Forcing Axioms

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1 Notation

In this notes, $f[A]$ (resp. $f^{-1}[A]$) will denote the set $f[A] = \{f(x) : x \in A\}$ (resp. with f^{-1}). We will use $[X]^\kappa$ (resp. $[X]^{<\kappa}$) to denote the set of all subsets of X of size κ (resp. less than κ). M_α will be the stage α of the cumulative hierarchy in M, and $H(\kappa)$ will be the class of all sets hereditarily of cardinality $< \kappa$. We shall write ϕ^M to mean the interpretation of ϕ in the model M.

If M is a transitive model of ZFC with $\mathbb{P} \in M$, $M^{\mathbb{P}}$ will be the set of $\mathbb{P}$-names in M, and $M[G]$ will be the forcing extension of M with a filter G that is M-generic for some $\mathbb{P}$. We will use $\dot{A}$ to denote a $\mathbb{P}$-name for $A \in M[G]$, $\check{A}$ to denote the standard $\mathbb{P}$-name for $A \in M$, and $\mathrm{val}_G(\dot{A})$ to denote the evaluation of the $\mathbb{P}$-name $\dot{A}$ with an M-generic filter G.

We recall that given a poset $\mathbb{P}$, a set $D \subseteq \mathbb{P}$ is *dense* iff for every $p \in \mathbb{P}$ there exists a $q \in D$, $q \leq p$; and a filter G is *M-generic for* $\mathbb{P}$ iff $G \cap D \cap M \neq \emptyset$ for every $D \in M$ dense subset of $\mathbb{P}$.

2 Generalized Stationarity

In this section we shall introduce a generalization of the notion of stationarity for subsets of cardinals to subsets of any set. This concept has been proved useful in many contexts, and is needed in our purpose to state the strong reflection principle SRP. Reference texts for this section are [2], [3, Chapter 2].

Definition 2.1. Let X be an uncountable set. A set C is a *club* on $\mathscr{P}(X)$ iff there is a function $f_C : X^{<\omega} \to X$ such that C is the set of elements of $\mathscr{P}(X)$ closed under f_C, i.e.

$$C = \{Y \in \mathscr{P}(X) : f_C[Y]^{<\omega} \subseteq Y\}$$

A set S is *stationary* on $\mathscr{P}(X)$ iff it intersects every club on $\mathscr{P}(X)$.

Example 2.2. The set $\{X\}$ is always stationary since every club contains X. Also $\mathscr{P}(X)\setminus\{X\}$ and $[X]^\kappa$ are stationary for any $\kappa \le |X|$ (following the proof of the well-known downwards Löwhenheim-Skolem Theorem). Notice that every element of a club C must contain $f_C(\emptyset)$, a fixed element of X.

Remark 2.3. The reference to the support set X for clubs or stationary sets may be omitted, since every set S can be club or stationary only on $\bigcup S$.

There is one more property of stationary sets that is worth to mention. Given any first-order structure M, from the set M we can define a Skolem function $f_M : M^{<\omega} \to M$ (i.e., a function coding solutions for all existential first-order formulas over M). Then the set C of all elementary submodels of M contains a club (the one corresponding to f_M). Henceforth, every set S stationary on X must contain an elementary submodel of any first-order structure on X.

Definition 2.4. A set S is *subset modulo club* of T, in symbols $S \subseteq^* T$, iff $\bigcup S = \bigcup T = X$ and there is a club C on X such that $S \cap C \subseteq T \cap C$. Similarly, a set S is *equivalent modulo club* to T, in symbols $S =^* T$, iff $S \subseteq^* T \wedge T \subseteq^* S$.

Definition 2.5. The *club filter* on X is $\mathrm{CF}_X = \{C \subset \mathscr{P}(X) : C \text{ contains a club}\}$. Similarly, the *non-stationary ideal* on X is $\mathrm{NS}_X = \{A \subset \mathscr{P}(X) : A \text{ not stationary}\}$.

Remark 2.6. If $|X| = |Y|$, then $\mathscr{P}(X)$ and $\mathscr{P}(Y)$ are isomorphic and so are CF_X and CF_Y (or NS_X and NS_Y): then we can suppose $X \in \mathbf{ON}$ or $X \supseteq \omega_1$ if needed.

Lemma 2.7. CF_X *is a σ-complete filter on $\mathscr{P}(X)$, and the stationary sets are exactly the* CF_X*-positive sets.*

Proof. CF_X is closed under supersets by definition. Given a family of clubs C_i, $i < \omega$, let f_i be the function corresponding to the club C_i. Let $\pi : \omega \to \omega^2$ be a surjection, with components π_1 and π_2, such that $\pi_2(n) \le n$. Define $g : X^{<\omega} \to X$ to be $g(s) = f_{\pi_1(|s|)}(s \restriction \pi_2(|s|))$. It is easy to verify that $C_g = \bigcap_{i<\omega} C_i$. □

Definition 2.8. Given a family $\{S_a \subseteq \mathscr{P}(X) : a \in X\}$, the *diagonal union* of the family is $\nabla_{a\in X} S_a = \{z \in \mathscr{P}(X) : \exists a \in z\ z \in S_a\}$, and the *diagonal intersection* of the family is $\Delta_{a\in X} S_a = \{z \in \mathscr{P}(X) : \forall a \in z\ z \in S_a\}$.

Lemma 2.9 (Fodor)**.** CF_X *is normal, i.e. is closed under diagonal intersection. Equivalently, every function* $f : \mathscr{P}(X) \to X$ *that is regressive on a* CF_X*-positive set is constant on a* CF_X*-positive set.*

Proof. Given a family C_a, $a \in X$ of clubs, with corresponding functions f_a, let $g(a^\frown s) = f_a(s)$. It is easy to verify that $C_g = \Delta_{a\in X} C_a$.

Even though the second part of our thesis is provably equivalent to the first one for any filter $\mathscr{F}$, we shall opt here for a direct proof. Assume by contradiction that $f : \mathscr{P}(X) \to X$ is regressive (i.e., $f(Y) \in Y$) in a CF_X-positive (i.e., stationary) set, and $f^{-1}[a]$ is non-stationary for every $a \in X$. Then, for every $a \in X$ there is a function $g_a : [X]^{<\omega} \to X$ such that the club C_{g_a} is disjoint from $f^{-1}[a]$. Without loss of generality, suppose that $C_{g_a} \subseteq C_a = \{Y \subseteq X : a \in Y\}$. As in the first part of the lemma, define $g(a^\frown s) = g_a(s)$. Then for every $Z \in C_g$ and every $a \in Z$, Z is in C_{g_a} hence is not in $f^{-1}[a]$ (i.e., $f(Z) \neq a$). So $f(Z) \notin Z$ for any $Z \in C_g$, hence C_g is a club disjoint with the stationary set in which f is regressive, a contradiction. □

Remark 2.10. The club filter is never ω_2-complete, unlike its well-known counterpart on cardinals. Let $Y \subseteq X$ be such that $|Y| = \omega_1$, and C_a be the club corresponding to $f_a : [X]^{<\omega} \to \{a\}$; then $C = \bigcap_{a\in Y} C_a = \{Z \subseteq X : Y \subseteq Z\}$ is disjoint from the stationary set $[X]^\omega$, hence is not a club.

This generalized notion of club and stationary set is closely related to the well-known one defined for subsets of cardinals.

Lemma 2.11. $C \subseteq \omega_1$ *is a club in the classical sense if and only if* $C \cup \{\omega_1\}$ *is a club in the generalized sense.* $S \subseteq \omega_1$ *is stationary in the classical sense if and only if it is stationary in the generalized sense.*

Proof. Let $C \subseteq \omega_1 + 1$ be a club in the generalized sense. Then C is closed: given any $\alpha = \sup \alpha_i$ with $f[\alpha_i]^{<\omega} \subseteq \alpha_i$, $f[\alpha]^{<\omega} = \bigcup_i f[\alpha_i]^{<\omega} \subseteq \bigcup_i \alpha_i = \alpha$. Furthermore, C is unbounded: given any $\beta_0 < \omega_1$, define a sequence β_i by taking $\beta_{i+1} = \sup f[\beta_i]^{<\omega}$. Then $\beta_\omega = \sup \beta_i \in C$.

Let now $C \subseteq \omega_1$ be a club in the classical sense. Let $C = \{c_\alpha : \alpha < \omega_1\}$ be an enumeration of the club. For every $\alpha < \omega_1$, let $\left\{d_i^\alpha : i < \omega\right\} \subseteq c_{\alpha+1}$ be a cofinal sequence in $c_{\alpha+1}$ (eventually constant), and let $\left\{e_i^\alpha : i < \omega\right\} \subseteq \alpha$ be an enumeration of α. Define f_C to be $f_C((c_\alpha)^n) = d_n^\alpha$, $f_C(0^\frown \alpha^n) = e_n^\alpha$, and $f_C(s) = 0$ otherwise. The sequence e_i^α forces all closure points of f_C to be ordinals, while the sequence d_i^α forces the ordinal closure points of f_C being in C. □

Lemma 2.12. *If κ is a cardinal with cofinality at least ω_1, $C \subseteq \kappa$ contains a club in the classical sense if and only if $C \cup \{\kappa\}$ contains the ordinals of a club in the generalized sense. $S \subseteq \kappa$ is stationary in the classical sense if and only if it is stationary in the generalized sense.*

Proof. If C is a club in the generalized sense, then $C \cap \kappa$ is closed and unbounded by the same reasoning of Lemma 2.11. Let now C be a club in the classical sense, and define $f : \kappa^{<\omega} \to \kappa$ to be $f(s) = \min\{c \in C : \sup s < c\}$. Then $C_f \cap \kappa$ is exactly the set of ordinals in $C \cup \{\kappa\}$ that are limits within C. □

Remark 2.13. If S is stationary in the generalized sense on ω_1, then $S \cap \omega_1$ is stationary (since $\omega_1 + 1$ is a club by Lemma 2.11), while this is not true for $\kappa > \omega_1$. In this case, $\mathscr{P}(\kappa) \setminus (\kappa + 1)$ is a stationary set: given any function f, the closure under f of $\{\omega_1\}$ is countable, hence not an ordinal.

Lemma 2.14 (Lifting and Projection)**.** *Let $X \subseteq Y$ be uncountable sets. If S is stationary on $\mathscr{P}(Y)$, then $S \downarrow X = \{B \cap X : B \in S\}$ is stationary. If S is stationary on $\mathscr{P}(X)$, then $S \uparrow Y = \{B \subseteq Y : B \cap X \in S\}$ is stationary.*

Proof. For the first part, given any function $f : [X]^{<\omega} \to X$, extend it in any way to a function $g : [Y]^{<\omega} \to Y$. Since S is stationary, there exists a $B \in S$ closed under g, hence $B \cap X \in S \downarrow X$ is closed under f.

For the second part, fix an element $x \in X$. Given any function $f : [Y]^{<\omega} \to Y$, replace it with a function $g : [Y]^{<\omega} \to Y$ such that for any $A \subset Y$, $g[A]$ contains $A \cup \{x\}$ and is closed under f. To achieve this, fix a surjection $\pi : \omega \to \omega^2$ (with projections π_1 and π_2) such that $\pi_2(n) \leq n$ for all n, and an enumeration $\langle t_i^n : i < \omega \rangle$ of all first-order terms with n variables, function symbols f_i for $i \leq n$ (that represent an i-ary application of f) and a constant x. The function g can now be defined as $g(s) = t_{\pi_1(|s|)}^{\pi_2(|s|)}(s \restriction \pi_2(|s|))$. Finally, let $h : [X]^{<\omega} \to X$ be defined by $h(s) = g(s)$ if $g(s) \in X$, and $h(s) = x$ otherwise. Since S is stationary, there exists a $B \in S$ with $h[B] \subseteq B$, but $h[B] = g[B] \cap X$ (since x is always in $g[B]$) and $g[B] \supset B$, so actually $h[B] = g[B] \cap X = B \in S$. Then, $g[B] \in S \uparrow Y$ and $g[B]$ is closed under f (by definition of g). □

Remark 2.15. Following the same proof, a similar result holds for clubs. If C_f is club on $\mathscr{P}(X)$, then $C_f \uparrow Y = C_g$ where $g = f \cup \mathrm{Id}_{Y \setminus X}$. If C_f is club on $\mathscr{P}(Y)$ such that $\bigcap C_f$ intersects X in x, and g, h are defined as in the second part of Theorem 2.14, $C_f \downarrow X = C_h$ is club. If $\bigcap C_f$ is disjoint from X, $C_f \downarrow X$ is not a club, but is still true that it contains a club (namely, $\left(C_f \cap C_{\{x\}}\right) \downarrow X$ for any $x \in X$).

Theorem 2.16 (Ulam)**.** *Let κ be an infinite cardinal. Then for every stationary set $S \subseteq \kappa^+$, there exists a partition of S into κ^+ many disjoint stationary sets.*

Proof. For every $\beta \in [\kappa, \kappa^+)$, fix a bijection $\pi_\beta : \kappa \to \beta$. For $\xi < \kappa$, $\alpha < \kappa^+$, define $A^\xi_\alpha = \left\{\beta < \kappa^+ : \pi_\beta(\xi) = \alpha\right\}$ (notice that $\beta > \alpha$ when $\alpha \in \mathrm{ran}(\pi_\beta)$). These sets can be fit in a $(\kappa \times \kappa^+)$-matrix, called *Ulam Matrix*, where two sets in the same row or column are always disjoint. Moreover, every row is a partition of $\bigcup_{\alpha<\kappa^+} A^\xi_\alpha = \kappa^+$, and every column is a partition of $\bigcup_{\xi<\kappa} A^\xi_\alpha = \kappa^+ \setminus (\alpha+1)$.

Let S be a stationary subset of κ^+. For every $\alpha < \kappa^+$, define $f_\alpha : S \setminus (\alpha + 1) \to \kappa$ by $f_\alpha(\beta) = \xi$ if $\beta \in A^\xi_\alpha$. Since $\kappa^+ \setminus (\alpha+1)$ is a club, every f_α is regressive on a stationary set, then by Fodor's Lemma 2.9 there exists a $\xi_\alpha < \kappa$ such that $f^{-1}_\alpha[\{\xi_\alpha\}] = A^{\xi_\alpha}_\alpha \cap S$ is stationary. Define $g : \kappa^+ \to \kappa$ by $g(\alpha) = \xi_\alpha$, g is regressive on the stationary set $\kappa^+ \setminus \kappa$, again by Fodor's Lemma 2.9 let $\xi^* < \kappa$ be such that $g^{-1}[\{\xi^*\}] = T$ is stationary. Then, the row ξ^* of the Ulam Matrix intersects S in a stationary set for stationary many columns T. So S can be partitioned into $S \cap A^{\xi^*}_\alpha$ for $\alpha \in T \setminus \{\min(T)\}$, and $S \setminus \bigcup_{\alpha \in T\setminus\{\min(T)\}} A^{\xi^*}_\alpha$. □

Remark 2.17. In the proof of Theorem 2.16 we actually proved something more: the existence of a Ulam Matrix, i.e. a $\kappa \times \kappa^+$-matrix such that every stationary set $S \subseteq \kappa^+$ is compatible (i.e., has stationary intersection) with stationary many elements of a certain row.

3 More on Stationarity

In this section we present some notable definition and results about stationary sets that are not strictly needed for the rest of the notes. Reference text for this section is [3, Chapter 2].

Definition 3.1. Let X be an uncountable set, $\kappa < |X|$ be a cardinal. A set C is a *club* on $[X]^\kappa$ (resp. $[X]^{<\kappa}$) iff there is a function $f_C : X^{<\omega} \to X$ such that C is the set of elements of $[X]^\kappa$ (resp. $[X]^{<\kappa}$) closed under f_C, i.e.

$$C = \left\{Y \in [X]^\kappa : f_C[Y]^{<\omega} \subseteq Y\right\}$$

A set S is *stationary* on $[X]^\kappa$ (respectively $[X]^{<\kappa}$) iff it intersects every club on $[X]^\kappa$ (respectively $[X]^{<\kappa}$).

This definition is justified by the observation that $[X]^\kappa$ (resp. $[X]^{<\kappa}$) is stationary on X for every $\kappa < |X|$. As in the unrestricted case, the club sets on $[X]^\kappa$ (resp. $[X]^{<\kappa}$) form a normal σ-complete filter on $[X]^\kappa$ (resp. $[X]^{<\kappa}$). We can also state an analogous formulation of Lemma 2.14, with additional care in the case $[X]^\kappa$: in that case, the lifting $[X]^\kappa \uparrow [Y]^\kappa$ may not be a club on $[Y]^\kappa$ if $|X| < |Y|$. For example, such a set is not a club if there exists a Completely Jónsson cardinal above $|Y|$ since its complement $[Y]^\kappa \setminus \left([X]^\kappa \uparrow [Y]^\kappa\right) = [X]^{<\kappa} \uparrow [Y]^\kappa$ is stationary.

Lemma 3.2 (Lifting and Projection)**.** *Let $X \subseteq Y$ be uncountable sets, $\kappa < |X|$ be a cardinal. If C contains a club on $[Y]^\kappa$ (resp. $[Y]^{<\kappa}$), then $C \downarrow [X]^\kappa = (C \downarrow X) \cap [X]^\kappa$ (resp. $C \downarrow [X]^{<\kappa}$) contains a club on $[X]^\kappa$ (resp. $[X]^{<\kappa}$). If C contains a club on $[X]^{<\kappa}$, then $C \uparrow [Y]^{<\kappa} = (C \uparrow Y) \cap [Y]^{<\kappa}$ contains a club on $[Y]^{<\kappa}$.*

If S is stationary on $[Y]^{<\kappa}$, then $S \downarrow [X]^{<\kappa}$ is stationary on $[X]^{<\kappa}$. If S is stationary on $[X]^\kappa$ (resp. $[X]^{<\kappa}$), then $S \uparrow [Y]^\kappa$ is stationary on $[Y]^\kappa$ (resp. with $[Y]^{<\kappa}$).

We can now define a natural ordering on stationary sets, that can be used to define a poset of notable relevance in set theory.

Definition 3.3. Let S, T be stationary sets. We write $S \leq T$ iff $\bigcup S \supseteq \bigcup T$ and $S \subseteq T \uparrow \left(\bigcup S\right)$.

Definition 3.4. The *full stationary tower* up to α is the poset $\mathbb{P}_{<\alpha}$ of all the stationary sets $S \in V_\alpha$ ordered by $S \leq T$ as defined above. The stationary tower restricted to size κ up to α is the poset $\mathbb{Q}^\kappa_{<\alpha} = \left\{S \in V_\alpha : S \subseteq \left[\bigcup S\right]^\kappa \text{ stationary}\right\}$ ordered by the same relation.

4 Forcing Axioms

Forcing is well-known as a versatile tool for proving consistency results. The purpose of forcing axioms is to turn it into a powerful tool for proving theorems: this intuition is partly justified by the following *Cohen's Absoluteness Lemma* 4.2.

In the following notes we will use the notation $M \prec_n N$ to mean $M \prec_{\Sigma_n} N$ (or equivalently $M \prec_{\Pi_n} N$, $M \prec_{\Delta_{n+1}} N$). Reference text for this section is [1, Chapter 3]. We first recall the following lemma.

Lemma 4.1 (Levi's Absoluteness)**.** *Let $\kappa > \omega$ be a cardinal. Then $H(\kappa) \prec_1 V$.*

Proof. Given any Σ_1 formula $\phi = \exists x\ \psi(x, p_1, \ldots, p_n)$ with parameters $p_1, \ldots, p_n$ in $H(\kappa)$, if $V \models \neg\phi$ also $H(\kappa) \models \neg\phi$ since $H(\kappa) \subseteq V$ and ψ is Δ_0 hence absolute for transitive models. Suppose now that $V \models \phi$, so there exists a q such that $V \models \psi(q, p_1, \ldots, p_n)$. Let λ be large enough so that $q \in H(\lambda)$. By downward Löwenheim Skolem Theorem there exists an $M \prec H(\lambda)$ such that $q \in M$, $\mathrm{trcl}(p_i) \subseteq M$ for all $i < n$, and $|M| = \omega \cup \left|\bigcup_{i<n} \mathrm{trcl}(p_i)\right| < \kappa$. Let N be the Mostowski Collapse of M, with $\pi : M \to N$ corresponding isomorphism. Since $H(\lambda) \models \psi(q, p_1, \ldots, p_n)$, the same does M and $N \models \psi(\pi(q), p_1, \ldots, p_n)$. Since N is transitive of cardinality less than κ, $N \subseteq H(\kappa)$ so $\pi(q) \in H(\kappa)$ and $H(\kappa) \models \phi$. □

Lemma 4.2 (Cohen's Absoluteness)**.** *Let T be any theory extending* ZFC*, and ϕ be any Σ_1 formula with a parameter p such that $T \vdash p \subseteq \omega$. Then $T \vdash \phi(p)$ if and only if $T \vdash \exists \mathbb{P}\ (\mathbb{1}_{\mathbb{P}} \Vdash \phi(p))$.*

Proof. The left to right implication is trivial (choosing a poset like $\mathbb{P} = 2$). For the reverse implication, suppose that $V \vDash \exists \mathbb{P}\,(\mathbb{1}_{\mathbb{P}} \Vdash \phi(\check{p}))$, let $\mathbb{P}$ be any such poset and θ be such that $p, \mathbb{P} \in V_\theta$ and V_θ satisfies a finite fragment of T large enough to prove basic ZFC and $\mathbb{1}_{\mathbb{P}} \Vdash \phi(p)$. Let M, N be defined as in the previous lemma (considering p as the parameter, $\mathbb{P}$ as the variable), then $N \vDash \left(\mathbb{1}_{\mathbb{Q}} \Vdash \phi(p)\right)$ where $\mathbb{Q} = \pi(\mathbb{P})$. Let G be N-generic for $\mathbb{Q}$, so that $N[G] \vDash \phi(p)$. Since ϕ is Σ_1, ϕ is upward absolute for transitive models, hence $V \vDash \phi(p)$. The thesis follows by completeness of first-order logic. □

Cohen's Absoluteness Lemma can be generalized to the case $p \subseteq \kappa$ for any cardinal κ. However, to achieve that we need the following definition.

Definition 4.3. We write $\mathrm{FA}_\kappa(\mathbb{P})$ as an abbreviation for the sentence "for every $\mathscr{D} \subset \mathscr{P}(\mathbb{P})$ family of open dense sets of $\mathbb{P}$ with $|\mathscr{D}| \leq \kappa$, there exists a filter $G \subset \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathscr{D}$".

In an informal sense, assuming the forcing axiom for a broad class of posets suggests that a number of different forcing has already been done in our model of set theory. This intuitive insight is reflected into the following equivalence.

Theorem 4.4. *Let $\mathbb{P}$ be a poset and $\theta > 2^{|\mathbb{P}|}$ be a cardinal. Then* $\mathrm{FA}_\kappa(\mathbb{P})$ *holds iff there exists an $M \prec H(\theta)$, $|M| = \kappa$, $\mathbb{P} \in M$, $\kappa \subset M$ and a G filter M-generic for $\mathbb{P}$.*

Proof. First, suppose that $\mathrm{FA}_\kappa(\mathbb{P})$ holds and let $M \prec H(\theta)$ be such that $\mathbb{P} \in M$, $\kappa \subset M$, $|M| = \kappa$. There are at most κ dense subsets of $\mathbb{P}$ in M, hence by $\mathrm{FA}_\kappa(\mathbb{P})$ there is a filter G meeting all those sets. However, G might not be M-generic since for some $D \in M$, the intersection $G \cap D$ might be disjoint from M. Define:

$$N = \left\{x \in H(\theta) :\ \exists \tau \in M \cap V^{\mathbb{P}}\ \exists q \in G\ \ (q \Vdash \tau = \check{x})\right\}$$

Clearly, N cointains M (hence contains κ), and the cardinality $|N| \leq \left|M \cap V^{\mathbb{P}}\right| = \kappa$ since every τ can be evaluated in an unique way by the elements of the filter G. To prove that $N \prec H(\theta)$, let $\exists x \phi(x, a_1, \ldots, a_n)$ be any formula with parameters $a_1, \ldots, a_n \in N$ which holds in V. Let $\tau_i \in M^{\mathbb{P}}$, $q_i \in G$ be such that $q_i \Vdash \tau_i = \check{a}_i$ for all $i < n$. Define $Q_\phi = \{p \in \mathbb{P} :\ p \Vdash \exists x \in V\ \phi(x, \tau_1, \ldots, \tau_n)\}$, this set is definable in M hence $Q_\phi \in M$. Furthermore, $Q_\phi \cap G$ is not empty since it contains any $q \in G$ below all q_i. By fullness in $H(\theta)$, we have that:

$$\begin{aligned} &H(\theta) \vDash \forall p \in Q_\phi\ p \Vdash \exists x \in V\ \phi(x, \tau_1, \ldots, \tau_n) \Rightarrow \\ &H(\theta) \vDash \exists \tau\ \forall p \in Q_\phi\ p \Vdash \tau \in V \wedge \phi(\tau, \tau_1, \ldots, \tau_n) \Rightarrow \\ &M \vDash \exists \tau\ \forall p \in Q_\phi\ p \Vdash \tau \in V \wedge \phi(\tau, \tau_1, \ldots, \tau_n) \end{aligned}$$

Fix such a τ, by elementarity the last formula holds also in $H(\theta)$ and in particular for $q \in Q_\phi$. Since the set $\{p \in \mathbb{P} :\ \exists x \in H(\theta)\ p \Vdash \check{x} = \tau\}$ is an open dense set definable in M, there is a $q' \in G$ below q belonging to this dense set, and an $a \in H(\theta)$ such that $q' \Vdash \tau = \check{a}$. Then q', τ testify that $a \in N$ hence the original formula $\exists x \phi(x, a_1, \dots, a_n)$ holds in N.

Finally, we need to check that G is N-generic for $\mathbb{P}$. Let $D \in N$ be a dense subset of $\mathbb{P}$, and $\dot{D} \in M$ be such that $\mathbb{1}_{\mathbb{P}} \Vdash \dot{D}$ is dense $\wedge \dot{D} \in V$ and for some $q \in G$, $q \Vdash \dot{D} = D$. Since $\mathbb{1}_{\mathbb{P}} \Vdash \dot{D} \cap \dot{G} \neq \emptyset$, by fullness lemma there exists a $\tau \in H(\theta)$ such that $\mathbb{1}_{\mathbb{P}} \Vdash \tau \in \dot{D} \cap \dot{G}$, and by elementarity there is such a τ also in M. Let $q' \in G$ below q be deciding the value of τ, $q' \Vdash \tau = \check{p}$. Since q' forces that $\check{p} \in \dot{G}$, it must be $q' \leq p$ so that $p \in G$ hence $p \in G \cap D \cap N$ is not empty.

For the converse implication, let M, G be as in the hypothesis of the theorem, and fix a collection $\mathscr{D} = \langle D_\alpha : \alpha < \kappa \rangle$ of dense subsets of $\mathbb{P}$. Define:

$$S = \left\{ N \prec H(|\mathbb{P}|^+) : \ \kappa \subset N \ \wedge \ |N| = \kappa \ \wedge \ \exists G \text{ filter } N\text{-generic} \right\}$$

Note that S is definable in M then $S \in M$. Furthermore, since $\mathbb{P} \in M$ so is $H(|\mathbb{P}|^+)$ hence $M \cap H(|\mathbb{P}|^+) \prec H(|\mathbb{P}|^+)$ and $M \cap H(|\mathbb{P}|^+)$ is in S. Given any $C_f \in M$ club on $H(|\mathbb{P}|^+)$, since $f \in M$ we have that $M \cap H(|\mathbb{P}|^+) \in C_f$. Then $V \vDash S \cap C_f \neq \emptyset$ and by elementarity the same holds for M. Thus, S is stationary in M and again by elementarity S is stationary also in V.

Let $N \in S$ be such that $\mathscr{D} \in N$. Since $\kappa \subset N$ and $\mathscr{D}$ has size κ, $D_\alpha \in N$ for every $\alpha < \kappa$. Thus, the N-generic filter G will meet all dense sets in $\mathscr{D}$, verifying $\mathrm{FA}_\kappa(\mathbb{P})$ for this collection. □

Corollary 4.5. *Let $\mathbb{P}$ be a poset with $\mathscr{P}(\mathbb{P}) \in H(\theta)$. Then $\mathrm{FA}_\kappa(\mathbb{P})$ holds if and only if there are stationary many $M \prec H(\theta)$ such that $|M| = \kappa$, $\mathbb{P} \in M$, $\kappa \subset M$ and a G filter M-generic for $\mathbb{P}$.*

Proof. The forward implication has already been proved in the first part of the proof of the previous Theorem 4.4. The converse implication directly follows from the same theorem. □

Lemma 4.6 (Generalized Cohen's Absoluteness)**.** *Let T be any theory extending* ZFC*, κ be a cardinal, ϕ be a Σ_1 formula with a parameter p such that $T \vdash p \subseteq \kappa$. Then $T \vdash \phi(p)$ if and only if $T \vdash \exists \mathbb{P} \ \left(\mathbb{1}_{\mathbb{P}} \Vdash \phi(p) \ \wedge \ \mathrm{FA}_\kappa(\mathbb{P})\right)$.*

Proof. The forward implication is trivial; the converse implication follows the proof of Lemma 4.2. Given p, $\mathbb{P}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash \phi(p)$ and $\mathrm{FA}_\kappa(\mathbb{P})$ holds, by Corollary 4.5 let $M \prec H(\theta)$ be such that $|M| = \kappa$, $\mathbb{P} \in M$, $\kappa \subset M$ and there exists a G filter M-generic for $\mathbb{P}$. Since there are stationary many such M, we can assume that $p \in M$. Let $\pi : M \to N$ be the transitive collapse map of M, then $H = \pi[G]$ is N-generic for $\mathbb{Q} = \pi[\mathbb{P}]$ and $p \subseteq \kappa \subseteq M$ is not moved by π so that $N[H] \vDash \phi(p)$. Since ϕ is Σ_1, ϕ is upward absolute for transitive models, hence $V \vDash \phi(p)$. □

It is now clear how the forcing axiom makes forcing a strong tool for proving theorems. For $\kappa = \omega_1$, the forcing axiom $\mathrm{FA}_{\omega_1}(\mathbb{P})$ is widely studied for many different poset $\mathbb{P}$. In particular, for the classes of posets:

$$\text{c.c.c.} \subset \text{proper} \subset \text{semiproper} \subset \text{locally s.s.p.}$$

the forcing axiom is called respectively MA (Martin's Axiom), PFA (Proper Forcing Axiom), SPFA (Semiproper Forcing Axiom), MM (Martin's Maximum). In this notes we will be mostly interested in the latter.

Definition 4.7. A poset $\mathbb{P}$ is c.c.c. iff every antichain in $\mathbb{P}$ is countable.

Definition 4.8. A poset $\mathbb{P}$ is proper iff for every θ regular cardinal such that $\mathscr{P}(\mathbb{P}) \in H(\theta)$, countable elementary substructure $M \prec H(\theta)$ and $p \in \mathbb{P} \cap M$, there is a condition $q \leq p$ that is M-generic (i.e., for every $D \in M$ dense subset of $\mathbb{P}$ and $r \leq q$, r is compatible with an element of $D \cap M$).

Equivalently, a poset $\mathbb{P}$ is proper iff it preserves stationary sets on $[\lambda]^\omega$ for any λ uncountable cardinal.

Definition 4.9. A poset $\mathbb{P}$ is semiproper iff for every θ regular cardinal such that $\mathscr{P}(\mathbb{P}) \in H(\theta)$, countable elementary substructure $M \prec H(\theta)$ and $p \in \mathbb{P} \cap M$, there is a condition $q \leq p$ that is M-semigeneric (i.e., for every $\dot{\alpha} \in M$ name for a countable ordinal, $q \Vdash \exists \beta \in M\ \check{\beta} = \dot{\alpha}$).

Under SPFA every s.s.p. poset is semiproper and viceversa, hence SPFA is equivalent to MM.

Definition 4.10. A poset $\mathbb{P}$ is *stationary set preserving* (in short, s.s.p.) iff for every stationary set $S \subseteq \omega_1$, $\mathbb{1}_{\mathbb{P}} \Vdash \forall x \subseteq \check{\omega}_1 (x \text{ club} \Rightarrow x \cap \check{S} \neq \emptyset)$.

Definition 4.11. A poset $\mathbb{P}$ is *locally* s.s.p. iff there exists a $p \in \mathbb{P}$ such that $\mathbb{P} \upharpoonright p = \{q \in \mathbb{P} : q \leq p\}$ is an s.s.p. poset.

The class of locally s.s.p. posets play a special role in the development of forcing axioms: MM is the strongest possible form of forcing axiom for ω_1. This is the case as shown by the following theorem.

Theorem 4.12 (Shelah)**.** *If $\mathbb{P}$ is not locally s.s.p. then* $\mathrm{FA}_{\omega_1}(\mathbb{P})$ *is false.*

Proof. Given $\mathbb{P}$ that is not locally s.s.p. let S be a stationary set on ω_1 and $\dot{C} \in V^{\mathbb{P}}$ be such that $\mathbb{1}_{\mathbb{P}} \Vdash \dot{C} \subseteq \check{\omega}_1$ club, $\mathbb{1}_{\mathbb{P}} \Vdash \check{S} \cap \dot{C} = \check{\emptyset}$. Define:

$$\begin{aligned} D_\alpha &= \{p \in \mathbb{P} : p \Vdash \check{\alpha} \in \dot{C} \vee p \Vdash \check{\alpha} \notin \dot{C}\} \\ E_\beta &= \{p \in \mathbb{P} : p \Vdash \check{\beta} \notin \dot{C} \Rightarrow \exists \gamma < \beta\ p \Vdash \dot{C} \cap \check{\beta} \subseteq \check{\gamma}\} \\ F_\gamma &= \{p \in \mathbb{P} : \exists \alpha > \gamma\ p \Vdash \check{\alpha} \in \dot{C}\} \end{aligned}$$

Those sets are dense by the forcing theorem, since $\dot{C}$ is forced to be a club and the above formulas are true for clubs (hence forced by a dense set of conditions). Suppose by contradiction that $\mathrm{FA}_{\omega_1}(\mathbb{P})$ holds, and let G be a filter that intersects all the D_α, E_β, F_γ. Then the set $C = \left\{\alpha < \omega_1 : \exists p \in G \; p \Vdash \alpha \in \dot{C}\right\}$ is a club in V, so there is a $\beta \in S \cap C$. By definition of C, there exists a condition $q \in G$ such that $q \Vdash \beta \in \dot{C}$, and $\beta \in S \Rightarrow q \Vdash \beta \in \check{S} \cap \dot{C} \neq \check{\emptyset}$, a contradiction. □

5 More on Forcing Axioms

In this section we will state a few interesting results without proof, not directly involved in the development of MM and SRP. Reference texts for this section are [4], [5]. Cohen's Absoluteness Lemma 4.2 is a valuable result, but is limiting in two aspects. First, it involves only Σ_1 formulas, and second, forces the parameter to be a subset of ω (or of larger cardinals, assuming stronger and stronger versions of forcing axioms). The following Woodin's Absoluteness Lemma, with an additional assumption on large cardinals, enhances Cohen's result to any formula relativized to $L(\mathbb{R})$.

Theorem 5.1 (Woodin's Absoluteness)**.** *Let T be a theory extending* ZFC + *there are class many Woodin cardinals. Let ϕ be any formula with a parameter p such that $T \vdash p \subseteq \omega$. Then $T \vdash \phi(p)^{L(\mathbb{R})}$ if and only if $T \vdash \exists \mathbb{P}\,(\mathbb{1}_\mathbb{P} \Vdash \phi(\check{p})^{L(\mathbb{R})})$.*

We would expect to generalize Woodin's result from $L(\mathbb{R}) = L(\mathscr{P}(\omega))$ to some bigger class by means of forcing axioms, as we did with Cohen's. This happens to be possible, at least for $L([\mathbf{ON}]^{<\omega_2})$, by a result of Viale. To state it we need to introduce some common variations of the forcing axiom.

Definition 5.2. We write $\mathrm{BFA}_\kappa(\mathscr{B})$ as an abbreviation for the sentence "for every $\mathscr{D} \subset [\mathscr{B}]^{\leq\kappa}$ family of predense sets of $\mathscr{B}$ with $|\mathscr{D}| \leq \kappa$, there exists a filter $G \subset \mathscr{B}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathscr{D}$". If $\mathbb{P}$ is a poset, we write $\mathrm{BFA}_\kappa(\mathbb{P})$ to mean $\mathrm{BFA}_\kappa(\mathscr{B})$ for $\mathscr{B}$ the regular open algebra of $\mathbb{P}$.

The bounded forcing axiom $\mathrm{BFA}_\kappa(\mathbb{P})$ can be used to define weaker versions of the usual forcing axioms: BMA, BPFA, BMM. Furthermore, $\mathrm{BFA}_\kappa(\mathbb{P})$ has an interesting equivalent formulation in terms of elementary substructures: namely, $\mathrm{BFA}_\kappa(\mathbb{P})$ holds if and only if $H(\kappa^+) \prec_1 V^\mathbb{P}$.

Definition 5.3. We write $\mathrm{FA}^{++}_{\omega_1}(\mathbb{P})$ as an abbreviation for the sentence "for every $\mathscr{D} \subset \mathscr{P}(\mathbb{P})$ family of open dense sets of $\mathbb{P}$ with $|\mathscr{D}| \leq \omega_1$, there exists a filter $G \subset \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathscr{D}$ and $\mathrm{val}_G(\dot{S})$ is stationary for every $\dot{S} \in V^\mathbb{P}$ such that $\mathbb{1}_\mathbb{P} \Vdash \dot{S} \subseteq \omega_1$ stationary".

The forcing axiom $\mathrm{FA}^{++}_{\omega_1}(\mathbb{P})$ can be used to define analogous versions of the usual forcing axioms: MA^{++}, PFA^{++}, MM^{++}. It is also possible to find an equivalent formulation of $\mathrm{FA}^{++}_{\omega_1}(\mathbb{P})$ similar to Theorem 4.4.

While MA^{++} is provably equivalent to MA, MM^{++} is an actual strengthening of MM. These axioms also have distinct consistency strengths: for example, BPFA and BSPFA^{++} are consistent relative to a reflecting cardinal, while BMM is consistent relative to ω-many Woodin cardinals, and MM^{++} is consistent relative to a supercompact cardinal.

Theorem 5.4. *Let $\mathbb{P}$ be a poset with $\mathscr{P}(\mathbb{P}) \in H(\theta)$. Then $\mathrm{FA}^{++}_{\omega_1}(\mathbb{P})$ holds if and only if there exists an $M \prec H(\theta)$, $|M| = \omega_1$, $\mathbb{P} \in M$, $\omega_1 \subset M$ and a G filter M-generic for $\mathbb{P}$ such that for every $\dot{S} \in V^{\mathbb{P}} \cap M$ name for a stationary subset of ω_1, $\mathrm{val}_G(\dot{S})$ is stationary.*

We are now ready to state the concluding results of this section, generalizations of Woodin's Absoluteness Lemma.

Theorem 5.5 (Viale)**.** *Let T be a theory extending* $\mathrm{ZFC} + \mathrm{MM}^{++} +$ *there are class many Woodin cardinals. Let ϕ be any Σ_2 formula with a parameter p such that $T \vdash p \in H(\omega_2)$. Then $T \vdash \phi(p)^{H(\omega_2)}$ iff $T \vdash \exists \mathbb{P} \in \mathrm{SSP}\ \mathbb{1}_{\mathbb{P}} \Vdash \left(\phi(\check{p})^{H(\omega_2)} \wedge \mathrm{BMM}\right)$.*

Theorem 5.6 (Viale)**.** *Let T be a theory extending* $\mathrm{ZFC} + \mathrm{MM}^{++} +$ *there are class many supercompact cardinals limit of supercompact cardinals. Let ϕ be any formula with a parameter p such that $T \vdash p \in H(\omega_2)$. Then $T \vdash \phi(p)^{L([\mathbf{ON}]^{<\omega_2})}$ if and only if $T \vdash \exists \mathbb{P} \in \mathrm{SSP}\ \mathbb{1}_{\mathbb{P}} \Vdash \left(\phi(\check{p})^{L([\mathbf{ON}]^{<\omega_2})} \wedge \mathrm{MM}^{++}\right)$.*

6 Strong Reflection Principle

In the study of the consequences of MM, there are certain statements that have been proved useful in isolating many of the characteristics of MM: among those, the most prominents are the strong reflection principle SRP, the open coloring axiom OCA, and the P-ideal dichotomy PID. Reference text for this section is [1, 5A]. In this section we shall state the first one, prove it under MM and examine its consequences. We first need the following definition.

Definition 6.1. A set $S \subseteq [X]^{\omega}$ is *projectively stationary* iff it is stationary, $\omega_1 \subseteq X$, and its restriction $S \downarrow \omega_1 = \{A \cap \omega_1 : A \in S\}$ contains a club on $[\omega_1]^{\omega}$.

The property of being projectively stationary will be mostly used by means of the following lemma.

Lemma 6.2. *Let $S \subseteq [X]^{\omega}$ be projectively stationary, and $T \subset \omega_1$ be stationary. Then $S \cap (T \uparrow X)$ is stationary.*

Proof. Given a club C on X, $S' = S \cap C$ is clearly projectively stationary. Let α be in $T \cap (S' \downarrow \omega_1)$, and $A \in S'$ such that $A \cap \omega_1 = \alpha$. Then $A \in S \cap (T \uparrow X) \cap C$. □

Definition 6.3. A stationary set $S \subseteq \mathscr{P}(X)$ *reflects on* Z iff $Z \subseteq X$ and $S \cap \mathscr{P}(Z)$ is stationary (notice that $S \downarrow Z$ is necessarily stationary while $S \cap \mathscr{P}(Z)$ may not). A stationary set $S \subseteq \mathscr{P}(X)$ *strongly reflects on* Z iff $S \cap [Z]^\omega$ contains a club on $[Z]^\omega$.

Definition 6.4. We call *strong reflection principle* on X and write SRP(X) as an abbreviation for the sentence "every projectively stationary set on $[X]^\omega$ strongly reflects on some $Z \supseteq \omega_1$ of size ω_1". We say *strong reflection principle* (and write SRP) to mean "SRP(X) for all $X \supseteq \omega_1$".

The reflection property can be restated in the following equivalent way.

Lemma 6.5. SRP(X) *holds iff for every projectively stationary* $S \subset [X]^\omega$ *there exists a continuous increasing function* $f : \omega_1 \to S$ *with* $\bigcup \operatorname{ran}(f) \supseteq \omega_1$.

Proof. First, suppose that SRP(X) holds and let $S \subset [X]^\omega$ be a projectively stationary set. Let $Z \supset \omega_1$ be such that S strongly reflects on Z. Fix an enumeration $\langle z_\alpha : \alpha < \omega_1 \rangle$ of Z, and let $Z_\alpha = \left\{ z_\beta : \beta < \alpha \right\}$. The set $C_1 = \{Z_\alpha : \alpha < \omega_1\}$ is a club on $[Z]^\omega$ (by a similar argument to the one for ω_1 club in Lemma 2.11). Since S strongly reflects on Z, $S \cap C_1 = \{Z_\alpha : Z_\alpha \in S\}$ contains a club C_2. Thus, the increasing enumeration of C_2 is a continuous increasing function $f : \omega_1 \to S$ with $\bigcup \operatorname{ran}(f) = Z \supseteq \omega_1$, as required.

Conversely, suppose there exists a function $f : \omega_1 \to S$ as above, and define $Z = \bigcup \operatorname{ran}(f)$. Then $S \cap [Z]^\omega$ contains ran(f) that is a club on $[Z]^\omega$ by the same argument as above. □

Notice that the requirement $\operatorname{ran}(f) \supseteq \omega_1$ prevents f to be eventually constant. To prove that SRP is a consequence of MM, we shall define a poset $\mathbb{P}_S$ that forces a projectively stationary set S to strongly reflect on some $Z \supseteq \omega_1$, and argue that this poset is s.s.p. for any S.

Definition 6.6. Given S a projectively stationary set, $\mathbb{P}_S$ is the poset of all the continuous increasing functions $f : \alpha + 1 \to S$ with $\alpha < \omega_1$ ordered by reverse inclusion.

Lemma 6.7. *The following sets are open dense in* $\mathbb{P}_S$ *for* $\alpha < \omega_1$, $a \in \bigcup S$*:*

$$\begin{aligned} D_\alpha &= \{f \in \mathbb{P}_S : \alpha \in \operatorname{dom}(f)\} \\ E_a &= \left\{f \in \mathbb{P}_S : a \in \bigcup \operatorname{ran}(f)\right\} \end{aligned}$$

Proof. For the first part, given any $f \in \mathbb{P}_S$, $f : \beta + 1 \to S$ define $g \in D_\alpha$ below f to be constant after β, i.e. $g(\gamma) = f(\beta)$ for every $\gamma \in \alpha + 1 \setminus \beta$, $g(\gamma) = f(\gamma)$ otherwise.

For the second part, given any $f \in \mathbb{P}_S$, $f: \beta+1 \to S$ let A be any set in the intersection of S with the club $C_{f(\beta)\cup\{a\}} = \left\{Y \subseteq X : f(\beta) \cup \{a\} \subseteq Y\right\}$. Then $g = f \cup \langle \beta+1, A\rangle \in E_a$ extends f and is in E_a. □

Lemma 6.8. *$\mathbb{P}_S$ is an s.s.p. poset.*

Proof. Let $T \subseteq \omega_1$ be a stationary set, and $\dot{C}$ be a $\mathbb{P}_S$-name for a club. Given any $p \in \mathbb{P}_S$, we need to find a $q \leq p$, $\delta \in T$ such that $q \Vdash \check{\delta} \in \dot{C}$.

Let M be a countable elementary submodel of $H(\theta)$ such that $p, S, T, \dot{C} \in M$ and $M \cap \bigcup S \in S$, $M \cap \omega_1 = \delta \in T$ (such an M exists by Lemma 6.2 and lifting). Fix an enumeration $\langle A_n : n < \omega\rangle$ of the $\mathbb{P}_S$-dense sets in M, and define a sequence p_n such that $p_0 = p$, $p_{n+1} \in A_n$ and $p_{n+1} \leq p_n$. Then $p_\omega = \bigcup_{n<\omega} p_n$ is a function from δ to S, since p_ω is below all D_α as in Lemma 6.7 for $\alpha \in M \cap \omega_1 = \delta$. Furthermore, $\bigcup p_\omega[\delta] = M \cap \bigcup S$, since p_ω is below all E_a as in Lemma 6.7 for $a \in M \cap \bigcup S$. Then $q = p_\omega \cup \langle \delta, M \cap \bigcup S\rangle$ is continuous, hence $q \in \mathbb{P}_S$. Moreover, $q \Vdash \check{\delta} \in \dot{C}$: given any generic filter G containing q, G is generic also for M hence $M[G] \vDash \mathrm{val}_G(\dot{C})$ club on ω_1, but $M[G] \cap \omega_1 = \delta$ so $\mathrm{val}_G(\dot{C}) \cap \delta$ is unbounded and $\delta \in \mathrm{val}_G(\dot{C})$. This holds for any $G \ni q$ hence $q \Vdash \check{\delta} \in \dot{C}$, $\delta \in T$. □

Theorem 6.9 (Todorcevic)**.** MM $\Rightarrow$ SRP.

Proof. Let S be a projectively stationary set, and $\mathbb{P}_S$ be defined as in Lemma 6.8. For every $\alpha < \omega_1$, D_α, E_α are open dense sets by Lemma 6.7. From Lemma 6.8 we know that $\mathbb{P}_S$ is s.s.p., so using MM we get a filter G meeting all D_α, E_α for $\alpha < \omega_1$. Define $g = \bigcup G : \omega_1 \to S$, then g is a continuous increasing function with $\bigcup \mathrm{ran}(g) \supseteq \omega_1$ hence by Lemma 6.5, SRP holds. □

The strong reflection principle has a number of interesting consequences. The most known is the following result on cardinal arithmetic.

Theorem 6.10. *Assume* SRP(κ) *with κ regular cardinal. Then $\kappa^{\omega_1} = \kappa^\omega = \kappa$.*

Proof. Let $\langle E_\alpha : \alpha < \kappa\rangle$ be a partition of $\{\alpha \in \kappa : \mathrm{cf}(\alpha) = \omega\}$ into stationary sets by Ulam Theorem 2.16. Similarly, let $\langle D_\alpha : \alpha < \omega_1\rangle$ be a partition of $\omega_1 \setminus \{0\}$ into stationary sets such that $\min D_\alpha > \alpha$. To accomplish this, from $\langle B_\alpha : \alpha < \omega_1\rangle$ partition of ω_1 into stationary sets define $A_\alpha = B_\alpha \setminus \alpha+1$, $A_0 = (\omega_1 \setminus \{0\}) \setminus \bigcup_{0<\alpha<\omega_1} A_\alpha)$. Given $f : \omega_1 \to \kappa$, define $S_f = \left\{X \in [\kappa]^\omega : \forall \alpha\; X \cap \omega_1 \in D_\alpha \Leftrightarrow \sup(X) \in E_{f(\alpha)}\right\}$.

Lemma 6.10.1. *S_f is projectively stationary for any f.*

Proof of Lemma. Let $A \subseteq \omega_1$ be stationary, and C_g be the club corresponding to the function $g : \kappa^{<\omega} \to \kappa$. We shall define an $X \in S_f \cap C_g \cap (A \uparrow \kappa)$ that testifies the projective stationarity of S_f. Let $h : A \setminus \{0\} \to \omega_1$ be defined by $h(\alpha) = \beta$

iff $\alpha \in D_\beta$. Since $\min(D_\beta) > \beta$, h is a regressive function on the stationary set $A \setminus \{0\}$. By Fodor's Lemma 2.9 let γ be such that $f^{-1}\left[\{\gamma\}\right] = A \cap D_\gamma$ is stationary.

Let $\langle M_\alpha :\ \alpha < \kappa \rangle$ be a continuous strictly increasing sequence of elementary substructures of $H(\theta)$ (for some large θ) of size less than κ, such that $g \in M_0$, $M_\alpha \in M_{\alpha+1}$, $\alpha \subset M_{\alpha+1}$. Since $M_\alpha \cap \kappa$ is an ordinal in club many $\alpha < \kappa$, by restricting to a subsequence we can assume that $M_\alpha \cap \kappa$ is an ordinal for all $\alpha < \kappa$.

Then $C_1 = \{M_\alpha \cap \kappa :\ \alpha < \kappa\}$ is a club subset of κ, so there is a $\delta \in E_{f(\gamma)} \cap C_1$, hence a structure M_ξ such that $M_\xi \cap \kappa = \delta \in E_{f(\gamma)}$. Since δ is in $E_{f(\gamma)}$, $\operatorname{cf}(\delta) = \omega$ and we can define an increasing sequence $\langle \delta_i :\ i < \omega \rangle$ converging to δ.

Let $\langle N_\alpha :\ \alpha < \omega_1 \rangle$ be defined by letting $N_\alpha \in C_g$ be the closure under g of the set $\{\delta_i :\ i < \omega\} \cup \alpha$. Since this set is a subset of M_ξ and g is in M_ξ (that is closed under g), for all α the set N_α is a subset of M_ξ hence $\sup(N_\alpha) = M_\xi \cap \kappa = \delta \in E_{f(\gamma)}$. Furthermore, the set $C_2 = \{\alpha < \omega_1 :\ N_\alpha \cap \omega_1 = \alpha\}$ is a club: closed by continuity of the sequence, and unbounded since given α_0 we can define $\alpha_{i+1} = \sup(N_{\alpha_i} \cap \omega_1)$ so that $\alpha_\omega = \sup_{i<\omega} \alpha_i \in C_2$.

Thus, there exists a β in the intersection of C_2 with the stationary set $A \cap D_\gamma$. The corresponding N_β will be such that $N_\beta \cap \omega_1 = \beta \in A \cap D_\gamma$, and $N_\beta \in C_g$, $\sup(N_\beta) = \delta \in E_{f(\gamma)}$. So N_β is in S_f, completing the proof of Lemma 6.10.1. □

Claim 6.10.2. *Given $f, g :\ \omega_1 \to \kappa$, if there exists $h_f : \omega_1 \to S_f$, $h_g : \omega_1 \to S_g$ continuous increasing functions such that* $\bigcup \operatorname{ran}(h_f) \supseteq \omega_1$, $\bigcup \operatorname{ran}(h_g) \supseteq \omega_1$ *and* $\sup\left(\bigcup \operatorname{ran}(h_f)\right) = \sup\left(\bigcup \operatorname{ran}(h_g)\right)$, *then* $f = g$.

Proof of Claim. Note that by Lemma 6.5 functions h_f, h_g satisfying all but the last condition exist. Let $C_1 = \left\{\alpha < \omega_1 :\ h_f(\alpha) \cap \omega_1 = h_g(\alpha) \cap \omega_1 = \alpha\right\}$ be a club.

Define $\delta_f^\alpha = \sup\left(h_f(\alpha)\right)$, $\delta = \sup_{\alpha<\omega_1} \delta_f^\alpha$. Given any $\alpha \in D_\xi \cap C_1$ (for some ξ), there exists a $\beta > \alpha$ with $\beta \in D_\zeta \cap C_1$ (for some $\zeta \neq \xi$), so by definition of S_f we have that $\delta_f^\alpha \in E_{f(\xi)}$, $\delta_f^\beta \in E_{f(\zeta)}$ and $\delta_f^\alpha \neq \delta_f^\beta$ (since $E_{f(\xi)} \cap E_{f(\zeta)} = \emptyset$). Then, the sequence $\langle \delta_f^\alpha :\ \alpha < \omega_1 \rangle$ is continuously increasing and not eventually constant, so the limit δ has cofinality ω_1 and the sequence $\langle \delta_f^\alpha :\ \alpha < \omega_1 \rangle$ is club on δ.

The same argument holds for $\langle \delta_g^\alpha :\ \alpha < \omega_1 \rangle$, $\delta = \sup_{\alpha<\omega_1} \delta_g^\alpha$ (by hypothesis) and $C_2 = \left\{\alpha < \omega_1 :\ \delta_f^\alpha = \delta_g^\alpha\right\} \cap C_1$ is a club: closed by continuity, unbounded since given any $\alpha_0 < \omega_1$ we can define $\alpha_{2i+1} = \min\left\{\beta \in C_1 :\ \delta_f^\beta \geq \delta_g^{\alpha_{2i}}\right\}$, and $\alpha_{2i+2} = \min\left\{\beta \in C_1 :\ \delta_g^\beta \geq \delta_f^{\alpha_{2i+1}}\right\}$, so that $\alpha_\omega = \sup_{i<\omega} \alpha_i$ is in C_2.

Suppose by contradiction that $f \neq g$, and let β be such that $f(\beta) \neq g(\beta)$, and $\gamma \in C_2 \cap D_\beta$. Then $f(\gamma) \cap \omega_1 = \gamma \in D_\beta$, $f(\gamma) \in S_f$ implies that $\delta_f^\gamma \in E_{f(\beta)}$. The same argument for g implies that $\delta_g^\gamma \in E_{g(\beta)}$, but $\delta_f^\gamma = \delta_g^\gamma$ and $E_{f(\beta)}$ is disjoint from $E_{g(\beta)}$, a contradiction. □

Proof of Theorem 6.10. Define a map $\pi :\ {}^{\omega_1}\kappa \to \kappa$ to be $\pi(f) = \delta$ for δ least such that $\delta = \sup\left(\bigcup \operatorname{ran}(h_f)\right)$ for some continuous increasing $h_f : \omega_1 \to S_f$. By Claim 6.10.2, π is well-defined and injective so $|\kappa| \geq |{}^{\omega_1}\kappa|$ hence $\kappa^{\omega_1} = \kappa$. □

Corollary 6.11. $\text{MM} \Rightarrow 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.

Proof. Since MM implies MA_{ω_1}, we know that $2^{\aleph_0} \geq \aleph_2$. But MM also implies $\text{SRP}(\omega_2)$, then $2^{\aleph_0} \leq \aleph_2^{\aleph_0} = \aleph_2$. Similarly, $2^{\aleph_1} \leq \aleph_2^{\aleph_1} = \aleph_2$ hence $2^{\aleph_1} = \aleph_2$. □

Remark 6.12. The purpose of cardinal arithmetic is to determine the value of λ^κ. Assuming MM we can determine the result at least for $\kappa \leq \aleph_2$ with κ regular: in this case, $\lambda^\kappa = \max(\lambda, \aleph_2)$. Unfortunately, the consequences of MM in cardinal arithmetic for regular cardinals stop there (for example, the value of $\aleph_0^{\aleph_2}$ can be changed by forcing). However, MM implies the singular cardinal hypothesis SCH. Our proof actually shows that assuming SRP $\lambda^\kappa = \lambda^+ + 2^\kappa$ for all $\lambda \geq \kappa \geq \operatorname{cf}(\lambda)$.

The following corollary gives us an interesting example of projectively stationary set.

Corollary 6.13. *Let S be a stationary set on κ restricted to cofinality ω. Then $E(S) = \{X \in [\kappa]^\omega : \sup(X) \in S\}$ is projectively stationary.*

Proof. The proof mimics the one of Lemma 6.10.1. Let A, C_g, $\langle M_\alpha : \alpha < \kappa \rangle$, C' be defined as in the lemma above. Since C' is a club, we can find a $\delta \in S \cap C'$, hence a structure M_ξ such that $M_\xi \cap \kappa = \delta \in S$ so that $\operatorname{cf}(\delta) = \omega$. Let $\langle \delta_i : i < \omega \rangle$, $\langle N_\alpha : \alpha < \omega_1 \rangle$, C'' be defined as in Lemma 6.10.1. Recall that for all α the set N_α is a subset of M_ξ in C_g such that $\sup(N_\alpha) = M_\xi \cap \kappa = \delta \in S$ (i.e., $N_\alpha \in E(S)$). Since C'' is club, let β be in $C'' \cap S$: the corresponding N_β is in $E(S) \cap C_g \cap (A \uparrow \kappa)$. □

The last consequence of SRP that we shall examine is the following Theorem 6.15 about the structure of NS_{ω_1}.

Definition 6.14. An ideal I on κ is *saturated* iff $\mathscr{P}(\kappa)/I$ is a κ^+-c.c. poset.

Theorem 6.15. $\text{SRP}(\omega_2) \Rightarrow \text{NS}_{\omega_1}$ *saturated.*

Proof. $\text{SRP}(\omega_2)$ implies that $\omega_2^{\omega_1} = \omega_2$ hence also $|\mathscr{P}(\omega_1)| = 2^{\omega_1} = \omega_2$, so that NS_{ω_1} is necessarily ω_3-cc. Assume by contradiction that NS_{ω_1} is not saturated, then there exists a maximal antichain $\mathscr{A} = \langle A_\alpha : \alpha < \omega_2 \rangle$ in $\mathscr{P}(\omega_1)/\text{NS}_{\omega_1}$. Define $S = \{X \in [\omega_2]^\omega : \exists \delta \in X\ X \cap \omega_1 \in A_\delta\}$. We claim that S is projectively stationary.

Given any stationary $T \subseteq \omega_1$, and $g : \omega_2^{<\omega} \to \omega_2$ with corresponding club C_g, we need to find an $X \in S \cap C_g$ (to prove the stationarity) such that $X \cap \omega_1 \in T$ (to prove the projective stationarity). By maximality of $\mathscr{A}$, let $\alpha < \omega_2$ be such that T is compatible with A_α (i.e., $T \cap A_\alpha$ is stationary). Let $\langle M_\beta : \beta < \omega_1 \rangle$ be a continuous strictly increasing sequence of countable elementary substructures of $H(\omega_3)$

such that $\mathscr{A}, T, \alpha, g \in M_0$ and $\beta \in M_{\beta+1}$. Then $C = \left\{\beta < \omega_1 : M_\beta \cap \omega_1 = \beta\right\}$ is a club: closed by continuity of $\langle M_\beta : \beta < \omega_1 \rangle$, unbounded since for any β_0 in ω_1 if $\beta_{i+1} = \sup(M_{\beta_i} \cap \omega_1)$, then $M_{\beta_\omega} \cap \omega_1 = \beta_\omega$ for $\beta_\omega = \sup_{i<\omega} \beta_i$. Let ξ be in $T \cap A_\alpha \cap C$, then $M_\xi \in C_g$ since $g \in M_\xi$. Furthermore, $M_\xi \cap \omega_2 \in S$ since $M_\xi \cap \omega_1 = \xi \in A_\alpha \cap T$ (this proves also the projectivity) and $\alpha \in M_\xi$. This completes the proof that S is projectively stationary.

Since S is projectively stationary on ω_2 and $\mathrm{SRP}(\omega_2)$ holds, there is a $Z \supseteq \omega_1$ of size ω_1 such that $S \cap [Z]^\omega$ is club. Let β be in $\omega_2 \setminus Z$, and define $T = S \cap \left(A_\beta \uparrow Z\right)$ stationary set on Z. Let $g : T \to Z$ be defined by $g(X) = \delta$ for a δ as in the definition of S (i.e., such that $X \cap \omega_1 \in A_\delta$ and $\delta \in X$). The function g is regressive on the stationary set T, then by Fodor's Lemma 2.9 there exists a fixed $\gamma \in Z$ (hence $\gamma \neq \beta$) such that $T' = g^{-1}[\gamma]$ is a stationary subset of T. Since $T' = \left\{X \in [Z]^\omega : \gamma \in X \wedge X \cap \omega_1 \in A_\gamma \cap A_\beta\right\}$, $T' \downarrow \omega_1$ is a stationary subset of $A_\gamma \cap A_\beta$, contradicting that $\mathscr{A}$ is an antichain. □

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