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CHARACTERIZATION OF SET-GENERIC EXTENSIONS

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CONTENTS

ABSTRACT

This thesis is divided in three parts. In the first one we follow the exposition of forcing via boolean ultrapowers; in the second one we discuss the classification of forcing extensions by means of first-order properties; in the last one we attempt to define an inner model with the κ-*approximation* property (Definition 2.2), that could be related to the topic of inner models for large cardinals.

In the first chapter we start with some general algebraic notions (Section 1.1, 1.2 and 1.3), introducing the concepts of κ-*complete boolean algebra* (Definition 1.33) and free boolean algebra (Definition 1.35), together with some basic results (Theorems 1.37 and 1.41). A brief exposition of *forcing* with boolean-valued models follows (Sections from 1.4 to 1.7), together with some more recent results (Section 1.6, Theorem 1.57) about boolean ultrapowers. In the last part of the chapter (Section 1.8), we briefly relate what we presented to the classical exposition of *forcing* with posets.

In the second chapter, after an introduction (Section 2.1) to some first-order properties that an inner model M may have with respect to V, we use these properties to prove the following three theorems:

Theorem (Laver, 2.16)**.** *If V is a forcing extension of M, then M is a definable class in V with parameters in M.*

Theorem (Bukovsky, 2.31)**.** *Let $M \subset V$ be models of* ZFC*. Then V is a generic extension of M by a κ-cc forcing iff M globally κ-covers V (Definition 2.5).*

Theorem (Friedman, 2.34)**.** *Let $M \subset V$ be models of* ZFC*. Then V is a generic extension of M by a forcing of size at most κ iff M globally κ^+-covers and κ-decomposes (Definition 2.3) V.*

In the last part of the chapter (Section 2.5), we use these results to obtain a first-order formulation of the *Ground Axiom*, in a similar way to the one found in [6, Reitz].

In the last chapter we first discuss the properties of sets κ-*approximated* in M (Section 3.1), proving that the class $\mathscr{A}_M$ of such sets is closed for all Gödel operations except for the couple, and thus satisfies all axioms of ZFC except for the *existence of couples*, the *power set axiom*, and a weakened version of *comprehension* (i.e., comprehension for formulas of the kind ϕ^M, Theorem 3.6). We shall then use these results to prove that, assuming $\mathscr{P}_\kappa^M(\lambda)$ stationary in $\mathscr{P}_\kappa^V(\lambda)$ for some λ, the class:

$$\overline{M}^\kappa = \bigcup \left\{ \Pi_R[X] : \ R \in \mathscr{A}_M \text{ well-founded relation } R \subset X^2 \right\}$$

with Π_R transitive collapse of R in V, is closed for all Gödel operations (Theorem 3.13). We have found a difficulty in the fact that the class $\overline{M}^\kappa$ might not be almost universal, thus not a model of ZFC. With the additional assumption that $\overline{M}^\kappa$ is almost universal, we obtain (Theorem 3.18) that $\overline{M}^\kappa$ is the minimal transitive class $\overline{M} \subset V$ such that:

1. $M \subset \overline{M}$,
2. $\overline{M} \vDash$ ZFC,
3. $\overline{M}$ κ-approximates V.

We assume that the reader knows the basic results about first-order logic and set theory. Reference texts for these topics are [5, Chapters 1-4], [1, Chapters 1, 3].

To read Chapter 2 one need to know the topics exposed in Chapter 1. To read Chapter 3 it is enough to read Section 2.1.

INTRODUZIONE

Questa tesi è divisa in tre parti: nella prima parte si seguirà una presentazione del forcing tramite modelli a valori booleani e delle ultrapotenze booleane, nella seconda si discuterà della classificazione di estensioni di forcing mediante proprietà del primo ordine, mentre nell'ultima parte si cercherà di costruire un modello interno con la proprietà di *κ-approssimazione* (Definizione 2.2), modello che potrebbe essere utile nell'ambito della ricerca di modelli interni per grandi cardinali.

Nel primo capitolo, dopo aver introdotto alcune nozioni algebriche generali (Sezione 1.1, 1.2 e 1.3), tra cui i concetti di *algebra di boole κ-completa* (Definizione 1.33) e di *algebra di boole libera* (Definizione 1.35) e alcuni risultati di base (esistenza e unicità, Teoremi 1.37 e 1.41), si passerà a una breve esposizione del *forcing* tramite i modelli a valori booleani (Sezioni da 1.4 a 1.7). Durante questa trattazione si presenteranno alcuni risultati più recenti (Sezione 1.6 e Teorema 1.57) sulle ultrapotenze booleane non generiche. Nell'ultima parte del capitolo (Sezione 1.8), si farà un accenno alla presentazione classica del forcing tramite posets.

Nel secondo capitolo, dopo aver introdotto (Sezione 2.1) alcune proprietà del primo ordine che un modello interno M può avere nei confronti del modello esterno V, le si utilizzeranno per dimostrare i seguenti tre risultati:

Teorema (Laver, 2.16)**.** *Se V è una estensione di forcing di M, allora M è una classe definibile in V con parametri in M.*

Teorema (Bukovsky, 2.31)**.** *Dati $M \subset V$ modelli di* ZFC*, V è una estensione generica di M mediante un forcing κ-cc se e solo se M κ-ricopre globalmente V (Definizione 2.5).*

Teorema (Friedman, 2.34)**.** *Dati $M \subset V$ modelli di* ZFC*, V è una estensione generica di M mediante un forcing di taglia massimo κ se e solo se M κ-decompone (Definizione 2.3) e κ^+-ricopre globalmente V.*

L'ultimo risultato (Friedman) è una recente generalizzazione del precedente teorema di Bukovsky. Il primo è un risultato recente, cha il merito di aver dato avvio allo studio dei *ground model* e alla cosiddetta *set-theoretic geology*: nell'ultima parte (Sezione 2.5) si tratterà brevemente l'argomento dei ground model, esponendo una definizione simile a quella proposta originalmente in [6, Reitz].

Nell'ultimo capitolo si studieranno dapprima le proprietà degli insiemi κ-*approssimati* in M (Sezione 3.1), dimostrando che la classe $\mathscr{A}_M$ di questi insiemi è chiusa per tutte le operazioni di Gödel eccetto la coppia, e che soddisfa quindi tutti gli assiomi di ZFC eccetto per *coppia*, *insieme potenza*, e una versione più debole dell'assioma di *comprensione* (i.e., comprensione per formule relativizzate in M, Teorema 3.6). Successivamente si utilizzeranno questi risultati per dimostrare che, assumendo che $\mathscr{P}_\kappa^M(\lambda)$ sia stazionario in $\mathscr{P}_\kappa^V(\lambda)$ per qualche λ, la classe:

$$\overline{M}^\kappa = \bigcup \left\{ \Pi_R[X] : \ R \in \mathscr{A}_M \text{ relazione ben fondata } R \subset X^2 \right\},$$

dove Π_R è la mappa di collasso transitivo di R in V, è chiusa per operazioni di Gödel (Teorema 3.13). È stata poi individuata una difficoltà nel fatto che la classe $\overline{M}^\kappa$ potrebbe non essere quasi universale, e quindi non essere un modello di ZFC. Con l'assunzione addizionale che $\overline{M}^\kappa$ sia quasi universale, si otterrà tuttavia (Teorema 3.18) che $\overline{M}^\kappa$ è caratterizzabile come la minima classe transitiva $\overline{M} \subset V$ tale che:

1. $M \subset \overline{M}$,
2. $\overline{M} \models$ ZFC,
3. $\overline{M}$ κ-approssima V.

Lo scopo di questa tesi è duplice. Per quanto riguarda la prima parte, lo scopo è esporre in maniera più chiara e completa alcuni risultati molto interessanti ma ancora poco noti sulla definibilità al primo ordine di proprietà di estensioni di forcing. Per quanto riguarda la seconda parte, lo scopo è studiare nuovi modi per costruire modelli interni che potrebbero essere utili nell'ambito della teoria dei grandi cardinali.

Assumiamo che il lettore conosca le basi della logica del primo ordine e della teoria degli insiemi, testi di riferimento in cui si può trovare questo materiale sono rispettivamente [5, Capitoli 1-4] e [1, Capitoli 1 e 3].

Per leggere il Capitolo 2 è necessario conoscere gli argomenti trattati nel Capitolo 1, mentre per leggere il Capitolo 3 è sufficiente la sola Sezione 2.1, senza i teoremi sulle estensioni di forcing.

NOTATIONS AND CONVENTIONS

$\alpha, \beta, \dots$	and $i, j, \dots$ will be used for ordinals;
κ, λ, θ	will be used for cardinals;
$A, B, \dots$	will be used for sets;
M, N, V	will be used for sets or classes that are models of ZFC;
$p, q, \dots$	will be used for conditions in $\mathbb{P}$;
$Q, R, \dots$	will be used for sets of conditions in $\mathbb{P}$;

$\phi, \psi, \dots$	will be used for formulas;
Φ, Ψ	will be used for sets of formulas;
$\Rightarrow, \Leftrightarrow$	will be used as logical relations in the metatheory;
$\rightarrow, \leftrightarrow$	will be used as operations in boolean algebras (or formulas);
$\triangle$	is the symmetrical difference $x \triangle y = (x \wedge \neg y) \vee (\neg x \wedge y)$

$f^{-1}(A)$	is the set of preimages $f^{-1}(A) = \{x : f(x) = A\}$;
$f[A]$, $f^{-1}[A]$	is the set of images $f[A] = \{f(x) : x \in A\}$ (resp. with f^{-1});
$\mathrm{ran}(A)$	is the range of a function;
$\mathrm{dom}(A)$	is the domain of a function or a structure (the dom operator will be implied when clear from the context);

$\mathscr{P}_\kappa(X)$	is the set of al subsets of X of size $< \kappa$;
$\mathrm{int}, \mathrm{cl}$	are the closure and interior topological operations;
$\mathrm{pred}(x, R)$	are the predecessors of x in the transitive closure of the order relation R;

M_α	is the stage α of the cumulative hierarchy in M;
$H(\kappa)$	is the class of all sets hereditarily of cardinality $< \kappa$;
ϕ^M	is the interpretation of ϕ in the model M;

$\dot{A}$	is a $\mathbb{P}$-name for A;
$\check{A}$	where $A \in M$ is the standard $\mathbb{P}$-name for A in M;
$M[A]$	is the ZFC model generated by M and A;
$M^{\mathbb{P}}$	is the class of all $\mathbb{P}$-names in M;

CHAPTER 1

PRELIMINARIES

In this chapter we shall briefly show some basic results on posets, boolean algebras, and forcing extensions. The main results are Theorem 1.41 on boolean algebras, Theorems 1.44, 1.48, 1.55, 1.71 on forcing extensions, and Theorems 1.57, 1.67 on boolean ultrapowers.

1.1 Posets

The theory of posets and boolean algebras (Section 1.2) has played a major role in set theory in the last decades, allowing the development of the method of *forcing*. We shall now expose the basic definitions and properties about posets that will be useful later on.

Definition 1.1. *A* poset *(partially ordered set) is a set* $\mathbb{P}$ *together with a binary relation* $\leq$ *on* $\mathbb{P}$ *which is transitive, reflexive and antisymmetric.*

We are also interested in some special subsets of posets with certain properties, as the following.

Definition 1.2. *A subset* $A \subset \mathbb{P}$ *is a* chain *if and only if is totally ordered in* $\mathbb{P}$*.*

Definition 1.3. *Given* $a, b \in \mathbb{P}$*, we say that* a *and* b *are* incompatible*, in formulas* $a \perp b$*, if and only if:*

$$a \perp b \quad \Leftrightarrow \quad \neg \exists c : \; c \leq a \wedge c \leq b$$

Similarly, we say that a, b *are* compatible*, in formulas* $a \parallel b$*, iff* $\neg(a \perp b)$.

Definition 1.4. *A subset $A \subset \mathbb{P}$ is an* antichain *if and only if every two element in A are incompatible: $\forall a, b \in A: \ a = b \vee a \perp b$.*

Definition 1.5. *An antichain $A \subset \mathbb{P}$ is* maximal *if and only if no subset $B \subset \mathbb{P}$, $B \supset A$ is an antichain.*

Definition 1.6. *An antichain $A \subset \mathbb{P}$* refines *an antichain $B \subset \mathbb{P}$ if and only if for every $a \in A$ there exists a $b \in B$ such that $b < a$.*

Definition 1.7. *A set $D \subset \mathbb{P}$ is dense in $\mathbb{P}$ iff for every $p \in \mathbb{P}$ exists $d \in D$ with $d \leq p$, i.e. D is dense in the order topology.*

There is a close connection between the definitions of dense sets and antichains, as shown in the following result.

Theorem 1.8. *Every dense set $D \subset \mathbb{P}$ contains a maximal antichain $A \subset D$. Conversely, the downward closure of every maximal antichain is dense.*

Proof. Given $D \subset \mathbb{P}$, by Zorn's Lemma let $A \subset D$ be a maximal antichain with respect to D. This same antichain must be maximal also in $\mathbb{P}$: any $x \perp A$ would have an $y < x$ in D, which would be $y \perp A$, contradicting maximality of A in D.

Conversely, given $A \subset \mathbb{P}$ maximal antichain, let $D = \{p \in \mathbb{P} : \exists a \in A \ p \leq a\}$. Every $p \in \mathbb{P}$ must be $p \parallel a$ for some $a \in A$ by maximality of A, hence has a r_p with $r_p \leq p$, $r_p \leq a \Rightarrow r_p \in D$. Thus D is dense. □

We shall now define the most common properties of posets, that we will need later on.

Definition 1.9. *A poset $\mathbb{P}$ is separative iff for all $p \in \mathbb{P}$, there exist $q, r \in \mathbb{P}$ with $q \leq p$, $r \leq p$, $q \perp r$.*

Separativity is often used as a "non triviality" property, since for most purposes any set of non-separable elements can be collapsed into a single element.

Proposition 1.10. *Let $\mathbb{P}$ be a poset. There exists a separative poset $\mathbb{Q}$ and a mapping h of $\mathbb{P}$ onto $\mathbb{Q}$ such that:*

$$\begin{aligned} x \leq y \quad &\Rightarrow \quad h(x) \leq h(y) \\ x \perp y \quad &\Leftrightarrow \quad h(x) \perp h(y) \end{aligned}$$

Proof. $\mathbb{Q}$ is the quotient of $\mathbb{P}$ under the equivalence relation:

$$x \sim y \Leftrightarrow \forall z \in \mathbb{P} \left(z \perp x \longleftrightarrow z \perp y\right)$$

with ordering given by:

$$[x] \prec [y] \Leftrightarrow (\forall z \leq x) z \parallel y$$ □

Definition 1.11. *A poset* $\mathbb{P}$ *is* κ-closed *if and only if every decreasing chain* $\langle p_\alpha : \alpha < \lambda\rangle$ *with* $\lambda < \kappa$ *is bounded by some element of* $\mathbb{P}$.

Definition 1.12. *A poset* $\mathbb{P}$ *is* κ-cc *(satisfies the* κ-chain condition*) if and only if every antichain has size* $< \kappa$.

Definition 1.13. *A poset* $\mathbb{P}$ *is* κ-distributive *if and only if every family* $\mathscr{F}$ *of* $< \kappa$ *dense sets in* $\mathbb{P}$ *has dense intersection* $\bigcap \mathscr{F}$.

The last two properties for a fixed κ excludes each other, as we can see in the following proposition.

Proposition 1.14. *A* κ*-cc separative poset* $\mathbb{P}$ *is not* κ^+*-distributive.*

Proof. Assume that $\mathbb{P}$ is instead κ^+-distributive. Let D_0 be dense in $\mathbb{P}$, and define by transfinite induction two sequences A_i, D_i: for every $i \leq \kappa$, let A_i be a maximal antichain in D_i, $D_{i+1} = \{p \in \mathbb{P} : \ \exists q \in A_i \ p < q\}$ and $D_\alpha = \bigcap_{i<\alpha} D_i$ for α limit ordinal. The limit step for $i \leq \kappa$ is allowed by κ^+-distributivity. Define

$$\mathscr{F} = \{f \subset (\kappa \times \mathbb{P}) : \ f \text{ is a function } \wedge f(i) \in A_i \wedge (i < j \to f(i) > f(j))\}$$

Since $\mathbb{P}$ is separative, there is an injection of 2^κ in $\mathscr{F}$: for every i successor, there are at least 2 choiches for the value of $f(i)$ given $f|_i$. So $|\mathscr{F}| \geq 2^\kappa$.

For every $p \in A_\kappa$, define $f_p \in \mathscr{F}$ as the unique function such that $f_p(i) > p$ for every $i < \kappa$. This defines a map $h : A_\kappa \to \mathscr{F}$, $h(p) = f_p$.

This map is surjective, otherwise let $g \notin h[A_\kappa]$. Since A_κ is a maximal antichain, for all $i < \kappa$ let $p_i \in A_\kappa$ be the only one such that $g(i) \parallel p_i$ (in fact $g(i) > p_i$, since A_κ is below A_i). Since $|A_\kappa| < \kappa$, there must be a $q \in A_\kappa$ such that the set $\{i < \kappa : \ p_i = q\}$ has size κ. This set must be unbounded in κ, so for every $i < \kappa$, there is a $j > i$ such that $g(i) > g(j) > p_j = q$, so $h(q) = g$ against the hypothesis.

Then $|\mathscr{F}| \leq |A_\kappa| < \kappa$, which contradicts $|\mathscr{F}| \geq 2^\kappa$. Hence the assumption that $\mathbb{P}$ is κ^+-distributive must be false. □

The following definitions are a crucial step in classical development of forcing: although we will not follow that approach, they will still be needed later on.

Definition 1.15. *A set* $I \subset \mathbb{P}$ *is an* ideal *in* $\mathbb{P}$ *iff* $(a \in I \wedge b < a) \to b \in I$ *and* $a, b \in I \to \exists c \in I : (a \leq c \wedge b \leq c)$.

Definition 1.16. *A set* $F \subset \mathbb{P}$ *is a* filter *in* $\mathbb{P}$ *iff* $(f \in F \wedge f < g) \to g \in F$ *and* $f, g \in F \to \exists h \in F : (h \leq f \wedge h \leq g)$.

Definition 1.17. *An ideal I is* principal *iff $I = I_p = \{q \in \mathbb{P} : q < p\}$ for some $p \in \mathbb{P}$. Similarly, a filter F is principal iff $F = F_p = \{q \in \mathbb{P} : p < q\}$ for some $p \in \mathbb{P}$.*

We shall mostly be interested in non-principal ideals, and sometimes in much stronger non-triviality properties, as the following.

Definition 1.18. *Let M be a transitive model of* ZFC *and $\mathbb{P} \in M$ be a poset. A filter $G \subset \mathbb{P}$ is M-generic for $\mathbb{P}$ if and only if it intersects every dense set $D \in M$.*

Equivalently, a filter G is M-generic if it intersects every maximal antichain. The same definition can be stated also for non-canonical models (M, E).

Definition 1.19. *Let (M, E) be a model of* ZFC *and $\mathbb{P} \in M$ be such that $M \vDash \mathbb{P}$ is a poset. A filter $G \subset \mathbb{P}$ is M-generic for $\mathbb{P}$ if and only if $(G \cap D)^M \neq \emptyset$ for every $D \subset \mathbb{P}$ in M such that $M \vDash D$ is dense .*

If $\mathbb{P}$ is a separative poset, M-generic filters cannot be found in M itself.

Theorem 1.20. *If $\mathbb{P} \in M$ is a separative poset, then no filter $G \subset \mathbb{P}$ in M is M-generic.*

Proof. Since $\mathbb{P} \in M$ is a separative poset, if G is a filter in $\mathbb{P}$ then $D_G = \mathbb{P} \setminus G$ is trivially dense and $G \cap D_G = \emptyset$. Then if $G \in M$, also $D_G \in M$ hence G is not M-generic. □

The previous theorem raises metamathematical problems regarding the existence of generic filters, which for example do not exist if $M = V$. To overcome this issues we will focus on the boolean-valued model approach to forcing, in which generic filters are not explicitly needed.

1.2 Boolean Algebras

Boolean Algebras are a special kind of posets that generalize the concept of *truth value*, from the classical two-valued notion to a many-valued notion. Our reference text for the results on boolean algebras is [4].

Definition 1.21. *A* lattice *is a poset in which any two elements a, b have a unique supremum $a \vee b$ (least upper bound, called their join) and infimum $a \wedge b$ (greatest lower bound, called their meet).*

Definition 1.22. *A* distributive lattice *is a lattice in which the operations of join and meet distribute over each other.*

Definition 1.23. *A* bounded lattice *is a lattice with a least element (called $\mathbb{0}$) and a greatest element (called $\mathbb{1}$).*

Definition 1.24. *A* complemented lattice *is a bounded lattice in which every element a has a complement, i.e. an element b satisfying $a \vee b = \mathbb{1}$ and $a \wedge b = \mathbb{0}$.*

Definition 1.25. *A* boolean algebra *is a complemented distributive lattice.*

A standard example of boolean algebra is the *regular open algebra* of a topological space X.

Definition 1.26. *Let X be a topological space. The* regular open algebra *of X is* $\mathrm{RO}(X) = \{a \subset X : a = \mathrm{int}\ \mathrm{cl}(a)\}$ *ordered by set-theoretical inclusion.*

All the definitions we stated about poset in section 1.1 can be extended to boolean algebras, by considering the corresponding poset $\mathbb{P}_{\mathscr{B}} = \mathscr{B} \setminus \{\mathbb{0}\}$. For example, two elements of a boolean algebra are incompatible if $a \wedge b = \mathbb{0}$.

This process can be applied also to the ideal and filter definitions. However, to gain a little more insight into what this definition really mean for boolean algebras, we first need to mention the similar concept of *boolean ring*.

Definition 1.27. *A* boolean ring *is an algebraic ring that consists only of* idempotent *elements, i.e. elements such that $x^2 = x$.*

Remark 1.1. There is a bijection between boolean rings and boolean algebras, given by $x \cdot y = x \wedge y$, $x + y = x \triangle y$, $x \vee y = x + y + x \cdot y$.

Definition 1.28. *A set $I \subset \mathscr{B}$ is an* ideal *in $\mathscr{B}$ if and only if it is closed for $\vee$ in I, and arbitrary $\wedge$:*

$$\begin{aligned} a \in I, b \in I &\quad\Rightarrow\quad a \vee b \in I \\ a \in I, b \in \mathscr{B} &\quad\Rightarrow\quad a \wedge b \in I \end{aligned}$$

Equivalently, I is an ideal of the boolean ring corresponding to $\mathscr{B}$ (or an ideal of the algebraic structure $(\mathscr{B}, \vee, \wedge)$, which however is not a ring).

Definition 1.29. *If I is an ideal of $\mathscr{B}$, the* quotient $\mathscr{B}/I$ *is the quotient of $\mathscr{B}$ with respect to the equivalence relation defined by $a =_I b \Leftrightarrow a \triangle b \in I$.*

The quotient is always well-defined, since correspond to algebraic quotient in the boolean ring interpretation.

Definition 1.30. *A set $F \subset \mathscr{B}$ is an* filter *in $\mathscr{B}$ if and only if it is closed for $\wedge$ in F, and arbitrary $\vee$:*

$$\begin{aligned} a \in F, b \in F &\quad\Rightarrow\quad a \wedge b \in F \\ a \in F, b \in \mathscr{B} &\quad\Rightarrow\quad a \vee b \in F \end{aligned}$$

Thus F is the dual of an ideal (hence an ideal of the algebraic structure $(\mathscr{B}, \wedge, \vee)$). We stress that this last definitions are all equivalent with those in section 1.1 applied to $\mathbb{P}_{\mathscr{B}} = \mathscr{B} \setminus \{\mathbb{0}\}$.

Definition 1.31. *A set $U \subset \mathscr{B}$ is an* ultrafilter *in $\mathscr{B}$ if and only if U is a filter and*

$$\forall p \in \mathscr{B} :\ p \in U \vee \neg p \in U$$

1.3 Infinite Operations and Free Constructions on Boolean Algebras

The operations of join and meet can be extended to arbitrary subsets of a boolean algebra.

Definition 1.32. *Let $\mathscr{B}$ be a boolean algebra. For $A \subset \mathscr{B}$, $\bigvee A$ ($\bigwedge A$) is the least upper bound (the greatest lower bound) of A in the partial order $\langle \mathscr{B}, \leq \rangle$, if it exists.*

In the most general case, the meet (resp. join) of a set A does not need to exist; we are interested in the boolean algebras in which it does.

Definition 1.33. *A boolean algebra $\mathscr{B}$ is complete (κ-complete) if both $\bigvee A$, $\bigwedge A$ exist for every $A \subset \mathscr{B}$ (for every $A \subset \mathscr{B}$ with $|A| < \kappa$).*

Let $\mathbf{C}_\kappa$, $\mathbf{C}_\infty$ denote the classes of κ-complete (respectively complete) boolean algebras. We define the notion of completeness for homomorphisms of boolean algebras:

Definition 1.34. *An homomorphism h of boolean algebras is complete (κ-complete) if $h(\bigvee A) = \bigvee h[A]$, $h(\bigwedge A) = \bigwedge h[A]$ for every $A \subset \mathscr{B}$ (for every $A \subset \mathscr{B}$ with $|A| < \kappa$) for which $\bigvee A$, $\bigwedge A$ exist.*

Naively, a boolean algebra $\mathscr{B}$ should be free over X if and only if X is a set of algebraically independent generators in $\mathscr{B}$. This intuition can be made precise by means of the following definition.

Definition 1.35. *A* free boolean algebra over X *is a pair $\langle e, \mathscr{B} \rangle$, with $\mathscr{B}$ boolean algebra and $e : X \to \mathscr{B}$, such that for every map $f : X \to \mathscr{B}'$ into a boolean algebra $\mathscr{B}'$ there is a unique morphism $g : \mathscr{B} \to \mathscr{B}'$ satisfying $g \circ e = f$.*

Definition 1.36. *A* free κ-complete boolean algebra over X *is a pair $\langle e, \mathscr{B} \rangle$, with $\mathscr{B}$ κ-complete boolean algebra and $e : X \to \mathscr{B}$, such that for every map $f : X \to \mathscr{B}'$ into a κ-complete boolean algebra $\mathscr{B}'$ there is a unique κ-complete morphism $g : \mathscr{B} \to \mathscr{B}'$ satisfying $g \circ e = f$.*

Theorem 1.37. *If $\langle e_1, \mathscr{B}_1 \rangle$, $\langle e_2, \mathscr{B}_2 \rangle$ are free (free κ-complete) boolean algebras over X_1, X_2, and $|X_1| = |X_2|$, then $\mathscr{B}_1$ is isomorphic to $\mathscr{B}_2$.*

Proof. Let $f : X_1 \to X_2$ be a bijection from X_1 onto X_2. Since $\mathscr{B}_1$ is free over X_1 let $\phi_1 : \mathscr{B}_1 \to \mathscr{B}_2$ be the unique morphism such that $\phi_1 \circ e_1 = e_2 \circ f$. Since also $\mathscr{B}_2$ is free over X_2, let $\phi_2 : \mathscr{B}_2 \to \mathscr{B}_1$ be the unique morphism such that $\phi_2 \circ e_2 = e_1 \circ f^{-1}$.

The composition $\phi_2 \circ \phi_1 : \mathscr{B}_1 \to \mathscr{B}_1$ is a morphism of $\mathscr{B}_1$ with $\phi_2 \circ \phi_1 \circ e_1 = e_1$; by freeness of $\mathscr{B}_1$ such a morphism is unique, hence $\phi_2 \circ \phi_1$

must be the identity on $\mathscr{B}_1$. Similarly, $\phi_1 \circ \phi_2$ must be the identity on $\mathscr{B}_2$. Then ϕ_1, ϕ_2 are inverses of each other, hence $\mathscr{B}$ is isomorphic to $\mathscr{B}'$. □

Since all free boolean algebras over X are isomorphic, we can choose $\mathrm{Fr}(X)$ to be any of them. Similarly, we can define $\mathrm{Fr}_\kappa(X)$ as the unique, up to isomorphisms, free κ-complete boolean algebra over X.

We now aim to show that $\mathrm{Fr}(X)$ and $\mathrm{Fr}_\kappa(X)$ exists for every set X.

Definition 1.38. *Let κ be an infinite cardinal, $\mathscr{B}$ a boolean algebra, $A \subset \mathscr{B}$. Then:*

$$\begin{array}{lcl} \langle A \rangle & = & \bigcap \{B : A \subset B \subset \mathscr{B} \wedge B \text{ a subalgebra of } \mathscr{B}\} \\ \langle A \rangle^{\kappa\text{-cm}} & = & \bigcap \{B : A \subset B \subset \mathscr{B} \wedge B \text{ a } \kappa\text{-complete subalgebra of } \mathscr{B}\} \\ \langle A \rangle^{\text{cm}} & = & \bigcap \{B : A \subset B \subset \mathscr{B} \wedge B \text{ a complete subalgebra of } \mathscr{B}\} \end{array}$$

is the subalgebra (resp. κ-complete subalgebra, complete subalgebra) generated by A in $\mathscr{B}$.

Proposition 1.39. *Let $\mathscr{B}$ be a κ-complete boolean algebra, κ regular, $A \subset \mathscr{B}$. If $\mathscr{B}$ is κ-completely generated by A (i.e. $\mathscr{B} = \langle A \rangle^{\kappa\text{-cm}}$), then $|\mathscr{B}| \le (|A| + \omega)^{<\kappa}$.*

Proof. Define by induction subsets A_i of $\mathscr{B}$ for $i \le \kappa$: let

$$\begin{array}{lcl} A_0 & = & A \cup \{\mathbb{0}, \mathbb{1}\} \\ A_{i+1} & = & A_i \cup \{\neg a : a \in A_i\} \cup \left\{\bigvee B : B \in \mathscr{P}_\kappa(A)\right\} \\ A_\alpha & = & \bigcup_{i<\alpha} A_i \end{array}$$

It is easily checked by induction on $i \le \kappa$ that $A_\kappa \subset \langle A \rangle^{\kappa\text{-cm}}$. Moreover, by regularity of κ we have that A_κ is a κ-complete subalgebra of $\mathscr{B}$ (it is trivially closed under $\neg$ and $\bigvee$, hence under $\bigwedge$), so $\langle A \rangle^{\kappa\text{-cm}} \subset A_\kappa$, thus $\langle A \rangle^{\kappa\text{-cm}} = A_\kappa$.

An easy induction on $i \le \kappa$ shows that $|A_\kappa| \le (|A| + \omega)^{<\kappa}$, hence $|\mathscr{B}| = \left|\langle A \rangle^{\kappa\text{-cm}}\right| = |A_\kappa| \le (|A| + \omega)^{<\kappa}$. □

Corollary 1.40. *If $\mathrm{Fr}_\kappa(X)$ exists, then $|\mathrm{Fr}_\kappa(X)| = |X|^{<\kappa}$.*

Proof. Let $\langle e, \mathscr{B} \rangle = \mathrm{Fr}_\kappa(X)$, $\mathscr{B}' = \langle X \rangle^{\kappa\text{-cm}}$ in $\mathscr{B}$. Since $\mathscr{B}' \subset \mathscr{B}$ and $\mathscr{B}$ is free over X, the map e can be extended to a κ-complete morphism $f : \mathscr{B} \to \mathscr{B}'$. Then the identity map I_B of B and f are both endomorphisms of $\mathscr{B}$ that extend e. So it must be that $f = I_B$ hence $\mathscr{B} = \mathscr{B}'$ and by Property 1.39, $|\mathscr{B}| = |\mathscr{B}'| \le |X|^{<\kappa}$.

We now prove that $|\mathscr{B}| \ge |X|^{<\kappa}$. Otherwise, there should be two different $x, y \in \mathscr{P}_\kappa(X)$ such that $\bigwedge e[x] = \bigwedge e[y]$. Suppose without loss of generality that $y \not\subseteq x$. Then the map $e' : X \to \mathbb{2}$ defined by $e'[x] = \{\mathbb{1}\}$, $e'[y \setminus x] = \{\mathbb{0}\}$, cannot be extended to a morphism ϕ from $\mathscr{B}$ to $\mathbb{2}$ since

$$\begin{array}{lcl} \mathbb{1} & = & \bigwedge e'[x] = \bigwedge \phi[e[x]] = \phi(\bigwedge e[x]) = \\ & = & \phi(\bigwedge e[y]) = \bigwedge \phi[e[y]] = \bigwedge e'[y] \le \bigwedge e'[y \setminus x] = \mathbb{0} \end{array}$$

that implies $\mathbb{1} \leq \mathbb{0}$, a contradiction. □

Theorem 1.41. $\mathrm{Fr}_\kappa(X)$ *exists for any set* X *and regular* κ.

Proof. Let $\lambda = |X|$, and suppose $\lambda \geq \omega$, otherwise $\mathrm{Fr}_\kappa(X) = \mathrm{Fr}(X)$ is the finite boolean algebra with $2^{2^{|X|}}$ elements. Let $\mathscr{F}$ be the family of κ-complete boolean algebras whose domain is contained in $\lambda^{<\kappa}$ generated by an image of X:

$$\mathscr{F} = \left\{\langle h, B\rangle : \; B \in \mathbf{C}_\kappa \wedge \mathrm{dom}(B) \subset \lambda^{<\kappa} \wedge h : X \to B \wedge \langle h[X]\rangle^{\kappa\text{-cm}} = B\right\}$$

If we enumerate $\mathscr{F} = \{\langle h_i, B_i\rangle : i < \theta\}$, let $B' = \prod_{i<\theta} B_i$, and set $h' : X \to B'$ such that $h'(x) = \langle h_i(x) : i < \theta\rangle$.

Consider in the boolean algebra B' the subalgebra $\mathscr{B} = \langle h'[X]\rangle^{\kappa\text{-cm}}$. We show that $\langle h', \mathscr{B}\rangle$ is a free κ-complete boolean algebra over X.

Given C a κ-complete boolean algebra, $k : X \to C$, let $C' = \langle k[X]\rangle^{\kappa\text{-cm}}$. By Property 1.39 $|C'| \leq \lambda^{<\kappa}$, so there must be a bijection $\phi : C' \to C'' \subset \lambda^{<\kappa}$, then $\left\langle \phi \circ k, C''\right\rangle = \langle h_i, B_i\rangle \in \mathscr{F}$ for some $i < \theta$. Let $\pi_i : B' \to B_i$ be the canonical projection onto B_i, such that $\pi_i \circ h' = h_i$. Then $\phi^{-1} \circ \pi_i$ is a κ-complete homomorphism from $\mathscr{B}$ to C satisfying $\phi^{-1} \circ \pi_i \circ h' = \phi^{-1} \circ h_i = k$ (since $h_i = \phi \circ k$). □

Remark 1.2. With abuse of notation, the set X can be identified with its image $e[X]$, hence regarding a boolean algebra $\mathscr{B}$ free over X as a superset of X.

1.4 Boolean-Valued Models

In this section, we shall briefly expose a generalization of first-order models with boolean algebras, important for its wide applications in consistency proofs. Our reference text for statements in the next few sections is [2, 14].

Boolean-valued models can be defined for any signature σ (even if we will be interested only in models of set theory with canonical signature) and complete boolean algebra $\mathscr{B}$. Such a model will consist of a set $\mathscr{M}$ of *names*, together with an interpretation $\overline{f} : \mathscr{M}^n \to \mathscr{M}$ for every n-ary function symbol $f \in \sigma$, and an interpretation $\overline{R} : \mathscr{M}^n \to \mathscr{B}$ for every n-ary relation symbol $R \in \sigma$ (including equality) satisfying some axioms we will see later.

The interpretation function induce an assignment of a *truth value* $[\![\phi]\!] \in \mathscr{B}$ to every closed formula ϕ, by means of:

1. $[\![R(t_1, \ldots, t_n)]\!] = \overline{R}(t_1, \ldots, t_n)$
2. $[\![\neg\phi]\!] = \neg\, [\![\phi]\!]$
3. $[\![\phi \vee \psi]\!] = [\![\phi]\!] \vee [\![\psi]\!]$

4. $[\![\exists x\phi(x)]\!] = \bigvee_{t\in\mathcal{M}} [\![\phi(t)]\!]$

As previously stated, the interpretation function is required to satisfy the following axioms.

1. $[\![s = s]\!] = \mathbb{1}$
2. $[\![s = t]\!] = [\![t = s]\!]$
3. $[\![s = t]\!] \wedge [\![t = u]\!] \leq [\![s = u]\!]$
4. $[\![\vec{s} = \vec{t}]\!] \wedge [\![R(\vec{s})]\!] \leq [\![R(\vec{t})]\!]$
5. $[\![\vec{s} = \vec{t}]\!] \wedge [\![u = f(\vec{s})]\!] \leq [\![u = f(\vec{t})]\!]$

Axioms 1-3 are a generalization of equality axioms to boolean-valued models, axioms 4-5 are substitution axioms with R relation symbol and f function symbol in σ.

Note how the notion of a boolean-valued model generalizes the notion of a model, and the boolean value generalizes the satisfaction predicate $\vDash$. If $\mathcal{B} = \mathbb{2}$, then a boolean-valued model is just a (two-valued) model; considering $\mathcal{M}/\equiv$ where $t \equiv s \Leftrightarrow [\![t = s]\!] = \mathbb{1}$.

In most cases, boolean-valued models can be used almost like models, by replacing the notion of truth with the notion of *validity*.

Definition 1.42. *A formula ϕ is* valid *in $\mathcal{M}$ if $[\![\phi]\!] = \mathbb{1}$. A theory T is* valid *in $\mathcal{M}$ if for every $\phi \in T$, ϕ is valid.*

Theorem 1.43. *If a formula ϕ is provable from T and T is valid in $\mathcal{M}$, then ϕ is valid in $\mathcal{M}$.*

Proof. We have to check validity of logical axioms and rules of inference for first-order logic. We shall use the deductive system reported in [5].

The propositional axiom $(\neg A \vee A)$ hold since $[\![\neg A \vee A]\!] = \neg [\![A]\!] \vee [\![A]\!] = \mathbb{1}$. The other propositional rules need similar trivial verifications.

Expansion Rule ($A \vdash B \vee A$): $[\![A]\!] = \mathbb{1} \Rightarrow [\![B \vee A]\!] = [\![B]\!] \vee \mathbb{1} = \mathbb{1}$.

Contraction Rule ($A \vee A \vdash A$): $[\![A \vee A]\!] = \mathbb{1} \Rightarrow [\![A]\!] = [\![A]\!] \vee [\![A]\!] = [\![A \vee A]\!] = \mathbb{1}$.

Association Rule ($A \vee (B \vee C) \vdash (A \vee B) \vee C$): if $[\![A \vee (B \vee C)]\!] = \mathbb{1}$, we have $[\![(A \vee B) \vee C]\!] = [\![A]\!] \vee [\![B]\!] \vee [\![C]\!] = [\![A \vee (B \vee C)]\!] = \mathbb{1}$.

Cut Rule ($A \vee B, \neg A \vee C \vdash B \vee C$): the hypothesis $[\![A \vee B]\!] = \mathbb{1}$, $[\![\neg A \vee C]\!] = \mathbb{1}$ implies that $B \geq \neg A$, $C \geq A$ hence $[\![B \vee C]\!] \geq [\![\neg A \vee A]\!] = \mathbb{1}$.

The first-order axioms and rules can now be similarly verified.

Substitution Axiom ($A(a) \to \exists x A(x)$): $[\![\exists x A(x)]\!] = \bigvee_{t \in \mathcal{M}} [\![A(t)]\!] \geq A(a)$.

Identity Axiom ($x = x$): is axiom 1 of boolean valued models.

Equality Axioms ($\vec{x} = \vec{y} \to f(\vec{x}) = f(\vec{y})$, $\vec{x} = \vec{y} \to R(\vec{x}) \to R(\vec{y})$): are axioms 4-5 of boolean valued models.

$\exists$-Introduction Rule ($\forall x\,(A(x) \to B) \vdash (\exists x A(x)) \to B$): if $[\![\forall x\,(A(x) \to B)]\!] = \mathbb{1}$, then $[\![(\exists x A(x)) \to B]\!] = [\![(\forall x \neg A(x)) \vee B]\!] = \left(\bigwedge_{t \in \mathcal{M}} [\![\neg A(t)]\!]\right) \vee [\![B]\!] = \bigwedge_{t \in \mathcal{M}} ([\![\neg A(t)]\!] \vee [\![B]\!]) = [\![\forall x\,(\neg A(x) \vee B)]\!] = [\![\forall x\,(A(x) \to B)]\!] = 1$. □

The last result is crucial, because allows us to use boolean-valued models instead of models for consistency proofs.

Corollary 1.44. *If T is valid in $\mathcal{M}$ and $[\![\phi]\!] > \mathbb{0}$, then ϕ is consistent with T.*

Proof. If ϕ where not consistent with T, $\neg\phi$ would be provable from T hence valid in $\mathcal{M}$; then $[\![\phi]\!] = \neg\,[\![\neg\phi]\!] = \neg\mathbb{1} = \mathbb{0}$, a contradiction. □

A relevant property of boolean-valued models is the following.

Definition 1.45. *A boolean-valued model $\mathcal{M}$ is* full *if it satisfies the fullness lemma, i.e., for every formula ϕ with parameters in $\mathcal{M}$,*

$$\exists t \in \mathcal{M} : \; [\![\exists x\; \phi(x)]\!] = [\![\phi(t)]\!]$$

Boolean-valued models with this property can be transformed into two-valued models, by means of the following construction.

Definition 1.46. *Let $\mathcal{M}$ be a boolean-valued model, and $U \subset \mathcal{B}$ be an ultrafilter. Then $\equiv_U$ is a two-valued relation on $\mathcal{M}$ defined by:*

$$t_1 \equiv_U t_2 \iff [\![t_1 = t_2]\!] \in U$$

Definition 1.47. *Let $\mathcal{M}$ be a boolean-valued model and $U \subset \mathcal{B}$ an ultrafilter. Then $\mathcal{M}_U$ is the two-valued model with support $\mathcal{M}/\equiv_U$, functions and relations f_U, R_U induced by the canonical projection $\pi_U : \mathcal{M} \to \mathcal{M}/\equiv_U$. More precisely,*

$$\begin{aligned} f_U(t^1_U, \ldots, t^n_U) &= \pi_U\left(f(t^1, \ldots, t^n)\right) \\ R_U(t^1_U, \ldots, t^n_U) &\iff R(t^1, \ldots, t^n) \in U \end{aligned}$$

where $t^1, \ldots, t^n$ are any representatives for the equivalence classes $t^1_U, \ldots, t^n_U$.

The fact that f_U, R_U are well-defined is easily checked using axioms 4-5 of boolean-valued models. From now on, we will use the notation x_U to mean $\pi_U(x)$ for any $x \in \mathscr{M}$. Notice that the equivalence classes in $\mathscr{M}_U$ might be quite large, e.g. when $\mathscr{M}$ is a model of set theory these classes can not be sets. However, this technical problem can be circumvented by means of the well-known *Scott's Trick* (i.e. by choosing as representatives of the equivalence classes only the elements with minimal rank).

When the boolean-valued model $\mathscr{M}$ is full, there is a close connection between $\mathscr{M}$ and $\mathscr{M}_U$, explained by the following Theorem 1.48 (generalization of Łoś Theorem for generic boolean-valued models).

Theorem 1.48 (Łoś)**.** *Let $\mathscr{M}$ be full, U ultrafilter on $\mathscr{B}$. For any formula $\phi(x_1, \ldots, x_n)$,*

$$\mathscr{M}_U \vDash \phi([x_1], \ldots, [x_n]) \iff [\![\phi(x_1, \ldots, x_n)]\!] \in U$$

Proof. If ϕ is atomic, the thesis is true by definition. If ϕ is a negation or a conjunction, it follows from the basic properties of ultrafilters and truth value.

Let now ϕ be $\exists x \psi(x)$. Let $a \in \mathscr{M}$ be such that $[\![\psi(a)]\!] = [\![\exists x \psi(x)]\!]$ by fullness; then $[\![\exists x \psi(x)]\!] \in U \Rightarrow \exists a \ [\![\psi(a)]\!] \in U \Rightarrow \mathscr{M}_U \vDash \psi(a) \Rightarrow \mathscr{M}_U \vDash \exists x \psi(x)$. Also, $\mathscr{M}_U \vDash \exists x \psi(x) \Rightarrow \exists a \ [\![\psi(a)]\!] \in U \Rightarrow \bigvee_{a \in \mathscr{M}} [\![\psi(a)]\!] = [\![\exists x \psi(x)]\!] \in U$. □

1.5 The Model $V^{\mathscr{B}}$

We are now interested in defining a boolean-valued model of ZFC with canonical signature $\sigma = (=, \in)$. In particular, we are interested in a canonical construction of a boolean-valued model of ZFC starting from a model M and a complete boolean algebra $\mathscr{B}$.

Throughout this paper, we shall also use additional symbols for definable classes when convenient. Although, this symbols can be eliminated then is sufficient to study the case $\sigma = (=, \in)$.

Definition 1.49. *For M a model of* ZFC *and $\mathscr{B} \in M$ a complete boolean algebra, we define:*

$$\begin{array}{rcl} M_0^{\mathscr{B}} & = & \emptyset \\ M_{\alpha+1}^{\mathscr{B}} & = & \left\{\tau \in M : \tau \subset M_\alpha^{\mathscr{B}} \times \mathscr{B} \wedge \tau \textit{ is a partial function}\right\} \\ M_\alpha^{\mathscr{B}} & = & \bigcup_{\beta<\alpha} M_\beta^{\mathscr{B}} \qquad \text{for } \alpha \text{ limit} \\ M^{\mathscr{B}} & = & \bigcup_{\alpha \in \mathbf{ON}} M_\alpha^{\mathscr{B}} \end{array}$$

We call the elements of $M^{\mathscr{B}}$ $\mathscr{B}$-names.

Definition 1.50. *Given $\tau \in M^{\mathscr{B}}$, define the* rank *of τ as:*

$$\operatorname{rank}(\tau) = \min\left\{\alpha \in \mathbf{ON} :\ \tau \in M^{\mathscr{B}}_{\alpha+1}\right\}$$

Definition 1.51. *Given $x \in M$, define the canonical $\mathscr{B}$-name $\check{x}$ recursively by:*

$$\check{x} = \{\langle \check{y}, \mathbb{1}\rangle : y \in x\}$$

Theorem 1.52. *$M^{\mathscr{B}}$ is a boolean-valued model of set theory, where:*

$$\begin{array}{rcl} [\![\tau_1 \in \tau_2]\!] &=& \bigvee_{\langle\sigma,p\rangle\in\tau_2} ([\![\tau_1 = \sigma]\!] \wedge p) \\ [\![\tau_1 \subseteq \tau_2]\!] &=& \bigwedge_{\langle\sigma,p\rangle\in\tau_1} (\neg p \vee [\![\sigma \in \tau_2]\!]) \\ [\![\tau_1 = \tau_2]\!] &=& [\![\tau_1 \subseteq \tau_2]\!] \wedge [\![\tau_2 \subseteq \tau_1]\!] \end{array}$$

Proof. We need to check axioms 1-4, since we don't have any function symbols (axiom 5).

For axiom 1, it suffices to show that $[\![\tau \subseteq \tau]\!] = \mathbb{1}$. By induction on the rank of $\mathscr{B}$-names we have that $[\![\emptyset \subseteq \emptyset]\!] = \mathbb{1}$ and:

$$\begin{array}{rcl} [\![\tau \subseteq \tau]\!] &=& \bigwedge_{\langle\sigma_1,p_1\rangle\in\tau} \neg p_1 \vee [\![\sigma_1 \in \tau]\!] \\ &=& \bigwedge_{\langle\sigma_1,p_1\rangle\in\tau} \left[\neg p_1 \vee \bigvee_{\langle\sigma_2,p_2\rangle\in\tau} ([\![\sigma_2 = \sigma_1]\!] \wedge p_2)\right] \\ &\geq& \bigwedge_{\langle\sigma_1,p_1\rangle\in\tau} [\neg p_1 \vee ([\![\sigma_1 = \sigma_1]\!] \wedge p_1)] = \bigwedge_{\langle\sigma_1,p_1\rangle\in\tau} [\neg p_1 \vee p_1] = \mathbb{1} \end{array}$$

Axiom 2 holds trivially since the formula for computing the boolean value for equality is symmetrical.

Axiom 3 and 4 can be similarly shown to hold by simultaneous induction on the rank of $\mathscr{B}$-names. A complete proof can be found in [2, Lemma 14.16]. □

From now on, V will be a model of ZFC with canonical signature $(=, \in)$, $\mathscr{B}$ a complete boolean algebra in V, and we will concern only about the boolean-valued model $V^{\mathscr{B}}$.

Lemma 1.53 (Mixing)**.** *If $A \subset \mathscr{B}$ is an antichain, $\langle \tau_a :\ a \in A\rangle$ are names in $V^{\mathscr{B}}$, there exists a name τ such that*

$$\forall a \in A\ [\![\tau = \tau_a]\!] \geq a$$

Proof. Let $\tau = \bigcup_{a\in A} \{\langle \sigma, p \wedge a\rangle :\ \langle \sigma, p\rangle \in \tau_a\}$, then τ witnesses the truth of the property above. □

Lemma 1.54 (Fullness)**.** *$V^{\mathscr{B}}$ is full.*

Proof. Given ϕ formula with parameters in M, let A be a maximal antichain below $\{[\![\phi(t)]\!] : t \in V^{\mathscr{B}}\}$. For all $a \in A$, let $\tau_a \in V^{\mathscr{B}}$ be such that $[\![\phi(\tau_a)]\!] \geq a$. By Mixing Lemma 1.53, let τ be such that for all $a \in A$, $[\![\tau = \tau_a]\!] \geq a$.

Then $[\![\phi(\tau)]\!] \geq [\![\tau = \tau_a]\!] \wedge [\![\phi(\tau_a)]\!] \geq a$ for all $a \in A$, hence $[\![\phi(\tau)]\!] \geq \bigvee A$. Thus, by maximality of A, we have $[\![\phi(\tau)]\!] = [\![\exists x\ \phi(x)]\!]$. □

From this well-known lemmas, one may prove the following main result.

Theorem 1.55. *For every axiom ϕ in* ZFC, $[\![\phi]\!] = \mathbb{1}$.

Proof. We need to verify nine axioms. Complete proofs for the following results can be found in [2, Theorem 14.24].

Extensionality: If ϕ is extensionality, for any x, y:

$$\begin{aligned}
[\![\phi(x,y)]\!] &= [\![\forall z\,(z \in x \longleftrightarrow z \in y) \to x = y]\!] \\
&= \neg\,[\![\forall z\,(z \in x \longleftrightarrow z \in y)]\!] \vee [\![x = y]\!] \\
&= \neg \textstyle\bigwedge_{z \in V^{\mathscr{B}}} [\![z \in x \longleftrightarrow z \in y]\!] \vee [\![x = y]\!]
\end{aligned}$$

We have that:

$$\begin{aligned}
\textstyle\bigwedge_{z \in V^{\mathscr{B}}} [\![z \in x \to z \in y]\!] &= \textstyle\bigwedge_{z \in V^{\mathscr{B}}} [\![\neg z \in x \vee z \in y]\!] \\
&\leq \textstyle\bigwedge_{\langle z,p\rangle \in x} \neg [\![z \in x]\!] \vee [\![z \in y]\!] \\
&\leq \textstyle\bigwedge_{\langle z,p\rangle \in x} \neg p \vee [\![z \in y]\!] \\
&= [\![x \subseteq y]\!]
\end{aligned}$$

The third passage follows from $\langle z,p\rangle \in x \Rightarrow [\![z \in x]\!] \geq p$. Then,

$$\begin{aligned}
[\![\phi(x,y)]\!] &= \neg \textstyle\bigwedge_{z \in V^{\mathscr{B}}} [\![z \in x \longleftrightarrow z \in y]\!] \vee [\![x = y]\!] \\
&\geq \neg\,([\![x \subseteq y]\!] \wedge [\![y \subseteq x]\!]) \vee [\![x = y]\!] \\
&= \neg\,[\![x = y]\!] \vee [\![x = y]\!] = \mathbb{1}
\end{aligned}$$

Pairing: Given $x, y \in V^{\mathscr{B}}$, $z = \{\langle x, \mathbb{1}\rangle, \langle y, \mathbb{1}\rangle\} \in V^{\mathscr{B}}$ is a witness for the pairing axiom.

Separation: Given $x \in V^{\mathscr{B}}$ and a formula with parameters $\phi(z)$, $y = \{\langle z, p \wedge [\![\phi(z)]\!]\rangle : \langle z,p\rangle \in x\} \in V^{\mathscr{B}}$ is a witness for the separation axiom.

Union: Given $x \in V^{\mathscr{B}}$, $y = \mathrm{dom}\left(\bigcup \mathrm{dom}(x)\right) \times \{\mathbb{1}\}$ is a witness for the union axiom, in the weak form.

Power Set: Given $x \in V^{\mathscr{B}}$, $y = \{z \in V^{\mathscr{B}} : z : \mathrm{dom}(x) \to \mathscr{B}\} \times \{\mathbb{1}\}$ is a witness for the power set axiom, in the weak form.

Infinity: $\check{\omega}$ is a witness for the axiom of infinity.

Replacement: It suffices to verify the Collection Principle, hence that for every $x \in V^{\mathscr{B}}$ there is an $y \in V^{\mathscr{B}}$ such that:

$$[\![\forall u \in x\,(\exists v\,\phi(u,v) \to \exists v \in y\,\phi(u,v))]\!] = \mathbb{1}$$

Here we let $y = \bigcup \{y_u : u \in \mathrm{dom}(x)\} \times \{\mathbb{1}\}$, where y_u is a set such that:

$$\bigvee_{v \in V^{\mathscr{B}}} [\![\phi(u,v)]\!] = \bigvee_{v \in y_u} [\![\phi(u,v)]\!]$$

Foundation: Suppose by contradiction that there exists an $x \in V^{\mathscr{B}}$, such that:

$$[\![\exists u\,(u \in x) \wedge (\forall y \in x)\,(\exists z \in y)\,z \in x]\!] = b > \mathbb{0}$$

Let $y \in V^{\mathscr{B}}$ be of least rank such that $[\![y \in x]\!] \wedge b > \mathbb{0}$. Since $[\![y \in x]\!] \wedge b \leq [\![(\exists z \in y)\,z \in x]\!]$, there is a $z \in \mathrm{dom}(y)$ such that $[\![z \in x]\!] \wedge [\![y \in x]\!] \wedge b > \mathbb{0}$, which contradicts the minimality of the rank of y.

Choice: Given $x \in V^{\mathscr{B}}$, let $f = \left\{\langle y, \check{y}\rangle_{V^{\mathscr{B}}} : y \in \mathrm{dom}(x)\right\} \times \{\mathbb{1}\}$. Since

$$f[x] \subset \check{\mathrm{dom}}(x) = \{\langle \check{y}, \mathbb{1}\rangle : y \in \mathrm{dom}(x)\}$$

define $\prec$ on x as $[\![y \prec z]\!] = [\![f(y)\,\check{R}\,f(z)]\!]$, where R is any well-order of $\mathrm{dom}(x)$ in V. Then $[\![x \text{ is well-ordered by } \prec]\!] = \mathbb{1}$. □

As a consequence of Corollary 1.44, we can now prove that for any formula ϕ, whenever we can find a complete boolean algebra $\mathscr{B}$ such that $[\![\phi]\!] > \mathbb{0}$ in $V^{\mathscr{B}}$, the formula ϕ is consistent with ZFC. However, we are also interested in actually defining a two-valued ZFC model in which ϕ is true.

Corollary 1.56. *For any ultrafilter $U \subset \mathscr{B}$, $V^{\mathscr{B}}_U$ is a model of* ZFC.

Proof. $V^{\mathscr{B}}$ is full by Theorem 1.54. Then, this result is an easy application of Łoś Theorem 1.48 and Theorem 1.55. □

If U is any ultrafilter such that $[\![\phi]\!] \in U$, from Łoś Theorem 1.48 $[\![\phi]\!] \in U$ implies that $V^{\mathscr{B}}_U \vDash \phi$, thus $V^{\mathscr{B}}_U$ is the required two-valued model. However, in most cases $V^{\mathscr{B}}_U$ will not be well-founded, hence not isomorphic to any transitive model of ZFC in V.

Theorem 1.57. *$V^{\mathscr{B}}_U$ is well-founded if and only if U is ω_1-closed.*

Proof. First, suppose that U is not ω_1-closed. Let $\langle u_i : i \in \omega\rangle$ be a decreasing chain in U with $\bigwedge_{i \in \omega} u_i = u_\omega \notin U$. For every $n \in \omega$, let $A_n = \left\{\left\langle \check{i}, u_{n+i}\right\rangle : i \in \omega\right\}$.

Notice that $[\![A_n \subseteq \omega]\!] = \mathbb{1}$, $\left[\!\left[A_n \subseteq \check{i}\right]\!\right] = \neg u_{n+i+1}$, $\left[\!\left[A_n \supseteq \check{i}\right]\!\right] = u_{n+i}$. Then $\langle \pi_U(A_n) : n \in \omega \rangle$ is an ill-founded chain:

$$\begin{aligned}
[\![A_{n+1} \in A_n]\!] &= \bigvee_{i\in\omega} \left(\left[\!\left[A_{n+1} = \check{i}\right]\!\right] \wedge u_{n+i}\right) \\
&= \bigvee_{i\in\omega} \left(\left[\!\left[A_{n+1} \subseteq \check{i}\right]\!\right] \wedge \left[\!\left[\check{i} \subseteq A_{n+1}\right]\!\right] \wedge u_{n+i}\right) \\
&= \bigvee_{i\in\omega} (\neg u_{n+1+i} \wedge u_{n+i} \wedge u_{n+i}) \\
&= \bigvee_{i\in\omega} (u_{n+i} \wedge \neg u_{n+i+1}) \\
&= \bigvee_{i\in\omega} \bigvee_{j<i} \left(u_{n+j} \wedge \neg u_{n+j+1}\right) \\
&= \bigvee_{i\in\omega} (u_n \wedge \neg u_{n+i}) = u_n \wedge \neg \bigwedge_{i\in\omega} u_{n+i} = u_n \wedge \neg u_\omega \in U
\end{aligned}$$

The equality between the last two rows is easily proved by induction using the fact that $[u_n \wedge \neg u_{n+i}] \vee [u_{n+i} \wedge \neg u_{n+i+1}] = u_n \wedge \neg u_{n+i+1}$.

Finally, suppose that U is ω_1-closed but $V_U^{\mathscr{B}}$ is ill-founded. Let $\langle \pi_U(A_n) :\ n \in \omega \rangle$ be an infinite decreasing chain in $V_U^{\mathscr{B}}$, i.e. $[\![A_{n+1} \in A_n]\!] \in U$ for every $n \in \omega$. Let $u = \bigwedge_{n\in\omega} [\![A_{n+1} \in A_n]\!]$, we have that $u \in U$ since U is ω_1-closed, hence $u > \mathbb{0}$.

From Theorem 1.55, we know that $[\![\text{foundation axiom}]\!] = \mathbb{1}$, hence $[\![A \text{ is well-founded}]\!] = \mathbb{1}$ for any set $A \in V^{\mathscr{B}}$ for which $\left[\!\left[A \neq \check{\emptyset}\right]\!\right] = \mathbb{1}$, in particular for $A = \{\langle A_i, \mathbb{1}\rangle :\ i \in \omega\}$.

$$\begin{aligned}
[\![A \text{ is ill-founded}]\!] &= [\![(\forall y \in A)\,(\exists z \in y)\, z \in A]\!] \\
&= \bigwedge_{y\in V^{\mathscr{B}}} [\![y \in A]\!] \to [\![(\exists z \in y)\, z \in A]\!] \\
&= \bigwedge_{y\in V^{\mathscr{B}}, i<\omega} \neg [\![y = A_i]\!] \vee [\![(\exists z \in y)\, z \in A]\!] \\
&\geq \bigwedge_{y\in V^{\mathscr{B}}, i<\omega} \neg [\![y = A_i]\!] \vee [\![(\exists z \in A_i) z \in A]\!] \\
&\geq \bigwedge_{y\in V^{\mathscr{B}}, i<\omega} \neg [\![y = A_i]\!] \vee [\![A_{i+1} \in A_i \wedge A_{i+1} \in A]\!] \\
&= \bigwedge_{y\in V^{\mathscr{B}}, i<\omega} \neg [\![y = A_i]\!] \vee [\![A_{i+1} \in A_i]\!] \\
&\geq \bigwedge_{y\in V^{\mathscr{B}}, i<\omega} (\neg [\![y = A_i]\!] \vee u) \ \geq\ u
\end{aligned}$$

Then $[\![A \text{ is well-founded}]\!] \leq \neg u < \mathbb{1}$, a contradiction. □

1.6 Boolean Ultrapowers

The results in the next few sections and Theorem 1.57 are based on a talk held by J. D. Hamkins at the Young Set Theory Workshop in Bonn, 2011, where he presented some of the material that will appear in [9], which at the present moment is still in preparation.

In this section we shall investigate the relationship between the boolean-valued model $V^{\mathscr{B}}$ and the original two-valued model V. We can introduce in $V^{\mathscr{B}}$ a "name for V" (which might be thought as a new defined symbol in the language), and an elementary map $j : V \to \check{V}$, that in some cases will induce an isomorphism (in a sense that will be made precise later).

Definition 1.58. *Let $\check{V}$ be a new relation symbol defined by:*

$$\left[\!\left[\tau \in \check{V}\right]\!\right] = \bigvee_{x \in V} [\![\tau = \check{x}]\!]$$

Note that $\check{V}$ is not simply the class $\{\check{x} : x \in V\}$ (as e.g. contains every mixing of a sequence $x_1, \ldots, x_n$ by an antichain), and that $\check{V}$ is a boolean-valued subclass of $V^{\mathscr{B}}$, in the sense that there are $\tau \in V^{\mathscr{B}}$ with $\mathbb{0} < \left[\!\left[\tau \in \check{V}\right]\!\right] < \mathbb{1}$. The symbol $\check{V}$ can be represented by its characteristic function $\chi_{\check{V}} : V^{\mathscr{B}} \to \mathscr{B}$, $\chi_{\check{V}}(x) = \left[\!\left[x \in \check{V}\right]\!\right]$.

Theorem 1.59. *For every sets $x_1, \ldots, x_n$ in V and formula ϕ, the following holds:*

$$V \vDash \phi(x_1, \ldots, x_n) \iff \left[\!\left[\phi^{\check{V}}(\check{x_1}, \ldots, \check{x_n})\right]\!\right] = \mathbb{1}$$

Proof. We shall prove this fact by induction on the complexity of the formula ϕ. Note that the inductive hypothesis applied to both ϕ and $\neg\phi$ implies that $\left[\!\left[\phi^{\check{V}}\right]\!\right] \in \{\mathbb{0}, \mathbb{1}\}$, fact that will be needed later.

If ϕ is atomic, it follows directly from the definition of canonical $\mathscr{B}$-name (by easy induction on the rank of $\mathscr{B}$-names and extensionality).

The verification for $\phi = \neg\psi$ is trivial, whence if $\phi = \psi_1 \wedge \psi_2$ it follows from the fact that:

$$[\![\psi_1 \wedge \psi_2]\!] = \mathbb{1} \Leftrightarrow [\![\psi_1]\!] = \mathbb{1} \wedge [\![\psi_2]\!] = \mathbb{1}$$

If ϕ is $\exists x \psi(x)$, we have that:

$$\begin{aligned} V \vDash \exists x \psi(x) &\Rightarrow (\exists x \in V)\, V \vDash \psi(x) \\ &\Rightarrow (\exists x \in V) \left[\!\left[\psi^{\check{V}}(\check{x})\right]\!\right] = \mathbb{1} \\ &\Rightarrow \left[\!\left[\left(\exists x \in \check{V}\right) \psi^{\check{V}}(x)\right]\!\right] = \mathbb{1} \end{aligned}$$

and the converse also holds by fullness of $V^{\mathscr{B}}$:

$$\begin{aligned} \left[\!\left[\left(\exists x \in \check{V}\right) \psi^{\check{V}}(x)\right]\!\right] = \mathbb{1} &\Rightarrow \exists x \in V^{\mathscr{B}} \left[\!\left[x \in \check{V} \wedge \psi^{\check{V}}(x)\right]\!\right] = \mathbb{1} \\ &\Rightarrow \bigvee_{y \in V} \left[\!\left[\psi^{\check{V}}(\check{y})\right]\!\right] = \mathbb{1} \\ &\Rightarrow \left(\exists y \in V\right) \left[\!\left[\psi^{\check{V}}(\check{y})\right]\!\right] = \mathbb{1} \Rightarrow V \vDash \exists x \psi(x) \end{aligned}$$

□

Corollary 1.60. *If $\check{V}$ is defined as above, the following holds:*

$$\left[\!\left[\check{V} \text{ is a transitive inner model of ZFC with all ordinals}\right]\!\right] = \mathbb{1}$$

Proof. Since $V \vDash \phi$ for every ϕ axiom of ZFC, from Theorem 1.59 we already know that $\check{V}$ is an inner model of ZFC. Furthermore,

$$\begin{aligned}
\left[\!\left[\check{V} \text{ is transitive} \right]\!\right] &= \left[\!\left[\left(\forall x \in \check{V}\right) (\forall y \in x)\ y \in \check{V} \right]\!\right] \\
&= \bigwedge_{x,y \in V^{\mathscr{B}}} \left[\!\left[x \in \check{V} \to \left(y \in x \to y \in \check{V} \right) \right]\!\right] \\
&= \bigwedge_{x,y \in V^{\mathscr{B}}, z \in V} \left[\!\left[x = \check{z} \to \left(y \in x \to y \in \check{V} \right) \right]\!\right] \\
&\geq \bigwedge_{x,y \in V^{\mathscr{B}}, z \in V} \left[\!\left[x = \check{z} \to \left(y \in \check{z} \to y \in \check{V} \right) \right]\!\right] \\
&= \bigwedge_{x,y \in V^{\mathscr{B}}, z \in V, w \in z} \neg [\![x = \check{z}]\!] \vee \left[\!\left[y = \check{w} \to y \in \check{V} \right]\!\right] \\
&\geq \bigwedge_{x,y \in V^{\mathscr{B}}, z \in V, w \in z} \neg [\![x = \check{z}]\!] \vee \left[\!\left[y = \check{w} \to \check{w} \in \check{V} \right]\!\right] \\
&= \bigwedge_{x,y \in V^{\mathscr{B}}, z \in V, w \in z} \neg [\![x = \check{z}]\!] \vee [\![y = \check{w} \to \mathbb{1}]\!] = \mathbb{1}
\end{aligned}$$

The inequalities used are derived from the converse of the substitution axiom for boolean-valued models (axiom 4).

We still have to check that for every x in $V^{\mathscr{B}}$:

$$\left[\!\left[x \text{ is an ordinal } \to x \in \check{V} \right]\!\right] = \mathbb{1}$$

A complete proof of this fact can be found in [2, Lemma 14.23]. □

The last results suggests us that $\check{V}$ is an inner model of $V^{\mathscr{B}}$ that behaves almost like V, so that the map $j : V \to \check{V}$ defined by $j(x) = \check{x}$ should be an elementary-like embedding. Since we have no definition for what an elementary embedding from a two-valued model to a boolean-valued model should be, we shall now consider the quotient $V_U^{\mathscr{B}}$ instead, in Definition 1.47.

Corollary 1.61. *Let $\check{V}_U = \left\{ \pi_U(x) \colon \left[\!\left[x \in \check{V} \right]\!\right] \in U \right\}$ and $j_U : V \to \check{V}_U$ be the map defined by $j_U = j \circ \pi_U$ (i.e., $j_U(x) = \check{x}_U$). Then j_U is an elementary embedding.*

Proof. By Theorem 1.59 and 1.48:

$$\begin{aligned}
V \vDash \phi(x_1, \ldots, x_n) &\Rightarrow \left[\!\left[\phi^{\check{V}}(\check{x}_1, \ldots, \check{x}_n) \right]\!\right] = \mathbb{1} \in U \\
&\Rightarrow V_U^{\mathscr{B}} \vDash \phi^{\check{V}_U}([\check{x}_1]_U, \ldots, [\check{x}_n]_U) \\
&\Rightarrow \check{V}_U \vDash \phi(j_U(x_1), \ldots, j_U(x_n))
\end{aligned}$$

The reverse implication follows from taking $\phi' = \neg\phi$. □

The class $\check{V}_U$ is called the *boolean ultrapower* of V with the ultrafilter U. We shall later see that when the ultrafilter U is V*-generic*, j_U is in fact an isomorphism (i.e., $\check{V}_U$ is a trivial ultrapower); however, in the most general case j_U will not be an isomorphism: e.g. when U is not ω_1-closed (see Theorem 1.57).

The name "boolean ultrapower" is justified, as we shall see in the next few theorems, by its close relationship with the classical model ultrapower, the two concepts being coincident when the boolean algebra $\mathscr{B}$ is an algebra of sets.

Theorem 1.62. *Let $\mathscr{B}$ be a boolean algebra, $A \subset \mathscr{B}$ a maximal antichain, U ultrafilter on $\mathscr{B}$, $U_A = \left\{X \in \mathscr{P}(A) : \bigvee X \in U\right\}$ ultrafilter on $\mathscr{P}(A)$. Then there exists an elementary map k_A for which the following diagram commutes:*

$$\begin{tikzcd} V \arrow[rr, "j_U"] \arrow[dr, "j_{U_A}"'] & & \check{V}_U \\ & V^A/U_A \arrow[ur, "k_A"'] & \end{tikzcd}$$

Proof. First, notice that U_A is indeed an ultrafilter: it is trivially closed by supersets, since $B \subseteq C \Rightarrow \bigvee B \leq \bigvee C$, and is also closed by intersections since $\bigvee B \wedge \bigvee C = \bigvee_{b\in B,\ c\in C}(b \wedge c)$ that equals to $\bigvee(B \cap C)$ for $B, C \subset A$ antichain.

Let $k_A : V^A/U_A \to \check{V}_U$ be such that $k_A([f]_{U_A}) = \left[\tau_f\right]_U$ where τ_f is the mixing of the values $\check{f(a)}$ along the antichain A, as in Lemma 1.53. We first need to prove that k_A is well-defined. Let f, g in V^A be such that $[f]_{U_A} = [g]_{U_A}$, and $I \in U_A$ be such that $f\restriction_I = g\restriction_I$. Then, by substitution,

$$\begin{array}{rcl} \left[\!\left[\tau_f = \tau_g\right]\!\right] & \geq & \bigvee_{a\in A}\left(\left[\!\left[\check{f(a)} = \check{g(a)}\right]\!\right] \wedge \left[\!\left[\tau_f = \check{f(a)}\right]\!\right] \wedge \left[\!\left[\tau_g = \check{g(a)}\right]\!\right]\right) \\ & \geq & \bigvee_{a\in A}\left(\left[\!\left[\check{f(a)} = \check{g(a)}\right]\!\right] \wedge a \wedge a\right) \\ & \geq & \bigvee_{a\in I} \mathbb{1} \wedge a = \bigvee I \in U \end{array}$$

Furthermore, the diagram commutes since $\left[\!\left[\tau_{j_{U_A}(x)} = \check{x}\right]\!\right] \geq \bigvee(\left[\!\left[\check{x} = \check{x}\right]\!\right] \wedge a) = \bigvee A = \mathbb{1}$. We still need to check that k_A is elementary. By substitution, Theorem 1.59 and Łoś Theorem 1.48,

$$\begin{array}{rcl} V^A/U_A \vDash \phi([f]_{U_A}) & \Rightarrow & \{a \in A : V \vDash \phi\left(f(a)\right)\} \in U_A \\ & \Rightarrow & \bigvee\{a \in A : V \vDash \phi\left(f(a)\right)\} \in U \\ & \Rightarrow & \bigvee\left\{a \in A : \left[\!\left[\phi^{\check{V}}\left(\check{f(a)}\right)\right]\!\right] = \mathbb{1}\right\} = p \in U \\ & \Rightarrow & \left[\!\left[\phi^{\check{V}}(\tau_f)\right]\!\right] \geq \bigvee_{a\in A}\left(\left[\!\left[\phi^{\check{V}}\left(\check{f(a)}\right)\right]\!\right] \wedge a\right) \geq p \in U \\ & \Rightarrow & \check{V}_U \vDash \phi(\left[\tau_f\right]_U) \end{array}$$

The reverse implication follows from taking $\phi' = \neg\phi$. □

Theorem 1.63. *Let $\mathscr{B}$ be a boolean algebra, $A \subset \mathscr{B}$ a maximal antichain, $B \subset \mathscr{B}$ a maximal antichain that refines A, U ultrafilter on $\mathscr{B}$, U_A, U_B ultrafilter on $\mathscr{P}(A)$, $\mathscr{P}(B)$ defined as above. Then there exists an elementary map $k_{A,B}$ for* *which the following diagram commutes:*

$$
\begin{tikzcd}
V \arrow[rr, "j_U"] \arrow[dr, "j_{U_A}"] \arrow[ddr, bend right, "j_{U_B}"'] & & \check{V}_U \\
& V^A/U_A \arrow[ur, "k_A"] \arrow[d, "k_{A,B}"] & \\
& V^B/U_B \arrow[uur, "k_B"'] &
\end{tikzcd}
$$

Proof. Let $k_{A,B}: V^A/U_A \to V^B/U_B$ be defined by $k_{A,B}([f]_{U_A}) = [f \downarrow B]_{U_B}$, where $(f \downarrow B)(b) = f(a)$ where a is the unique element of $A$with $a \geq b$. We first need to prove that $k_{A,B}$ is well-defined. Let f, g in V^A be such that $[f]_{U_A} = [g]_{U_A}$, and $I \in U_A$ be such that $f \restriction_I = g \restriction_I$. Then $k_{A,B}(f)$ coincides with $k_{A,B}(g)$ on $I \downarrow B = J = \{b \in B : \exists a \in I \; a \geq b\}$. Since $\bigvee(B \setminus J) \leq \bigvee(A \setminus I) \notin U$, we have that $\bigvee J \in U$ hence $[f]_{U_B} = [g]_{U_B}$.

The next step is to check that the diagram commutes. From Theorem 1.62, we only need to verify that $k_{A,B} \circ j_{U_A} = j_{U_B}$ and $k_B \circ k_{A,B} = k_A$. The first verification is trivial since $k_{A,B}$ maps constant functions to constant functions. We now check the second equality.

$$
\begin{aligned}
[\![\tau_{k_{A,B}(f)} = \tau_f]\!] &\geq \bigvee_{a \in A} \left([\![\tau_{k_{A,B}(f)} = \check{f(a)}]\!] \wedge [\![\tau_f = \check{f(a)}]\!]\right) \\
&\geq \bigvee_{a \in A} \left(\bigvee_{b \in (a \downarrow B)} [\![\tau_{k_{A,B}(f)} = (f \downarrow \check{B})(b)]\!] \wedge a\right) \\
&\geq \bigvee_{a \in A} \left(\bigvee_{b \in (a \downarrow B)} b \wedge a\right) = \bigvee_{a \in A} (a \wedge a) = \bigvee A = \mathbb{1} \in U
\end{aligned}
$$

We still need to show that $k_{A,B}$ is elementary:

$$
\begin{aligned}
V^A/U_A \vDash \phi([f]_{U_A}) &\Leftrightarrow \{a \in A : \; V \vDash \phi\,(f(a))\} = I \in U_A \\
&\Leftrightarrow (I \downarrow B) \in U_B \\
&\Leftrightarrow \{b \in B : \; V \vDash \phi\,(k_{A,B}(f)(b))\} \in U_B \\
&\Leftrightarrow V^B/U_B \vDash \phi\left([k_{A,B}(f)]_{U_B}\right)
\end{aligned}
$$

This completes the proof. □

Definition 1.64. *A* directed set *D is a poset such that for every two elements a, b in D, there exists an upper bound $c \in D$, $a \leq c$, $b \leq c$.*

Definition 1.65. *A* directed system *of algebraic structures $S = \langle X_a, i_{a,b}\rangle$ over D, is a collection of algebraic structures X_a and homomorphisms between structures $i_{a,b}: \; X_a \to X_b$ for $a, b \in D$, $a \leq b$, such that $i_{a,c} = i_{b,c} \circ i_{a,b}$ for every $a, b, c \in D$ with $a \leq b \leq c$.*

Definition 1.66. *The* direct limit $\varinjlim S$ *of a directed system $S = \langle X_a, i_{a,b}\rangle$ is the unique up to isomorphisms algebraic structure X such that there exist maps i_a :*

$X_a \to X$ with $i_a = i_b \circ i_{a,b}$ for all $a, b \in D$, $a \le b$, and for every other algebraic structure Y with analogous maps $j_a : X_a \to Y$ there exists a map $j : X \to Y$ such that $j_a = j \circ i_a$.

Theorem 1.67. *Let $\mathscr{B}$ be a boolean algebra, U ultrafilter on $\mathscr{B}$. Then the boolean ultrapower $\check{V}_U$ is precisely the limit of the directed system $\langle V^A/U_A, k_{A,B}\rangle$ with $A, B \subset \mathscr{B}$ antichains, where $A \le B$ if A refines B and the homomorphisms $k_{A,B}$ are defined as in Theorem 1.63.*

Proof. From Theorem 1.62 we can define maps k_A such that $k_A = k_A \circ k_{A,B}$. Let now X be a structure with maps $i_A : V^A/U_A \to X$ such that $i_A = i_B \circ k_{A,B}$, we want to define a map $i : \check{V}_U \to X$ such that $i_A = i \circ k_A$.

Let τ be such that $[\tau]_U \in \check{V}_U$. Observe that $A = \{[\![\tau = \check{x}]\!] : x \in V\}$, is an antichain such that $\bigvee A = [\![\tau \in \check{V}]\!] \in U$. Let $f_\tau : A \to V$ be such that $f_\tau(a) = x_a$ where x_a is the unique set such that $[\![\tau = \check{x}_a]\!] \ge a$ for $a \in A$, and $x_a = \emptyset$ otherwise. Then,

$$\begin{aligned} [\![k_A(f_\tau) = \tau]\!] &\ge \bigvee_{a\in A}\left([\![\tau = \check{f_\tau(a)}]\!] \wedge [\![k_A(f_\tau) = \check{f_\tau(a)}]\!]\right) \\ &\ge \bigvee_{a\in A}([\![\tau = \check{x}_a]\!] \wedge a) \ge \bigvee_{a\in A}(a \wedge a) = \bigvee A \in U \end{aligned}$$

For every $[\tau]_U \in \check{V}_U$, let $i([\tau]_U) = i_A([f_\tau]_U)$, so that $i_A = i \circ k_A$ by definition. Furthermore, i is elementary since for every $\phi([\tau_1]_U, \ldots, [\tau_n]_U)$ we can pick an antichain A such that $\tau_m \in k_A[V^A/U_A]$ for every m, and restricted to $k_A[V^A/U_A]$, $i = i_A \circ k_A^{-1}$. □

Corollary 1.68. *If $\mathscr{B} = \mathscr{P}(A)$ is an algebra of sets, U ultrafilter on $\mathscr{B}$, A is a maximal antichain of atoms that is minimal in the order defined above, then $\check{V}_U$ is isomorphic to V^A/U_A.*

1.7 Generic Extensions

We have proved in Theorem 1.20 that V-generic ultrafilters (see Definition 1.18) cannot exist for non-trivial algebras $\mathscr{B}$. However, in some sense we can define in the model $V^{\mathscr{B}}$ a canonical name for a V-generic ultrafilter.

Theorem 1.69. *Let $\dot{G} = \{\langle \check{p}, p\rangle : p \in \mathscr{B}\}$, then $[\![\dot{G}$ is a $\check{V}$-generic ultrafilter$]\!] = \mathbb{1}$.*

Proof. First, notice that $[\![\check{p} \in \dot{G}]\!] = \bigvee_{q\in\mathscr{B}}([\![\check{p} = \check{q}]\!] \wedge q) = p$.

We prove that $[\![\dot{G}$ is an ultrafilter $]\!] = \mathbb{1}$. For every $p \le q$ in $\mathscr{B}$,

$$[\![\check{p} \in \dot{G} \to \check{q} \in \dot{G}]\!] = \neg [\![\check{p} \in \dot{G}]\!] \vee [\![\check{q} \in \dot{G}]\!] = \neg p \vee q \ge \neg q \vee q = \mathbb{1}$$

$$[\![\check{p} \in \dot{G} \wedge \check{q} \in \dot{G} \to (p \check{\wedge} q) \in \dot{G}]\!] = \neg [\![\check{p} \in \dot{G} \wedge \check{q} \in \dot{G}]\!] \vee (p \wedge q) = \neg(p \wedge q) \vee (p \wedge q) = \mathbb{1}$$

$$\left[\!\left[\check{p} \in \dot{G} \vee \neg\check{p} \in \dot{G}\right]\!\right] = \left[\!\left[\check{p} \in \dot{G}\right]\!\right] \vee \left[\!\left[\neg\check{p} \in \dot{G}\right]\!\right] = p \vee \neg p = \mathbb{1}$$

Furthermore, if D is a dense subset of $\mathscr{B}$ in V,

$$\left[\!\left[\exists b \in \dot{G} \cap \check{D}\right]\!\right] = \bigvee_{b\in D} \left[\!\left[\check{b} \in \dot{G}\right]\!\right] = \bigvee_{b\in D} b = \mathbb{1}$$

since every dense set contains a maximal antichain, and every maximal antichain in a complete boolean algebra is a partition of $\mathbb{1}$. This completes the verification that $\dot{G}$ is a $\check{V}$-generic ultrafilter. □

Notice that in $V_U^{\mathscr{B}}$ we have that $\dot{G}_U \subset \check{\mathscr{B}}_U$ and $\check{p}_U \in \dot{G}_U \Leftrightarrow p \in U$. Although, $\dot{G}_U$ is globally different from $\check{U}_U$ when U is non-principal, since:

$$\begin{array}{rcl} \left[\!\left[\dot{G} = \check{U}\right]\!\right] & = & \left[\!\left[\dot{G} \subset \check{U}\right]\!\right] \wedge \left[\!\left[\check{U} \subset \dot{G}\right]\!\right] \\ & = & \bigwedge_{p\in\mathscr{B}} \left(\neg p \vee \left[\!\left[\check{p} \in \check{U}\right]\!\right]\right) \wedge \bigwedge_{p\in U} \left[\!\left[\check{p} \in \dot{G}\right]\!\right] \\ & = & \bigwedge_{p\in\mathscr{B}\setminus U} \neg p \wedge \bigwedge_{p\in U} p \\ & = & \bigwedge_{p\in U} p = \bigwedge U = \mathbb{0} \end{array}$$

Definition 1.70. *Let M be a transitive model of* ZFC *and $G \subset M$. Then $M[G]$ is the least transitive model of* ZFC *containing both M and G.*

If G is an M-generic ultrafilter for some boolean algebra $\mathscr{B}$ complete in M, we will say that $M[G]$ is a *generic extension* of M.

Theorem 1.71. *For every ultrafilter $U \subset \mathscr{B}$, $\dot{G}_U$ is $\check{V}_U$-generic for $\check{\mathscr{B}}_U$ and*

$$V_U^{\mathscr{B}} \vDash \check{V}_U[\dot{G}_U] = V_U^{\mathscr{B}}$$

Proof. From Łoś Theorem 1.48 and previous Theorem 1.69, we have that $\left[\!\left[\dot{G} \text{ is a } \check{V}\text{-generic ultrafilter}\right]\!\right] = \mathbb{1} \Rightarrow V_U^{\mathscr{B}} \vDash \dot{G}_U$ is a $\check{V}_U$-generic ultrafilter.

We already know that $V_U^{\mathscr{B}}$ is a model of ZFC that contains $\check{V}_U$ and $\dot{G}_U$, and from Corollary 1.60 and Łoś Theorem 1.48 we have that:

$$\begin{array}{l} \left[\!\left[\check{V} \text{ is a transitive inner model of ZFC with all ordinals}\right]\!\right] = \mathbb{1} \Rightarrow \\ \qquad V_U^{\mathscr{B}} \vDash \check{V}_U \text{ is a transitive inner model of ZFC with all ordinals} \end{array}$$

We still have to check that $V_U^{\mathscr{B}}$ is minimum, i.e. whenever M contains $\check{V}_U$ and $\dot{G}_U$, M must contain all $V_U^{\mathscr{B}}$. By induction on the rank of τ, we can show that $\left[\!\left[\tau = \mathrm{val}(\check{\tau}, \dot{G})\right]\!\right] = \mathbb{1}$, where the function val is defined by recursion as:

$$\mathrm{val}(\tau, G) = \left\{\mathrm{val}(\sigma, G) : \langle\sigma, p\rangle \in \tau \wedge p \in G\right\}$$

The proof of this fact is left as an exercise for the reader.

Now, let M be a class definable model of ZFC that contains $\check{V}_U$ and $\dot{G}_U$, and let τ_U be any set in $V_U^{\mathscr{B}}$. Then, M contains both $\check{\tau}_U \in \check{V}_U$ and $\dot{G}_U$ hence contains $\sigma_U = \mathrm{val}(\check{\tau}_U, \dot{G}_U)$, but $\left[\!\left[\tau = \mathrm{val}(\check{\tau}, \dot{G})\right]\!\right] = \mathbb{1} \Rightarrow \tau_U = \mathrm{val}(\check{\tau}_U, \dot{G}_U)$ hence $\tau_U = \sigma_U \in M$ and $M = V_U^{\mathscr{B}}$. □

Theorem 1.72. *U is V-generic if and only if j_U is an isomorphism.*

Proof. If U is V-generic, let $\tau_U \in \check{V}_U$. Then $q = \left[\!\left[\tau \in \check{V}\right]\!\right] = \bigvee_{x \in V} [\![\tau = \check{x}]\!] \in U$, and $D = \{[\![\tau = \check{x}]\!] : x \in V\}$ is a maximal antichain below q. If $p \in D \cap U$, there is an $x \in V$ such that $[\![\tau = \check{x}]\!] = p \in U$, thus $j_U(x) = \tau_U$ and j_U is onto.

Vice versa, suppose j_U is onto, and let $A \subset \mathscr{B}$ be a maximal antichain. By mixing, let τ be such that $[\![\tau = \check{a}]\!] \geq a$ for every $a \in A$. So $\left[\!\left[\tau \in \check{V}\right]\!\right] = \bigvee A = \mathbb{1}$, let $x \in V$ be such that $j_U(x) = \tau_U$. Then $[\![\check{x} = \tau]\!] \in U \Rightarrow [\![\check{x} = \tau]\!] = x \in U$, U meets A hence is V-generic. □

If U is not V-generic, j_U will not be an isomorphism. We shall show that there is close relationship between the size of dense sets missed by U and the size of the initial segment of V preserved by j_U.

Definition 1.73. *Let j be an elementary embedding, the* critical point *of j is:*

$$\mathrm{cp}(j) = \min\{\kappa \in \mathbf{ON} : j(\kappa) \neq j[\kappa]\}$$

Theorem 1.74. *If U is not V-generic,*

$$\mathrm{cp}(j_U) = \min\{|A| : A \textit{ maximal antichain } \wedge A \cap U = \emptyset\}$$

Proof. Suppose $A = \{a_\alpha : \alpha < \kappa\}$ maximal antichain with $A \cap U = \emptyset$, $\dot{\beta} = \{\langle \check{\alpha}, a_\alpha \rangle : \alpha < \kappa\}$ so that $\left[\!\left[\dot{\beta} = \check{\alpha}\right]\!\right] = a_\alpha$. Then $\left[\!\left[\dot{\beta} \in \check{\kappa}\right]\!\right] = \bigvee_{\alpha<\kappa} \left[\!\left[\dot{\beta} = \check{\alpha}\right]\!\right] = \bigvee A = \mathbb{1}$, while $\left[\!\left[\dot{\beta} = \check{\alpha}\right]\!\right] = a_\alpha \notin U$ for all α. Thus $\dot{\beta}_U \in j_U(\kappa) \setminus j_U[\kappa]$ hence $j_U(\kappa) \neq j_U[\kappa]$.

Conversely, suppose that $\dot{\beta}_U \in j_U(\kappa) \setminus j_U[\kappa]$, and let $A = \left\{\left[\!\left[\dot{\beta} = \check{\alpha}\right]\!\right] : \alpha < \kappa\right\}$. This is an antichain of size κ such that $\bigvee A \in U$, and $A \cap U = \emptyset$ otherwise $\dot{\beta}_U$ would be in $j_U[\kappa]$. Hence $A' = A \cup \left\{\neg\left(\bigvee A\right)\right\}$ is a maximal antichain of size κ with $A' \cap U = \emptyset$. □

Proposition 1.75. *U is κ-complete if and only if $U \cap A \neq \emptyset$ for every A maximal antichain with $|A| < \kappa$.*

Proof. First, suppose by contradiction that U is κ-complete but there exists an A with $|A| < \kappa$ and $U \cap A = \emptyset$. Then $\{\neg a : a \in A\} \subseteq U$ hence $\mathbb{0} = \neg\mathbb{1} = \neg\bigvee A = \bigwedge\{\neg a : a \in A\} \in U$, a contradiction.

Conversely, suppose by contradiction that U meets every maximal antichain of size less than κ and is λ-complete but not λ^+-complete for some $\lambda < \kappa$. Let

$B = \{x_i : i < \lambda\} \subset U$ with $\bigwedge B = b \notin U$. Then $y_i = \neg b \wedge \bigwedge_{j<i} x_j$ is a decreasing chain in U with infimum $\mathbb{0}$. Let $\langle y'_i : i < \lambda' \rangle$, $\lambda' < \lambda$ be a strictly decreasing subsequence of $\langle y_i : i < \lambda \rangle$. Then $z_i = y'_i \wedge \neg y'_{i+1}$ is an antichain disjoint from U, with

$$\begin{aligned} \bigvee_{i<\lambda'} z_i &= \bigvee_{i<\lambda} \bigvee_{j<i} y'_j \wedge \neg y'_{j+1} \\ &= \bigvee_{i<\lambda'} y'_0 \wedge \neg y'_i \\ &= y'_0 \wedge \neg \bigwedge_{i<\lambda'} y'_i = \neg b \wedge \neg \mathbb{0} = \neg b \in U \end{aligned}$$

Hence $A = \{z_i : i < \lambda'\} \cup \{b\}$ is a maximal antichain of size $\lambda' < \kappa$ with $A \cap U = \emptyset$, a contradiction. □

1.8 Forcing Relation and Posets

In the last sections we have examined generic extensions for a model V of ZFC and $\mathcal{B}$ a boolean algebra. In this section we will briefly analyse the construction of a generic extension of a model M for an arbitrary poset $\mathbb{P}$, as in the classical approach to forcing. Our reference text for the results in this section is [1, VII].

If $\mathbb{P}$ is not a boolean algebra, it is not yet possible to define the truth value $[\![\phi]\!]_{\mathbb{P}}$. Although, the truth value of ϕ can be analysed via the *forcing relation.*

Definition 1.76. *Given M a transitive model of* ZFC*, $\mathbb{P} \in M$ a poset and $p \in \mathbb{P}$, we say that p* forces $\phi(\tau_1, \ldots, \tau_n)$ with respect to M*, in formulas:*

$$p \Vdash^M_{\mathbb{P}} \phi(\tau_1, \ldots, \tau_n)$$

iff for every M-generic filter $G \subset \mathbb{P}$ with $p \in G$, $M[G] \vDash \phi([\tau_1]_G, \ldots, [\tau_n]_G)$.

The reference to the model M or the poset $\mathbb{P}$ in the symbol $\Vdash$ can be omitted when clear from the context.

Theorem 1.77 (Cohen)**.** *Let M be a model of* ZFC*, $\mathcal{B} \in M$ a boolean algebra, $\mathbb{P} = \mathcal{B} \setminus \mathbb{0}$. Then $p \Vdash^M_{\mathbb{P}} \phi(\tau_1, \ldots, \tau_n)$ if and only if $M \vDash p \leq [\![\phi(\tau_1, \ldots, \tau_n)]\!]$.*

Proof. From Theorems 1.71 and 1.72, we have that $M^{\mathcal{B}}_G = \check{M}_G[\dot{G}_G] = M[G]$. Then from Łoś Theorem 1.48 we know that $M[G] \vDash \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ if and only if $[\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$. So, if $p \leq [\![\phi(\tau_1, \ldots, \tau_n)]\!]$,

$$\begin{aligned} p \in G &\Rightarrow [\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G \\ &\Rightarrow M[G] \vDash \phi([\tau_1]_G, \ldots, [\tau_n]_G) \end{aligned}$$

hence $p \Vdash_{\mathbb{P}} \phi(\tau_1, \ldots, \tau_n)$.

Otherwise, if $p \nleq [\![\phi(\tau_1, \ldots, \tau_n)]\!]$, we can find a generic filter G that contains $p \wedge \neg [\![\phi(\tau_1, \ldots, \tau_n)]\!] \neq \mathbb{0}$, so that $p \in G$ but $[\![\phi(\tau_1, \ldots, \tau_n)]\!] \notin G$. Then $M[G] \nvDash \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ and $p \nVdash_{\mathbb{P}} \phi(\tau_1, \ldots, \tau_n)$. □

Although Definition 1.76 makes sense only in V when M-generic filters exist, the forcing relation can always be defined inside of M: i.e., for every formula ϕ there is a formula ϕ' such that $p \Vdash^M_{\mathbb{P}} \phi(\tau_1, \dots, \tau_n) \Leftrightarrow M \vDash \phi'(p, \tau_1, \dots, \tau_n)$, and the map $\phi \to \phi'$ is recursive. The forcing relation can be used to define the quotient $M^{\mathbb{P}}_G$ and prove an analogous form of the theorems in the last sections: complete proofs for this facts can be found in [1, VII], but are not needed in the following exposition and will not be reported here. Furthermore, every generic extension with a poset $\mathbb{P}$ is equivalent to one with a complete boolean algebra.

Definition 1.78. *Let $\mathbb{P}$, $\mathbb{Q}$ be posets and $i : \mathbb{P} \to \mathbb{Q}$. i is a* dense embedding *iff*

1. $\forall p_1, p_2 \in \mathbb{P} \ (p_1 \leq p_2 \to i(p_1) \leq i(p_2))$
2. $\forall p_1, p_2 \in \mathbb{P} \ (p_1 \perp p_2 \to i(p_1) \perp i(p_2))$
3. *$i[\mathbb{P}]$ is dense in $\mathbb{Q}$.*

Proposition 1.79. *Every poset $\mathbb{P}$ can be embedded densely in a complete boolean algebra $\mathscr{B}$, the* completion *of $\mathbb{P}$.*

Proof. Let A_p, $p \in \mathbb{P}$ be a basis for the order topology, i.e. $A_p = \{q \in \mathbb{P} : q \leq p\}$. Let $\mathscr{B} = \mathrm{RO}(\mathbb{P})$, $i(p) = \mathrm{int}\ \mathrm{cl}(A_p)$. It can now be verified that $\mathscr{B}$ is indeed a complete boolean algebra, and i is a dense embedding (see [1, II, Lemma 3.3]). □

Theorem 1.80. *Suppose i, $\mathbb{P}$, and $\mathbb{Q}$ are in M, $i : \mathbb{P} \to \mathbb{Q}$, and i is a dense embedding. If G is M-generic for $\mathbb{P}$, and H is the filter generated by $i[G]$ (or conversely, H is M-generic for $\mathbb{Q}$ and $G = i^{-1}[H]$), then $M[G] = M[H]$.*

Proof. A complete proof for this statement can be found in [1, VII, Theorem 7.11]. □

Corollary 1.81. *Every generic extension with a poset $\mathbb{P}$ is equivalent to the generic extension with $\mathscr{B}$ the completion of $\mathbb{P}$.*

Thus, generic extensions with posets $\mathbb{P}$ is not a new tool for consistency proofs. It is however more convenient in some practical cases to define a poset $\mathbb{P}$ that produces the desired generic extension rather than defining a boolean algebra $\mathscr{B}$: but this will not be the case in the remainder of this paper.

CHAPTER 2

CHARACTERIZATION OF SET-GENERIC EXTENSIONS

Set-generic extensions have been characterized for the first time by the work of L. Bukovsky [7]. The current presentation will follow a recent work of S. D. Friedman [10].

We will start with an introduction with some useful tools (Section 2.1), then present Laver's Theorem [8] (Section 2.2) and Bukovsky's Theorem (Section 2.3). Afterwards, we shall see a recent generalization of Bukovsky's Theorem by Friedman (Section 2.4), and finally use this results to prove that being a ground model is a first-order definable property (Section 2.5).

Throughout this chapter M and V will be models of ZFC, with $M \subset V$ and $\mathbf{ON}^M = \mathbf{ON}^V$.

2.1 Basic Properties

In this chapter we will use extensively some properties of forcing extensions; the basic results that we will need have been collected here.

Definition 2.1. *Let M be a transitive class in V. A set A is κ-*approximated *in M if and only if $A \subset M$ and for any $B \in M$ of size $< \kappa$, the intersection $A \cap B$ is in M,*

$$\forall B \in M : |B|^M < \kappa \to A \cap B \in M$$

Definition 2.2. *M κ-*approximates *V if and only if M contains every set A that is*

κ-approximated in M,

$$\forall A \in V\, (\forall B \in M : |B|^M < \kappa \to A \cap B \in M) \to A \in M$$

Definition 2.3. M κ-decomposes V *if and only if any subset of M in V is the union of at most κ many subsets in M,*

$$\forall A \in V\, (A \subset M) \to \exists B \subset M\, (|B|^V \leq \kappa \wedge A = \bigcup B)$$

Definition 2.4. M κ-covers V *if and only if any subset of M in V of size $< \kappa$ is covered by such a set in M,*

$$\forall A \in V\, (A \subset M \wedge |A|^V < \kappa) \to \exists B \in M\, (A \subset B \wedge |B|^M < \kappa)$$

Definition 2.5. M globally κ-covers V *if and only if for any $f : \alpha \to M$ in V there is a $g : \alpha \to M$ in M such that $f(i) \in g(i)$ and $\forall i : \left|g(i)\right|^M < \kappa$,*

$$\begin{gathered} \forall \alpha \in \mathbf{ON}\, \forall f \in V\, \left[f \subset (\alpha \times M) \wedge f \text{ is a function } \right] \to \\ \exists g \in M\, \left[g \subset (\alpha \times M) \wedge g \text{ is a function } \wedge \forall i < \alpha\, (f(i) \in g(i) \wedge \left|g(i)\right| < \kappa) \right] \end{gathered}$$

The last definition is a slight strengthening of the previous one.

Proposition 2.6. *If M globally κ-covers V then M κ-covers V.*

Proof. Let $A \subset M$ be of size $< \kappa$, and let $f : \alpha \to A \subset M$, $\alpha = |A|^V$, be a bijection in V. By global κ-covering let $g : \alpha \to M$ be such that $f(i) \in g(i)$.

By substitution $C = \bigcup \operatorname{ran}(g)$ is in M and $A \subset C$ since for all i, $f(i) \in g(i)$. Furthermore $\alpha < \kappa$ and for all i, $\left|g(i)\right|^M < \kappa$, so $|C|^M < \kappa \cdot \kappa = \kappa$ as required. □

Except for κ-covering, all these properties hold monotonically in κ.

Proposition 2.7. *If M κ-approximates (risp. κ-decomposes, globally κ-covers) V and $\lambda > \kappa$, then M λ-approximates (risp. λ-decomposes, globally λ-covers) V.*

Proof. Follows trivially from the definitions. Remark that a set λ-approximated in M is also κ-approximated in M. □

All this four properties hold for set-generic extensions, for some κ depending on the poset $\mathbb{P}$ used.

Proposition 2.8. *Let V be a $\mathbb{P}$-generic extension of M. If $|\mathbb{P}|^M < \kappa$, then M κ-approximates V.*

Proof. Suppose towards a contradiction that $A \in V \setminus M$, $A \subset M$, and every κ-approximation of A is in M. Let $\dot{A}$ any $\mathbb{P}$-name for A. Since $A \notin M$, for all $p \in \mathbb{P}$ we may choose (by definability of $\Vdash$) a set $x_p \in M$ such that p does not decide the membership of x_p in $\dot{A}$, let $B = \left\{x_p \in M : p \in \mathbb{P}\right\}$. Since $|B|^M < \kappa$ and $B \in M$ by choice and definability of $\Vdash$, we must have $A \cap B = C \in M$. Let $q \in \mathbb{P}$ be such that $q \Vdash \dot{A} \cap \check{B} = \check{C}$. For construction q does not decide whether $x_q \in \dot{A}$, but since $\dot{A} \cap \check{B} = \check{C}$ and $x_q \in \check{B}$, $x_q \in \dot{A} \Leftrightarrow x_q \in \check{C}$ so q must not decide whether $x_q \in \check{C}$. But $\mathbb{1}$ decides whether $x_q \in \check{C}$, a contradiction. □

Proposition 2.9. *Let V be a $\mathbb{P}$-generic extension of M. If $|\mathbb{P}|^M \leq \kappa$, then M κ-decomposes V.*

Proof. Let $V = M[G]$, where G is $\mathbb{P}$-generic over M. Given $A \in V$, $A \subset M$, fix a $\mathbb{P}$-name $\dot{A}$ and for each $p \in G$ let $A_p = \left\{x \in M : \ p \Vdash \check{x} \in \dot{A}\right\}$, $B = \left\{A_p : \ p \in G\right\}$. Then $A = \bigcup B$ is the desired κ-decomposition. □

Proposition 2.10. *Let V be a $\mathbb{P}$-generic extension of M. If $\mathbb{P}$ is κ-cc, then M globally κ-covers V, hence κ-covers V (by Proposition 2.6).*

Proof. Given $f : \alpha \to M$ in V as in the definition of global κ-covering, let $\dot{f}$ be any $\mathbb{P}$-name for f, and

$$Q = \left\{p \in \mathbb{P} : \ p \Vdash \dot{f} \text{ is a function } \wedge \operatorname{dom} \dot{f} = \check{\alpha} \wedge \operatorname{ran} \dot{f} \subset \check{M}\right\}$$

From f and Q define $g : \alpha \to M$ as:

$$g = \left\{\langle i, x\rangle \in \alpha \times M : \ y \in x \longleftrightarrow \exists p_y \in Q \ \ p_y \Vdash \dot{f}(\check{i}) = \check{y}\right\}$$

By definability in M of $\Vdash$ we have $g \in M$, and $f(i) \in g(i)$ since there exists a $p \Vdash \dot{f}(\check{i}) = \check{a}$ with $a = f(i)$.

Finally, for every element $x \in g(i)$ choose a condition $p_x \Vdash \dot{f}(\check{i}) = \check{x}$, and define $Q_i = \{p_x : x \in g(i)\}$. The set Q_i must be an antichain, since $(q < p_x) \wedge (q < p_y)$ implies $q \Vdash (\dot{f}(\check{i}) = \check{x}) \wedge (\dot{f}(\check{i}) = \check{y})$ hence $x = y$ and $p_x = p_y$. But $\mathbb{P}$ is κ-cc so $|Q_i|^M = \left|g(i)\right|^M < \kappa$. □

Later in this chapter we will need to write down fist-order sentences, where class quantification is not allowed. Then it will be useful to lift the properties above from classes to subsets that satisfy a sufficiently large fragment of ZFC.

Proposition 2.11. *If M κ-approximates (risp. κ-decomposes, κ-covers, globally κ-covers) V, and $\lambda > \kappa$ is regular in V, then $H(\lambda)^M$ κ-approximates (risp. κ-decomposes, κ-covers, globally κ-covers) $H(\lambda)^V$.*

Proof. For κ-approximation, given $A \in H(\lambda)^V$ with $A \subset H(\lambda)^M$, $A \subset H(\lambda)^M$ implies that for any $B' \in M$ with $|B'|^M < \kappa$, $A \cap B' = A \cap B' \cap H(\lambda)^M = A \cap C$ with $C = B' \cap H(\lambda)^M$. We have then $|C|^M \leq |B'|^M < \kappa < \lambda$, $C \in M$, so $C \subset H(\lambda)^M$

implies that $C \in H(\lambda)^M$, and by κ-approximation in M it must be $A \in M$. But $|A|^V < \lambda \Rightarrow |A|^M < \lambda$, $A \in M$ so $A \subset H(\lambda)^M \Rightarrow A \in H(\lambda)^M$.

For κ-decomposition, given $A \in H(\lambda)^V$ with $A \subset H(\lambda)^M$, let $A = \bigcup B$, $|B|^V \leq \kappa$. Since every $x \in B$ is $x \subset A \subset H(\lambda)^M$, $x \in M$, $|x|^V \leq |A|^V < \lambda \Rightarrow |x|^M < \lambda$, we have $x \in H(\lambda)^M$ hence $B \subset H(\lambda)^M$; also, $|B|^V \leq \kappa < \lambda$ hence $B \in H(\lambda)^V$.

For κ-covering, given $A \in H(\lambda)^V$ with $A \subset H(\lambda)^M$, $|A| < \kappa$, let $A \subset B \in M$ by κ-covering in M. If $C = B \cap H(\lambda)^M$, we have $|C|^M \leq |B|^M < \kappa < \lambda$ and $C \in M$ so $C \subset H(\lambda)^M \Rightarrow C \in H(\lambda)^M$.

For global κ-covering, given $f \in H(\lambda)^V$, $f : \alpha \to H(\lambda)^M$, define $g \in M$, $g : \alpha \to M$ by globally κ-covering in M, and let g' be such that

$$g' = g \cap (\alpha \times H(\lambda)^M)$$

Then g' covers f, $g' \subset H(\lambda)^M$, and $\alpha < \lambda$ hence $\left|g'\right|^M < \lambda \Rightarrow g' \in H(\lambda)^M$. □

The converse also holds, as we will see below.

Proposition 2.12. *If there exists λ_0 such that for all regular $\lambda > \lambda_0$, $H(\lambda)^M$ κ-approximates (risp. κ-decomposes, κ-covers, globally κ-covers) $H(\lambda)^V$, then M κ-approximates (risp. κ-decomposes, κ-covers, globally κ-covers) V.*

Proof. For every property, the verification for a set A is trivially achieved by letting λ be such that $A \in H(\lambda)^V$. □

2.2 Laver's Theorem

In this section we will prove that if V is a set-generic extension of M, then M is definable in V with parameters in M (Laver's Theorem 2.16).

This will be achieved by first showing a well-known basic result (Theorem 2.13), then we'll make use of a simultaneous covering property to obtain (Theorem 2.15) that inner models satisfying the κ-covering and κ-approximation properties are unique fixed the set $H(\kappa^+)^M$.

Theorem 2.13 (ZFC − P)**.** *If M, N are inner models of V that share the same sets of ordinals, then $M = N$.*

Proof. Let $A \in M$, we will show that also $A \in N$. Let $B = \mathrm{trcl}(\{A\})$ and κ be such that $|B|^M \leq \kappa$, and $f \in M$ be any 1-1 map $f : B \to \kappa$. Define a relation R on κ by:

$$\alpha R \beta \Leftrightarrow \alpha, \beta \in \mathrm{ran}(f) \wedge f^{-1}(\alpha) \in f^{-1}(\beta)$$

With the bijection $\delta : \kappa \times \kappa \to \kappa$ ($\delta \in M \cap N$ since it is Δ_0-definable), we obtain that $\delta[R] \in M \cap N$ since is a set of ordinals, then $\delta^{-1}[\delta[R]] = R \in M \cap N$. The transitive collapse of a well-founded relation R on κ, $\Pi_R : \kappa \to H(\kappa)$ is Δ_1-definable then $\Pi_R \in M \cap N$, hence $\Pi_R(f(A)) = A \in M \cap N$. □

Lemma 2.14 (ZFC − P)**.** *If M, N are inner models of V that κ-cover and κ-approximate V, then for every $A \in V$ subset of size $< \kappa$ of a $C \in M \cap N$ of arbitrary size is covered by a set $B \in M \cap N$ with $|B| \leq \kappa$.*

Proof. Given $A \in V$, $A \subset C \in M \cap N$, $|A|^V < \kappa$, define a κ-chain $A = A_0 \subseteq A_1 \subseteq \dots$ of sets such that $|A_i| < \kappa$, $A_i \subset C$, $A_{2j+1} \in M$, $A_{2j+2} \in N$. This is possible by κ-covering for M, N taking each time the intersection with C. Let $B = \bigcup_i A_i$, $|B| \leq \kappa$. Any κ-approximation of B with a set in M (risp. N) must lie in M (resp. N), because for any $|D| < \kappa$ the sequence $A_i \cap D$ must equal $B \cap D$ from some index on (by regularity). So by κ-approximation $B \in M$, $B \in N$. □

Theorem 2.15 (ZFC − P)**.** *If M, N are inner models of V which κ-cover and κ-approximate V such that $H(\kappa^+)^M = H(\kappa^+)^N$, then $M = N$.*

Proof. We show that any $A \in M$ is also in N, the converse is specular. By Theorem 2.13, we let A be a set of ordinals without loss of generality.

First, let $A \in M$ be a set of $< \kappa$ ordinals. By Lemma 2.14, let $B \supset A$ be of size $\leq \kappa$, $B \subset \mathbf{ON}$. Let $\Pi_B : \operatorname{trcl} B \to \kappa$ be the transitive collapse of B, $\Pi_B \in M \cap N$ since $B \in M \cap N$ and Π_B is Δ_1-definable. N, M have the same subsets of κ, so $C = \Pi_B[A]$ must be in $M \cap N$, and $\Pi_B^{-1}[C] = A$ then $A \in M \cap N$.

Now let $A \in M$ be a set of ordinals of arbitrary size. By the result above every $B \in M$ of size $< \kappa$ is such that $B \cap \mathbf{ON} \in M \cap N$, so from $A \cap B \in M$ and $|A \cap B| < \kappa$ follows that $A \cap B \in M \cap N$. Thus, by κ-approximation $A \in N$. □

From the uniqueness Theorem 2.15, we can now obtain the following.

Theorem 2.16 (Laver)**.** *If V is a set-generic extension of M, then M is a definable inner model of V with parameters in M.*

Proof. By Proposition 2.8 and 2.10 M κ-approximates and κ-covers V for κ greater than $|\mathbb{P}|^M$, so by Theorem 2.15 M is the unique such model given $K = H(\kappa^+)^M$. Let us put down this intuition into a first-order statement.

Notice that:

$$x \in M \Longleftrightarrow \exists \lambda : \lambda \text{ is a regular cardinal} \wedge x \in H(\lambda)^M$$

Then it suffices to give a definition of $H(\lambda)^M$ for λ regular. The sets $H(\lambda)^M$ and $H(\lambda)^V$ are both models of ZFC−P, and by Proposition 2.11 $H(\lambda)^M$ κ-approximates and κ-covers $H(\lambda)^V$. By Theorem 2.15, $H(\lambda)^M$ is then definable as the unique set that κ-covers and κ-approximates $H(\lambda)^V$ given $K = H(\kappa^+)^M$.

Then $x \in H(\lambda)^M \Leftrightarrow \phi'(x, K, \kappa, \lambda)$, where $\phi'(x, y, \kappa, \lambda)$ is

$$\left[\exists A : x \in A \wedge A\ \kappa\text{-cover and } \kappa\text{-approximate } H(\lambda)^V \wedge H(\kappa^+)^A = y\right]$$

Thus M is definable from κ, K as $x \in M \Leftrightarrow \phi(x, K, \kappa)$ where:

$$\phi(x, y, z) = [\exists \lambda : \lambda \text{ is a regular cardinal} \wedge \phi'(x, y, z, \lambda)]$$ □

2.3 Bukovsky's Theorem

In this section we will characterize κ-cc generic extensions as extensions with the global κ-cover property. The left-to-right implication is in fact Proposition 2.10, the right-to-left one is Bukovsky's Theorem 2.31.

This will be achieved by first showing that for any A, $M[A]$ is a κ-cc forcing extension, and then showing that V itself is of the form $M[A]$ for some carefully chosen A.

Since we are willing to define a forcing extension $M[A]$, we need a poset in M in which A could be a generic filter. Let $\left\langle e_\kappa, \mathscr{B}_\kappa^\lambda \right\rangle = \mathrm{Fr}_\kappa(\lambda) \in M$ be a free κ-complete boolean algebra generated by λ in M. Remark that for every $\kappa' > \kappa$, we can embed $\mathscr{B}_\kappa^\lambda$ as a subalgebra of $\mathscr{B}_{\kappa'}^\lambda$ by taking the κ-complete morphism f such that $f \circ e_\kappa = e_{\kappa'}$ (since $\mathscr{B}_{\kappa'}^\lambda$ is κ'-complete hence κ-complete). From now on, we will then suppose without loss of generality that $\mathscr{B}_\kappa^\lambda \subset \mathscr{B}_{\kappa'}^\lambda$ for every $\kappa < \kappa'$.

The boolean algebra $\mathscr{B}_\kappa^\lambda$ can be identified with the boolean algebra of propositional logic formulas built from the atomic formulas "$\alpha \in x$", $\alpha < \lambda$, where x represents an unknown subset $A \subset \lambda$, using negation, conjunction and disjunction for sets of size $< \kappa$.

Theorem 2.17. *There is a bijection between subsets of λ and κ-complete ultrafilters in $\mathscr{B}_\kappa^\lambda$.*

Proof. Let $h : \mathrm{Ult}_\kappa(\mathscr{B}_\kappa^\lambda) \to \mathscr{P}(\lambda)$ be defined by $h(U) = e_\kappa^{-1}[U]$. We show that h is a bijection.

For every subset $A \subset \lambda$ let $f_A : \lambda \to 2$ be such that $(f_A(i) = \mathbb{1}) \longleftrightarrow (i \in A)$. Since $\mathscr{B}_\kappa^\lambda$ is free over λ, f_A can be extended uniquely to a κ-complete morphism $g_A : \mathscr{B}_\kappa^\lambda \to 2$, $g_A \circ e_\kappa = f_A$. Then $U_A = g_A^{-1}[\mathbb{1}]$ is a κ-complete ultrafilter, and

$$h(U_A) = e_\kappa^{-1}[U_A] = e_\kappa^{-1} \circ g_A^{-1}[\mathbb{1}] = f_A^{-1}[\mathbb{1}] = A$$

Furthermore, any κ-complete ultrafilter U'_A such that $e_\kappa^{-1}[U'_A] = A$ induces a κ-complete morphism $g'_A : \mathscr{B}_\kappa^\lambda \to 2$, $g'_A(b) = \mathbb{1} \Leftrightarrow b \in U'_A$, with $g'_A \circ e_\kappa = f_A$. Then $g'_A = g_A$ (by uniqueness of g_A) hence $U'_A = U_A$. □

Definition 2.18. *Given $A \subset \lambda$ in V, let $U_A^\kappa \in V$ be the ultrafilter corresponding to A in the boolean algebra $\mathscr{B}_\kappa^\lambda$.*

In the propositional logic identification, the ultrafilter U_A^κ is the set of formulas true for $x = A$. Notice that if $A \notin M$ also $U_A^\kappa \notin M$.

The set $\mathscr{B}_\kappa^\lambda$ seems to have the expressive power needed to interpret $M[A]$ as a generic extension for any $A \subset \lambda$. In fact, this is not the case since for example it is not κ-cc, fact that will be crucial later in the proof of Theorem 2.31.

To obtain a poset that meets all this requirements we will take an appropriate ideal I_A of $\mathscr{B}_\kappa^\lambda$ and define $\mathbb{P}_A = \mathscr{B}_\kappa^\lambda / I_A - \{\mathbb{0}\}$, by means of the global κ-covering property.

Definition 2.19. *Let $f_A : \mathscr{P}(\mathscr{B}_\kappa^\lambda)^M \to \mathscr{B}_\kappa^\lambda$ in V be such that:*

$$\forall \Phi \in \mathscr{P}(\mathscr{B}_\kappa^\lambda)^M : \quad \left(\Phi \cap U_A^\kappa \neq \emptyset\right) \to \left(f_A(\Phi) \in \left[U_A^\kappa \cap \Phi\right]\right)$$

Such an f_A must exist: for example, we can take the composition of f' such that $f'(\Phi) = \Phi \cap U_A^\kappa$ with any choice function $\mathscr{P}(U_A^\kappa) \to U_A^\kappa$. Since $f_A \in V$, we can use the global κ-covering property to define:

Definition 2.20. *Let $g'_A : \mathscr{P}(\mathscr{B}_\kappa^\lambda)^M \to \mathscr{P}(\mathscr{B}_\kappa^\lambda)^M$ in M such that:*

$$\forall \Phi \in \mathscr{P}(\mathscr{B}_\kappa^\lambda)^M : \quad \left|g'_A(\Phi)\right| < \kappa \ \wedge \ f_A(\Phi) \in g'_A(\Phi)$$

Let $g_A(\Phi) = g'_A(\Phi) \cap \Phi$. We now define the theory T_A of the formulas used in the definition of g_A: since $g_A \in M$, also T_A will be in M.

Definition 2.21. *Let $T_A \subset \mathscr{B}_\theta^\lambda$, with $\theta = \left(2^{|\mathscr{B}_\kappa^\lambda|}\right)^+ = \left(2^{\lambda^{<\kappa}}\right)^+$, in M such that:*

$$T_A = \left\{ \left(\bigvee \Phi \to \bigvee g_A(\Phi)\right) : \quad \Phi \subseteq \mathscr{B}_\kappa^\lambda \right\}$$

The set T_A is included in U_A^θ: if $\bigvee \Phi \in U_A^\theta$, then $\Phi \cap U_A^\theta = \Phi \cap U_A^\kappa \neq \emptyset$ (by θ-completeness of U_A^θ), so $f_A(\Phi) \in U_A^\kappa$ and $\bigvee g_A(\Phi) \in U_A^\theta$. Thus $\bigwedge T_A \neq \mathbb{0}$.

Definition 2.22. *Let $I_A \subset \mathscr{B}_\kappa^\lambda$ in M be such that:*

$$I_A = \left\{ \phi \in \mathscr{B}_\kappa^\lambda : \quad \bigwedge (T_A \cup \{\phi\}) = \mathbb{0} \ \text{ in } \ \mathscr{B}_\theta^\lambda \right\}$$

The set I_A is disjoint from U_A^κ, otherwise if $\phi \in I_A \cap U_A^\kappa$, the θ-complete ultrafilter U_A^θ would contain $\mathbb{0} = \bigwedge (T_A \cup \{\phi\})$, since $T_A \subset U_A^\theta$.

Lemma 2.23. *I_A is an ideal in $\mathscr{B}_\kappa^\lambda$.*

Proof. Given ϕ, ψ in I_A, $\bigwedge(T_A \cup \{\phi \vee \psi\}) = \bigwedge(T_A \cup \{\phi\}) \vee \bigwedge(T_A \cup \{\psi\}) = \mathbb{0} \vee \mathbb{0} = \mathbb{0}$, hence $(\phi \vee \psi) \in I_A$. Given $\rho \in \mathscr{B}^\lambda_\kappa$, $\bigwedge(T_A \cup \{\phi \wedge \rho\}) = \bigwedge(T_A \cup \{\phi\}) \wedge \rho = \mathbb{0} \wedge \rho = \mathbb{0}$, hence $(\phi \wedge \rho) \in I_A$. This completes the proof. □

Definition 2.24. *Let* $\mathbb{P}_A = \mathscr{B}^\lambda_\kappa / I_A - \{\mathbb{0}\}$.

In the propositional logic identification, the poset $\mathbb{P}_A$ is the set of T_A-consistent formulas in $\mathscr{B}^\lambda_\kappa$ modulo T_A-provability.

Lemma 2.25. $\mathbb{P}_A$ *is* κ*-cc.*

Proof. Let Q be a maximal antichain in $\mathbb{P}_A$, and $\Phi \subset \mathscr{B}^\lambda_\kappa$ a set of representatives of Q, so that $Q = \{[\phi]_{I_A} : \phi \in \Phi\}$. By definition of g_A, $g_A(\Phi) \subseteq \Phi$ and $\left|g_A(\Phi)\right| < \kappa$: we will show that $g_A(\Phi) = \Phi$ which implies $|\Phi| < \kappa$ hence $|Q| < \kappa$. It suffices to show that any $\phi \in \Phi$ is I_A-compatible with some $\phi' \in g_A(\Phi) \subset \Phi$, since Φ being an antichain implies that $\phi \wedge \phi' \notin I_A \Rightarrow [\phi \wedge \phi']_{I_A} \neq \mathbb{0} \Rightarrow [\phi]_{I_A} = [\phi']_{I_A} \Rightarrow \phi = \phi'$.

Every $\phi \in \Phi$ is not in I_A, hence there is a lower bound $\rho > \mathbb{0}$ for $T_A \cup \{\phi\}$. Since ρ is nonzero, it must be contained in a θ-complete ultrafilter $U^\theta_B \subset \mathscr{B}^\lambda_\theta$, and $\rho \in U^\theta_B$ implies that $T_A \subset U^\theta_B$ and $\phi \in U^\theta_B$.

Since $T_A \subset U^\theta_B$ includes $\left(\bigvee \Phi \to \bigvee g_A(\Phi)\right)$, and U^θ_B contains $\rho < \phi < \bigvee \Phi$, we must have $\bigvee g_A(\Phi) \in U^\theta_B$. Then by θ-completeness some $\phi' \in g_A(\Phi)$ must be in U^θ_B, so $\phi \wedge \phi' \geq \rho \wedge \phi' = \rho' \in U^\theta_B$ and $\rho' > \mathbb{0}$. Then ρ' is a lower bound for $T_A \cup \{\phi \wedge \phi'\}$ hence $\phi \wedge \phi' \notin I_A$ and $[\phi]_{I_A}$ is compatible with $[\phi']_{I_A}$. □

Lemma 2.26. $G_A = U^\kappa_A / I_A$ *is* $\mathbb{P}_A$*-generic over* M.

Proof. Let Φ be a set of representatives of a maximal antichain $Q \subset \mathbb{P}_A$. Since $\mathbb{P}_A$ is κ-cc, $|\Phi| < \kappa$ then let $\bigvee \Phi = \psi \in \mathscr{B}^\lambda_\kappa$ by κ-completeness. Then $[\psi]_{I_A} = [\mathbb{1}]_{I_A}$, otherwise $\{[\neg\psi]_{I_A}\} \cup Q$ would be an antichain extending Q, contradicting maximality of Q.

Then $\bigvee \Phi = \psi \in [\mathbb{1}]_{I_A} \subset U^\kappa_A$ since U^κ_A is disjoint from I_A, and some $\phi \in \Phi$ must be in U^κ_A. Hence $[\phi]_{I_A} \in G_A \cap Q$ and G_A is $\mathbb{P}_A$-generic over M. □

We can now prove the following main lemma of the Bukovsky's Theorem.

Theorem 2.27. *If* M *globally* κ*-covers* V*, and* $A \in V$ *is a subset of* λ*, then* $M[A]$ *is a* κ*-cc generic extension.*

Proof. Define $\mathbb{P}_A$ and G_A as before. $M[G_A]$ is a κ-cc generic extension by Lemma 2.26, so it is enough to show that $M[A] = M[G_A]$.

First we have $A \in M[G_A]$, since the set A is named by $\tau = \{\langle \check{\alpha}, [\alpha]_{I_A} \rangle : \alpha < \lambda\}$, thus $M[A] \subseteq M[G_A]$.

Furthermore, G_A is constructible from A, $\mathbb{P}_A$ hence $M[A] \supseteq M[G_A]$, then $M[A] = M[G_A]$ as required. □

The last result can be extended to any A subset of M, as the following shows.

Lemma 2.28. *Any extension $M[A]$ with $A \subset M$, is equivalent to a $M[B] = M[A]$ where B is a set of ordinals.*

Proof. Since $A \in V$, also $\alpha = \bigcup\{\text{rank}(x)+1 : \ x \in A\}$ is a ordinal in V (hence in M), and $A \subset C = V_\alpha \cap M = M_\alpha \in M$.

Now, let $\phi \in M$ be a bijection $\phi : C \to \lambda = |C|$, and fix $B = \phi[A]$ in $M[A]$. Since ϕ is in M, the class $M[A]$ must contain B as well as $M[B]$ must contain A, so $M[A] = M[B]$ with B set of ordinals. □

Lemma 2.29. *If M globally κ-covers V, $A \subset M$, then $M[A]$ globally κ-covers V.*

Proof. Given $f : \alpha \to M[A]$, $f \in V$, by Theorem 2.27 and 2.28 $M[A]$ is a κ-cc generic extension, then for all $x \in M[A]$ let $\tau_x \in M^{\mathbb{P}}$ be such that $x = \text{val}_A(\tau_x)$. Let $f' : \alpha \to M$ in V be such that $f'(i) = \tau_{f(i)}$.

By global κ-covering in M, let $g' : \alpha \to M$ that covers f', and $g'' : \alpha \to \mathscr{P}(M^{\mathbb{P}})$ be the intersection of g' with $\alpha \times \mathscr{P}(M^{\mathbb{P}})$ by constructibility of $M^{\mathbb{P}}$.

f is then covered by $g : \alpha \to M[A]$ in $M[A]$, defined by $g(i) = \text{val}_A[g''(i)]$, hence $M[A]$ globally κ-covers V. □

Theorem 2.30. *Any nontrivial κ-cc generic extension adds a subset of κ.*

Proof. By Lemma 1.14, a κ-cc algebra $\mathscr{B}$ is not κ^+-distributive. So let D_i, $i < \kappa$ be dense sets such that $\bigcap D_i = \emptyset$, and define recursively two sequences A_i, D'_i:

- $D'_0 = D_0$,
- $A_i = \left\{p_{ij} \in \mathbb{P} : j < \theta_i < \kappa\right\}$ maximal antichain in D'_i,
- $D'_{i+1} = \left\{p \in D_{i+1} : \ \exists j < \theta_i \ \ p < p_{ij}\right\}$, dense below A_i,
- $D'_\alpha = \bigcap_{i<\alpha} D'_i$, for α limit.

The construction stops for some $\theta \le \kappa$, when $\bigcap_{i<\theta} D'_i = \emptyset$.
Let $\dot{f} = \left\{\left\langle \left\langle i, \check{p}_{ij} \right\rangle, p_{ij} \right\rangle : \ i < \theta, j < \theta_i\right\}$, $f = [\dot{f}]_G$. f can not be in M, otherwise $\text{ran}(f)$ would be a chain in $\mathscr{B}$, and $\bigwedge \text{ran}(f)$ would be in every D'_α, $\alpha < \theta$, hence in $D'_\theta = \emptyset$. Then f is a new subset of $\kappa \times \kappa$, so given $\delta : (\kappa \times \kappa) \to \kappa$, $\delta[f]$ is a new subset of κ. □

Theorem 2.31 (Bukovsky)**.** *If $M \subset V$ are models of* ZFC, *where M globally κ-covers V, then V is a κ-cc generic extension of M.*

Proof. Let $A = \mathscr{P}^V(\kappa)$. Since $M[A]$ is transitive, it contains all subsets of κ in V.

By Lemma 2.27 and 2.28, every extension $M[B]$ with $B \subset M$ is a κ-cc generic extension. Then $M[A]$ is κ-cc generic and, by Lemma 2.29, globally κ-cover V. By Theorem 2.27 in $M[A]$ every extension $M[A,B]$ is also κ-cc generic, then by Theorem 2.30 must be trivial since it does not add any subset of κ (all such sets in V are already in $M[A]$).

Then every $B \subset M$, $B \in V$, is also in $M[A]$, hence $M[A]$ and V are models of ZFC sharing the same sets of ordinals, hence by Theorem 2.13 $V = M[A]$. □

2.4 Forcing Extensions by Size

The results in the previous section characterize κ-cc forcing extensions as extensions with the global κ-covering property. A reasonable question is whether it should be possible to characterize forcing extensions also by size. The answer is affirmative, and we will show in this section that size at most κ forcing extensions match with model extensions with both the κ-decomposition and global κ^+-covering property.

The left-to-right implication is in fact Proposition 2.9 and 2.10 (together with the trivial fact that size at most κ implies κ^+-cc), the right-to-left one will be shown at the end of this section (Theorem 2.34).

Since we have required the globally κ^+-cover property, we can use Bukovsky's Theorem 2.31 to take a κ^+-cc poset $\mathbb{P} \in M$ and a $\mathbb{P}$-generic filter G such that $V = M[G]$. Without loss of generality, we can assume that $\mathbb{P}$ is indeed $\mathbb{P} = \mathscr{B} \setminus \{\mathbb{0}\}$ with $\mathscr{B}$ a complete κ^+-cc boolean algebra (the completion of $\mathbb{P}$). However, $\mathbb{P}$ may not have the required limitation of size, so we need to find a smaller poset to force with. The easier way is to take any dense set $D \subset \mathbb{P}$, $D \in M$ of the required size, since forcing with the sub-order induced on D is equivalent to forcing with the whole $\mathbb{P}$.

Lemma 2.32. *If $D \in M$, $D \subset \mathbb{P}$ is dense in $\mathbb{P}$, and G is a $\mathbb{P}$-generic filter, then $M[G] = M[G \cap D]$.*

Proof. Let us show that $G \cap D$ is indeed a D-generic filter, by first proving that $G \cap D$ is dense in G. So let $g \in G$, and $D' = \{p \in \mathbb{P} : \ p \perp g\} \cup \{p \in D : \ p \leq g\}$. The set D' is dense in M, since if p is compatible with g they have a common extension q and, by density of D, $\exists r \ q \geq r \in D$ so $r \leq g$ hence $r \in D'$, $r \leq p$. Then $\exists s \in G \cap D'$, and $s \in G \Rightarrow s \parallel g$ hence $s \in D' \Rightarrow s \leq g$, that completes the proof of density of $G \cap D$ in G.

Moreover, $G \cap D$ is a filter in D, as closure by $\leq$ is obviously inherited from G, and given any $p, q \in G \cap D$ they must have a common extension $q \in G$, and by density of $G \cap D$, a common extension $q \geq r \in G \cap D$. The set $G \cap D$ is also D-generic, since any D' dense in D is also dense in $\mathbb{P}$ hence intersects G (and $G \cap D$ since $D' \subset D$).

Finally, $M[G \cap D] \subset M[G]$ since $G, D \in M[G] \Rightarrow G \cap D \in M[G]$. Vice versa, $G \cap D, \mathbb{P} \in M[G \cap D]$ implies that $G = \{g \in \mathbb{P} : \ \exists q \in G \cap D \ q \leq g\}$ is in $M[G \cap D]$ hence $M[G] \subset M[G \cap D]$ (the last equality holds since $G \cap D$ is dense in G, and G is closed by $\leq$). □

The dense set D will be defined by taking the infimum of every element of a decomposition of G; so we will first need to prove that this infimum exists in $\mathbb{P}$.

Lemma 2.33. *If G is $\mathscr{B}$-generic over M and $A \subset G$ is in M, then $\bigwedge A > \mathbb{0}$.*

Proof. Suppose instead that $\bigwedge A = \mathbb{0}$, and define

$$D = \{p \in \mathscr{B} : \ \exists q \in A \ \ p \wedge q = \mathbb{0}\}$$

D is dense in $\mathscr{B} \setminus \{\mathbb{0}\}$: given $r \in \mathscr{B} \setminus \{\mathbb{0}\}$, $\bigwedge A = \mathbb{0}$ then $\exists b \in A$ with $r \nleq b$, and $q = (r \wedge \neg b)$ is an extension of r in D (since $q \wedge b = \mathbb{0}$), and $q \neq \mathbb{0}$ since $(r \wedge \neg b) = \mathbb{0}$ would imply $r \leq b$.

Since D is dense, $D \in M$, $G \cap D \neq \emptyset$ then let $p \in G \cap D$. Since $p \in D$, there is a $b \in A$ with $p \wedge b = \mathbb{0}$, but $b \in A \subset G$ so G contains two elements b, p that are incompatible hence it is not a filter, a contradiction. □

We can now define a dense set D of size at most κ, and then prove the following Theorem 2.34 (Friedman, [10]).

Theorem 2.34. *If $M \subset V$ are models of* ZFC, *where M κ-decomposes and globally κ^+-covers V, then V is a size at most κ generic extension over M.*

Proof. As mentioned above, by Theorem 2.31 V is a generic extension with a κ^+-cc boolean algebra $\mathscr{B}$, $\mathbb{P} = \mathscr{B} \setminus \{\mathbb{0}\}$. Let us define a dense set $D \subset \mathbb{P}$ of size at most κ.

Since M κ-decomposes V, for every G generic exists a κ-decomposition $G = \bigcup_{i<\kappa} g(i)$, then

$$M[G] \models \exists x [x \subset (\kappa \times M) \wedge x \text{ is a function } \wedge G = \bigcup_{i<\kappa} x(i)]$$

By definition of $\Vdash$,

$$\mathbb{1} \Vdash \exists x [x \subset (\check{\kappa} \times \check{M}) \wedge x \text{ is a function } \wedge \dot{G} = \bigcup_{i<\kappa} x(i)]$$

Then, by fullness lemma, it exists a name $\dot{g}$ such that

$$\mathbb{1} \Vdash [\dot{g} \subset (\check{\kappa} \times \check{M}) \land \dot{g} \text{ is a function } \land \dot{G} = \bigcup_{i<\kappa} \dot{g}(i)]$$

For every $i < \kappa$, $\mathbb{1} \Vdash \dot{g}(i) \in \check{M}$, so the set

$$D_i = \left\{ p : \exists A \in M \;\; p \Vdash \dot{g}(i) = \check{A} \right\}$$

of conditions that decide membership in $\dot{g}(i)$ is dense. Let $X_i = \left\{ p_{ij} : j < \lambda_i \right\}$ be a maximal antichain in D_i, and let $A_{ij} \in M$ such that $p_{ij} \Vdash \dot{g}(i) = \check{A_{ij}}$. Since $\mathbb{P}$ is κ^+-cc, $\lambda_i \leq \kappa$.

Now, for every $i < \kappa, j < \lambda_i \leq \kappa$, $p_{ij} \Vdash \dot{g}(i) = \check{A_{ij}} \Rightarrow p_{ij} \Vdash \check{A_{ij}} \subset \dot{G}$; so by Lemma 2.33 there must be a condition $q_{ij} > \mathbb{0}$ such that $p_{ij} \Vdash \bigwedge \check{A_{ij}} = \check{q_{ij}}$, hence $\bigwedge A_{ij} = q_{ij}$. Let $D = \left\{ q_{ij} : i, j < \kappa \right\}$, $|D| = \kappa \cdot \kappa = \kappa$.

Let us show that D is dense: given any $r \in \mathbb{P}$, $r \Vdash \check{r} \in \dot{G} = \bigcup_{i<\kappa} \dot{g}(i)$. So $r \Vdash \exists i \; \check{r} \in \dot{g}(i)$ and by the fullness lemma there must be a $s \leq r$, $\alpha < \kappa$ such that $s \Vdash \check{r} \in \dot{g}(\alpha)$. Since X_α is a maximal antichain, there is a $\beta < \kappa$ such that $p_{\alpha\beta} \parallel s$, so $(s \land p_{\alpha\beta}) \Vdash \check{r} \in \check{A_{\alpha\beta}} \Rightarrow r \in A_{\alpha\beta}$ and so $r \geq q_{\alpha\beta} \in D$ and D is dense.

Since D is dense, $D \in M$ and $|D| \leq \kappa$, by Lemma 2.32 $V = M[G] = M[G \cap D]$ is a size at most κ generic extension over M. □

2.5 The Ground Axiom

In the last two sections we have characterized forcing extensions of various kind with properties of the models, verifiable without need of the poset $\mathbb{P}$ or the generic G. This results, combined with Laver's Theorem and the basic results at the beginning of the chapter, can be used to obtain a first-order expression of "being some kind of a forcing extension of some model M".

A remarkable case of this approach is the *Ground Axiom*.

Ground Axiom. *The set-theoretic universe V is not a forcing extension of any inner model M.*

The Ground Axiom seems to be second-order hence not a valid axiom for canonical set theory: we then need to write down a first-order equivalent of it. Before discussing the first-order transposition of the Ground Axiom, we will transpose the properties studied in Sections 2.3 and 2.4.

Theorem 2.35. *There is a first-order sentence $\phi_1(\kappa)$ that holds in V if and only if V is a κ-cc generic extension of some inner model M.*

Proof. By Theorem 2.31, we already know that V is a κ-cc generic extension of some inner model M if and only if $\psi(\kappa) = \exists M : M \models \text{ZFC} \wedge M$ globally κ-covers V. The sentence $\psi(\kappa)$ is still second-order, so we need some further manipulation. First, it is obvious that $\psi(\kappa)$ is equivalent to $\psi'(\kappa)$:

$$\exists\lambda\exists K\exists M : M \models \text{ZFC} \wedge H(\lambda^+)^M = K \wedge M \text{ globally } \kappa\text{-covers and } \lambda\text{-approximates } V$$

$\psi'(\kappa)$ is also second-order, and is equivalent to $\psi(\kappa)$ since we can choose for M the same M as in ψ, for λ, K any $\lambda > |\mathbb{P}|^V$, $K = H(\lambda^+)^M$. The advantage of $\psi'(\kappa)$ is that it is closer to the first-order sentence $\phi_1(\kappa)$:

$$\begin{aligned}\exists\lambda\,\exists K\,\forall\gamma > \lambda \text{ regular } \exists W : \quad & W \models \text{ZFC} - \text{P} \wedge H(\lambda^+)^W = K \\ & \wedge W \text{ globally } \kappa\text{-covers } H(\gamma)^V \\ & \wedge W\ \lambda\text{-approximates } H(\gamma)^V\end{aligned}$$

First, we have that $\psi'(\kappa) \Rightarrow \phi_1(\kappa)$, since for any $\gamma > \lambda$ regular we can take for W the structure $H(\gamma)^M$ with M satisfying $\psi'(\kappa)$. This way, $W \models \text{ZFC} - \text{P}$ and $H(\lambda^+)^W = K$ hold trivially, while covering and approximation hold by Proposition 2.11.

We now prove that $\phi_1(\kappa) \Rightarrow \psi'(\kappa)$. By the uniqueness Theorem 2.15 (and the fact that globally κ-cover imply λ-cover by Proposition 2.7 and 2.6), given γ the model W is unique; let us call it W_γ. Then $H(\delta)^{W_\gamma} = W_\delta$, for $\delta < \gamma$, since by Proposition 2.11 both the two sets meet the hypotheses of Theorem 2.15 for λ, $H(\delta)^V$.

Let $M = \bigcup W_\gamma$, since the sequence W_γ is coherent. $M \models \text{ZFC}$ follows by moving to a large enough $H(\gamma)^M$, $H(\lambda^+)^M = H(\lambda^+)^{W_\gamma} = K$ holds trivially, covering and approximation holds by Proposition 2.12, thus M witnesses the truth of $\psi'(\kappa)$. □

Theorem 2.36. *There is a first-order sentence $\phi_2(\kappa)$ that holds in V if and only if V is a size at most κ generic extension of some inner model M.*

Proof. The procedure follows exactly the one of the previous proof. By Theorem 2.34, we already know that V is a size at most κ generic extension of some inner model M if and only if

$$\psi(\kappa) = \exists M : M \models \text{ZFC} \wedge M \text{ globally } \kappa^+\text{-covers and } \kappa\text{-decomposes } V$$

$\psi(\kappa)$ is trivially equivalent to $\psi'(\kappa)$:

$$\begin{aligned}\exists\lambda\exists K\exists M : \quad & M \models \text{ZFC} \wedge H(\lambda^+)^M = K \wedge M \text{ globally } \kappa^+\text{-covers } V \\ & \wedge M\ \kappa\text{-decomposes and } \lambda\text{-approximates } V\end{aligned}$$

The first-order equivalent $\phi_2(\kappa)$ will then be:

$$\begin{aligned}\exists\lambda\,\exists K\;\forall\gamma>\lambda \text{ regular }\;\exists W:\quad & W\models \mathrm{ZFC}-\mathrm{P}\;\wedge\; H(\lambda^+)^W=K\\ & \wedge\; W \text{ globally } \kappa^+\text{-covers } H(\gamma)^V\\ & \wedge\; W\;\kappa\text{-decomposes } H(\gamma)^V\\ & \wedge\; W\;\lambda\text{-approximates } H(\gamma)^V\end{aligned}$$

First, $\psi'(\kappa)\Rightarrow\phi_2(\kappa)$, since by Proposition 2.11 we can take $W=H(\gamma)^M$.

We now prove that $\phi_2(\kappa)\Rightarrow\psi'(\kappa)$. By the uniqueness Theorem 2.15 and Proposition 2.11, given γ the model W must be unique and the sequence W_γ is coherent; then let $M=\bigcup W_\gamma$. M witnesses the truth of $\psi'(\kappa)$ by moving each step to a large enough $H(\gamma)^M$, and using Proposition 2.12. □

Given $\phi_1(\kappa)$ or $\phi_2(\kappa)$, we can now write down a first-order equivalent for the Ground Axiom.

Corollary 2.37. *There is a first-order sentence ϕ_3 that holds in V if and only if V is a not a generic extension of any inner model M.*

Proof. Such sentences are:

$$\phi_3=\big(\neg\exists\kappa\;\;\kappa \text{ regular cardinal }\wedge\phi_i(\kappa)\big)$$

with $i=1,2$. □

CHAPTER 3

APPROXIMATED EXTENSIONS

Throughout this chapter we will try to define, given $M \subset V$ transitive models of ZFC with $\mathbf{ON}^M = \mathbf{ON}^V$ such that M κ-covers V, another model N of ZFC such that $M \subset N \subset V$ that will be the "closure" of M by κ-approximation (i.e., such that N κ-approximate V). To verify that a class N is a model of ZFC we will use the Theorem 3.3 below [2, Theorem 13.9].

Definition 3.1. *G_i is a Gödel operation if*

$$\begin{array}{rcl}
G_1(X,Y) & = & \{X,Y\} \\
G_2(X,Y) & = & X \times Y \\
G_3(X,Y) & = & \in(X,Y) = \{\langle x,y\rangle \in X \times Y : x \in y\} \\
G_4(X,Y) & = & X \setminus Y \\
G_5(X,Y) & = & X \cap Y \\
G_6(X) & = & \bigcup X \\
G_7(X) & = & \operatorname{dom}(X) \\
G_8(X) & = & \{\langle x,y\rangle : \langle y,x\rangle \in X\} \\
G_9(X) & = & \{\langle x,y,z\rangle : \langle x,z,y\rangle \in X\} \\
G_{10}(X) & = & \{\langle x,y,z\rangle : \langle y,z,x\rangle \in X\}
\end{array}$$

A class N is closed under Gödel operations *if $G_i(X,Y) \in N$ whenever $X,Y \in N$.*

Definition 3.2. *A class N is* almost universal *in V iff for every subset $A \subset N$ in V there is a $C \in N$ superset of A.*

Theorem 3.3. *A transitive class N is an inner model of* ZF *if and only if it is almost universal and closed under Gödel operations.*

Proof. Since Gödel operations are absolute for transitive models, an inner model is necessarily closed under Gödel operations. If X is a subset of an inner model N, then $X \subset V_\alpha \cap N = N_\alpha$ for some α. Thus the condition is necessary.

Now let N be a transitive almost universal class that is closed under Gödel operations, and examine the axioms of ZF.

extensionality, foundation, infinity. N is transitive therefore extensional, infinity holds since ω is absolute for transitive models, foundation holds in any class.

pairing, union. Follow from the fact N is closed under G_1, G_6.

replacement, power set. Follow from almost universality of N.

comprehension. The axiom of comprehension requires that for every $X \in N$, the set $Y = \{u \in X : \phi^N(u)\}$ is in N. If ϕ is a Δ_0 formula, Y can be obtained from X by a careful application of Göedel operations: the verification is lengthy but easy (can be found in [2, Theorem 13.4]).

Otherwise, if ϕ has k quantifiers, let $\phi'(u_1, \dots, u_n, Y_1, \dots, Y_n)$ be the Δ_0 formula obtained by replacing each $\exists x$ (or $\forall x$) in ϕ by $\exists x \in Y_i$ (or $\forall x \in Y_i$) for $i = 1..k$. We shall prove, by induction on k, that for every $\phi(u_1, \dots, u_n)$ with k quantifiers, for every $X \in N$ there exists $Y_1, \dots, Y_k \in N$ such that:

$$\forall u_1, \dots, u_n \in X : \quad \phi^N(u_1, \dots, u_n) \Leftrightarrow \phi'(u_1, \dots, u_n, Y_1, \dots, Y_k)$$

then the existence of $Y \in N$ will follow from Δ_0-comprehension.

If $k = 0$ then $\phi = \phi'$. For the induction step, let $\phi(u)$ be $\exists v\ \psi(u, v)$ where ψ has k quantifiers. Thus ϕ' is $\exists v \in Y_{k+1}\ \psi'(u, v, Y_1, \dots, Y_k)$. By collection in V, there exists a set $A \in V$ such that:

$$\exists v \in N\ \psi^N(u, v) \Leftrightarrow \exists v \in A\ v \in N \wedge \psi^N(u, v) \Leftrightarrow \exists v \in (A \cap N)\ \psi^N(u, v)$$

Since N is almost universal, take $Y_{k+1} \in N$ such that $(A \cap N) \cup X \subset Y_{k+1}$. The above equivalence then will hold also with Y_{k+1} in place of $A \cap N$. By the induction hypothesis, given $Y_{k+1} \in N$, there exists $Y_1, \dots, Y_k \in N$ such that for all $u, v \in Y_{k+1}$:

$$\psi^N(u, v) \Leftrightarrow \psi'(u, v, Y_1, \dots, Y_k)$$

Since $X \subset Y_{k+1}$, we have for all $u \in X$:

$$\begin{aligned} \phi^N(u) &\iff \exists v \in N\ \psi^N(u,v) \\ &\iff \exists v \in Y_{k+1}\ \psi^N(u,v) \\ &\iff \exists v \in Y_{k+1}\ \psi'(u,v,Y_1,\dots,Y_k) \\ &= \phi'(u,Y_1,\dots,Y_{k+1}) \end{aligned}$$

This completes the proof. □

3.1 Approximated Sets

The intuition suggests us that a model N "closed by κ-approximation" should at least contain all sets κ-approximated in M.

Definition 3.4. *Let* $\mathscr{A}_M = \{X \in V : X\ \kappa\text{-approximated in } M\}$.

Clearly $\mathscr{A}_M$ is $M \subset \mathscr{A}_M \subset V$ as required, however it is not a model of ZFC: $\mathscr{A}_M$ satisfy all axioms in ZFC except for pairing, power set and comprehension. This fact reflects on Gödel operations in $\mathscr{A}_M$.

Theorem 3.5. *The class $\mathscr{A}_M$ of sets κ-approximated in M is closed under all Gödel operations except pair.*

Proof. Let X, Y be sets in $\mathscr{A}_M$, $z \in M$ be of size $< \kappa$, and $C \in M$ be such that $X \cup Y \subset C$. We will examine the operations $G_2, \dots, G_{10}$ in a convenient order.

2. $(X \times Y) \cap z = [(X \cap \mathrm{dom}(z)) \times (Y \cap \mathrm{ran}(z))] \cap z \in M$ hence $G_2(X,Y) \in \mathscr{A}_M$.

4. $(X \setminus Y) \cap z = (X \cap z) \setminus (Y \cap z) \in M$ hence $G_4(X,Y) \in \mathscr{A}_M$.

5. $(X \cap Y) \cap z = (X \cap z) \cap (Y \cap z) \in M$ hence $G_5(X,Y) \in \mathscr{A}_M$.

3. $\in (X,Y) = (X \times Y) \cap \{\langle x,y \rangle \in C^2 : x \in y\}$ hence $G_3(X,Y) \in \mathscr{A}_M$ by the previous point.

7. Let $\mathrm{ran}(X) \cap z = A$, by choice in V let B such that $X[B] = A$ and X restricted to B is 1-1; so $|B| = |A| < \kappa$. Then by κ-covering let $B \subset B' \in M$ with $|B'| < \kappa$. Now $\mathrm{ran}(X) \cap z = \mathrm{ran}(X \cap B') \cap z \in M$ then $\mathrm{ran}(X) \in \mathscr{A}_M$.

   By the same way can be obtained $G_7(X) = \mathrm{dom}(X) \in \mathscr{A}_M$. Also, $f[X] \in \mathscr{A}_M$ when $f, X \in \mathscr{A}_M$, since $f[X] = \mathrm{ran}(f \cap (X \times \mathrm{ran}(f))) \in \mathscr{A}_M$.

6. $\bigcup X = f[X]$ where $f = \{\langle x,y \rangle \in C^2 : y \in x\} \in M$, hence $G_6(X) \in \mathscr{A}_M$ by the previous point.

8. $G_8(X) = f[X]$ where $f : \mathrm{dom}(C) \times \mathrm{ran}(C) \to \mathrm{ran}(C) \times \mathrm{dom}(C)$ is such that $f(\langle a,b \rangle) = \langle b,a \rangle$, hence $G_8(X) \in \mathscr{A}_M$ by the previous point. The other converses operations G_9, G_{10} are obtained similarly.

Notice that $\mathscr{A}_M$ is also closed by finite unions,

$$(X \cup Y) \cap z = (X \cap z) \cup (Y \cap z) \in M \text{ hence } (X \cup Y) \in \mathscr{A}_M$$

This property usually follows from closure under G_1 and G_6, but $\mathscr{A}_M$ is not closed under G_1 so this property is in fact independent from the others. □

Since $\mathscr{A}_M$ is almost closed under Gödel operations, it turns out to satisfy a restricted form of comprehension, that will be useful later.

Theorem 3.6. *Let M be a inner definable class model of* ZFC. *The class $\mathscr{A}_M$ of sets κ-approximated in M satisfies comprehension relativized to M, i.e. for every $A, B_1, \dots B_m \in \mathscr{A}_M$, ϕ formula, exists $X \in \mathscr{A}_M$ such that*

$$X = \left\{ x \in A : \phi^M(x, B_1, \dots, B_m) \right\}$$

Proof. Let us prove that

$$X = \left\{ \langle x_1, \dots, x_n \rangle \in A_1 \times \dots \times A_n : \phi^M(x_1, \dots, x_n, B_1, \dots, B_m) \right\}$$

is in $\mathscr{A}_M$, with ϕ as in the hypothesis, $A_1, \dots, A_n, B_1, \dots B_m$ in $\mathscr{A}_M$. We will prove it by induction on the length of ϕ, using the previous Theorem 3.5.

Let $\bigcup A_i \subset C \in M$. If ϕ is

- $x_i = x_j$, $x_i \in x_j$: then $\phi = \phi^M$ and

$$X = \left(\prod A_i \right) \cap \left\{ \langle x_1, \dots, x_n \rangle \in C^n : x_i = x_j \text{ (resp. } x_i \in x_j) \right\}$$

- $B_i = x_j$, $B_i \in x_j$: then $\phi = \phi^M$ and

$$X = \left(\prod A_i \right) \cap \left\{ \langle x_1, \dots, x_n \rangle \in C^n : B_i = x_j \text{ (resp. } B_i \in x_j) \right\}$$

  if B_i is in M, $X = \emptyset$ otherwise, since all x_i will always vary in M.

- $x_i \in B_j$: then $\phi = \phi^M$ and $X = A_1 \times \dots \times A_{i-1} \times (A_i \cap B_j) \times A_{i+1} \times \dots \times A_n$.
- $\neg\psi$: then $X = \left(\prod A_i \right) \setminus \{ \langle x_1, \dots, x_n \rangle \in A_1 \times \dots \times A_n : \psi^M \}$.
- $\psi_1 \wedge \psi_2$: then

$$X = \left\{ \langle x_1, \dots, x_n \rangle \in A_1 \times \dots \times A_n : \psi_1^M \right\} \cap \left\{ \langle x_1, \dots, x_n \rangle \in A_1 \times \dots \times A_n : \psi_2^M \right\}$$

- $\exists x_{n+1}\ \psi$: by definition, we have

$$X=\left\{\langle x_1,\ldots,x_n\rangle\in A_1\times\ldots\times A_n:\ (\exists x_{n+1}\in M)\ \psi^M\right\}$$

By collection in V, given $A_1,\ldots,A_n$ there must be an $Y\in V$ such that

$$\begin{aligned}\exists x_{n+1}\,(x\in M\wedge\psi^M)\ &\Leftrightarrow\ \exists x_{n+1}\in Y\ \ (x\in M\wedge\psi^M)\\ &\Leftrightarrow\ \exists x_{n+1}\in(Y\cap M)\ \ \psi^M\\ &\Leftrightarrow\ \exists x_{n+1}\in A_{n+1}\ \ \psi^M\end{aligned}$$

where $A_{n+1}\in M$ is such that $Y\cap M\subset A_{n+1}$, by almost universality of M in V. Thus we can obtain X as

$$X=\mathrm{dom}_{1..n}(\left\{\langle x_1,\ldots,x_{n+1}\rangle\in A_1\times\ldots\times A_{n+1}:\ \psi^M\right\})\qquad\square$$

Remark 3.1. This same result could have been proved for any class $\mathscr{C}$ closed under all Gödel operations except pair, such that $M\subset\mathscr{C}\subset\mathscr{P}^V(M)\subset V$.

3.2 ZFC in $\overline{M}^{\kappa}$

The class $\mathscr{A}_M$ can not be a candidate for a κ-approximation closure of M, as it contains only subsets of M. Instead of $\mathscr{A}_M$ we will then consider the following.

Definition 3.7. *Let* $\overline{M}^{\kappa}=\bigcup\{\Pi_R[X]:\ R\in\mathscr{A}_M$ *well-founded relation* $R\subset X^2\}$, *with* Π_R *transitive collapse of* R *in* V.

Intuitively, the class $\overline{M}^{\kappa}$ should contain $\mathscr{A}_M$ in order to be a κ-approximation closure of M.

Lemma 3.8. $\mathscr{A}_M\subset\overline{M}^{\kappa}$.

Proof. Let A be in $\mathscr{A}_M$, $\mathrm{trcl}(A)\subset C\in M$. Then $R=(A\times\{C\})\cup\in(C,C)$ is in $\mathscr{A}_M$ and has transitive collapse $\Pi_R:(C\cup\{C\})\to\overline{M}^{\kappa}$ with $\Pi_R(C)=A$; then $A\in\overline{M}^{\kappa}$. $\square$

In order to prove that $\overline{M}^{\kappa}$ is a model of ZFC, we will assume from now on that $\mathscr{P}_{\kappa}^M(\lambda)$ is stationary in $\mathscr{P}^V(\lambda)$, for some fixed λ (hence for every λ', by lifting and projection lemma).

Lemma 3.9. *For every ϕ formula, with parameters $A_1,\ldots,A_m\in N$ that appears only at the right side of* $\in$, *and* $x_1,\ldots,x_n$ *free:*

$$\phi^N(x_1,\ldots,x_n,A_1\cap N,\ldots A_m\cap N)\longleftrightarrow\phi^N(x_1,\ldots,x_n,A_1,\ldots A_m)$$

Proof. Follows by induction on ϕ, from the fact that every x_i appearing in ϕ^N will always vary in N, hence $x_i \in A_j \cap N \longleftrightarrow x_i \in A_j$. □

Definition 3.10. *Given A well-founded relation, ϕ formula, let $R_{A,\phi}$ be defined by transfinite recursion on A using ϕ as follows:*

$$R_{A,\phi} = \left\{ \langle x, y\rangle \in \mathrm{dom}(A)^2 : \phi(x, y, A, R_{A,\phi} \cap [\mathrm{pred}(x, A) \times \mathrm{pred}(y, A)]) \right\}$$

Theorem 3.11. *If A is a well-founded relation in $\mathscr{A}_M$, ϕ is a formula with parameters only at the right side of $\in$, then $R_{A,\phi}$ is in $\mathscr{A}_M$.*

Proof. Let θ be a regular cardinal with $A, R_{A,\phi} \in H(\theta)^V$. Given $N \in V$ such that $N \preceq H(\theta)^V$, $A, R_{A,\phi} \in N$, $|N| < \kappa$, we first prove that $R_{A\cap N^2,\phi} = R_{A,\phi} \cap N^2$.

Suppose by contradiction that $\langle x, y\rangle$ is a minimal pair (with respect to A-rank) in N^2 with $(x R_{A,\phi} y) \longleftrightarrow \neg(x R_{A\cap N^2,\phi} y)$. Then:

$$\begin{aligned} x R_{A\cap N^2,\phi} y \longleftrightarrow &\quad \phi^N(x, y, A\cap N^2, R_{A\cap N^2,\phi} \cap [\mathrm{pred}(x, A\cap N^2) \times \mathrm{pred}(y, A\cap N^2)]) \\ \longleftrightarrow &\quad \phi^N(x, y, A\cap N^2, R_{A\cap N^2,\phi} \cap [\mathrm{pred}(x, A) \times \mathrm{pred}(y, A)] \cap N^2) \end{aligned}$$

by minimality of $\langle x, y\rangle$, $R_{A,\phi}$ and $R_{A\cap N^2,\phi}$ coincide below $\langle x, y\rangle$, then:

$$\begin{aligned} x R_{A\cap N^2,\phi} y &\longleftrightarrow \phi^N(x, y, A\cap N^2, R_{A\cap N^2,\phi} \cap [\mathrm{pred}(x, A) \times \mathrm{pred}(y, A)] \cap N^2) \\ &\longleftrightarrow \phi^N(x, y, A\cap N^2, R_{A,\phi} \cap [\mathrm{pred}(x, A) \times \mathrm{pred}(y, A)] \cap N^2) \\ &\longleftrightarrow \phi^N(x, y, A, R_{A,\phi} \cap [\mathrm{pred}(x, A) \times \mathrm{pred}(y, A)]) \end{aligned}$$

The last equivalence followed from Lemma 3.9. Now by $N \preceq H(\theta)^V$:

$$\begin{aligned} x R_{A\cap N^2,\phi} y &\longleftrightarrow \phi^N(x, y, A, R_{A,\phi} \cap [\mathrm{pred}(x, A) \times \mathrm{pred}(y, A)]) \\ &\longleftrightarrow N \vDash \phi(x, y, A, R_{A,\phi} \cap [\mathrm{pred}(x, A) \times \mathrm{pred}(y, A)]) \\ &\longleftrightarrow H(\theta)^V \vDash \phi(x, y, A, R_{A,\phi} \cap [\mathrm{pred}(x, A) \times \mathrm{pred}(y, A)]) \\ &\longleftrightarrow x R_{A,\phi} y \end{aligned}$$

in contrast with $(x R_{A,\phi} y) \longleftrightarrow \neg(x R_{A\cap N^2,\phi} y)$. Then $R_{A\cap N^2,\phi} = R_{A,\phi} \cap N^2$.

Now since $\mathscr{P}^M_\kappa(\lambda)$ is stationary, given $z \in M$, $|z| < \kappa$, by lifting let N be as before with $\mathrm{dom}(z) \cup \mathrm{ran}(z) \subset N$, $\lambda \cap N \in M$. Then $z \subset N^2$, $A \subset \lambda^2$, and:

$$R_{A,\phi} \cap z = R_{A,\phi} \cap N^2 \cap z = R_{A\cap N^2,\phi} \cap z = R_{A\cap\lambda^2\cap N^2} \cap z = R_{A\cap(\lambda\cap N)^2} \cap z \in M$$

since $\lambda \cap N \in M$ of size $< \kappa$ and $A \in \mathscr{A}_M$. Then $R_{A,\phi} \in \mathscr{A}_M$, as required. □

Corollary 3.12. *If $R \in \mathscr{A}_M$ is a well-founded relation $R \subset X^2 \in M$, there exists an extensional well-founded relation $R' \in \mathscr{A}_M$ such that $R' \subset X^2$, $\Pi_{R'} \subset \Pi_R$ and $\Pi_{R'}[X] = \Pi_R[X]$.*

Proof. Without loss of generality, suppose $X = \lambda$. Let T be defined by recursion on R as:

$$\begin{aligned} T &= \{\langle a,b\rangle \in \lambda^2 : [\forall a' \in \lambda\, \langle a',a\rangle \in R \rightarrow \exists b' \in \lambda\, \langle b',b\rangle \in R \wedge \langle a',b'\rangle \in T] \\ &\quad \wedge [\forall b' \in \lambda\, \langle b',b\rangle \in R \rightarrow \exists a' \in \lambda\, \langle a',a\rangle \in R \wedge \langle a',b'\rangle \in T]\} \end{aligned}$$

The relation T above is such that $\langle a,b\rangle \in T \Leftrightarrow \Pi_R(a) = \Pi_R(b)$, and by Theorem 3.11 $T \in \mathscr{A}_M$. From T we can now define a set $A \subset \lambda$ that contains exactly one preimage for every $x \in \Pi_R[\lambda]$, as

$$A = \{i \in \lambda : \forall j \in \lambda\, (i < j \rightarrow \langle i,j\rangle \notin T)\}$$

This set is in $\mathscr{A}_M$ by Theorem 3.6. We can now define $R' \subset A^2$ as

$$R' = \left\{\langle a,b\rangle \in A^2 : \exists a' \in \lambda\, \exists b' \in \lambda\, \left(\langle a',a\rangle \in T \wedge \langle b',b\rangle \in T \wedge \langle a',b'\rangle \in R\right)\right\}$$

R' is in $\mathscr{A}_M$ by Theorem 3.6, and is therefore an extensional well-founded relation with $\Pi_{R'}[\lambda] = \Pi_R[\lambda]$. □

Lemma 3.13. *The class $\overline{M}^{\kappa}$ is closed under all Gödel operations.*

Proof. Without loss of generality, let X, Y be such that $X = \Pi_R(\alpha)$, $Y = \Pi_S(\beta)$, where $R, S \in \mathscr{A}_M$ are well-founded relations $R, S \subset \lambda^2$.

1. Let T be defined by:

$$\begin{aligned} T &= \{\langle\langle 0,i\rangle,\langle 0,j\rangle\rangle : \langle i,j\rangle \in R\} \cup \{\langle\langle 1,i\rangle,\langle 1,j\rangle\rangle : \langle i,j\rangle \in S\} \\ &\quad \cup \{\langle 0,\alpha\rangle,\langle 1,\beta\rangle\} \times \{\langle 2,0\rangle\} \end{aligned}$$

   T is in $\mathscr{A}_M$ by Theorem 3.6, hence $G_1(X,Y) = \Pi_T(\langle 2,0\rangle) \in \overline{M}^{\kappa}$.

2. Let T be defined by:

$$\begin{aligned} T &= \{\langle\langle 0,i\rangle,\langle 0,j\rangle\rangle : \langle i,j\rangle \in R\} \cup \{\langle\langle 1,i\rangle,\langle 1,j\rangle\rangle : \langle i,j\rangle \in S\} \\ &\quad \cup \{\langle\langle 0,i\rangle,\langle 2,i\rangle\rangle : i < \lambda\} \\ &\quad \cup \{\langle\langle 0,i\rangle,\langle 3,i,j\rangle\rangle : i,j < \lambda\} \cup \{\langle\langle 1,j\rangle,\langle 3,i,j\rangle\rangle : i,j < \lambda\} \\ &\quad \cup \{\langle\langle 2,i\rangle,\langle 4,i,j\rangle\rangle : i,j < \lambda\} \cup \{\langle\langle 3,i,j\rangle,\langle 4,i,j\rangle\rangle : i,j < \lambda\} \\ &\quad \cup \{\langle 4,i,j\rangle : \langle i,\alpha\rangle \in R \wedge \langle j,\beta\rangle \in S\} \times \{\langle 5,0\rangle\} \end{aligned}$$

   T is in $\mathscr{A}_M$ by Theorem 3.6, hence $G_2(X,Y) = \Pi_T(\langle 5,0\rangle) \in \overline{M}^{\kappa}$.

3. Let T be defined as in the previous point. Let $T' \in \mathscr{A}_M$ be extensional with the same collapse as T by Theorem 3.12, and γ' be the unique element in

dom(T') with $\Pi_T(\gamma') = \Pi_T(\langle 5,0\rangle)$.
Let T'' be defined by:

$$T'' = T' \cup \{x \in \mathrm{dom}(T') \colon \langle x,\gamma'\rangle \in T' \wedge (\exists y,z \in \mathrm{dom}(T')\ \phi(x,y,z))\} \times \{\langle 5,1\rangle\}$$

where $\phi(x,y,z)$ is

$$\begin{aligned}\phi(x,y,z) = \langle y,z\rangle \in T' \quad &\wedge \quad [\forall w \in \mathrm{dom}(T')\ \langle w,x\rangle \in T' \to \langle y,w\rangle \in T'] \\ &\wedge \quad [\exists w \in \mathrm{dom}(T')\ \langle w,x\rangle \in T' \wedge \langle z,w\rangle \in T']\end{aligned}$$

T'' is in $\mathscr{A}_M$ by Theorem 3.6, hence $G_3(X,Y) = \Pi_T(\langle 5,1\rangle) \in \overline{M}^\kappa$.

4. Let T be defined by:

$$T = \{\langle\langle 0,i\rangle,\langle 0,j\rangle\rangle \colon \langle i,j\rangle \in R\} \cup \{\langle\langle 1,i\rangle,\langle 1,j\rangle\rangle \colon \langle i,j\rangle \in S\}$$

Let $T' \in \mathscr{A}_M$ be extensional with the same collapse as T by Theorem 3.12, and α', β', be the unique elements in dom(T') with $\Pi_T(\alpha') = \Pi_T(\langle 0,\alpha\rangle)$, $\Pi_T(\beta') = \Pi_T(\langle 1,\beta\rangle)$.
Let T'' be defined by:

$$T'' = T' \cup \{x \in (2\times\lambda) \colon \langle x,\alpha'\rangle \in T' \wedge \langle x,\beta'\rangle \notin T'\} \times \{\langle 2,0\rangle\}$$

T'' is in $\mathscr{A}_M$ by Theorem 3.6, hence $G_4(X,Y) = \Pi_{T''}(\langle 2,0\rangle) \in \overline{M}^\kappa$.

5. Let T, T', α', β' be defined as in the previous point. Let T'' be defined by:

$$T'' = T' \cup \{x \in (2\times\lambda) \colon \langle x,\alpha'\rangle \in T' \wedge \langle x,\beta'\rangle \in T'\} \times \{\langle 2,0\rangle\}$$

T'' is in $\mathscr{A}_M$ by Theorem 3.6, hence $G_5(X,Y) = \Pi_{T''}(\langle 2,0\rangle) \in \overline{M}^\kappa$.

6. Let T be defined by:

$$T = R \cup \mathrm{dom}(R \cap (\lambda \times \mathrm{dom}(R \cap (\lambda \times \{\alpha\})))) \times \{\lambda\}$$

T is in $\mathscr{A}_M$ by Theorem 3.5, hence $G_6(X) = \Pi_T(\lambda) \in \overline{M}^\kappa$.

7. Let $R' \in \mathscr{A}_M$ be extensional with the same collapse as R by Theorem 3.12, and α' be the unique element in dom(R') with $\Pi_R(\alpha') = \Pi_R(\alpha)$.
Let T be defined by:

$$T = R' \cup \{i \in \lambda \colon \exists j \in \lambda\ \phi(i,j)\} \times \{\lambda\}$$

Where $\phi(i,j) = \langle j,\alpha'\rangle \in R' \wedge \forall k \in \lambda\ (\langle k,j\rangle \in R' \to \langle i,k\rangle \in R')$. T is in $\mathscr{A}_M$

by Theorem 3.6. If X is a relation, $G_7(X) = \mathrm{dom}(X) = \Pi_T(\lambda) \in \overline{M}^{\kappa}$.

If X is not a relation, replace ϕ in the definition of T with

$$\begin{aligned} \psi(i,j) \;=\; \phi(i,j) & \;\wedge\; \exists^{=2} k \in \lambda \; (\langle k, j\rangle \in R') \\ & \;\wedge\; \exists^{=2} h \in \lambda \; (\exists k \in \lambda \; \langle h,k\rangle \in R' \wedge \langle k,j\rangle \in R') \\ & \;\wedge\; \exists^{=1} h \in \lambda \, \exists^{=1} k \in \lambda \; (\langle h,k\rangle \in R' \wedge \langle k,j\rangle \in R') \end{aligned}$$

Now $G_7(X) = \Pi_T(\lambda) \in \overline{M}^{\kappa}$.

8. Let R', α' be defined as in the previous point. Let T be defined by:

$$\begin{aligned} T \;=\; & \{\langle\langle 0,i\rangle, \langle 0,j\rangle\rangle : \langle i,j\rangle \in R'\} \\ & \cup \{\langle\langle 0,i\rangle, \langle 1,i\rangle\rangle : i \in \lambda\} \\ & \cup \{\langle\langle 0,i\rangle, \langle 2,j\rangle\rangle : \langle i,j\rangle \in R' \wedge \exists^{\geq 2} k \in \lambda \; \langle k,j\rangle \in R'\} \\ & \cup \{\langle\langle 1,i\rangle, \langle 2,j\rangle\rangle : \exists^{=1} k \in \lambda \; \langle i,k\rangle \in R' \wedge \langle k,j\rangle \in R'\} \\ & \cup \{\langle 2,i\rangle : \langle i,\alpha'\rangle \in R'\} \times \{\langle 3,0\rangle\} \end{aligned}$$

T is in $\mathscr{A}_M$ by Theorem 3.6. If X is a relation, $G_8(X) = \Pi_T(\langle 3,0\rangle) \in \overline{M}^{\kappa}$.

If X is not a relation, we can also obtain $G_8(X) \in \overline{M}^{\kappa}$ in the same way we did in the previous point.

Finally, the fact that $G_9(X)$, $G_{10}(X)$ are in $\overline{M}^{\kappa}$ is a minor variant of the above proof for $G_8(X)$. □

It is not known if the class $\overline{M}^{\kappa}$ can be almost universal, hence $\overline{M}^{\kappa}$ might fail to be a model for ZF. From now on we shall say $U(M)$ for the sentence "$\overline{M}^{\kappa}$ is almost universal".

Theorem 3.14. *Assume M κ-covers V and $U(M)$, then $\overline{M}^{\kappa} \models$ ZFC.*

Proof. The class $\overline{M}^{\kappa}$ is trivially transitive, as a union of transitive sets.

Furthermore, by Lemma 3.13 $\overline{M}^{\kappa}$ is closed under Gödel operations hence $\overline{M}^{\kappa} \models$ ZF by Theorem 3.3.

Let us verify the axiom of choice in $\overline{M}^{\kappa}$. Given $A \in \overline{M}^{\kappa}$, $A \in \Pi_R[X]$, R is in $\mathscr{A}_M$ hence in $\overline{M}^{\kappa}$ by Lemma 3.8, and also $\Pi_R \in \overline{M}^{\kappa}$ since $\overline{M}^{\kappa} \models$ ZF. Fix a well-order $(X,<)$ in M, and define a corresponding well-order $(A,<)$: given $a,b \in A$,

$$a < b \longleftrightarrow \min\left(\Pi_R^{-1}(a)\right) < \min\left(\Pi_R^{-1}(b)\right)$$

This is a well-order on A definable in $\overline{M}^{\kappa}$, hence $\overline{M}^{\kappa} \models$ ZFC. □

3.3 Characterization of $\overline{M}^{\kappa}$

The model $\overline{M}^{\kappa}$ satisfies some more properties, inherited from M:

Proposition 3.15. *Assume M κ-covers V and $U(M)$, then $\overline{M}^\kappa$ κ-covers V.*

Proof. Let $A \subset \overline{M}^\kappa$, $|A| < \kappa$. Let $B \in \overline{M}^\kappa$ with $A \subset B$ and $f : B \to \lambda$ bijection in $\overline{M}^\kappa$. $f[A]$ is a subset of M of size $< \kappa$ so $f[A] \subset C \in M$, $|C| < \kappa$.

Then $A \subset f^{-1}[C] \in \overline{M}^\kappa$, $\left|f^{-1}[C]\right| < \kappa$. □

Proposition 3.16. *Assume M κ-covers V, then $\mathscr{P}^M_\kappa \mathbf{ON} = \mathscr{P}^{\overline{M}^\kappa}_\kappa \mathbf{ON}$.*

Proof. First let A be in $\mathscr{P}^{\overline{M}^\kappa}_\kappa \kappa$, $B = \mathrm{trcl}(\{A\})$. Since $A \in \overline{M}^\kappa$, let $A \in \mathrm{ran}(\Pi_R)$.

Since $|B| < \kappa$, we have $\Pi_R^{-1}[B] \subset C \in M$, $|C| < \kappa$ by κ-covering. Then $R \cap (C \times C)$ is in M, and $A \in \mathrm{ran}(\Pi_{R\cap(C\times C)}) \in M$, as required.

Now let A be in $\mathscr{P}^{\overline{M}^\kappa}_\kappa \mathbf{ON}$. By κ-covering let $B \in M$ be superset of A of size $< \kappa$, and $f : B \to \kappa$ be a 1-1 function in M. $f[A]$ is a subset of κ, then $f[A] \in \mathscr{P}^{\overline{M}^\kappa}_\kappa \kappa$ and $f[A] \in M$ as proved before. Then also $A = f^{-1}[f[A]] \in M$. □

Furthermore, $\overline{M}^\kappa$ is "closed by κ-approximation", as required:

Lemma 3.17. *Assume M κ-covers V and $U(M)$, then $\overline{M}^\kappa$ κ-approximates V.*

Proof. Let $A \in \mathscr{A}_{\overline{M}^\kappa}$, $A \subset C \in \overline{M}^\kappa$ and $f : C \to \lambda$ bijection in $\overline{M}^\kappa$. As in Theorem 3.5, $B = f[A] \subset \mathbf{ON}$ is in $\mathscr{A}_{\overline{M}^\kappa}$, and then also in $\mathscr{A}_M$: in fact, $B \cap z = B \cap z \cap \mathbf{ON}$, $B \cap z \cap \mathbf{ON} \in \mathscr{P}^{\overline{M}^\kappa}_\kappa \mathbf{ON} = \mathscr{P}^M_\kappa \mathbf{ON}$ by Lemma 3.16, hence $B \in \mathscr{A}_M$.

By Lemma 3.8, $B \in \mathscr{A}_M \subset \overline{M}^\kappa$ then also $A = f^{-1}[B] \in \overline{M}^\kappa$, as required. □

In fact, $\overline{M}^\kappa$ can be characterized as the minimum class having such properties, hence justifying our claim to be a "closure" of M:

Theorem 3.18. *Assume M κ-covers V and $U(M)$, then $\overline{M}^\kappa$ is the smallest transitive class $\overline{M} \subset V$ such that:*

- $M \subset \overline{M}$,
- $\overline{M} \vDash$ ZFC,
- $\overline{M}$ *κ-approximates V.*

Proof. From Lemma 3.14 and 3.17 we have that $\overline{M}^\kappa$ meets the last two requirements asked for $\overline{M}$. From $M \subset \mathscr{A}_M$ and Lemma 3.8 $\overline{M}^\kappa$ meets also the first one, so it suffices to prove that for every such $\overline{M}$, $\overline{M}^\kappa \subset \overline{M}$.

First prove that $\mathscr{A}_M \subset \mathscr{A}_{\overline{M}}$. Given $A \in \mathscr{A}_M$, $z \in \overline{M}$, $|z| < \kappa$, by κ-covering $(z \cap M) \subset C \in M$, so $A \cap z = A \cap M \cap z = (A \cap C) \cap z \in \overline{M}$ hence $A \in \mathscr{A}_{\overline{M}}$.

Now let R be a well-founded relation in $\mathscr{A}_M \subset \mathscr{A}_{\overline{M}}$. Since $\overline{M}$ κ-approximates V, $\mathscr{A}_{\overline{M}} = \overline{M}$ then $R \in \overline{M}$, $\Pi_R \in \overline{M}$ hence $\overline{M}^\kappa \subset \overline{M}$. □

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