Generic absoluteness, strong forcing axioms, MM^{+++}

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This talk present researches motivated by two broad questions:

QUESTIONS:

1. Why forcing is such an efficient semantic for ZFC?
2. Why forcing axioms are so effective in settling most of the problems of set theory or of mathematics which are undecidable on the basis of ZFC alone?

FORCING

Forcing was introduced in 1963 by Paul Cohen to prove the independence with respect to ZFC of the continuum hypothesis, the first in the list of 23 Hilbert's problems.
It soon emerged that forcing is a very powerful tool to prove independence results in ZFC and also in many other branches of pure mathematics, for example:

- Group theory (Whitehead's problem) Shelah 1974,
- General topology: Todorčević and Moore's results on the S-space and the L-space problems,
- Functional analysis: many results of Todorčević on Banach spaces,
- Operator algebras: Farah's works on the automorphisms of the Calkin algebra which develops on Shelah and Veličković's analysis of the automorphism group of $P(\omega)/\mathrm{FIN}$,
- ....

I'm surely forgetting many fundamental results and contributions, hopefully by people not in this room.

FORCING

Forcing can be seen as a "computable" function:

$$(M, \mathbb{B}) \mapsto M^{\mathbb{B}}$$

- M is a (transitive) model of (a large enough fragment of) ZFC.
- $\mathbb{B} \in M$ is a non-atomic complete boolean algebra which M models to be complete *(being a non-atomic boolean algebra is absolute for transitive models while being complete is not)*.
- $M^{\mathbb{B}}$ is a boolean valued model of ZFC and is a definable class in M.
- There is a definable in M injective map $\check{i}_{\mathbb{B}} : M \to M^{\mathbb{B}}$

  $$\check{i}_{\mathbb{B}} : M \to M^{\mathbb{B}}$$
  $$a \mapsto \check{a}$$

  such that M will be naturally identified with $\check{i}_{\mathbb{B}}[M] \subset M^{\mathbb{B}}$.
- Truth in $M^{\mathbb{B}}$ depends on
  - The first order theory of M.
  - The combinatorial properties that M gives to $\mathbb{B}$.
- Truth in $M^{\mathbb{B}}$ is (almost) definable in M with parameter $\mathbb{B}$.

The forcing relation

To evaluate the semantics of $M^{\mathbb{B}}$ one introduces a (definable in M) boolean evaluation of formulas:

Definition

Given

- $\tau_1, \dots, \tau_n \in M^{\mathbb{B}}$,
- $\phi(x_1, \dots, x_n)$ formula in the language of set theory in the free variables $x_1, \dots, x_n$,

a boolean value $[\![\phi(\tau_1, \dots, \tau_n)]\!]_{\mathbb{B}}$ in $\mathbb{B}$ is assigned as follows:

- $[\![\phi \wedge \psi]\!]_{\mathbb{B}} = [\![\phi]\!]_{\mathbb{B}} \wedge_{\mathbb{B}} [\![\psi]\!]_{\mathbb{B}}$,
- $[\![\neg\phi]\!]_{\mathbb{B}} = \neg_{\mathbb{B}} [\![\phi]\!]_{\mathbb{B}}$,
- $[\![\forall x \phi(x)]\!]_{\mathbb{B}} = \bigwedge_{\mathbb{B}} \{[\![\phi(\tau)]\!]_{\mathbb{B}} : \tau \in M^{\mathbb{B}}\}$.

What is difficult is to define $[\![\tau_1 \in \tau_2]\!]_{\mathbb{B}}$ and $[\![\tau_1 = \tau_2]\!]_{\mathbb{B}}$.

What holds in $M^{\mathbb{B}}$

Theorem (Cohen)

Assume M is a (transitive) model of ZFC *and* $\mathbb{B} \in M$ *is a non atomic complete boolean algebra in M. Then*

$$M \models ([\![\phi]\!]_{\mathbb{B}} = 1_{\mathbb{B}})$$

for any axiom ϕ *of* ZFC.

What else holds in $M^{\mathbb{B}}$ depends on the choice of $\mathbb{B}$ and the first order properties of M.

Baire's category theorem and forcing

Given a boolean algebra $\mathbb{B}$, $\mathbb{B}^+$ is the set of its positive elements (i.e. $\mathbb{B}^+ = \mathbb{B} \setminus \{0_{\mathbb{B}}\}$).
$G \subset \mathbb{B}^+$ is a ultrafilter on $\mathbb{B}$ if:

- for all $p \in G$ and $q \geq_{\mathbb{B}} p$, $q \in G$,
- $1_{\mathbb{B}} \in G$,
- for all $p, q \in G$, $p \wedge_{\mathbb{B}} q \in G$,
- for all $p \in \mathbb{B}^+$, $p \in G$ or $\neg_{\mathbb{B}} p \in G$.

$D \subset \mathbb{B}^+$ is dense in $\mathbb{B}$ if for all $p \in \mathbb{B}$, there is $q \in D$ such that $q \leq p$.

$A \subset \mathbb{B}^+$ is open in $\mathbb{B}$ if whenever $p \in A$ and $q \leq p$, $q \in A$ as well.

Baire's category theorem and forcing

$\mathsf{FA}_\lambda(\mathbb{B})$ holds if for all family $\{D_\xi : \xi < \lambda\}$ of dense open subsets of $\mathbb{B}$, there is a filter G which meets all these dense sets.

Theorem (Baire's category theorem)

For all non atomic complete boolean algebras $\mathbb{B}$, $\mathsf{FA}_{\aleph_0}(\mathbb{B})$ holds.

Towards Cohen's forcing Theorem

From now on, we assume V exists and is the "true" universe of sets. Else (if one does not want to be platonist) one has to reformulate everything with more care.

Let $(M, \in) \in V$ be a model of a large enough fragment of ZFC.

Typically:

- $M \prec H_\lambda$ or $M \prec V_\alpha$ for some large enough regular cardinal λ or some big enough ordinal α,
- $M = \pi_N[N]$ where $N \prec H_\lambda$ ($N \prec V_\alpha$) and π_N is the transitive collapse of the structure $(N, \in \cap N^2)$.

Let $\mathbb{B} \in M$ be such that M models $\mathbb{B}$ is a boolean algebra.

Definition

An ultrafilter $G \subset \mathbb{B}$ is an M-generic filter for $\mathbb{B}$ if

$G \cap D \cap M$ is non-empty for all $D \in M$ dense open subset of $\mathbb{B}$.

Cohen's forcing Theorem I

Theorem (Los Theorem for boolean valued models)

Assume:

- (M, E) *is any model of* ZFC
- $(M, E) \models \mathbb{B}$ *is a boolean algebra,*
- G *is an ultrafilter on* $\mathbb{B}$.

Then we can define the quotient structure $M^{\mathbb{B}}/G$ *letting*

$$[\tau]_G R_G [\sigma]_G$$

if and only if $[\![\tau R \sigma]\!]_{\mathbb{B}} \in G$ *and we get that*

$$(M^{\mathbb{B}}/G, E_G) \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$$

if and only if

$$[\![\phi(\tau_1, \ldots, \tau_n)]\!]_{\mathbb{B}} \in G.$$

Separating Baire's category theorem from consistency issues

Given any statement ϕ,

for the purpose of the existence of a Tarski model of ZFC $+ \phi$, what matters is only this form of Los theorem for boolean valued models and their quotients.

Baire's category theorem is irrelevant for the construction of a Tarski model of ZFC $+ \phi$......

Cohen's forcing Theorem: II

Theorem (Cohen's forcing Theorem)

Assume:

- *M is a **transitive** model of ZFC,*
- *$\mathbb{B} \in M$ is a complete non-atomic boolean algebra in M.*
- *G is an M**-generic ultrafilter** for $\mathbb{B}$.*

*Then we also have that the transitive collapse of $M^{\mathbb{B}}/G$ is the **transitive structure** $M[G]$ and with this identification the evaluation map*

$$\sigma_G : M^{\mathbb{B}} \to M[G]$$

is such that:

- *$\sigma_G(\tau) = [\tau]_G$ for all $\tau \in M^{\mathbb{B}}$,*
- *$\sigma_G(\check{a}) = a = [\check{a}]_G$ for all $a \in M$.*
- *If $N \models$ ZFC, $G \in N$, and $M \subset N$, then $M[G] \subset N$,*

A CAVEAT FOR SET THEORIST

Any partial order P gives rise in a natural way to its boolean completion $\mathbb{B}(P)$ which is a non-atomic complete boolean algebra if P has no minimal elements (i.e. P is a non trivial notion of forcing).
Forcing in its usual development focuses on partial orders and not on their boolean completions, however:

Fact

For M a transitive model of ZFC and P and Q in M non trivial forcing notions TFAE:

- *M models that $\mathbb{B}(P)$ and $\mathbb{B}(Q)$ are isomorphic.*
- *For every G M-generic for $\mathbb{B}(P)$ there is H M-generic for $\mathbb{B}(Q)$ such that $M[G] = M[H]$ and conversely.*

Non-atomic complete boolean algebras are sufficient to capture the class of models produced by forcing.

Cohen's absoluteness Lemma

For the purpose of absoluteness results, Baire's category theorem is essential:

Corollary

Assume that:

- $\phi(x, r)$ *is a* Δ_0*-formula with real parameter* r.
- $\mathbb{B} \in V$ *is a Boolean algebra such that*

$$V \models (1_{\mathbb{B}} = [\![\exists x \phi(x, \check{r})]\!]_{\mathbb{B}}).$$

Then $H_{\omega_1} \models \exists x \phi(x, r)$.

Proof:

Assume $\mathbb{B} \in V$ is a Boolean algebra such that

$$V \models (1_{\mathbb{B}} = [\![\exists x \phi(x, \check{r})]\!]_{\mathbb{B}}).$$

To simplify matters assume there is an inaccessible λ such that $\mathbb{B} \in V_\lambda$ (redundant assumption).
Then $V_\lambda \models$ ZFC and

$$V_\lambda \models (1_{\mathbb{B}} = [\![\exists x \phi(x, \check{r})]\!]_{\mathbb{B}}).$$

Pick $N \prec V_\lambda$ countable such that $\mathbb{B} \in N$.
Let $M = \pi_N[N]$ and $\mathbb{Q} = \pi_N(\mathbb{B})$. Notice that $r \in P(\omega)$ and $\pi_N(\omega) = \omega$,
Thus $\pi_N(r) = r$.

Proof continued

Since $\pi_N : N \to M$ is an isomorphism and $\mathbb{Q} = \pi_N(\mathbb{B})$, $\pi_N(r) = r$,

$$M \models (1_{\mathbb{Q}} = [\![\exists x \phi(x, \check{r})]\!]_{\mathbb{Q}}).$$

Now M is countable and transitive, $\mathbb{Q} \in M$ and $\mathsf{FA}_{\aleph_0}(\mathbb{Q})$ holds in V. Thus there is $G \in V$ which is an M-generic filter for $\mathbb{Q}$.

Proof continued

By Cohen's forcing Theorem we can define $\sigma_G : M^{\mathbb{Q}} \to M[G]$ surjective such that

- $\sigma_G(\check{a}) = a$ for all $a \in M$,
- $M[G]$ is transitive,
- $M[G] \models \psi$ iff $[\![\psi]\!]_{\mathbb{Q}} \in G$.

In particular since $\sigma_G(\check{r}) = r$

$$M[G] \models \exists x \phi(x, r).$$

Proof continued

Thus there is $a \in M[G]$ such that $M[G] \models \phi(a, r)$.
Since $M[G]$ is countable and transitive, $M[G] \in H_{\omega_1}$ and $M[G] \subset H_{\omega_1}$, thus $a, r \in H_{\omega_1}$.
Since $\phi(a, r)$ is a Σ_0-formula with parameters in $M[G] \subset H_{\omega_1}$:

$$M[G] \models \phi(a, r) \iff H_{\omega_1} \models \phi(a, r).$$

In particular a witnesses that $H_{\omega_1} \models \exists x \phi(x, r)$. $\square$

Cohen's absoluteness Lemma reformulated

Actually if one doesn't want to commit to any philosophical position on the onthology of sets Cohen's absoluteness Lemma can be formulated as follows:

Corollary (Cohen)

Let T be any first order theory which extends ZFC *and* $\phi(x, r)$ *be a* Σ_0 *formula with a parameter* r *such that* $T \vdash r \subseteq \omega$. *TFAE:*

- $T \vdash \exists x \phi(x, r)$.
- $T \vdash$ *There exists a boolean algebra* $\mathbb{B}$ *such that* $1_{\mathbb{B}} = [\![\exists x \phi^{H_{\omega_1}}(x, \check{r})]\!]_{\mathbb{B}}$.

KEY OBSERVATION:

Forcing gives a provably correct and complete semantics for the Σ_1-fragment of the theory of H_{ω_1}.

Forcing is a powerful tool to prove theorems and transforms, for certain class of problems, a proof of the consistency of a solution, in the solution.

Woodin's absoluteness

Theorem (Woodin)

Assume there are class many Woodin cardinals which are limit of Woodin cardinals in V, then for every formula ϕ with real parameters:

$$L(\mathrm{Ord}^\omega)^V \models \phi$$

if and only if there exists a boolean algebra $\mathbb{B} \in V$ such that

$$V \models (1_{\mathbb{B}} = [\![\phi^{L(\mathrm{Ord}^\omega)}(\check{r})]\!]_{\mathbb{B}}).$$

Notice that we had to relativize the formulas to $L(\mathrm{Ord}^\omega)$ to obtain the absoluteness results.
This is an unavoidable consequence of the fact that formulas which are not Σ_0 are neither upward absolute nor downward absolute between transitive structures.
(We needed the upward absoluteness of $\phi(a, r)$ to conclude the proof of Cohen's absoluteness.)

Effects of large cardinals on the theory of $L(\mathrm{Ord}^\omega)$

If one investigates with care Woodin's proof, the assumption that V is transitive is redundant.
In particular Woodin actually proved:

Theorem (Woodin)

Let T be any theory which extends ZFC+there are class many Woodin cardinals which are limits of Woodin cardinals and r be a parameter such that $T \vdash r \subset \omega$. Then for any formula $\phi(x)$ TFAE:

- $T \vdash [L(\mathrm{Ord}^\omega) \models \phi(r)]$,
- $T \vdash$ *There is a boolean algebra* $\mathbb{B}$ *such that* $1_{\mathbb{B}} = [\![\phi^{L(\mathrm{Ord}^\omega)}(\check{r})]\!]_{\mathbb{B}}$.

Stepping up to $L(\mathrm{Ord}^{\omega_1})$

We shall assume a platonistic stance towards set theory. We have one canonical model V of ZFC of which we try to uncover the truths.
We may allow ourselves to use all model theoretic techniques that produce new models of the truths of $Th(V)$ on which we are confident, which (if we are platonists) certainly include ZFC and all the axioms of large cardinals.
We may start our quest for uncovering the truth in V by first settling the theory of $H^V_{\omega_1}$, then the theory of $H^V_{\omega_2}$ and so on, so forth covering step by step all infinite cardinals.
Woodin's absoluteness results show that large cardinal axioms give a correct and complete semantics with respect to first order derivability and forceability for the theory of $H_{\aleph_1} \subset L(\mathrm{Ord}^\omega)$ with real parameters.
A key ingredient of Woodin's result is the fact that $\mathrm{FA}_{\aleph_0}(\mathbb{B})$ holds for the largest possible class of boolean algebras $\mathbb{B}$, i.e. for all $\mathbb{B}$.

The elementary diagram of $H_{\aleph_2}$

We want to find a natural extension of ZFC$+$*large cardinals* which makes "complete" the theory of $H_{\aleph_2}$.

Definition

Let V be a model of ZFC. The Σ_0-diagram of $H^V_{\omega_2}$ is given by the theory

$$\{\phi(p) : p \in H^V_{\omega_2},\ \phi(p) \text{ a } \Sigma_0\text{-formula true in } V\}.$$

The Σ_0-diagram of $H_{\aleph_2}$ captures undeniable truths of $H_{\aleph_2}$ (from a platonistic stand point).

$\Omega_{\aleph_1}$ and absoluteness

Lemma (Generalized Cohen's absoluteness Lemma)

Assume $\mathsf{FA}_{\aleph_1}(\mathbb{B})$ *and* $\phi(x,y)$ *is a* Σ_0*-formula. The following are equivalent for some* $a \in H_{\omega_2}$*:*

1. $H_{\omega_2} \models \exists x \phi(x,a)$,
2. $[\![\exists x \phi(x,a)]\!]_{\mathbb{B}} = 1_{\mathbb{B}}$.

Definition

$\Omega^V_{\aleph_1}$ is the class of non-atomic complete boolean algebras $\mathbb{B} \in V$ such that $\mathsf{FA}_{\aleph_1}(\mathbb{B})$ holds.

The above Lemma shows that to decide the Σ_1-theory of $H^V_{\aleph_2}$ it is enough to look at $M \supset V$ which are obtained as forcing extensions by a forcing in $\Omega^V_{\aleph_1}$.

KEY OBSERVATION: $\Omega^V_{\aleph_1}$ gives a complete and correct semantics for the Σ_1-theory of $H^V_{\aleph_2}$.

Stationary set preserving forcings and absoluteness

Fact

Assume that $M \supset V$ is a model of ZFC *which maintains the truth of the Σ_0-diagram of V. Then* $\mathrm{NS}^M_{\omega_1} \cap V = \mathrm{NS}_{\omega_1}$.

Definition

A boolean algebra $\mathbb{B}$ is stationary set preserving (SSP) if for all S stationary subset of ω_1

$$[\![\check{S} \text{ is stationary}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

Fact (Shelah)

Assume $\mathbb{B} \notin$ SSP and G is V-generic for $\mathbb{B}$. Then $V[G]$ falsifies the Σ_0-elementary diagram of $H^V_{\omega_2}$.

Corollary

$\Omega_{\aleph_1} \subseteq \mathrm{SSP}$

Enlarging $\Omega_{\aleph_1}$ to become SSP.

Martin's maximum MM asserts that $\Omega_{\aleph_1} = \mathrm{SSP}$.
The consistency proof of Martin's maximum builds over V a "saturated" model M of ZFC with respect to the Σ_1-types of the Σ_0-elementary diagram of $H^M_{\aleph_2}$ which are forced by an SSP partial order.
Such a model M will realize simultaneously all Σ_1-types over H_{ω_2} which can be forced by an SSP forcing.

The natural strategy to build such a "saturated" extension M of V is to "iterate" all possible stationary set preserving forcings.

Using Shelah's results on iterated forcing, Foreman Magidor and Shelah have built such models M.

Following these patterns of ideas strong forcing axioms have been discovered and proved consistent.

Martin's maximum and generic absoluteness

MM settles almost all relevant problems of third order arithmetic or of the theory of H_{ω_2} (and denies CH).

WHY?

One plausible reason is that it asserts $\mathsf{FA}_{\aleph_1}(B)$ for the largest possible class of $\mathbb{B}$ i.e. all $\mathbb{B} \in \mathsf{SSP}$.

Baire's category theorem (i.e. $\mathsf{FA}_{\aleph_0}(\mathbb{B})$ holds for all $\mathbb{B}$) has been a key ingredient in Woodin's absoluteness result for $L(\mathrm{Ord}^\omega)$.
MM is the natural extension to ω_1 of Baire's category theorem.
This makes plausible that Woodin's absoluteness result can be stepped up to $L(\mathrm{Ord}^{\omega_1})$ for some theory extending

$$\mathsf{ZFC} + \mathsf{MM} + \text{ large cardinals.}$$

In what follows we shall show that this is indeed the case.

We shall present MM^{+++}, a natural strengthening of MM, and show that it makes the theory of $L(\mathrm{Ord}^{\omega_1})$ provably complete with respect to SSP-forcings and first order calculus.

On the other hand, Aspero, Larson and Moore have shown that no such generic absoluteness result can be produced for any theory extending $\mathsf{ZFC} + \mathsf{CH}$.

The category of stationary set preserving forcings

We can formulate the logic notions of elementary extension by SSP-forcing and of saturation with respect to SSP-consistent Σ_1-types in the language of categories:

- Generic extensions by stationary set preserving forcings corresponds to complete boolean algebras which are stationary set preserving.
- Iterations of stationary set preserving forcings correspond to a natural family of directed systems of complete homomorphisms.

Definition

$\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$ is the category whose objects are SSP complete boolean algebras $\mathbb{B}$ and whose arrows are non-atomic complete (*but possibly non-injective*) homomorphisms

$$i : \mathbb{B} \to \mathbb{Q}$$

such that

$$[\![\mathbb{Q}/i[\dot{G}_{\mathbb{B}}] \in \mathsf{SSP}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

Non triviality of $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$

Any category can be seen a partial order whose elements are its objects and whose order relation is induced by its arrows.

Fact

If Γ is the class of all complete non-atomic boolean algebras and Θ is the class of all non-atomic complete homomorphisms between elements of Γ we have that any two elements $P, Q \in \Gamma$ are compatible in $\mathbb{U}^{\Gamma,\Theta}$ as witnessed by $Col(\omega, <\delta)$ for any large enough $\delta > |P|, |Q|$.

Fact

If $P = Col(\omega_1, \omega_2)$ and Q is Namba forcing on $\aleph_2$, $\mathbb{B}(P)$ and $\mathbb{B}(Q)$ are incompatible conditions of $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$.

Proof: If not in some generic extension of an SSP forcing which absorbs both of them we would have that ω_2^V has at the same time countable and uncountable cofinality.

TOTAL RIGIDITY

Definition

$\mathbb{B} \in \mathsf{SSP}$ is totally rigid if any of the following equivalent condition is met by $\mathbb{B}$:

1. For all $i_0 : \mathbb{B} \to \mathbb{Q}$, $i_1 : \mathbb{B} \to \mathbb{Q}$ arrows of $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$ we have that $i_0 = i_1$.
2. For all $b \in \mathbb{B}^+$, $\mathbb{B} \upharpoonright b$ and $\mathbb{B} \upharpoonright \neg_{\mathbb{B}} b$ are incompatible in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$.
3. For all $\mathbb{Q} \leq_{\mathsf{SSP}} \mathbb{B}$, whenever G is a V-generic filter for $\mathbb{Q}$, in $V[G]$ there is only one H which is V-generic for $\mathbb{B}$.

Theorem (V.)

Assume δ is supercompact and P_δ is any of the standard methods to produce a model of MM^{++} *collapsing δ to become ω_2. Then P_δ is totally rigid.*

Fact

If $\mathbb{U}_\delta \in \mathsf{SSP}$, $\mathbb{U}_\delta$ is totally rigid.

Theorem (V.)

Assume there are class many supercompact cardinals. Then the class

$$D_0 = \{\mathbb{B} : \mathbb{B} \text{ is totally rigid}\}$$

is dense in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$.

Theorem (A variation on Woodin's work)

Assume there are class many Woodin cardinals. Then the following are equivalent

1. MM^{++},
2. $D_1 = \{\mathbb{B} \in \mathsf{SSP} : \mathbb{B} \text{ is a presaturated tower}\}$ *is dense in* $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$.

Definition

MM^{+++} asserts that $D_0 \cap D_1$ is dense in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$.

What is a presaturated tower?

We don't have the time to define this notion of forcing. Basically a preaturated tower allows to define a generic "almost huge" ultrapower embedding with small critical point. For the purposes of this talk we just need to know this of presaturated towers:

Fact

Let $\mathbb{B} \in \mathrm{SSP}$ be a presaturated tower and G be V-generic for $\mathbb{B}$. Then

$$\langle L(\mathrm{Ord}^{\omega_1})^V, \in, P(\omega_1)^V\rangle \prec \langle L(\mathrm{Ord}^{\omega_1})^{V[G]}, \in, P(\omega_1)^V\rangle$$

MM^{+++} and generic absoluteness for $L(\mathrm{Ord}^{\omega_1})$

Theorem (V.)

Assume

$$T \supseteq \mathrm{ZFC} + \mathrm{MM}^{+++} \text{ there are class many superhuge cardinals,}$$

ϕ is any formula and $a \subset \omega_1$. Then the following are equivalent:

1. $T \vdash \phi^{L(\mathrm{Ord}^{\omega_1})}(a)$
2. *T proves that there is some $\mathbb{B} \in \mathrm{SSP}$ such that*

$$[\![\phi^{L(\mathrm{Ord}^{\omega_1})}(a)]\!]_{\mathbb{B}} = [\![\mathrm{MM}^{+++}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

Similarity properties of $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$

Observe that if $P \in \mathsf{SSP}$ and G is V-generic for P

$$(\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}})^{V[G]} = \{\mathbb{B}/i[G] : V \models i : P \to \mathbb{B} \text{ is a complete embedding }\}$$

Proposition

Let $\mathbb{U}_\delta = \mathbb{U}^{\mathsf{SSP},\mathsf{SSP}} \cap V_\delta$ and $\mathbb{B} \in \mathbb{U}_\delta$. Assume $i : \mathbb{B} \to \mathbb{B}(\mathbb{U}_\delta)$ is a complete homomorphism with a stationary set preserving quotient. Then

$$[\![\dot{\mathbb{U}}_\delta = (\mathbb{U}_\delta \upharpoonright \mathbb{B})/i[\dot{G}_{\mathbb{B}}]]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

This is the same similarity property that is peculiar of $Col(\omega, <\delta)$.

How to get there? I

Definition

δ is superhuge if for all $\lambda > \delta$ there is $j : V \to M$ with $M^{j(\delta)} \subset M \subset V$ and $j(\delta) > \lambda$.

Theorem (V.)

MM^{+++} *is consistent relative to the existence of class many superhuge cardinals.*

Actually any of the known iteration of length a super-huge δ which produces a model of MM^{++}, will produce a model of MM^{+++}.

How to get there? *II*

Theorem (V.)

Assume MM^{+++} *and that there are class many superhuge cardinals* δ*. Then for any such* δ*:*

- $\mathbb{U}_\delta \in \mathsf{SSP}$ *is a totally rigid presaturated tower (i.e.* $\mathbb{U}_\delta \in D_0 \cap D_1$*).*
- $\mathbb{B} \geq_{\mathsf{SSP}} \mathbb{U}_\delta \restriction \mathbb{B}$ *for all* $\mathbb{B} \in \mathbb{U}_\delta$.

A sketch of proof of the generic absoluteness result

Assume $V \models \mathsf{MM}^{+++}$ and $P \in \mathsf{SSP}$ forces MM^{+++}. Let G be V-generic for $\mathbb{U}_\delta \upharpoonright P$ for some superhuge $\delta > |P|$.

Let $i : P \to \mathbb{U}_\delta \upharpoonright P$ be a complete embedding and $H = i^{-1}[G] \in V[G]$ be V-generic for P.

Then:

- $\mathbb{U}_\delta^{V[H]} = (\mathbb{U}_\delta \upharpoonright P)^V / i[H]$
- δ is superhuge in V, $V[H]$ which are both models of MM^{+++}.

Thus $\mathbb{U}_\delta \upharpoonright P$ is a presaturated tower in V and $\mathbb{U}_\delta^{V[H]}$ is a presaturated tower in $V[H]$.

This gives that:

$$\langle L(\mathrm{Ord}^{\omega_1})^V, \in, P(\omega_1)^V\rangle \prec \langle L(\mathrm{Ord}^{\omega_1})^{V[G]}, \in, P(\omega_1)^V\rangle.$$

$$\langle L(\mathrm{Ord}^{\omega_1})^{V[H]}, \in, P(\omega_1)^{V[H]}\rangle \prec \langle L(\mathrm{Ord}^{\omega_1})^{V[G]}, \in, P(\omega_1)^{V[H]}\rangle.$$

Thus

$$\langle L(\mathrm{Ord}^{\omega_1})^V, \in, P(\omega_1)^V\rangle \equiv \langle L(\mathrm{Ord}^{\omega_1})^{V[H]}, \in, P(\omega_1)^V\rangle.$$

Ideas for the proof of the consistency of MM^{+++}

Theorem

Assume δ is super huge and P_δ is any of the standard methods to produce a model of MM^{++} collapsing δ to become ω_2. Let G be V-generic for P_δ. Then $V[G]$ models that $j(P_\delta)/G$ is totally rigid and forcing equivalent to a presaturated normal tower.

Corollary

Assume δ is super huge and P_δ is any of the standard methods to produce a model of MM^{++} collapsing δ to become ω_2. Let G be V-generic for P_δ. Then in $V[G]$ the class $\mathsf{SPT} = D_0 \cap D_1$ of totally rigid presaturated normal towers is dense in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$, i.e. MM^{+++} holds.

Ideas for the proof that $\mathbb{U}_\delta$ is a strongly presaturated tower

Theorem

Assume SPT *is a dense class in* $\mathbb{U}^{\text{SSP,SSP}}$. *Let* δ *be an inaccessible cardinal such that* $\mathbb{U}_\delta \in$ SSP *and* SPT $\cap V_\delta$ *is dense in* $\mathbb{U}_\delta$. *Then* $\mathbb{U}_\delta$ *is forcing equivalent to a presaturated normal tower.*

Sketch of proof: We use the density of SPT in $\mathbb{U}_\delta$ to show that whenever G is V-generic for $\mathbb{U}_\delta$, we can patch together the generic filters for elements of $G \cap$ SPT to define in $V[G]$ a generic ultrapower embedding

$$j : V \to M$$

such that $M^{<\delta} \subset M$, $crit(j) = \omega_2$, $j(\omega_2) = \delta$.
This will give that $\mathbb{U}_\delta$ is forcing equivalent to a presaturated tower of normal filters.

Now we observe that if SPT is dense in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$ and δ is a strong cardinal which is a limit of $< \delta$-supercompact cardinal we have that $\mathbb{U}_\delta \in \mathsf{SSP}$ is totally rigid and forcing equivalent to a presaturated tower of normal ideals.

This concludes the sketch of all proofs.

SOME REFERENCES

In these two papers are presented the results I talked about:

- **Martin's maximum revisited**
- **Category forcings,** MM^{+++}**, and generic absoluteness for the theory of strong forcing axioms**

They're both available on my webpage:
`http://www2.dm.unito.it/paginepersonali/viale/`.

THANKS FOR YOUR PATIENCE AND ATTENTION!!