V^{\mathsf{B}}_{\leq\alpha}}}\Big(\Big(\bigvee_{\eta\in X}([\![\eta = \tilde{\sigma}]\!] \wedge \tau(\eta))\Big) \vee \neg\Big(\bigvee_{\eta\in X}([\![\eta = \tilde{\sigma}]\!] \wedge \tau(\eta))\Big)\Big) = \bigwedge_{\tilde{\sigma}\in\widetilde{V^{\mathsf{B}}_{\leq\alpha}}} 1_{\mathsf{B}} = 1_{\mathsf{B}}.$$
So $[\![\tau = \tilde{\tau}]\!]^{V^{\mathsf{B}}} = 1_{\mathsf{B}}$. □

To complete the proof that V^{B} and $V^{\mathrm{St(B)}}$ are boolean isomorphic, it suffices to find a boolean isomorfism of $\widetilde{V^{\mathsf{B}}}$ with $V^{\mathrm{St(B)}}$, since the above proposition shows that V^{B} and $\widetilde{V^{\mathsf{B}}}$ are boolean isomorphic. The stratified nature of $\widetilde{V^{\mathsf{B}}}$ and $V^{\mathrm{St(B)}}$ suggests to define again the isomorphism level by level. Since $\widetilde{V^{\mathsf{B}}_1} = \{\emptyset\}$ and $V_1^{\mathrm{St(B)}} = \{c_\emptyset\}$ where $c_\emptyset$ is the only function from $\mathrm{St}(\mathsf{B})$ to $\{\emptyset\}$, $\phi(\emptyset)$ must be $c_\emptyset$. At level 2 we have
$$\widetilde{V^{\mathsf{B}}_2} = \{\{\langle\emptyset, b\rangle\} \mid b \in B\},$$

while

$$V_2^{\mathrm{St}(\mathsf{B})} = \{f\colon \mathrm{St}(\mathsf{B}) \to 2^{\{c_\emptyset\}} \mid f \text{ is continuous}\}.$$

Both sets have a canonical bijection with B: the former is obvious; for the latter notice that if $f\colon \mathrm{St}(\mathsf{B}) \to 2^{\{c_\emptyset\}} \cong 2$ is continuous, then $f^{-1}[\{(c_\emptyset, 1)\}]$ must be a clopen subset of $\mathrm{St}(\mathsf{B})$, so it can be identified via the Stone isomorphism with the unique $b \in \mathsf{B}$ such that $f^{-1}[\{(c_\emptyset, 1)\}] = N_b = \{G \in \mathrm{St}(\mathsf{B}) \mid b \in G\}$. By composing these two maps we obtain a bijection between $\widetilde{V_2^{\mathsf{B}}}$ and $V_2^{\mathrm{St}(\mathsf{B})}$, explicitly let $\tau_b = \{\langle\emptyset, b\rangle\}$ and

$$f_b\colon \mathrm{St}(\mathsf{B}) \to 2^{\{c_\emptyset\}}, \quad G \mapsto \begin{cases}\{(c_\emptyset, 1)\} \text{ if } b \in G\\ \{(c_\emptyset, 0)\} \text{ if } b \notin G\end{cases}$$

for all $b \in \mathsf{B}$. Then

$$\begin{aligned}\phi\colon \widetilde{V_2^{\mathsf{B}}} &\to V_2^{\mathrm{St}(\mathsf{B})}\\ \tau_b &\mapsto f_b\end{aligned}$$

is the desired isomorphism of $\widetilde{V_2^{\mathsf{B}}}$ with the second level of $V^{\mathrm{St}(\mathsf{B})}$. In order to extend ϕ to the upper levels of $\widetilde{V^{\mathsf{B}}}$ and $V^{\mathrm{St}(\mathsf{B})}$ we look more closely on the definition of ϕ just given: in fact for every $b \in \mathsf{B}$ and every $G \in \mathrm{St}(\mathsf{B})$ it holds

$$f_b(G)(c_\emptyset) = 1 \iff \tau_b(\emptyset) \in G.$$

We will extend recursively this definition to obtain a total function $\phi\colon \widetilde{V^{\mathsf{B}}} \to V^{\mathrm{St}(\mathsf{B})}$.

Definition 3.2.2. $\phi\colon \widetilde{V^{\mathsf{B}}} \to V^{\mathrm{St}(\mathsf{B})}$ is defined by recursion on the ordinals as follows: Assume $\phi_\beta : \widetilde{V_\beta^{\mathsf{B}}} \to V_\beta^{\mathrm{St}(\mathsf{B})}$ has been defined for all $\beta < \alpha$ and is surjective.

- If α is limit, define $\phi_\alpha = \cup_{\beta<\alpha}\phi_\beta$.
- If $\alpha = \beta + 1$, let $\bar{\phi}_\beta = \bigcup_{\xi\leq\beta}\phi_\xi$ and $\bar{\phi}_\beta(\eta) = f_\eta$ for all $\eta \in \widetilde{V_{\leq\beta}^{\mathsf{B}}}$. Now define:

$$\begin{aligned}\phi_{\alpha+1}\colon \widetilde{V_{\alpha+1}^{\mathsf{B}}} &\to V_{\alpha+1}^{\mathrm{St}(\mathsf{B})}\\ \tau &\mapsto f_\tau\end{aligned}$$

  where for all $\eta \in \widetilde{V_{\leq\beta}^{\mathsf{B}}}$ and $G \in \mathrm{St}(\mathsf{B})$

$$f_\tau(G)(f_\eta) = 1 \iff \tau(\eta) \in G.$$

Finally $\phi\colon \widetilde{V^{\mathsf{B}}} \to V^{\mathrm{St}(\mathsf{B})}$ is given by

$$\phi = \bigcup_{\alpha\in\mathrm{Ord}} \phi_\alpha.$$

It remains to prove that ϕ is a boolean isomorphism. We will first prove that ϕ is even a bijection between $\widetilde{V^{\mathsf{B}}}$ and $V^{\mathrm{St}(\mathsf{B})}$, next we will show that ϕ preserves the boolen values (i.e. that $[\![\tau\ R\ \sigma]\!] = [\![\phi(\tau)\ R\ \phi(\sigma)]\!]$ for all $\sigma, \tau \in \widetilde{V^{\mathsf{B}}}$ and $R \in \{\in, =, \subseteq\}$). We start proving that ϕ is a bijection:

Definition 3.2.3. Let

$$\begin{aligned} \tau \colon V^{\mathrm{St}(\mathsf{B})} &\to \widetilde{V^{\mathsf{B}}} \\ f &\mapsto \tau_f \end{aligned}$$

be defined by the following rule: given $f \in V^{\mathrm{St}(\mathsf{B})}_{\alpha+1}$ (hence $f : \mathrm{St}(\mathsf{B}) \to 2^{V^{\mathrm{St}(\mathsf{B})}_{\leq\alpha}}$ is continuous),

$$\begin{aligned} \tau_f \colon \widetilde{V^{\mathsf{B}}_{\leq\alpha}} &\to \mathsf{B} \\ \tau_0 &\mapsto \{G \in \mathrm{St}(\mathsf{B}) \mid f(G)(f_{\tau_0}) = 1\}, \end{aligned}$$

where the latter set is identified, as a clopen subset of $\mathrm{St}(\mathsf{B})$ (see 3.1.5), with an element of B.

Lemma 3.2.4. *The following holds:*

1. *For every f in $V^{\mathrm{St}(\mathsf{B})}$ $\phi(\tau_f) = f$ (i.e. $\phi \colon \widetilde{V^{\mathsf{B}}} \to V^{\mathrm{St}(\mathsf{B})}$ is surjective).*

2. *If $\sigma \in \widetilde{V^{\mathsf{B}}_{\alpha}}$ then $\tau_{\phi(\sigma)} = \sigma$.*

Proof. We prove 1. We proceed by induction. Assume that 1 holds for all $h \in V^{\mathrm{St}(\mathsf{B})}_{\leq\alpha}$, and let $f \in V^{\mathrm{St}(\mathsf{B})}_{\alpha+1}$. For every $G \in \mathrm{St}(\mathsf{B})$ and $h \in V^{\mathrm{St}(\mathsf{B})}_{\leq\alpha}$ (by 1 applied to α) we have that $h = f_{\tau_h}$ for all $h \in V^{\mathrm{St}(\mathsf{B})}_{\leq\alpha}$; hence

$$\phi(\tau_f)(G)(h) = \phi(\tau_f)(G)(f_{\tau_h}) = 1 \iff \tau_f(\tau_h) \in G \iff f(G)(h) = 1$$

for all $h \in V^{\mathrm{St}(\mathsf{B})}_{\alpha}$; therefore $\phi(\tau_f) = f$.

We now prove 2, again by induction on the rank of $\sigma \in \widetilde{V^{\mathsf{B}}}$: assume the thesis holds for all $\tau \in \widetilde{V^{\mathsf{B}}_{\leq\alpha}}$, and let σ have rank α. By definition $\mathrm{dom}(\tau_{\phi(\sigma)}) = \widetilde{V^{\mathsf{B}}_{\leq\alpha}}$ and

$$\tau_{\phi(\sigma)}(\sigma_0) = \{G \in \mathrm{St}(\mathsf{B}) \mid \phi(\sigma)(G)(f_{\sigma_0}) = 1\} = \{G \in \mathrm{St}(\mathsf{B}) \mid \sigma(\sigma_0) \in G\} = N_{\sigma(\sigma_0)} = \sigma(\sigma_0).$$

□

Remark the following byproduct of the above Lemma:

Remark 3.2.5. $\mathrm{rank}(f_\tau) = \mathrm{rank}(\tau)$ for all $\tau \in \widetilde{V^{\mathsf{B}}}$, and $\mathrm{rank}(\tau_f) = \mathrm{rank}(f)$ for all $f \in V^{\mathrm{St}(\mathsf{B})}$.

The following assertion completes the proof that V^{B} and $V^{\mathrm{St}(\mathsf{B})}$ are isomorphic boolean valued models for Set Theory.

Lemma 3.2.6. *For every couple of* B*-names* σ *and* η *in* $\widetilde{V^{\mathsf{B}}}$ *and* $R \in \mathscr{L}$ *it holds*

$$[\![f_\sigma \; R \; f_\eta]\!]^{V^{\mathrm{St}(\mathsf{B})}} = [\![\sigma R \, \eta]\!]^{V^{\mathsf{B}}}.$$

Proof. By induction on the pair $(\mathrm{rank}(\sigma), \mathrm{rank}(\eta))$:

$$\begin{aligned}
[\![f_\sigma \; \in \; f_\eta]\!]^{V^{\mathrm{St}(\mathsf{B})}} &= \bigvee_{h \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(\eta)}} \left([\![h = f_\sigma]\!]^{V^{\mathrm{St}(\mathsf{B})}} \wedge \{G \in \mathrm{St}(\mathsf{B}) \mid f_\eta(G)(h) = 1\}\right) = \\
= \bigvee_{h \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(\eta)}} &\left([\![f_{\tau_h} = f_\sigma]\!]^{V^{\mathrm{St}(\mathsf{B})}} \wedge \tau_{\phi(\eta)}(\tau_h)\right) = \bigvee_{h \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(\eta)}} \left([\![\tau_h = \sigma]\!]^{V^{\mathsf{B}}} \wedge \eta(\tau_h)\right) = \\
= \bigvee_{\eta_0 \in \widetilde{V^{\mathsf{B}}_{\leq \mathrm{rank}(\eta)}}} &\left([\![\eta_0 = \sigma]\!]^{V^{\mathsf{B}}} \wedge \eta(\eta_0)\right) = [\![\sigma \in \eta]\!]^{V^{\mathsf{B}}},
\end{aligned}$$

where we used the inductive assumption to state that $[\![f_{\tau_h} = f_\sigma]\!]^{V^{\mathrm{St}(\mathsf{B})}} = [\![\tau_h = \sigma]\!]^{V^{\mathsf{B}}}$, plus the results proved in 3.2.4 for the other steps. In the same vein:

$$[\![f_\sigma \; \subseteq \; f_\eta]\!]^{V^{\mathrm{St}(\mathsf{B})}} = \bigwedge_{h \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(\sigma)}} \left([\![h \in f_\eta]\!]^{V^{\mathrm{St}(\mathsf{B})}} \vee \{G \in \mathrm{St}(\mathsf{B}) \mid f_\sigma(G)(h) = 0\}\right) =$$

$$= \bigwedge_{h \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(\sigma)}} \left([\![\tau_h \in \eta]\!]^{V^{\mathsf{B}}} \vee \neg\sigma(\tau_h)\right) = \bigwedge_{\sigma_0 \in \widetilde{V^{\mathsf{B}}_{\leq \mathrm{rank}(\sigma)}}} \left([\![\sigma_0 \in \eta]\!]^{V^{\mathsf{B}}} \vee \neg\sigma(\sigma_0)\right) = [\![\sigma \subseteq \eta]\!]^{V^{\mathsf{B}}}.$$

Finally

$$\begin{aligned}
&[\![f_\sigma = f_\eta]\!]^{V^{\mathrm{St}(\mathsf{B})}} = [\![f_\sigma \subseteq f_\eta]\!]^{V^{\mathrm{St}(\mathsf{B})}} \wedge [\![f_\eta \subseteq f_\sigma]\!]^{V^{\mathrm{St}(\mathsf{B})}} = \\
&= [\![\sigma \subseteq \eta]\!]^{V^{\mathsf{B}}} \wedge [\![\eta \subseteq \sigma]\!]^{V^{\mathsf{B}}} = [\![\sigma = \eta]\!]^{V^{\mathsf{B}}}.
\end{aligned}$$

□

Chapter 4

$V^{\mathsf{St(B)}}$ and sheaves

In this chapter we show the connection between boolean valued models for Set Theory and categories of sheaves. We first represent $V^{\mathsf{St(B)}}$ (or equivalently V^{B}) as a category, then we will see that any B-name in $V^{\mathsf{St(B)}}$ gives rise to a sheaf in a functorial way, and that $V^{\mathsf{St(B)}}$ can be identified with a full subcategory of $\mathsf{Sh}(\mathrm{St}(\mathsf{B}))$. Finally we will use Giraud's theorem to show that the category in question is in fact a Grothendieck Topos.

4.1 $V^{\mathsf{St(B)}}$ as a category

Set is the trivial sheaves category associated to V, likewise $V^{\mathsf{St(B)}}$ has an associated category of sheaves. This category will be denoted by Bname and it is defined as follows:

- an object in Bname is an equivalence class of B-names in $V^{\mathsf{St(B)}}$, with two elements x and y identified when $[\![x = y]\!] = \mathrm{St}(B)$;
- an arrow between two B-names C and D is an equivalence class of B-names f such that $[\![f\colon C \to D]\!] = \mathrm{St}(B)$. Two elements f and g of this kind are considered the same arrow if $[\![f = g]\!] = \mathrm{St}(B)$.

There is a foundational issue here to be addressed: literally our definition yelds that every object and every arrow is a proper class, making it impossible to even define the collection of all the objects, as well as the collection of all the arrows between two classes of B-names. We can resolve this prolem resorting to Scott's trick or to the global axiom of choice to grant that each object and each arrow is a set.

It follows from the definition and the analogous properties of V (or Set) that the above definition actually gives rise to a category. For example, the arrows are composable: if $[\![f\colon C \to D]\!] = \mathrm{St}(B)$ and $[\![g\colon D \to E]\!] = \mathrm{St}(B)$ then $[\![\phi := \exists h(h = g \circ f)]\!]^{V^{\mathsf{St(B)}}} = \mathrm{St}(\mathsf{B})$ because ϕ can be proved from the ZFC axioms, and by fullness of $V^{\mathsf{St(B)}}$ there is $h \in V^{\mathsf{St(B)}}$ for which $[\![h = g \circ f]\!] = \mathrm{St}(\mathsf{B})$. Similar arguments give the associativity of composition of arrows and the existence of identities.

In order to establish a connection between B-names and sheaves it is convenient to resort to a special kind of B-names.

Definition 4.1.1. A B-name $A \in V^{\mathrm{St(B)}}$ is *full* if it satisfies the following property:

> for every B-name $x \in V^{\mathrm{St(B)}}$ and every $b \in \mathsf{B}$ such that $[\![x \in A]\!] \geq N_b$, there exists some $\tilde{x} \in V^{\mathrm{St(B)}}_{\leq \mathrm{rank}(A)}$ such that $[\![\tilde{x} = x]\!] \geq N_b$ and $A(G)(\tilde{x}) = 1$ for every ultrafilter $G \in N_b$.

In other words, A is full if the supremum defining the boolean value

$$[\![x \in A]\!] = \bigvee_{y \in V^{\mathrm{St(B)}}_{\leq \mathrm{rank}(A)}} ([\![x = y]\!] \wedge \{G \in \mathrm{St}(\mathsf{B}) \mid A(G)(y) = 1\})$$

is actually a maximum for every $x \in V^{\mathrm{St(B)}}$.

The next result ensures that the class of full B-names preserves all the information carried by the boolean valued model for Set Theory.

Theorem 2. *For every $A \in V^{\mathrm{St(B)}}$ there is some B-name $\tilde{A}$ such that $\left[\!\left[A = \tilde{A}\right]\!\right] = \mathrm{St}(\mathsf{B})$ and $\tilde{A}$ is full.*

In order to find such an $\tilde{A}$, the natural idea is to "fill" A defining $\tilde{A}$ as

$$\tilde{A}\colon \mathrm{St}(\mathsf{B}) \to 2^{V^{\mathrm{St(B)}}_{\leq \mathrm{rank}(A)}}, \qquad \tilde{A}(G)(x) = 1 \iff G \in [\![x \in A]\!]. \tag{4.1}$$

For this construction to work, we have to make sure that for every $b \in \mathsf{B}$ and $x \in V^{\mathrm{St(B)}}$ such that $[\![x \in A]\!] \geq N_b$ there is some $y \in V^{\mathrm{St(B)}}_{\leq \mathrm{rank}(A)}$ with $[\![x = y]\!] \geq N_b$. We will use the following lemmas:

Lemma 4.1.2. *For every $A \in V^{\mathrm{St(B)}}$, if $\mathrm{rank}(A) = \alpha$ and $\beta > \alpha$ there is some $\bar{A} \in V^{\mathrm{St(B)}}_{\beta+1}$ such that $[\![A = \bar{A}]\!] = \mathrm{St}(\mathsf{B})$.*

Proof. Define $\bar{A}\colon \mathrm{St}(\mathsf{B}) \to 2^{V^{\mathrm{St(B)}}_{\leq \beta}}$ as

$$\bar{A}(G)(y) = \begin{cases} A(G)(y) & \text{if } \mathrm{rank}(y) < \alpha \\ 0 & \text{if } \mathrm{rank}(y) \geq \alpha. \end{cases}$$

for every $G \in \mathrm{St}(\mathsf{B})$. Since $\mathsf{ZFC} \vdash \forall A, \bar{A}\, [A = \bar{A} \leftrightarrow \forall x(x \in A \iff x \in \bar{A})]$, we can equivalently show that

$$[\![x \in A]\!] = [\![x \in \bar{A}]\!] \qquad \text{for all } x \in V^{\mathrm{St(B)}};$$

and this is the case because

$$[\![x \in \bar{A}]\!] = \bigvee_{y \in V^{\mathrm{St(B)}}_{\leq \beta}} ([\![y = x]\!] \wedge \{G \in \mathrm{St}(\mathsf{B}) : \bar{A}(G)(y) = 1\}) =$$

$$= \bigvee_{y \in V^{\mathrm{St(B)}}_{\leq \alpha}} ([\![y = x]\!] \wedge \{G \in \mathrm{St}(\mathsf{B}) : A(G)(y) = 1\}) = [\![x \in A]\!]. \qquad \square$$

Lemma 4.1.3. *Let x and y be two* B*-names in $V^{\mathrm{St(B)}}_{\beta+1}$ satisfying $x\restriction_{N_b}= y\restriction_{N_b}$ for some clopen subset N_b of* $\mathrm{St(B)}$*. Then $[\![x=y]\!]\geq N_b$.*

Proof.

$$[\![x\subseteq y]\!] = \bigwedge_{z\in V^{\mathrm{St(B)}}_{\leq\beta}} \Big([\![z\in y]\!] \vee \{G\in \mathrm{St(B)}:\, x(G)(z)=0\}\Big) =$$

$$= \bigwedge_{z\in V^{\mathrm{St(B)}}_{\leq\beta}} \Big(\bigvee_{w\in V^{\mathrm{St(B)}}_{\leq\beta}} \Big[[\![w=z]\!] \wedge \{G\in \mathrm{St(B)}:\, y(G)(w)=1\}\Big] \vee \{G\in \mathrm{St(B)}:\, x(G)(z)=0\}\Big) \geq$$

$$\geq \bigwedge_{z\in V^{\mathrm{St(B)}}_{\leq\beta}} \Big(\{G\in \mathrm{St(B)}:\, y(G)(z)=1\} \vee \{G\in \mathrm{St(B)}:\, x(G)(z)=0\}\Big).$$

Now, if $G\in N_b$ by hypotesis $x(G)(z)=1 \iff y(G)(z)=1$, so

$$\{G\in \mathrm{St(B)}:\, y(G)(z)=1\} \vee \{G\in \mathrm{St(B)} \mid x(G)(z)=0\} \geq N_b \qquad \text{for all } z\in V^{\mathrm{St(B)}}_{\leq\beta}$$

giving $[\![x\subseteq y]\!]\geq N_b$. The proof of $[\![y\subseteq x]\!]\geq N_b$ is exactly the same. □

Lemma 4.1.4. *Assume $\alpha=\beta+1$ is a successor ordinal. For any $A\in V^{\mathrm{St(B)}}_{\alpha+1}$ and $b\in$ B, and for all $x\in V^{\mathrm{St(B)}}$ such that $[\![x\in A]\!]\geq N_b$ there is some $\bar{x}\in V^{\mathrm{St(B)}}_{\alpha}$ with $[\![x=\bar{x}]\!]\geq N_b$.*

Proof. $[\![x\in A]\!]\geq N_b$ entails that $\bigvee_{y\in V^{\mathrm{St(B)}}_{\leq\alpha}}[\![x=y]\!]\geq N_b$. This means that the set

$$\downarrow\Big\{[\![x=y]\!] : y\in V^{\mathrm{St(B)}}_{\leq\alpha}\Big\}$$

is dense below N_b. Extract a maximal antichain $\{N_i : i\in I\}$, and for every $i\in I$ let $y_i\in V^{\mathrm{St(B)}}_{\leq\alpha}$ be such that $[\![x=y_i]\!]\geq N_i$. By 4.1.2 we can assume $y_i\in V^{\mathrm{St(B)}}_{\alpha}$ so $\mathrm{rank}(y_i)=\beta$ for all $i\in I$. Define

$$\bar{x}\restriction \bigcup_{i\in I} N_i = \bigcup_{i\in I}(y_i\restriction N_i).$$

Since $\bigcup_{i\in I} N_i$ is a dense open subset of N_b, $\bar{x}$ can be extended by continuity on the whole N_b as follows:

> Let $N_{b_{s,j}}$ be the closure of $\bigcup\{N_i : y_i(s)=j\}$ for all $s\in 2^{V^{\mathrm{St(B)}}_{\leq\beta}}$. Then $N_{b_{s,0}}\cup N_{b_{s,1}}\supseteq N_b$ and $N_{b_{s,0}}\cap N_{b_{s,1}}=\emptyset$ for all $s\in 2^{V^{\mathrm{St(B)}}_{\leq\beta}}$. Hence for all $G\in N_b$ and $s\in 2^{V^{\mathrm{St(B)}}_{\leq\beta}}$, just one among $b_{s,0}, b_{s,1}$ belongs to G; let $\bar{X}(G)(s)=j$ if and only if $b_{s,j}\in G$.

On the remainder $N_{\neg b}$, let $\bar{x}(G)=\bar{0}$, where $\bar{0}$ is the zero function in $2^{V^{\mathrm{St(B)}}_{\leq\beta}}$. By construction $[\![x=y_i]\!]\geq N_i$, and by 4.1.3 $[\![\bar{x}=y_i]\!]\geq N_i$, hence

$$[\![x=\bar{x}]\!]\geq \bigvee_{i\in I} N_i \geq N_b.$$

□

We can now prove Theorem 2:

Proof. 4.1.2 allows to assume WLOG that $\mathrm{rank}(A)$ is not a limit ordinal. Let $\tilde{A}$ be defined as in 4.1. We compute the boolean value $\left[\!\left[A = \tilde{A} \right]\!\right]$:

$$\left[\!\left[A \subseteq \tilde{A} \right]\!\right] = \bigwedge_{x \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}} \left(\left[\!\left[x \in \tilde{A} \right]\!\right] \vee \{G \in \mathrm{St}(\mathsf{B}) : \, A(G)(x) = 0\} \right) =$$

$$= \bigwedge_{x \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}} \Bigg(\bigvee_{y \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}} \Big[[\![x = y]\!] \wedge \{G \in \mathrm{St}(\mathsf{B}) : \, \tilde{A}(G)(y) = 1\} \Big] \vee \{G \in \mathrm{St}(\mathsf{B}) : \, A(G)(x) = 0\} \Bigg) \geq$$

$$\geq \bigwedge_{x \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}} (\{G \in \mathrm{St}(\mathsf{B}) : \tilde{A}(G)(x) = 1\} \vee \{G \in \mathrm{St}(\mathsf{B}) : A(G)(x) = 0\}) =$$

$$= \bigwedge_{x \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}} ([\![x \in A]\!] \vee \{G \in \mathrm{St}(\mathsf{B}) : A(G)(x) = 0\}) = \mathrm{St}(\mathsf{B})$$

because

$$[\![x \in A]\!] \geq \{G \in \mathrm{St}(\mathsf{B}) : A(G)(x) = 1\}$$

by definition for every $x \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}$. On the other hand

$$\left[\!\left[\tilde{A} \subseteq A \right]\!\right] = \bigwedge_{x \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}} \left([\![x \in A]\!] \vee \left\{ G \in \mathrm{St}(\mathsf{B}) : \, \tilde{A}(G)(x) = 0 \right\} \right) =$$

$$= \bigwedge_{x \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}} ([\![x \in A]\!] \vee \neg [\![x \in A]\!]) = \bigwedge_{x \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}} \mathrm{St}(\mathsf{B}) = \mathrm{St}(\mathsf{B}).$$

It remains to prove that $\tilde{A}$ is full; but this is the case because $\left[\!\left[A = \tilde{A} \right]\!\right] = \mathrm{St}(B)$ iff

$$[\![x \in A]\!] \geq N_b \iff \left[\!\left[x \in \tilde{A} \right]\!\right] \geq N_b \qquad \text{for all } x \in V^{\mathrm{St}(\mathsf{B})} \text{ and } b \in \mathsf{B},$$

hence 4.1 and 4.1.4 yeld the fullness of $\tilde{A}$. □

4.2 The sheaf associated to a full B-name

The definition of the B-names in $V^{\mathrm{St}(\mathsf{B})}$ as continuous functions from $\mathrm{St}(\mathsf{B})$ suggests that the concept of B-name is connected with the idea of a sheaf on $\mathrm{St}(\mathsf{B})$. Every B-name A whose rank is a successor ordinal corresponds to the two sets of continuous functions

$$U_0(A) = \left\{ x \in V^{\mathrm{St}(\mathsf{B})}_{\mathrm{rank}(A)} : [\![x \in A]\!] = \mathrm{St}(\mathsf{B}) \right\}$$

and

$$U_1(A) = \Big\{ x \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)} : \forall G \in \mathrm{St}(\mathsf{B})\; A(G)(x) = 1 \Big\};$$

related respectively to the presheaves

$$F_0(A) \colon \mathsf{B}^{\mathrm{op}} \to \mathsf{Set}, \qquad b \mapsto \big\{ x\restriction_{N_b} : x \in V^{\mathrm{St}(\mathsf{B})}_{\mathrm{rank}(A)} \wedge [\![x \in A]\!] \geq N_b \big\}$$

and

$$F_1(A) \colon \mathsf{B}^{\mathrm{op}} \to \mathsf{Set}, \qquad b \mapsto \{ [x\restriction_{N_b}]_\sim : \text{ for all } G \in N_b\; A(G)(x) = 1 \}$$

where $x\restriction_{N_b} \sim y\restriction_{N_b}$ if x and y can be taken so that $[\![x = y]\!] \geq N_b$.

It can be shown that $F_0(A)$ is always a sheaf. Suppose in fact that $\{b_i : i \in I\}$ is a maximal antichain in B, and for every $i \in I$ let $x_i \in V^{\mathrm{St}(\mathsf{B})}_\alpha$ be such that $[\![x_i \in A]\!] \geq N_{b_i}$. Let $x \in V^{\mathrm{St}(\mathsf{B})}_\alpha$ be the unique continuous function respecting on the dense open set $\bigcup_{i \in I} N_i$ the requirement $x(G) = x_i(G)$ if $b_i \in G$. We have

$$[\![x \in A]\!] \geq \bigvee_{i \in I} ([\![x_i \in A]\!] \wedge [\![x = x_i]\!]) \geq \bigvee_{i \in I} N_{b_i};$$

that is, the collation of local sections of $F_0(A)$ is again a local section of $F_0(A)$.

This is not the case for $F_1(A)$, and gives a motivation to focus on full B-names:

Theorem 3. *$F_1(A)$ is a sheaf if and only if A is full.*

Proof. Assume A is full, let $\{b_i : i \in I\}$ be an antichain in B. For every $i \in I$, consider a function $x_i \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}$ such that $A(G)(x_i) = 1$ for all $G \in N_{b_i}$. To prove that $F_1(A)$ is a sheaf it must be found an $x \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}$ with the property $[\![x = x_i]\!] \geq N_{b_i}$ for all i and $A(G)(x) = 1$ if $b_i \in G$ for some $i \in I$. We use 4.1.2 to find $\tilde{x}_i \in V^{\mathrm{St}(\mathsf{B})}_{\mathrm{rank}(A)}$ so that $[\![\tilde{x}_i = x_i]\!] = \mathrm{St}(\mathsf{B})$ for every $i \in I$, and define $\tilde{x}$ to be the unique continuous function satisfying for all $i \in I$ the condition:

$\tilde{x}(G) = \tilde{x}_i(G)$ *if and only if* $b_i \in G$.

It holds

$$[\![\tilde{x}_i \in A]\!] = [\![x_i \in A]\!] = \bigvee_{y \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)}} ([\![x_i = y]\!] \wedge \{ G \in \mathrm{St}(\mathsf{B}) \mid A(G)(y) = 1 \}) \geq$$

$$\geq [\![x_i = x_i]\!] \wedge \{ G \in \mathrm{St}(\mathsf{B}) \mid A(G)(x_i) = 1 \} \geq N_{b_i}.$$

Now, calculating the boolean value $[\![\tilde{x} \in A]\!]$:

$$[\![\tilde{x} \in A]\!] \geq \bigvee_{i \in I} ([\![\tilde{x} = x_i]\!] \wedge [\![x_i \in A]\!]) \geq \bigvee_{i \in I} N_{b_i} = \mathrm{St}(\mathsf{B}).$$

Furthermore $[\![\tilde{x} = x_i]\!] \geq N_{b_i}$, hence $\tilde{x}$ works.

Conversely, suppose $F_1(A)$ is a sheaf, and take $x \in V^{\mathrm{St}(\mathsf{B})}$ such that $[\![x \in A]\!] \geq N_b$. Arguing as in the proof of 4.1.4, we know that the set

$$\downarrow \big\{ [\![x = w]\!] \wedge \{ G \in \mathrm{St}(\mathsf{B}) \mid A(G)(w) = 1 \} : w \in V^{\mathrm{St}(\mathsf{B})}_{\leq \mathrm{rank}(A)} \big\}$$

is dense below N_b, so a maximal antichain $\{N_{b_i} : i \in I\}$ can be extracted, and for all $i \in I$ there is $y_i \in V^{\mathrm{St(B)}}_{\leq \mathrm{rank}(A)}$ such that $[\![x = y_i]\!] \wedge \{G \in \mathrm{St(B)} \mid A(G)(y_i) = 1\} \geq N_{b_i}$. Since $F_1(A)$ is a sheaf and $\{[y_i] : i \in I\}$ is a collating family in $F_1(A)$, there is $\tilde{x}$ such that $A(G)(\tilde{x}) = 1$ for every $G \in N_b$ and $[\![\tilde{x} = y_i]\!] \geq N_{b_i}$ for all $i \in I$, therefore

$$[\![\tilde{x} = x]\!] \geq \bigvee_{i \in I} ([\![\tilde{x} = y_i]\!] \wedge [\![y_i = x]\!]) \geq \bigvee_{i \in I} N_{b_i} \geq N_b$$

showing that A is full. $\square$

Remark 4.2.1. If one wants $F_1(A)$ to be a sheaf, the equivalence classes of the local sections of the B-names have to be considered. It can be shown that

$$\tilde{F}_1(A) \colon \mathsf{B}^{\mathrm{op}} \to \mathsf{Set}, \qquad b \mapsto \{x{\restriction}_{N_b} \colon \forall G \in N_b\ A(G)(x) = 1\}$$

may not be a sheaf even if A is full. To see this, take two distict B-names $x_1, x_2 \in V^{\mathrm{St(B)}}_\alpha$ for some $\alpha \in \mathrm{Ord}$, $b \in \mathsf{B}$ and let $x \in V^{\mathrm{St(B)}}_\alpha$ given by

$$x(G) = \begin{cases} x_1(G) & \text{if } b \in \mathsf{B} \\ x_2(G) & \text{if } b \notin \mathsf{B}. \end{cases}$$

Using 4.1.2, choose an $\tilde{x} \in V^{\mathrm{St(B)}}_{\alpha+1}$ such that $[\![\tilde{x} = x]\!] = \mathrm{St(B)}$, and define $A \in V^{\mathrm{St(B)}}_{\alpha+2}$ as

$$A(G)(y) = 1 \iff y = \tilde{x} \vee (b \in G \wedge y = x_1) \vee (b \notin G \wedge y = x_2).$$

Now, for every $z \in V^{\mathrm{St(B)}}$

$$\begin{aligned} [\![z \in A]\!] = & \bigvee_{y \in V^{\mathrm{St(B)}}_{\leq \mathrm{rank}(A)}} ([\![z = y]\!] \wedge \{G \in \mathrm{St(B)} \mid A(G)(y) = 1\}) = \\ & = ([\![z = x_1]\!] \wedge N_b) \vee ([\![z = x_2]\!] \wedge N_{\neg b}) \vee [\![z = \tilde{x}]\!] \leq \\ & \leq ([\![z = x_1]\!] \wedge [\![x_1 = \tilde{x}]\!]) \vee ([\![z = x_2]\!] \wedge [\![x_2 = \tilde{x}]\!]) \vee [\![z = \tilde{x}]\!] = \\ & \leq [\![z = \tilde{x}]\!]\,; \end{aligned}$$

so $\tilde{x}$ is the witness of the fullness of A. But $\tilde{F}_1(A)$ is not a sheaf because $A(G)(x) = 0$ for all $G \in \mathrm{St(B)}$ even if x is obtained by collating local sections of $\tilde{F}_1(A)$.

The following result gives a connection between the semantic notion of boolean equality of two B-names and the condition of isomorphism of sheaves.

Theorem 4.2.2. *Let A and B be two full* B*-names. Then* $[\![A = B]\!] = \mathrm{St(B)}$ *if and only if* $F_1(A) \simeq F_1(B)$.

Proof. Suppose it holds $[\![A = B]\!] = \mathrm{St(B)}$, that is $[\![x \in A]\!] = [\![x \in B]\!]$ for every $x \in V^{\mathrm{St(B)}}$. If $[x{\restriction}_{N_b}]_A \in F_1(A)(b)$, then $[\![x \in B]\!] = [\![x \in A]\!] \geq N_b$, so by the fullness of B there is $y \in V^{\mathrm{St(B)}}$ such that $[y{\restriction}_{N_b}]_B \in F_1(B)(b)$ and $[\![x = y]\!] \geq N_b$. If $\tilde{y}$ is another B-name with the property $[\tilde{y}{\restriction}_{N_b}]_B \in F_1(B)(b)$ and $[\![x = \tilde{y}]\!] \geq N_b$, then $[\![y = \tilde{y}]\!] \geq$

$[\![x = y]\!] \wedge [\![x = \tilde{y}]\!] \geq N_b$ yelding $[y{\restriction_{N_b}}]_B = [\tilde{y}{\restriction_{N_b}}]_B$ Then the collection $i = \{i_b : b \in \mathsf{B}\}$ with

$$i_b \colon F_1(A)(b) \to F_1(B)(b); \qquad i_b([x{\restriction_{N_b}}]_A) = [y{\restriction_{N_b}}]_B$$

for every $b \in \mathsf{B}$ defines a natural transformation from $F_1(A)$ to $F_1(B)$ (i trivially commutes with the restrictions). The same holds for $j = \{j_b : b \in \mathsf{B}\}$ with

$$j_b \colon F_1(A)(b) \to F_1(B)(b); \qquad j_b([z{\restriction_{N_b}}]_B) = [w{\restriction_{N_b}}]_A$$

where $w \in V^{\mathrm{St}(\mathsf{B})}$ is any B name satisfying $[\![z = w]\!] \geq N_b$ and $[w{\restriction_{N_b}}]_B \in F_1(B)(b)$. Moreover, for every $b \in \mathsf{B}$, i_b and j_b are inverse functions: if $[y{\restriction_{N_b}}]_B = i_b([x{\restriction_{N_b}}]_A)$ and $[w{\restriction_{N_b}}]_A = j_b([y{\restriction_{N_b}}]_B)$, then $[\![w = x]\!] \geq [\![w = y]\!] \wedge [\![x = y]\!] \geq N_b \wedge N_b = N_b$, so $[w{\restriction_{N_b}}]_A = [x{\restriction_{N_b}}]_A$. The same argument applies for $i_b \circ j_b$.

Conversely, suppose $F_1(A)$ and $F_1(B)$ are isomorphic sheaves, and let i and j be the inverse natural transformations witnessing the isomorphism. If $x \in V^{\mathrm{St}(\mathsf{B})}$ is such that $[\![x \in A]\!] \geq N_b$, there is $\tilde{x}$ with $[\tilde{x}{\restriction_{N_b}}]_A \in F_1(A)(b)$ and $[\![x = \tilde{x}]\!] \geq N_b$. Let $[y{\restriction_{N_b}}]_B = i_b([\tilde{x}{\restriction_{N_b}}]_A)$. Then $[\![y = \tilde{x}]\!] \geq N_b$ and $[\![y \in B]\!] \geq N_b$, yelding $[\![x \in B]\!] \geq N_b$. In the same way, for every $x \in V^{\mathrm{St}(\mathsf{B})}$, $[\![x \in B]\!] \geq N_b \implies [\![x \in A]\!] \geq N_b$. Then $[\![A = B]\!] = \mathrm{St}(\mathsf{B})$. $\square$

We proceed to show that the function $F_1 \colon V^{\mathrm{St}(\mathsf{B})} \to \widehat{O(\mathrm{St}(\mathsf{B}))}$ induces a functor

$$F \colon \mathsf{Bname} \to \mathsf{Sh}(\mathrm{St}(\mathsf{B})).$$

To see this, consider an object $[C] \in \mathsf{Bname}$, and define $F([C]) = F_1(\bar{C})$ where $\bar{C}$ is any full B-name in $[C]$ (no matter which for 4.2.2). If f is such that $[\![f \colon \bar{C} \to \bar{D}]\!] = \mathrm{St}(\mathsf{B})$ and $[x{\restriction_{N_b}}]_{\bar{C}} \in F_1(\bar{C})(N_b)$ then $[\![\exists! y(y \in D \wedge f(x) = y)]\!] \geq N_b$, and by the fullness of $V^{\mathrm{St}(\mathsf{B})}$ there is some $\tilde{z}$ such that $[\![\tilde{z} \in \bar{D}]\!] \geq N_b$ and $[\![f(x) = \tilde{z}]\!] \geq N_b$, and by the fullness of $\bar{D}$ there is z so that $[z{\restriction_{N_b}}]_{\bar{D}} \in F_1(\bar{D})$ and $[\![\tilde{z} = z]\!] \geq N_b$. If w is another B-name such that $[\![w = \tilde{z}]\!] \geq N_b$ and $[w{\restriction_{N_b}}]_{\bar{D}} \in F_1(\bar{D})$, then $[\![w = y]\!] \geq N_b$ so $[w{\restriction_{N_b}}]_{\bar{D}} = [y{\restriction_{N_b}}]_{\bar{D}}$.

So an arrow f (more precisely a representative of f) gives rise to a collection of functions

$$F(f)(b) \colon F_1(\bar{C})(b) \to \bar{F}_1(\bar{D})(b), \qquad [x{\restriction_{N_b}}]_{\bar{C}} \mapsto [z{\restriction_{N_b}}]_{\bar{D}}$$

defining a natural transformation between the sheaves $F([C])$ and $F([D])$ (naturality of $F(f)$ follows immediately from the definition). Taking the identity arrow of $[C]$ as f, one has

$$F(\mathrm{Id}_{[C]})(b)([x{\restriction_{N_b}}]_{\bar{C}}) = [\mathrm{Id}_{\bar{C}}(x){\restriction_{N_b}}]_{\bar{C}} = [x{\restriction_{N_b}}]_{\bar{C}}$$

so $F(\mathrm{Id}_{[C]})$ is the identity transformation of $F([C])$. Furthermore, if $[\![h = g \circ f]\!] = \mathrm{St}(\mathsf{B})$, $[\![y = f(x)]\!] \geq N_b$ and $[\![z = g(y)]\!] \geq N_b$ then $[\![z = g \circ f(x)]\!] \geq N_b$, giving $F(g \circ f) = F(g) \circ F(f)$.

It can be proved that F is not only a functor, but it also gives an immersion of categories between Bname and its image:

Theorem 4. $F \colon \mathsf{Bname} \to \mathsf{Sh}(\mathrm{St}(\mathsf{B}))$ *is a full and faithful functor, which is injective on objects.*

Proof. Injectivity on objects is an immediate consequence of 4.2.2. Let f and g be (representatives of) two different arrows from $[C]$ to $[D]$. $[\![f = g]\!] < \mathrm{St}(\mathsf{B})$ means that

$$\bigwedge_{x \in V^{\mathrm{St}(\mathsf{B})}} (x \in \bar{C} \to f(x) = g(x)) < \mathrm{St}(\mathsf{B})$$

so $\left[\!\left[x \in \bar{C} \to f(x) = g(x)\right]\!\right] = \neg \left[\!\left[x \in \bar{C}\right]\!\right] \vee [\![f(x) = g(x)]\!] < \mathrm{St}(\mathsf{B})$ for some $x \in V^{\mathrm{St(B)}}$, equivalently $\left[\!\left[x \in \bar{C} \wedge f(x) = g(x)\right]\!\right] < \left[\!\left[x \in \bar{C}\right]\!\right]$. Let b be the element in B such that $\left[\!\left[x \in \bar{C}\right]\!\right] = N_b$. By the fullness of $\bar{C}$ we can assume $[x \restriction_{N_b}]_{\bar{C}} \in F_1(\bar{C})(b)$, and

$$\left[\!\left[x \in \bar{C} \wedge f(x) = g(x)\right]\!\right] < N_b \implies [f(x) \restriction_{N_b}]_{\bar{D}} \neq [g(x) \restriction_{N_b}]_{\bar{D}},$$

then $F(f) \neq F(G)$. So F is faithful.

Now, let α be a natural trasformation from $F([C])$ to $F([D])$. From α it can be constructed a B-name f such that:

- $\left[\!\left[f : \bar{C} \to \bar{D}\right]\!\right] = \mathrm{St}(\mathsf{B})$,
- for every $x \in V^{\mathrm{St(B)}}$ with $[x \restriction_{N_b}]_{\bar{C}} \in F_1(\bar{C})(b)$ and $y \in V^{\mathrm{St(B)}}$ such that $[y \restriction_{N_b}]_{\bar{D}} = \alpha_b([x \restriction_{N_b}]_{\bar{C}})$ it holds $[\![f(x) = y]\!] \geq N_b$.

We define the B-name attached to α as follows:

- For two B-names x and y define

  $$\mathrm{ins}(x, y) \colon \mathrm{St}(B) \to 2^{V^{\mathrm{St}(\mathsf{B})}_{\leq \max(\mathrm{rank}(x), \mathrm{rank}(y))+1}}$$
  $$\mathrm{ins}(x, y)(G)(h) = 1 \iff h = x \vee h = y.$$

  It follows that $[\![\forall h (h \in \mathrm{ins}(x, y) \iff (h = x \vee h = y))]\!] = \mathrm{St}(\mathsf{B})$, or put in another way $[\![\mathrm{ins}(x, y) = \{x, y\}]\!] = \mathrm{St}(\mathsf{B})$;

- Define

  $$\mathrm{co}(x, y) \colon \mathrm{St}(B) \to 2^{V^{\mathrm{St}(\mathsf{B})}_{\leq \max(\mathrm{rank}(x), \mathrm{rank}(y))+2}}$$
  $$\mathrm{co}(x, y)(G)(h) = 1 \iff h = \mathrm{ins}(x, x) \vee h = \mathrm{ins}(x, y)$$

  It follows that $[\![\forall h (h \in \mathrm{co}(x, y) \iff (h = \mathrm{ins}(x, x) \vee h = \mathrm{ins}(x, y)))]\!] = \mathrm{St}(\mathsf{B})$, that is $[\![\mathrm{co}(x, y) = \{\{x\}, \{x, y\}\}]\!] = \mathrm{St}(\mathsf{B})$;

- Define f as $f \colon \mathrm{St}(B) \to 2^{V^{\mathrm{St}(\mathsf{B})}_{\leq \max(\mathrm{rank}(C), \mathrm{rank}(D))+3}}$ where

  $$f(G)(h) = 1 \iff h = \mathrm{co}(x, y) \wedge [y \restriction_{N_b}]_{\bar{C}} = \alpha_b([y \restriction_{N_b}]_{\bar{D}})$$

  for some $x, y \in V^{\mathrm{St(B)}}$ and $b \in G$. It follows that for all z with $[\![z \in f]\!] > \emptyset$ there exists $x, y \in V^{\mathrm{St(B)}}$ such that $[\![z = (x, y)]\!] = \mathrm{St}(\mathsf{B})$ and $\left[\!\left[x \in \bar{C}\right]\!\right] \leq [\![(x, y) \in f]\!]$, and there is such a z for all x with $\left[\!\left[x \in \bar{C}\right]\!\right] > \emptyset$. So $\left[\!\left[f : \bar{C} \to \bar{D}\right]\!\right] = \mathrm{St}(\mathsf{B})$, and $\alpha = F(f)$ by definition. $\square$

4.3 $V^{\mathrm{St(B)}}$ as a Grothendieck Topos

So far we have shown that certain B-names in $V^{\mathrm{St(B)}}$ have an interpretation as sheaves of continuous functions on $\mathrm{St}(B)$. The natural question is: can *all* $V^{\mathrm{St(B)}}$ be seen as a category of sheaves? The answer is positive because, as it will be proved soon, Bname satisfies all the clauses of Giraud's theorem.

To see this, the key observation is that the properties of the category Set (i.e. the universe V) are given by the ZFC axioms. Now, the hypoteses of Giraud's theorem characterizing a Grothendieck Topos (GT in the following) can be expressed as first order formulas in the language of Set Theory, and the fact that Set is a GT means that these sentences are provable from ZFC. Recalling that $V^{\mathrm{St(B)}}$ is a boolean valued model for Set Theory, this entails that the boolean values $[\![\psi]\!]$ are all equal to $\mathrm{St}(B)$ if ψ is a sentence expressing one of the hypoteses of Giraud's theorem. From this, with some technical reasoning, it follows that Bname is a GT, the clauses of Giraud's theorem being valid for it.

Theorem 5. *$V^{\mathrm{St(B)}}$ satisfies the hypotheses of Giraud's theorem, hence it is a Grothendieck Topos.*

Proof. We make use of the idea just exposed. For example, to see that Bname has a set of generators consider the formula

$$\psi := \exists x\forall y\forall z\forall f\forall g\forall h[(f,g\colon y\to z\wedge h\colon x\to y\wedge f\circ h=g\circ h)\implies f=g]$$

expressing exactly this property. Set, as a GT, has a set of generators, in particular the object $\{\emptyset\}$ is a generator for Set. So ψ is a statement provable from ZFC, hence valid in $V^{\mathrm{St(B)}}$, i.e. we have that $[\![\psi]\!]=\mathrm{St}(B)$. By fullness of $V^{\mathrm{St(B)}}$ we know there is some B-name $\overline{x}$ such that

$$[\![\forall y\forall z\forall f\forall g\forall h([f,g\colon y\to z\wedge h\colon \overline{x}\to y\wedge f\circ h=g\circ h)]\implies f=g)]\!]=\mathrm{St}(B).$$

Now, if f and g are two (representatives of) arrows between the B-names y and z for all arrows h with codomain $\overline{x}$ we have $[\![f\circ h=g\circ h]\!]\le[\![f=g]\!]$, and if $f\circ h$ and $g\circ h$ are all the same arrows then $[\![f\circ h=g\circ h]\!]=\mathrm{St}(B)$, so $[\![f=g]\!]=\mathrm{St}(B)$ and f and g are the same arrow and $[\overline{x}]$ is a generator for Bname. Furthermore the object $[\overline{x}]$ can be shown explicitly: since the generator for Set is $\{\emptyset\}$, $\overline{x}$ can be taken with the property $[\![p(\overline{x})]\!]=\mathrm{St}(B)$ where p is the formula saying that every element of $\overline{x}$ is the empty set. Such an $\overline{x}$ can easily be constructed:

$$\overline{x}\colon\ \mathrm{St}(B)\to 2^{V_1^{\mathrm{St(B)}}},\qquad G\mapsto c_\emptyset.$$

In the same way in can be shown that in Bname every epi arrow is a coequalizer. Being a Gr. topos, in Set every epimorphism is a coequalizer. The formula expressing this fact is

$$\begin{aligned}\psi := \forall x\forall y\forall f[f\colon x\to y\,\text{is epi}\implies \exists g\exists h\exists z(f,h\colon z\to x\wedge fg=fh\wedge\\ \forall d\forall w(d\colon x\to w\wedge dg=dh)\implies\exists e(d=ef))].\end{aligned}$$

Arguing as in the previous point, knowing that $[\![\psi]\!] = \mathrm{St}(B)$, if f is an epi arrow then by fullness of $V^{\mathrm{St(B)}}$ there are two arrows g and h such that $fg = fh$ universal with this property, so f is an equalizer. The other three clauses of Giraud's theorem are proved in the same vein. $\square$

Bibliography

[1] Steve Awodey, *Category theory*, second ed., Oxford Logic Guides, vol. 52, Oxford University Press, Oxford, 2010. MR 2668552

[2] P. J. Cohen, *The independence of the continuum hypothesis*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1143–1148. MR 0157890 (28 #1118)

[3] Michael P. Fourman, *Sheaf models for set theory*, J. Pure Appl. Algebra **19** (1980), 91–101. MR 593249

[4] T. Jech, *Set theory*, Springer Monographs in Mathematics, Springer, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513

[5] S. Koppelberg, R. Bonnet, and J. D. Monk, *Handbook of Boolean Algebras Vol. 1*, Elsevier, 1989.

[6] K. Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland, Amsterdam, 1980, An introduction to independence proofs. MR 597342

[7] Kenneth Kunen, *Set theory*, Studies in Logic (London), vol. 34, College Publications, London, 2011. MR 2905394

[8] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR 1712872

[9] Saunders Mac Lane and Samuel Eilenberg, *General theory of natural equivalences*, Transactions of The American Mathematical Society **58** (1945), 231–294.

[10] Saunders Mac Lane and Ieke Moerdijk, *Sheaves in geometry and logic*, Universitext, Springer-Verlag, New York, 1994, A first introduction to topos theory, Corrected reprint of the 1992 edition. MR 1300636

[11] Joseph R. Shoenfield, *Mathematical logic*, Association for Symbolic Logic, Urbana, IL; A K Peters, Ltd., Natick, MA, 2001, Reprint of the 1973 second printing. MR 1809685

[12] A. Vaccaro and M. Viale, *Generic absoluteness and boolean names for elements of a Polish space*, Boll Unione Mat Ital (2017).

[13] Matteo Viale, *Notes on forcing*, 2017.