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# Saturated structures constructed using forcing and applications

Relatore: Prof. Matteo Viale

Candidato: Moreno Pierobon

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**Abstract**

This thesis deals with a method to embed first order structures in saturated ones, by means of quotient of boolean valued models. To this extent, the basic framework of boolean valued models is introduced and the construction of the boolean ultrapower of a first order structure is presented. Secondarily, good ultrafilters are defined, and it is shown that the quotient of a full boolean valued model by a good ultrafilter is a saturated first order structure. Next, the optimal conditions to guarantee the existence of good ultrafilters are investigated. In particular, an explicit example of boolean algebra containing good ultrafilters is described.

The rest of the work is devoted to explore some examples of boolean valued models and to discuss the peculiarities of the quotients of such models by an ultrafilter. The first example analyzed is the construction of a space of ultrafunctions as a quotient of a boolean valued model. Consequently, the analysis focuses on the degree of saturation of this space.

The last class of examples are motivated by the analysis of the method of forcing. In particular, $\mathsf{B}$-names for the elements of a first order structure are defined, and the connection between this construction and boolean ultrapowers is made explicit. Moreover for a specific first order structure, the topological space $2^\omega$, a different characterization of its set of $\mathsf{B}$-names in terms of continuous functions is presented.

Finally it is outlined that the theory of sheaves provides a categorial setting where all these examples can be analyzed: we show for example that, among all the boolean valued models, the ones which satisfy the mixing property can be identified with the sheaves inside a certain class of presheaves. It is also shown that the quotient by an ultrafilter of a boolean valued model which satisfies the mixing property is exactly an equivalent description of the stalk of the sheaf corresponding to the model.

## Abstract

Questa tesi analizza vari metodi per immergere strutture del prim'ordine in strutture sature utilizzando quozienti di modelli booleani. In quest'ottica, vengono introdotti alcuni elementi della teoria dei modelli booleani e viene presentata la costruzione dell'ultrapotenza booleana di una struttura del prim'ordine. In secondo luogo, vengono definiti gli ultrafiltri good, e si dimostra che il quoziente di un modello booleano per un ultrafiltro good è una struttura satura. Vengono quindi investigate le condizioni ottimali su un'algebra di Boole che garantiscono l'esistenza di ultrafiltri good e, in particolare, si descrive un esempio esplicito di una tale algebra di Boole.
La restante parte del lavoro concerne la presentazione di alcuni esempi di modelli booleani e l'indagine della struttura dei rispettivi quozienti modulo un ultrafiltro. La costruzione di uno spazio di ultrafunzioni costituisce il primo di questi esempi.
La classe di esempi più consistente è motivata da una analisi del forcing, un metodo introdotto per produrre una grande varietà di modelli della teoria degli insiemi. Nello specifico, vengono definiti i $\mathsf{B}$-nomi per elementi di una struttura del prim'ordine, e viene indagata la correlazione tra questa costruzione e le ultrapotenze booleane. Infine, per una struttura specifica, lo spazio topologico $2^\omega$, viene studiata una caratterizzazione dei $\mathsf{B}$-nomi in termini di funzioni continue.
La teoria dei fasci fornisce un contesto categoriale nel quale inquadrare questi esempi: in questo lavoro si mostra come, nella classe dei modelli booleani, quelli che soddifano la mixing property corrispondono ai fasci di una certa famiglia di prefasci. Un ulteriore risultato mostra che il quoziente per un ultrafiltro di un modello booleano che soddisfa la mixing property risulta essere una spiga del corrispondente fascio.

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# Contents

# Introduction

This dissertation explores some connections between set theory and model theory. Its core part describes various methods to construct saturated structures using boolean valued models.
A structure $\mathcal{M}$ for the language $L$ is *saturated* if, for every finitely consistent family $\Phi$ of $L$-formulae with one free variable of size less than $|\mathcal{M}|$, there exists $a \in \mathcal{M}$ such that $\phi(a)$ is true for every $\phi(x) \in \Phi$. Saturated structures are of central interest in modern model theory, since they are universal models of a theory $T$ in which all the other models of $T$ of smaller size can be embedded.
Boolean valued models with their associated semantics provide an efficient language where to develop the forcing method, invented by Cohen to prove indipendence results for set theory. Boolean valued models generalize first order structures: in such models, sentences need not to be true or false, but they can have as truth values an element of a boolean algebra $\mathsf{B}$. It is possible to quotient a $\mathsf{B}$-valued $\mathcal{M}$ by an ultrafilter $U$ in the boolean algebra $\mathsf{B}$, the resulting structure $\mathcal{M}/_U$ is an ordinary first order structure: the ultrafilter $U$ decides which sentences will be true in the quotient $\mathcal{M}/_U$.
Chapter 1 introduces some basic facts from model theory and topology used in the remainder of the thesis. Furthermore, we will give a brief introduction to the theory of partial orders and boolean algebras.
Chapter 2 develops the main features of the theory of boolean valued models. We focus in particular on three examples. We devote an entire section to define the boolean power $\mathcal{M}^{\downarrow\mathsf{B}}$ of a first order structure $\mathcal{M}$ (a construction due to Mansfield [15]), producing a $\mathsf{B}$-valued model extending $\mathcal{M}$. We show that, for any ultrafilter $U$ in $\mathsf{B}$, the quotient $\mathcal{M}^{\downarrow\mathsf{B}}/_U$ is an elementary extension of $\mathcal{M}$. The second part of the chapter analyzes the construction of boolean valued models for set theory: if $V$ is a model for set theory, we define its $\mathsf{B}$-valued extension $V^{\mathsf{B}}$. Moreover, for every class $M$ in $V$, the correspondent class $M^{\mathsf{B}}$ in $V^{\mathsf{B}}$ is described, and its subclass $\check{M}^{\mathsf{B}}$ is introduced (this latter class will describe the family of $\mathsf{B}$-names for elements of the ground model $M$). Finally, for the specific case in which $M$ is the set $2^\omega$, we exhibit an isomorphism between $(2^\omega)^{\mathsf{B}}$ and the space given by continuous functions $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$; we also show that the image of $(\check{2^\omega})^{\mathsf{B}}$ under this isomorphism is the space $\mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$ given by locally constant continuous functions.
Chapter 3 studies how to produce saturated extensions of a first order structure $\mathcal{M}$ using boolean valued models and their quotients structures. We expand on the thesis of Parente [16], Mansfield's [15], and Balcar's and Franek's[1], generalizing many of the results on the properties of quotients of boolean valued models by ultrafilters appearing in [16]. First of all, we introduce the notion of *good ultrafilter*, and we prove that the quotient of a full boolean valued model with the mixing property by a good ultrafilter is a saturated first order structure. We obtain in this way Mansfield's result stating that for any complete boolean algebra $\mathsf{B}$, any first order structure $\mathcal{M}$ and any good ultrafilter $U$ on $\mathsf{B}$, the quotient $\mathcal{M}^{\downarrow\mathsf{B}}/_U$ is a saturated elementary extension of $\mathcal{M}$. We discuss and isolate the optimal hypothesis a boolean algebra has to satisfy in order to admit good ultrafilters

(a result appearing in [1]), and we show an explicit example of such a boolean algebra. The last part of the chapter rephrases in the language of boolean valued models a construction appearing in non-standard analysis due to Benci [2]: the so called *space of ultrafunctions*. This space enlarges a fixed functional space $V(\Omega)$ (such as $L^2(\Omega)$ with $\Omega$ an open subset of $\mathbb{R}^n$) to a much larger one $V_\Lambda(\Omega)$, which contains also the space of distributions. $V_\Lambda(\Omega)$ has the property that it admits limits also for nets with values in $V(\Omega)$ which do not converge even in the space of distributions. Benci and Luperi Baglini use $V_\Lambda(\Omega)$ to find non-standard solutions to many otherwise untractable problems concerning partial differential equations (see for example [4]). We show that the space of ultrafunctions $V_\Lambda(\Omega)$ can be constructed as a quotient of a specific boolean valued model. Furthermore, we discuss the relations between good ultrafilters in this setting, showing that the construction of $V_\Lambda(\Omega)$ obtained using a good ultrafilter ensure that the space $V_\Lambda(\Omega)$ admits limits also for most nets which take values in $V_\Lambda(\Omega) \setminus V(\Omega)$.
Chapter 4 outlines further interactions between model theoretic concepts and set theoretic concepts. We introduce the categorical language of sheaves and compare it with the language of boolean valued models. The results of Chapter 2 show that the interesting boolean valued models satisfy two fundamental properties: the mixing property and the fullness property, with the former implying the latter. We show that viewing boolean valued models as presheaves, the mixing property characterizes the boolean valued models which are sheaves (to our knowledge this result is original, or at least we are not able to trace it in the literature). We reformulate many results presented in previous chapters using the language of sheaves; for example we show that the isomorphisms of boolean valued models $\check{M}^{\mathsf{B}} \cong \mathcal{M}^{\downarrow\mathsf{B}}$, $(2^\omega)^{\mathsf{B}} \cong \mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$ and $(\check{2^\omega})^{\mathsf{B}} \cong \mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$ exhibited in Chapter 2, extend to categorial equivalences for the corresponding sheaves. This allows us to state, for instance, that the elementary extension $(2^\omega)^{\downarrow\mathsf{B}}/_U$ of the Cantor space $2^\omega$ described in Chapter 3, is simply the stalk in $U$ of the presheaf of locally constant functions from the Stone space of a complete boolean algebra $\mathsf{B}$ in $2^\omega$.

# Chapter 1

# Preliminaries

In this chapter we present a brief overview of the main tools we will employ in our work.

## 1.1 From pre-orders to boolean algebras

In this section we introduce basic definitions and facts from the theory of boolean algebras. For the proofs of all the theorems stated, we adress the reader to [19], for example. In particular, for a complete discussion of the subject, we suggest [8].

**Definition 1.1.1.** A *pre-ordered set* (or *pre-order*) is a pair $(P, \leq)$ where $P$ is a set and $\leq$ is a binary relation on $P$ that is reflexive and transitive. The formula $p < q$ means $p \leq q$ and $p \neq q$.
If $\leq$ is also antisymmetric, we call it a *partial order* (or, simply, an *order*).
We will often refer to a pre-order $(P, \leq)$ only with its underlying set $P$.
A *total order* is a partial order $P$ such that for every $p, q \in P$ either $p \leq q$ or $q \leq p$.
Two elements $p, q$ in a pre-order $P$ are *compatible* if there exists $r \in P$ such that $r \leq p$ and $r \leq q$.
Otherwise, we say that $p$ and $q$ are *incompatible*, denoted $p \perp q$.
Let $X \subseteq P$ and let $a \in P$. We say that:

- $a$ is an *upper bound* of $X$ if $x \leq a$ for every $x \in X$;
- $a$ is a *lower bound* of $X$ if $a \leq x$ for every $x \in X$;
- $a$ is the *gratest element* of $X$ ($a = \max X$) if $a$ is an upper bound of $X$ and $a \in X$;
- $a$ is the *least element* of $X$ ($a = \min X$) if $a$ is a lower bound of $X$ and $a \in X$;
- $a$ is the *supremum* of $X$ ($a = \sup X$) if $a$ is the least upper bound of $X$ (i.e. $a = \min \{c : c \text{ is an upper bound of } X\}$);
- $a$ is the *infimum* of $X$ ($a = \inf X$) if $a$ is the greatest lower bound of $X$ (i.e. $a = \max \{c : c \text{ is a lower bound of } X\}$).

If $a, b \in P$, we write $a \wedge b := \inf \{a, b\}$ and $a \vee b := \sup \{a, b\}$, if they exist.
A subset $C$ of a pre-order $P$ is a *chain* of $P$ if $(C, \leq \restriction C)$ is a total order.
A subset $A$ of a pre-order $C$ is an *antichain* if every two distinct elements of $A$ are incompatible.
Let $\lambda$ be a cardinal number. A pre-order $P$ satisfies the $< \lambda$*-chain condition* if every antichain in $P$ has cardinality less then $\lambda$. The $\aleph_1$-chain condition is called the *countable chain condition* (CCC).

Let $P, Q$ be two pre-orders. A map $f : P \to Q$ is a *morphism of pre-orders* if, for every $p, p' \in P$, $p \leq p'$ implies $f(p) \leq f(p')$. A morphism $f : P \to Q$ is an *embedding* if, for every $p, p' \in P$, $p \leq p'$ if and only if $f(p) \leq f(p')$. $f : P \to Q$ is a *dense embedding* if its image is dense in $Q$.

**Definition 1.1.2.** A partial order $P$ is an *upward-filtering order* if every pair in $P$ has an upper bound. *Downward-filtering orders* are defined analogously.
A partial order $P$ is a *lattice* if for every $a, b \in P$ $a \wedge b$ and $a \vee b$ exist.
A lattice $P$ is distributive if, for every $a, b, c \in P$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \textit{and} \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

A lattice $P$ is *complemented* if $P$ admits greatest and least elements (in this case, we will write $0 := \min P$ and $1 := \max P$) and if, for every $a \in P$, there exists $b \in P$ (called a *complement* for $a$) such that $a \wedge b = 0$ and $a \vee b = 1$.
We define a *boolean algebra* to be a complemented distributive lattice.

It is easy to see that each finite subset of a lattice admits supremum and infimum. Notice also that, if $P$ is a complemented distributive lattice, every $a \in P$ admits an unique complement, denoted by $\neg a$. If $\mathsf{B}$ is a boolean algebra and $a, b, c \in \mathsf{B}$, then

$$a \vee b = b \vee a \quad \text{and} \quad a \wedge b = b \wedge a, \tag{1.1}$$
$$a \vee (b \vee c) = (a \vee b) \vee c \quad \text{and} \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c, \tag{1.2}$$
$$(a \vee b) \wedge b = b \quad \text{and} \quad (a \wedge b) \vee b = b, \tag{1.3}$$
$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) \quad \text{and} \quad (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c), \tag{1.4}$$
$$a \vee \neg a = 1 \quad \text{and} \quad a \wedge \neg a = 0. \tag{1.5}$$

Therefore, we can also define a boolean algebra as follows.

**Definition 1.1.3.** A *boolean algebra* is a 6-uple $\mathsf{B} = (B, \wedge, \vee, \neg, 0, 1)$, where $B$ is a set, $\wedge, \vee$ are binary operations on $B$, $\neg$ is an unary operation on $B$ and $0, 1$ are two elements of $B$.
We require also that, for every $a, b, c \in B$, equations (1.1), (1.2), (1.3), (1.4) and (1.5) hold.
A *subalgebra* of $\mathsf{B}$ is a subset of $B$ containing $0, 1$ and closed under the operations.

If $\mathsf{B}$ is a boolean algebra in the sense of Definition 1.1.3, we can define an order letting $a \leq b$ if and only if $a \wedge b = a$. It is possible to check that $(B, \leq)$ is a complemented distributive lattice.
Moreover each powerset $\mathcal{P}(X)$ (endowed with the canonical set-theoretic operations of union, intersection and complement) is a boolean algebra.

**Definition 1.1.4.** Let $\mathsf{B}$ and $\mathsf{C}$ be two boolean algebras. A *morphism of boolean algebras* from $\mathsf{B}$ to $\mathsf{C}$ is a map $f : \mathsf{B} \to \mathsf{C}$ that preserves the operations $\vee, \wedge, \neg$ and such that $f(0_{\mathsf{B}}) = 0_{\mathsf{C}}$, $f(1_{\mathsf{B}}) = 1_{\mathsf{C}}$.
An *isomorphism* is a bijective morphism.

**Definition 1.1.5.** A subset $D \subset \mathsf{B}$ of a boolean algebra $\mathsf{B}$ is a *prefilter* if $b_1 \wedge \cdots \wedge b_n > 0_{\mathsf{B}}$ for every $b_1, \ldots, b_n \in D$.
Moreover, if for every $b_1, \ldots, b_n \in D$ we have that $0 < b_1 \wedge \cdots \wedge b_n \in D$, we say that $D$ satisfies the *finite intersection property*.
A prefilter $F$ that satisfies the finite intersection property and that is *upward closed*, i.e.

$$a \in F \textit{ and } a \leq b \textit{ implies } b \in F$$

is a *filter*.
A filter that is not properly contained in any other filter is an *ultrafilter*.
Equivalently, an ultrafilter is a filter $U$ in which either $b \in U$ or $\neg b \in U$, for any $b \in B$.

$\mathrm{St}(\mathsf{B})$ is the set of all the ultrafilters of the boolean algebra $\mathsf{B}$. If we assume Zorn's lemma, $\mathrm{St}(\mathsf{B})$ is always not empty. We endow it with the following topology: define for every $b \in \mathsf{B}$,

$$N_b := \{U \in \mathrm{St}(\mathsf{B}) : b \in U\}.$$

It is possible to see that the set $\{N_b : b \in \mathsf{B}\}$ is a base for a compact, Hausdorff, zero-dimensional topology on $\mathrm{St}(\mathsf{B})$ (a space is *zero-dimensional* if its topology admits a base of clopen sets) and, in particular, it can be proved that $\{N_b : b \in \mathsf{B}\}$ is exactly the base of clopen sets. In general, we will denote with $\mathrm{CLOP}(X)$ the set of clopen subsets of a topological space $X$. Using this notation $\mathrm{CLOP}(\mathrm{St}(\mathsf{B}))$ is a subalgebra of $\mathcal{P}(\mathrm{St}(\mathsf{B}))$.

**Theorem 1.1.6** (Stone's Representation Theorem)**.** *Let* $\mathsf{B}$ *be a boolean algebra. Then the map*

$$\begin{aligned}\mathsf{B} &\to \mathrm{CLOP}(\mathrm{St}(\mathsf{B}))\\ b &\mapsto N_b\end{aligned}$$

*is an isomorphism of boolean algebras.*
*Dually, let* $X$ *be a compact, zero-dimensional Hausdorff space. For each* $x \in X$ *define*

$$U_x := \{C \in \mathrm{CLOP}(X) : x \in C\}.$$

*Then, for every* $x \in X$*,* $U_x$ *is an ultrafilter in* $\mathrm{CLOP}(X)$ *and the map*

$$\begin{aligned}X &\to \mathrm{St}(\mathrm{CLOP}(X))\\ x &\mapsto U_x\end{aligned}$$

*is an homeomorphism.*

**Definition 1.1.7.** Let $\lambda$ be a cardinal number. A boolean algebra $\mathsf{B}$ is called a $<\lambda$*-complete boolean algebra* if, for every subset $X \subseteq \mathsf{B}$ such that $|X| < \lambda$, $\bigvee X := \sup X$ and $\bigwedge X := \inf X$ exist.
A subalgebra $\mathsf{B}$ of a powerset that is $<\aleph_1$-complete is called $\sigma$*-algebra*.
A boolean algebra $\mathsf{B}$ in which every subset admits supremum and infimum is called *complete*.

**Definition 1.1.8.** Let $X$ be a topological space and $A \subseteq X$. The *regularization* of $A$ is the interior of the closure of $A$ in $X$, i.e.

$$\mathrm{Reg}\,(A) := (\bar{A})^\circ.$$

A subset $A \subseteq X$ is *regular open* if $A = \mathrm{Reg}\,(A)$. The set of regular open subsets of $X$ is denoted by $\mathsf{RO}(X)$.

**Lemma 1.1.9.** *Let* $X$ *be a topological space. For any open subset* $A \subseteq X$ *we have:*

$$\mathit{Reg}\,(A) = \{x \in X : \text{exists an open neighborhood } U \text{ of } x \text{ s. t. } A \cap U \text{ is dense in } U\}.$$

*Proof.* See, for instance, [[19], Lemma 2.3.11]. □

In general, given a topological space $X$, we observe that $\mathrm{CLOP}(X) \subseteq \mathsf{RO}(X)$ and if the equality between the two sets holds we will say that $X$ is *extremally disconnected*.

**Theorem 1.1.10.** *Let $X$ be a topological space. Let $0 := \emptyset$, $1 := X$ and, for $U, V \in \mathsf{RO}(X)$, define*

$$
\begin{gathered}
U \vee V := \textit{Reg}\,(U \cup V)\,, \\
U \wedge V := U \cap V, \\
\neg U := X \setminus \bar{U}.
\end{gathered}
$$

*Moreover, for any family $\{U_i : i \in I\} \subseteq \mathsf{RO}(X)$, define*

$$
\begin{aligned}
\bigvee_{i \in I} U_i &:= \textit{Reg}\left(\bigcup_{i \in I} U_i\right), \\
\bigwedge_{i \in I} U_i &:= \textit{Reg}\left(\bigcap_{i \in I} U_i\right).
\end{aligned}
$$

*It can be proved that, endowed with these operations, $\mathsf{RO}(X)$ is a complete boolean algebra.*

**Corollary 1.1.11.** *A boolean algebra $\mathsf{B}$ is complete if and only if* $\mathrm{CLOP}(\mathrm{St}(\mathsf{B})) = \mathsf{RO}(\mathrm{St}(\mathsf{B}))$.

*Remark* 1.1.12. Let $\{N_a : a \in A\}$ be a family of basic open sets in the Stone space $\mathrm{St}(\mathsf{B})$ of a boolean algebra $\mathsf{B}$ such that $\bigvee A$ exists. Then $\bigcup_{a \in A} N_a$ is a dense open set in $\mathrm{St}(\mathsf{B})$ if and only if $\bigvee A = 1$. However, it is not true that $\bigvee A = 1$ implies $\bigcup_{a \in A} N_a = \mathrm{St}(\mathsf{B})$. If $\mathsf{B}$ is complete, what we can say is

$$
\mathrm{Reg}\left(\bigcup_{a \in A} N_a\right) = N_{\bigvee A}.
$$

**Definition 1.1.13.** Let $\mathsf{B}, \mathsf{C}$ be two complete boolean algebras. A *complete morphism of complete boolean algebras* from $\mathsf{B}$ to $\mathsf{C}$ is a morphism $f : \mathsf{B} \to \mathsf{C}$ such that, for every $X \subseteq \mathsf{B}$,

$$
f\Big[\bigvee X\Big] = \bigvee f[X] \quad \textit{and} \quad f\Big[\bigwedge X\Big] = \bigwedge f[X].
$$

The following definition is crucial in our analysis of good ultrfilters.

**Definition 1.1.14.** Let $\mathsf{B}$ be a boolean algebra.
For all functions $f, g : X \to \mathsf{B}$, we say that $f$ is a *refinement* of $g$ if $f(x) \leq g(x)$ for every $x \in X$. A function $g : X \to \mathsf{B}^+ := \mathsf{B} \setminus \{0\}$ is *disjoint* if $\mathrm{ran}(g)$ is an antichain and $g$ is injective. We say that $f$ *can be disjointed* if it admits a disjoint refinement.
A boolean algebra $\mathsf{B}$ is $< \kappa$*-disjointable* if every function $f : X \to \mathsf{B}^+$ with domain $X$ of size less than $\kappa$ can be disjointed[1].

*Remark* 1.1.15. Assume $\mathsf{B}$ is a $< \kappa$-complete boolean algebra with the property that for each $b \in \mathsf{B}^+$ and $\alpha < \kappa$ there is an antichain $\{c_\xi : \xi < \alpha\}$ with $\bigvee_{\xi < \alpha} c_\xi \leq b$. Then $\mathsf{B}$ is $< \kappa$-disjointable.

[1] In essence every subset of $\mathsf{B}^+$ of size less than $\kappa$ can be refined to an antichain of the same size.

*Proof.* Assume $f : X \to \mathsf{B}^+$ with $X = \{x_\alpha : \alpha < \gamma\}$ of size $\gamma$ and $f(x_\alpha) = c_\alpha$ for all $\alpha < \gamma$. Let $a_\alpha = c_\alpha \setminus \bigvee_{\beta<\alpha} c_\beta$, if the latter is positive. We have obtained an antichain $\{a_\alpha : \alpha < \delta\}$ for some $\delta \leq \gamma$. Now, for every $\alpha < \delta$, let $E_\alpha := \{a_\alpha^\beta : \beta < \gamma\}$ be an antichain of size $\gamma$ below $a_\alpha$. Now we only have to define $b_\alpha := a_\alpha^\alpha$ if $c_\alpha \setminus \bigvee_{\beta<\alpha} c_\beta > 0$. Otherwise, let $\beta(\alpha) < \alpha$ be the least $\beta$ such that $c_\alpha \wedge a_\beta > 0$ and define $b_\alpha := a_{\beta(\alpha)}^\alpha$.
The map $h : x_\alpha \mapsto b_\alpha$ is disjoint and refines $f$. □

We will also need the definition of the following two cardinals associated to a boolean algebra:

**Definition 1.1.16.** Given a boolean algebra $\mathsf{B}$ its density $\mathfrak{d}(\mathsf{B})$ is the smallest size of a dense subset of $(\mathsf{B}^+, \leq)$, while $\mathfrak{a}(\mathsf{B})$ is the supremum of the cardinals $\kappa$ such that $\mathsf{B}^+$ admits a maximal antichain of size $\kappa$.

### 1.1.1 Completion of a pre-order

If $(P, \leq)$ is a pre-order, we can endow it with the order (or downward) topology. To this extent for $X \subseteq P$ define

$$\downarrow X := \{p \in P : \textit{ there exists } x \in \textit{ such that } p \leq x\}.$$

$X \subseteq P$ is a *down-set* if $X = \downarrow X$.
The family DOWN$(P)$ of the down-sets of $P$ is a topology for $P$, the *downward topology*.

**Definition 1.1.17.** An embedding of pre-orders $f : P \to Q$ is *dense* if $\operatorname{ran}(f)$ is dense in $Q$ endowed with the downward topology.

**Definition 1.1.18.** A pre-order $P$ is *separative* if, for every $p, q \in P$, $p \nleq q$ implies that there exists $r \in P$ such that $r \leq p$ and $r \perp q$.

Observe the following:

- If $\mathsf{B}$ is a boolean algebra, then $\mathsf{B}^+ := \mathsf{B} \setminus \{0_\mathsf{B}\}$ with the induced order is a separative ordered set.
- If $P$ is a pre-order, we can always suriect it onto a separative pre-order letting for $p, q \in P$, $p \sim q$ if and only if
$$\forall r \leq p \neg(r \perp q) \wedge \forall r \leq q \neg(r \perp p).$$
$\sim$ is an equivalence relation, and the quotient $P/\sim$ has a well-defined order relation
$$[p]_\sim \leq [q]_\sim \iff p \leq q,$$
that makes $P/\sim$ a separative order: the *separative quotient* of $P$.

**Definition 1.1.19.** The *boolean completion* of a separative order $P$ is a pair $(B, e)$, where $\mathsf{B}$ is a complete boolean algebra and $e : P \to \mathsf{B}^+$ is a dense embedding.
The *boolean completion* of a boolean algebra $\mathsf{B}$ is the boolean completion of the separative order $\mathsf{B}^+$.
The *boolean completion* of a pre-order $P$ is a pair $(B, i)$ for which there exists $e$ such that $(B, e)$ is the boolean completion of the separative quotient $P/\sim$ and $i$ is defined as $i(p) := e([p]_\sim)$.

**Theorem 1.1.20.** *If $P$ is a separative order, then $(\mathsf{RO}(P), e)$ is its boolean completion, where*

$$\begin{aligned} e : P &\to \mathsf{RO}(P). \\ p &\mapsto \downarrow p \end{aligned}$$

**Corollary 1.1.21.** *Every boolean algebra can be densely embedded in a complete boolean algebra: the algebra of regular open sets of its Stone space*[2].

**Proposition 1.1.22.** *If the partial order $P$ satisfies the $< \kappa$-chain condition, then $|RO(P)| \leq |P|^{<\kappa}$.*

*Proof.* Let $e : P \to \mathsf{RO}(P)^+$ be a dense embedding. It suffices to show that if $\mathcal{A}$ is the set of the antichains of $P$, the map $a : \mathcal{A} \to \mathrm{RO}(P) \setminus \{0\}$ such that $a(A) := \bigvee e[A]$ is surjective. Let $b \in \mathrm{RO}(P) \setminus \{0\}$ and consider

$$D := e[P] \cap \{a \in \mathrm{RO}(P) \setminus \{0\} : a \leq b\}.$$

It can be seen that $D$ is dense below $b$. Let now $W$ be a maximal antichain in $D$ (its existence is granted by Zorn's Lemma); find $A \subseteq P$ maximal antichain such that $e[A] = W$; then $a(A) = b$. $\square$

## 1.2 Basic notions of model theory

We present some fundamental notions of model theory. A reference for this section is [6].

**Definition 1.2.1.** A *signature* or *similarity type* is a 4-uple $\tau = \langle I, J, K, \mathrm{ar}\rangle$, with $I, J, K$ pairwise disjoint sets and $\mathrm{ar} : I \cup J \to \omega \setminus \{0\}$.
A *first order language* $L$ is a pair $\langle S, \mathrm{ar}\rangle$ with the following properties: there exist sets $I, J, K$ such that $S = \mathrm{Rel}_L \cup \mathrm{Fun}_L \cup \mathrm{Const}_L \cup \{\neg, \vee, \exists, =\} \cup \mathrm{Vbl}$, where $\mathrm{Vbl} = \{x_n : n \in \omega\}$ (the set of *variables*), $\mathrm{Rel}_L = \{R_i : i \in I\}$ (the set of *relational symbols*), $\mathrm{Fun}_L = \{f_j : j \in J\}$ (the set of *functional symbols*), and $\mathrm{Const}_L = \{c_k : k \in K\}$ (the set of *constant symbols*). Moreover, $\mathrm{ar} : \mathrm{Rel}_L \cup \mathrm{Fun}_L \to \omega \setminus 0$ is a function (called *arity*).
A language where $J = \emptyset$ is called *relational*, a language where $I = \emptyset$ is called *functional*.
Since signatures and languages are in a canonical bijection, we will refer to a signature or to a language indifferently.

**Definition 1.2.2.** Let $L$ be a signature. An *$L$-structure* $\mathcal{M}$ consists of

1. a non-empty set $M$, called the *domain* of $\mathcal{M}$;
2. for each $n$-ary relational symbol $R \in L$, its *interpretation* $R^{\mathcal{M}} \subseteq M^n$;
3. for each $n$-ary functional symbol $f \in L$, its *interpretation* $f^{\mathcal{M}} : M^n \to M$;
4. for each constant symbol $c$, an element $c^{\mathcal{M}} \in M$.

We will interchangeably use $\mathcal{M}$ or $M$ to indicate an $L$-structure or its domain. Furthermore, the supscript $\mathcal{M}$ will be always omitted, whenever the structure is clear from the context.
If $\mathcal{M}$ is an $L$-structure and $A \subseteq M$, the *extension* of $L$ with $A$ is the segnature $L(A) := L \cup \{c_a : a \in A\}$, where $c_a$ is a constant symbol such that $c_a^{\mathcal{M}} = a$.

[2]The embedding is dense since any regular open set of $\mathrm{St}(\mathsf{B})$ contains a clopen set.

**Definition 1.2.3.** Let $\langle \mathcal{M}_i : i \in I \rangle$ be a family of $L$-structures. Its *direct product* is the $L$-structure $\mathcal{M} := \prod_{i \in I} \mathcal{M}_i$ such that:

1. the domain $M$ of $\mathcal{M}$ is the set-theoretic product of the domains;
2. if $R \in L$ is an $n$-ary relational symbol and $g_1, \dots, g_n \in M$, $\langle g_1, \dots, g_n \rangle \in R^{\mathcal{M}}$ if and only if, for every $i \in I$,
$$\langle g_1(i), \dots, g_n(i) \rangle \in R^{\mathcal{M}_i};$$
3. if $f \in L$ is an $n$-ary functional symbol and $g_1, \dots, g_n \in M$,
$$f^{\mathcal{M}}(g_1, \dots, g_n) = \langle f^{\mathcal{M}_i}(g_1(i), \dots, g_n(i)) : i \in I \rangle;$$
4. if $c$ is a constant symbol, $c^{\mathcal{M}} = \langle c_i^{\mathcal{M}_i} : i \in I \rangle$.

Let $F$ be a filter on $I$. For $f, g \in M$, we say that $f \sim_F g$ if
$$\{ i \in I : g(i) = f(i) \} \in F.$$

It is easy to see that, since $F$ is a filter, $\sim_F$ is an equivalence relation and we denote with $M/_F$ the quotient space with respect to this relation. The *reduced product modulo* $F$ of $\langle \mathcal{M}_i : i \in I \rangle$ is the $L$-structure $\prod_F \mathcal{M}_i$ that satisfies the following requirements:

1. its domain is $M_F$;
2. if $R \in L$ is an $n$-ary relational symbol and $[g_1], \dots, [g_n] \in M_F$, then
$$([g_1], \dots, [g_n]) \in R^{\prod_F \mathcal{M}_i} \quad \textit{if and only if} \quad \left\{ i \in I : (g_1(i), \dots, g_n(i)) \in R^{\mathcal{M}_i} \right\} \in F;$$
3. if $f \in L$ is an $n$-ary functional symbol and $[g_1], \dots, [g_n] \in M_F$,
$$f^{\prod_F \mathcal{M}_i}([g_1], \dots, [g_n]) := [\langle f^{\mathcal{M}_i}(g_1(i), \dots, g_n(i)) : i \in I \rangle];$$
4. if $c$ is a constant symbol, $c^{\prod_F \mathcal{M}_i} := [\langle c^{M_i} : i \in I \rangle]$.

If $M_i = N$ for every $i \in I$, $\mathcal{N}^I/_F := \prod_F \mathcal{N}$ is called the *reduced power of* $\mathcal{N}$ *modulo* $F$. If $U$ is an ultrafilter, we say that $\prod_U \mathcal{M}_i$ is an *ultraproduct* and $\mathcal{N}^I/_G$ is an *ultrapower*.

**Definition 1.2.4.** Let $\mathcal{M}$ and $\mathcal{N}$ be $L$-structures. A *homomorphism* from $\mathcal{M}$ to $\mathcal{N}$ is a map $h : M \to N$ such that:

- for every $n$-ary relational symbol $R \in L$ and $\langle a_1, \dots, a_n \rangle \in M^n$, if $\langle a_1, \dots, a_n \rangle \in R^{\mathcal{M}}$, then $\langle h(a_1), \dots, h(a_n) \rangle \in R^{\mathcal{N}}$;
- for every $n$-ary functional symbol $f \in L$ and $\langle a_1, \dots, a_n \rangle \in M^n$, $h(f^{\mathcal{M}}(a_1, \dots, a_n)) = f^{\mathcal{N}}(h(a_1), \dots, h(a_n))$;
- for every constant symbol $c \in L$, $h(c^{\mathcal{M}}) = c^N$.

An *embedding* is a homomorphism $h$ such that, for every relational symbol $R \in L$ and $\langle a_1, \dots, a_n\rangle \in M^n$, $\langle a_1, \dots, a_n\rangle \in R^{\mathcal{M}}$ if and only if $\langle h(a_1), \dots, h(a_n))\rangle \in R^{\mathcal{N}}$.
A surjective embedding $h : \mathcal{M} \to \mathcal{N}$ is called *isomorphism* and $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic*, we will use the notation $\mathcal{M} \cong \mathcal{N}$.
If $\mathcal{M}$ and $\mathcal{N}$ are $L$-structures, $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ if $M \subseteq N$ and the inclusion $M \to N$ is an embedding.

Let us now fix in advance a language $L$.

**Definition 1.2.5.** The *terms* of $L$ are defined as follows:

- every variable is a term;
- every constant symbol is a term;
- if $t_1, \dots, t_n$ are terms and $f \in L$ is an $n$-ary functional symbol, then $f(t_1, \dots, t_n)$ is a term.

**Definition 1.2.6.** The *atomic formulae* of $L$ are the following:

- if $t_1$ and $t_2$ are terms of $L$, then $t_1 = t_2$ is an atomic formula;
- if $t_1, \dots, t_n$ are terms of $L$ and $R \in L$ is an $n$-ary relational symbol, then $R(t_1, \dots, t_n)$ is an atomic formula.

The *formulae* of $L$ (or *$L$-formulae*) are the following:

- every atomic formula is a formula;
- if $\varphi$ is a formula, then $\neg\varphi$ is a formula;
- if $\varphi$ and $\psi$ are formulae, then $\varphi \vee \psi$ is a formula;
- if $\varphi$ is a formula, then $\exists x\varphi$ is a formula.

We will write $\varphi \wedge \psi$ for $\neg(\neg\varphi \vee \neg\psi)$, $\forall x\varphi$ for $\neg\exists x\neg\varphi$ and $\varphi \to \psi$ for $\neg\varphi \vee \psi$. We will often write $\varphi \in L$ to say that $\varphi$ is an $L$-formula.

**Definition 1.2.7.** Given an $L$-formula $\varphi$, we define the set $\mathrm{FV}(\varphi)$ of *free variables* of $\varphi$ as follows:

- if $\varphi$ is atomic, $\mathrm{FV}(\varphi)$ is the set of variables which appear in $\varphi$;
- $\mathrm{FV}(\neg\varphi) = \mathrm{FV}(\varphi)$;
- $\mathrm{FV}(\varphi \vee \psi) = \mathrm{FV}(\varphi) \cup \mathrm{FV}(\psi)$;
- $\mathrm{FV}(\exists x\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$.

A formula is an *$L$-sentence* if $\mathrm{FV}(\varphi) = \emptyset$, and a *theory* in $L$ (or an *$L$-theory*) is a set of $L$-sentences.

**Definition 1.2.8.** Given a language $L$ and an $L$-structure $\mathcal{M}$, an *assignment* is a map $\nu : \mathrm{Var} \to M$. If $\nu$ is an assignment and $t$ is a term of $L$, we define $t^{\mathcal{M}}[\nu]$ in the following way:

- if $x$ is a variable, then $x^{\mathcal{M}}[\nu] := \nu(x)$;
- if $c \in L$ is a constant symbol, $c^{\mathcal{M}}[\nu] := c^{\mathcal{M}}$;

- if $t_1, \dots, t_n$ are terms of $L$ and $f \in L$ is an $n$-ary functional symbol, then $f(t_1, \dots, t_n)^{\mathcal{M}}[\nu] := f^{\mathcal{M}}(t_1^{\mathcal{M}}[\nu], \dots, t_n^{\mathcal{M}}[\nu])$.

Moreover, if $\nu$ is an assignment, $x$ is a variable and $a \in M$, we define a new assignment $\nu_{a/x}$ setting $\nu_{a/x}(x) := a$ and, for every $y \in \mathrm{Var} \setminus \{x\}$, we define $\nu_{a/x}(y) := \nu(y)$.

**Definition 1.2.9.** Let $\mathcal{M}$ be an $L$-structure and let $\nu$ be an assignment. The relation of *satisfaction* $\mathcal{M} \vDash \varphi[\nu]$ is defined by recursion in the following way:

- $\mathcal{M} \vDash (t_1 = t_2)[\nu]$ for $t_1, t_2$ terms if and only if $t_1^{\mathcal{M}}[\nu] = t_2^{\mathcal{M}}[\nu]$;
- $\mathcal{M} \vDash (t_1, \dots, t_n)[\nu]$ for $t_1, \dots, t_n$ terms if and only if $\langle t_1^{\mathcal{M}}[\nu], \dots, t_n^{\mathcal{M}}[\nu] \rangle \in R^{\mathcal{M}}$;
- $\mathcal{M} \vDash \neg\varphi[\nu]$ if and only if it is not the case that $\mathcal{M} \vDash \varphi[\nu]$;
- $\mathcal{M} \vDash (\varphi \vee \psi)[\nu]$ if and only if either $\mathcal{M} \vDash \varphi[\nu]$ or $\mathcal{M} \vDash \psi[\nu]$;
- $\mathcal{M} \vDash (\exists x \varphi)[\nu]$ if and only if there exists $a \in M$ such that $\mathcal{M} \vDash \varphi[\nu_{a/x}]$.

Given a formula $\varphi(x_1, \dots, x_n)$, if $a_1 = \nu(x_1), \dots, a_n = \nu(x_n)$, then the fact that $\mathcal{M} \vDash \varphi[\nu]$ holds depends only on $a_1, \dots, a_n$. In this case we write $\mathcal{M} \vDash \varphi(a_1, \dots, a_n)$ to say $\mathcal{M} \vDash \varphi[\nu]$ for some (any) assignment $\nu$ such that $x_i \mapsto a_i$ for every $i = 1, \dots, n$.
If $\varphi$ is a sentence, since every assignment is irrelevant, we will write $\mathcal{M} \vDash \varphi$.

**Definition 1.2.10.** Let $T$ be an $L$-theory. A *model* of $T$ is an $L$-structure $\mathcal{M}$ such that $\mathcal{M} \vDash \varphi$ for every $\varphi \in T$.

If $\mathcal{M}$ is an $L$-structure, the set $T$ of the $L$-sentences that are true in $\mathcal{M}$ is a theory, and $\mathcal{M}$ is a model for $T$. For this reason, we will use the notions of structure and model interchangeably.

**Theorem 1.2.11** (Compactness)**.** *Let $L$ be a language and let $T$ be an $L$-theory. Then $T$ has a model if and only if every finite subset of $T$ has a model.*

This theorem is a corollary of the following.

**Theorem 1.2.12** (Łoś Theorem)**.** *Let $\langle \mathcal{M}_i : i \in I \rangle$ be a family of $L$-structures and let $U$ be an ultrafilter on $I$. Assume that, for every $i$, the domain of $\mathcal{M}_i$ is well-ordered. Then, for every $L$-formula $\varphi(x_1, \dots, x_n)$ and $g_1, \dots, g_n$ choice functions on $\langle \mathcal{M}_i : i \in I \rangle$,*

$$\prod_U \mathcal{M}_i \vDash \varphi([g_1], \dots, [g_n]) \quad \textit{if and only if} \quad \{i \in I : \mathcal{M}_i \vDash \varphi(g_1(i), \dots, g_n(i))\} \in U.$$

**Definition 1.2.13.** Let $h : \mathcal{M} \to \mathcal{N}$ be an embedding of $L$-structures. We say that $h$ is *elementary* if, for every $L$-formula $\varphi$ and for every $a_1, \dots, a_n \in M$,

$$\mathcal{M} \vDash \varphi(a_1, \dots, a_n) \quad \textit{if and only if} \quad \mathcal{N} \vDash \varphi(h(a_1), \dots, h(a_n)).$$

A structure $\mathcal{M}$ is an *elementary substructure* of a structure $\mathcal{N}$ if $\mathcal{M}$ is a substructure of $\mathcal{N}$ and if the inclusion $M \to N$ is an elementary embedding.

**Lemma 1.2.14** (Tarski-Vaught test)**.** *For every $L$-structure $N$ and for every subset $A \subset N$ are equivalent:*

1. *$A$ is the domain of an elementary substructure of $N$;*

2. *for every single free variable formula $\varphi(x)$ in the language $L(A)$, if $N \models \exists x\varphi(x)$, then $N \models \varphi(b)$ for some $b \in A$.*

**Definition 1.2.15.** Let $L$ be a language. A *type* is a set of $L$-formulas. We will write $p(x)$ to denote a type, where $x$ is a the tuple of all the variables occurring in $p$. If $x$ is a finite tuple, say $\langle x_1, \dots, x_n \rangle$, we will write $p(x_1, \dots, x_n)$ and we will say that $p$ is a $n$*-type*. Clearly, a 0-type is simply a theory.
Let $\mathcal{M}$ be an $L$-structure. A tuple $\langle a_1, \dots, a_n \rangle \in M^n$ *realizes* $p(x_1, \dots, x_n)$ if $\mathcal{M} \models \varphi(a_1, \dots, a_n)$ for every $\varphi(x_1, \dots, x_n) \in p(x_1, \dots, x_n)$.
If an $n$-type is not realized by any $n$-uple of elements of $M$, we say that $\mathcal{M}$ *omits* $p$.

**Definition 1.2.16.** Let $T$ be an $L$-theory. An $n$*-type of* $T$ is any type of $L$ realized in some model of $T$.
An $n$-type $p(x_1, \dots, x_n)$ is complete if, for every $L$-formula $\varphi(x_1, \dots, x_n)$, either $\varphi \in p$ or $\neg\varphi \in p$.
If $T$ is a theory of a given $L$-structure $\mathcal{M}$, a *type over* $B \subseteq M$ is a type of the theory of $\mathcal{M}$ in the language $L(B)$.

**Proposition 1.2.17.** *Let $\mathcal{M}$ be an $L$-structure, $B \subseteq M$ and let $p(x_1, \dots, x_n)$ be an $n$-type over $B$ in $L$. Then, every finite subset $q \subseteq p$ is realized in $\mathcal{M}$.*

**Definition 1.2.18.** Let $\mathcal{M}$ be an $L$-structure and $\lambda$ a cardinal number. We say that $\mathcal{M}$ is $\lambda$*-saturated* if, for every $B \subseteq M$ of cardinality less then $\lambda$, all complete 1-types over $B$ are realized in $\mathcal{M}$. In particular, $\mathcal{M}$ is *saturated* if it is $|M|$-saturated.

## 1.3 Remarkable topological facts

In this section we recall some topological definitions and facts that we will use later.

**Definition 1.3.1.** Let $X = (X, \tau)$ be a topological space. The *family of Borel sets* of $X$ is the $\sigma$-algebra generated by $\tau$.
The Borel hierarchy of $X$ is the family of sets $\Sigma^0_\alpha(X)$, $\Pi^0_\alpha(X)$ for $\alpha$ countable ordinal, defined inductively as follows:

- $\Sigma^0_1(X) := \tau = \{A \subseteq X : A \textit{ is open}\}$;
- $\Pi^0_1(X) := \{C \subseteq X : C \textit{ is closed}\}$;
- $\Sigma^0_\alpha(X) := \left\{\bigcup_{n\in\omega} A_n : A_n \in \bigcup_{\beta<\alpha} \Pi^0_\beta(X)\right\}$;
- $\Pi^0_\alpha(X) := \left\{\bigcap_{n\in\omega} A_n : A_n \in \bigcup_{\beta<\alpha} \Sigma^0_\beta(X)\right\} = \left\{X \setminus A : A \in \Sigma^0_\alpha(X)\right\}$.

**Definition 1.3.2.** Let $X$ be a topological space.
A subset $A \subseteq X$ is *nowhere dense* if its closure has empty interior.
A subset $A \subseteq X$ is *meager* if it is contained in a countable union of closed nowhere dense sets.
A subset $A \subseteq X$ has the *Baire property* if there exists an (unique) regular open set $U$ such that the simmetric difference $A \triangle U$ is meager.

**Proposition 1.3.3.** *Every Borel set has the Baire property.*

*Proof.* See, for example, [9, Lemma 11.15]. □

**Definition 1.3.4.** Let $X$ be a topological space. A *net* (or *generalized sequence*) in $X$ is a pair $(\Lambda, i)$, where $\Lambda$ is an upward-filtering order and $i$ is a map $\Lambda \to X$.
A *subnet* of a net $(\Lambda, i)$ in $X$ is a net $(M, j)$ in $X$ together with a map $h : M \to \Lambda$ such that $j = i \circ h$, and such that, for each $\lambda \in \Lambda$, there exists $\mu(\lambda) \in M$ such that $h(\mu) \geq \lambda$ for every $\mu \geq \mu(\lambda)$.

The standard notation for a net will be $(x_\lambda)_{\lambda \in \Lambda}$, where $x_\lambda = i(\lambda)$.
A net is a generalization of a sequence, being a sequence in $X$ simply a net $(\mathbb{N}, i)$. Sequences suffice to handle all convergence problems in spaces that satisfy the first axiom of countability. Certain spaces require the more general notion of net.

**Definition 1.3.5.** A net $(x_\lambda)_{\lambda \in \Lambda}$ in $X$ is *eventually* in a subset $Y \subseteq X$ if there exists $\lambda_0$ such that $x_\lambda \in Y$ for every $\lambda \geq \lambda_0$.
A net $(x_\lambda)_{\lambda \in \Lambda}$ in $X$ is *frequently* in a subset $Y \subseteq X$ if, for each $\lambda \in \Lambda$, there exists $\mu \geq \lambda$ such that $x_\mu \in Y$.
A net $(x_\lambda)_{\lambda \in \Lambda}$ in $X$ *converges* to $x \in X$ if it is eventually in every open neighborhood of $x$.
A point $x \in X$ is an *accumulation point* for a net $(x_\lambda)_{\lambda \in \Lambda}$ in $X$ if the net is frequently in every open neighborhood if $x$ or, equivalently, if there exists a subnet of $(x_\lambda)_{\lambda \in \Lambda}$ converging to $x$.

**Definition 1.3.6.** A net $(x_\lambda)_{\lambda \in \Lambda}$ in $X$ is *universal* if, for every $Y \subseteq X$, the net is either eventually in $Y$ or eventually in $X \setminus Y$.

For the proofs of the following results, we refer to [17, Chapter 1].

**Proposition 1.3.7.** *A point $x$ in a topological space $X$ belongs to the closure of a set $Y$ if and only if there is a net in $Y$ converging to $x$.*

**Proposition 1.3.8.** *Let $f : X \to Y$ be function between topological spaces. Then $f$ is continuous at $x$ if and only if, for each net $(x_\lambda)_{\lambda \in \Lambda}$ converging to $x$, the net $(f(x_\lambda))_{\lambda \in \Lambda}$ converges to $f(x)$.*

**Proposition 1.3.9.** *Every net in a topological space $X$ has a universal subnet.*

**Proposition 1.3.10.** *A topological space $X$ is Hausdorff if and only if each net converges to at most one point.*

**Proposition 1.3.11.** *A topological space $X$ is compact if and only if every universal net in $X$ is convergent.*

**Lemma 1.3.12.** *Let $X$ be a compact extremally disconnected Hausdorff topological space and let $Y$ be a compact Hausdorff space. Suppose moreover $f : W \to Y$ is a continuous function defined on a dense open subset $W \subseteq X$. Then $f$ can be extended to an unique continuous function $\hat{f} : X \to Y$.*

*Proof.* $\mathsf{B} := \mathsf{RO}(X) = \mathrm{CLOP}(X)$ is a complete boolean algebra and $X$ is homeomorphic to $\mathrm{St}(\mathsf{B})$, since $X$ is compact and extremally disconnected; hence we identify $X$ with $\mathrm{St}(\mathsf{B})$. Therefore we can assume that $W = \bigcup_{a \in A} N_a$, where $N_a \in \mathsf{RO}(\mathrm{St}(\mathsf{B}))$ and $\bigvee_{\mathsf{B}} A = 1$. Now, let

$x \in \mathrm{St}(\mathsf{B}) \setminus \bigcup_{a \in A} N_a$. Since $W$ is dense, and since every basic open neighborhood $N_a$ of $x$ is regular, $N_a \cap W$ is non-meager by Lemma 1.1.9. Define a net $(x_b)_{b \in \mathsf{B}}$ on $X$ such that

$$x_b \in N_b \cap W \text{ for every } b \in x.$$

By Proposition 1.3.9, it has an universal subnet $(z_\lambda)_{\lambda \in \Lambda}$, which has to converge to $x$. We can now consider the net $(f(z_\lambda))_{\lambda \in \Lambda}$, which is again universal on $Y$. Then, since $Y$ is compact, by Proposition 1.3.11, $(f(z_\lambda))_{\lambda \in \Lambda}$ has a limit point $y \in Y$, which is unique by Proposition 1.3.10. We can then define $\hat{f}(x) := y$.
This extension $\hat{f}$ is continuous: let $(x_\eta)_{\eta \in \Gamma}$ be any other net in $X$ converging to $x$. By considering a subnet, we can assume $(x_\eta)_{\eta \in \Gamma}$ to be universal. Then, for every $N_b$ basic open neighborhood of $x$ $(x_\eta)_{\eta \in \Gamma}$ has to be eventually in $N_b$. Hence, since $X$ is Hausdorff,

$$\{N_b \cap W : (x_\eta)_{\eta \in \Gamma} \textit{ is eventually in } N_b\} = \{N_c \cap W : (z_\lambda)_{\lambda \in \Lambda} \textit{ is eventually in } N_c\}.$$

The same holds for $(f(x_\eta))_{\eta \in \Gamma}$ and $(f(z_\lambda))_{\lambda \in \Lambda}$ in $Y$, since they are both universal nets on $Y$, i.e.:

$$\{A \subseteq Y : (f(x_\eta))_{\eta \in \Gamma} \textit{ is eventually in } A\} = \{A \subseteq Y : (f(z_\lambda))_{\lambda \in \Lambda} \textit{ is eventually in } A\}. \quad (1.6)$$

Moreover, by the same arguments used before, $(f(x_\eta))_{\eta \in \Gamma}$ has a unique limit point $y'$ in $Y$. Then $y = y'$: otherwise, being $Y$ Hausdorff, there would be disjoint basic open sets $A, B$ of $Y$ such that $y' \in A$ and $y \in B$, contradicting 1.6.
This extension is unique: two continuous functions defined on a Hausdorff space which agree on a dense subset of their common domain have to be the same map. □

# Chapter 2

# Boolean valued models

This chapter gives an introduction to the theory of boolean valued models.
In the first section we introduce the basic facts and definitions. A boolean valued model generalizes the notion of first order structure by giving to each sentence a value of truth represented by an element in a boolean algebra. References for what is presented can be [9] or [19].
The second section introduces a construction due to Mansfield, namely the boolean power of a first order structure: given a first order structure $\mathcal{M}$ and a complete boolean algebra $\mathsf{B}$ it is possible to define the boolean ultrapower $\mathcal{M}^{\downarrow\mathsf{B}}$. The key property of this object is the fact that, for every ultrafiter $U$ of $\mathsf{B}$, the quotient $\mathcal{M}^{\downarrow\mathsf{B}}/_U$ is an elementary extension of $\mathcal{M}$.
The third section clarifies the connection between the boolean power of a first order structure $\mathcal{M}$ for the language $\{=\}$ and the family of $\mathsf{B}$-names for elements of $\mathcal{M}$. The latter is a well-known object for those familiar with the theory of forcing and we analize the class $\check{M}^{\mathsf{B}}$ given by $\mathsf{B}$-names for elements of $\mathcal{M}$.
The last part of the Chapter explores in detail the relations between the boolean valued models $M^{\mathsf{B}}$ and $\check{M}^{\mathsf{B}}$, we focus on the specific case in which $\mathcal{M}$ is a first order structure with domain $2^\omega$. For this specific case we introduce a third $\mathsf{B}$-model: the space of continuous functions $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$. We define isomorphisms between $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$ and the family of $\mathsf{B}$-names for elements of $2^\omega$ existing in the boolean valued model for set theory $V^{\mathsf{B}}$; we finally discuss which subset of $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$ corresponds to $(\check{2^\omega})^{\mathsf{B}}$ (i.e. the family of $\mathsf{B}$-names in $V^{\mathsf{B}}$ for ground model elements of $2^\omega$).

## 2.1 Basics on boolean valued models

In this section we define boolean valued models for a generic relational language $L$. This is a standard approach to forcing. Reference text for what we present can be [9] or [19].
The definition of a boolean valued model can be given for a generic first order language, however in what follows we do not consider languages with function symbols in order to avoid some technical difficulties.

**Definition 2.1.1.** Let $L = \{R_i : i \in I\} \cup \{c_j . j \in J\}$ be a relational language, and $\mathsf{B}$ a boolean algebra. A $\mathsf{B}$-valued model for $L$ is a tuple $\mathcal{M} = \{M\} \cup \{=_{\mathsf{B}}^{\mathcal{M}}\} \cup \{R_{i\mathsf{B}}^{\mathcal{M}} : i \in I\} \cup \left\{c_j^{\mathcal{M}} : j \in J\right\}$, where:

1. $M$ is a non-empty set;

2. $=^{\mathcal{M}}_{\mathsf{B}}$ is the boolean value of the equality symbol, i.e. a function

$$=^{\mathcal{M}}_{\mathsf{B}}\colon M^2 \to \mathsf{B};$$
$$\langle x,y\rangle \mapsto [\![x = y]\!]^{\mathcal{M}}_{\mathsf{B}}$$

3. $R^{\mathcal{M}}_{i\mathsf{B}}$ is the interpretation of the relational symbol $R_i$. If $R_i$ has arity $n$,

$$R^{\mathcal{M}}_{i\mathsf{B}} : M^n \to \mathsf{B};$$
$$\langle x_1, \ldots ,x_n\rangle \mapsto [\![R_i(x_1, \ldots , x_)]\!]^{\mathcal{M}}_{\mathsf{B}}$$

4. $c^{\mathcal{M}}_j \in M$ is the interpretation of the constant symbol $c_j$.

We require that the following conditions hold:

- for all $x, y, z \in M$,

$$[\![x = x]\!]^{\mathcal{M}}_{\mathsf{B}} = 1_{\mathsf{B}}, \tag{2.1}$$

$$[\![x = y]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![y = x]\!]^{\mathcal{M}}_{\mathsf{B}}, \tag{2.2}$$

$$[\![x = y]\!]^{\mathcal{M}}_{\mathsf{B}} \wedge [\![y = z]\!]^{\mathcal{M}}_{\mathsf{B}} \leq [\![x = z]\!]^{\mathcal{M}}_{\mathsf{B}}; \tag{2.3}$$

- if $R \in L$ is a $n$-ary relational symbol, for every $\langle x_1, \ldots , x_n\rangle, \langle y_1, \ldots , y_n\rangle \in M^n$,

$$\left(\bigwedge_{i=1}^{n} [\![x_i = y_i]\!]^{\mathcal{M}}_{\mathsf{B}}\right) \wedge [\![R(x_1, \ldots , x_n)]\!]^{\mathcal{M}}_{\mathsf{B}} \leq [\![R(y_1, \ldots , y_n)]\!]^{\mathcal{M}}_{\mathsf{B}}. \tag{2.4}$$

From here on, if no confusion can arise, we avoid to put the superscript $\mathcal{M}$ and the subscript $\mathsf{B}$. Moreover, we will write $\mathcal{M}$ or $M$ equivalently to indicate a boolean valued model or its underlying set.

Let us now assume the boolean algebra $\mathsf{B}$ to be complete.

**Definition 2.1.2.** In this setting, we evaluate the formulae of $L(M) := L \cup \{c_a : a \in M\}$ without free variables in the following way:

- $[\![R(c_{x_1}, \ldots , c_{x_n})]\!] := [\![R(x_1, \ldots , x_n)]\!]$;
- $[\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \wedge [\![\psi]\!]$;
- $[\![\neg\varphi]\!] := \neg [\![\varphi]\!]$;
- $[\![\varphi \to \psi]\!] := \neg [\![\varphi]\!] \vee [\![\psi]\!]$;
- $[\![\exists x\varphi(x, c_{a_1}, \ldots , c_{a_n})]\!] := \bigvee_{b\in M} [\![\varphi(c_b, c_{a_1}, \ldots , c_{a_n})]\!]$;
- $[\![\forall x\varphi(x, c_{a_1}, \ldots , c_{a_n})]\!] := \bigwedge_{b\in M} [\![\varphi(c_b, c_{a_1}, \ldots , c_{a_n})]\!]$.

Observe that, if $\mathsf{B} = \{0, 1\}$, a $\mathsf{B}$-model is simply a Tarski structure for the language $L$, and the semantic we have just defined is the Tarski semantic.

**Definition 2.1.3.** A statement $\varphi$ in the language $L$ is *valid* in a $\mathsf{B}$-valued model $\mathcal{M}$ for $L$ if $[\![\varphi]\!]^{\mathcal{M}}_{\mathsf{B}} = 1_{\mathsf{B}}$. A theory $T$ is valid in $\mathcal{M}$ if every axiom of $T$ is valid in $\mathcal{M}$.

It can be proved (see the proof of [[19], Theorem 4.1.5]) that, if $\varphi(x_1, \dots, x_n)$ is a formula with free variables $x_1, \dots, x_n$ and $a_1, \dots, a_n, \mathsf{B}_1, \dots, b_n \in M$, then

$$[\![a_1 = \mathsf{B}_1]\!] \wedge \dots \wedge [\![a_n = b_n]\!] \wedge [\![\varphi(a_1, \dots, a_n)]\!] \leq [\![\varphi(b_1, \dots, b_n)]\!] . \tag{2.5}$$

From here on, we will consider this fact as granted.

**Definition 2.1.4.** Let $\mathsf{B}$ be a complete boolean algebra and let $L = \{R_i : i \in I\}$ be a first order relational language. Let $\mathcal{M} = \left\{M, R_i^{\mathcal{M}} : i \in I\right\}$ be a $\mathsf{B}$-valued model. Let $F$ be a filter in $\mathsf{B}$. We define the *quotient* $\mathcal{M}/_F = \{M/_F, R_i/_F\}$ of $\mathcal{M}$ by $F$ as follows:

- $M/_F := \{[f]_F : f \in M\}$, where $[f]_F := \{g \in M : [\![f = g]\!] \in F\}$;
- $[\![R_i([f_1]_F, \dots, [f_n]_F)]\!]^{\mathcal{M}/_F} := \left[[\![R_i(f_1, \dots, f_n)]\!]^{\mathcal{M}}\right]_F \in \mathsf{B}/_F$ for every $i \in I$.

It is possible to see that $\mathcal{M}/_F$ is a $\mathsf{B}/_F$-valued model. In particular, if $U$ is a ultrafilter, $\mathcal{M}/_U$ is a 2-valued model, i.e. a classical Tarski structure.
Let us stress the fact that the notions of reduced product and reduced power of first order structures are examples of quotient of boolean valued models. Indeed, let $\langle \mathcal{M}_i; i \in I\rangle$ be a family of $L$-structures for a relational language $L$, then the set-theoretic product $\prod_{i \in I} M_i$ of the domains has a natural structure of $\mathcal{P}(I)$-valued model for $L$: if $R \in L$ is an $n$-ary relational symbol and $g_1, \dots, g_n \in \prod_{i \in I} M_i$,

$$R^{\prod_{i \in I} M_i}(g_1, \dots, g_n) := \left\{i \in I : \langle g_1(i), \dots, g_n(i)\rangle \in R^{\mathcal{M}_i}\right\}$$

and, if $c \in L$ is a constant symbol, $c^{\prod_{i \in I} M_i} := \langle c^{\mathcal{M}_i} : i \in I\rangle$.
Finally, it is possible to check that

$$\prod_F \mathcal{M}_i = \Big(\prod_{i \in I} M_i\Big)/_F$$

for every filter $F$ on $I$.
In particular, every ultraproduct of first order structures is a quotient of a boolean valued model.

**Definition 2.1.5.** Let $\kappa$ be a cardinal, $L$ be a first order language, $\mathsf{B}$ a $\kappa$-complete boolean algebra, $\mathcal{M}$ a $\mathsf{B}$-valued model for $L$.

- $\mathcal{M}$ satisfies the *$\kappa$-mixing property* if for every antichain $A \subset \mathsf{B}$ of size at most $\kappa$, and for every subset $\{\tau_a : a \in A\} \subseteq M$, there exists $\tau \in M$ such that $a \leq [\![\tau = \tau_a]\!]$ for every $a \in A$.
- $\mathcal{M}$ satisfies the *$< \kappa$-mixing property* if it satisfies the $\lambda$-mixing property for all cardinals $\lambda < \kappa$.
- $\mathcal{M}$ satisfies the *mixing property* if it satisfies the $|\mathsf{B}|$-mixing property.

**Definition 2.1.6.** Let $L$ be a language and $\mathsf{B}$ a complete boolean algebra. A $\mathsf{B}$-valued model $\mathcal{M}$ for $L$ is *full* if, for every $L$-formula $\phi(x, y_1, \dots, y_n)$ and for every $(\sigma_1, \dots, \sigma_n) \in M^n$ there exists $\tau \in M$ such that

$$[\![\exists x \phi(x, \sigma_1, \dots, \sigma_n)]\!] = [\![\phi(\tau, \sigma_1, \dots, \sigma_n)]\!].$$

**Proposition 2.1.7.** *Let $\mathcal{M}$ be a* $\mathsf{B}$*-model for $L$ satisfying the mixing property. Then $\mathcal{M}$ is full.*

*Proof.* Fix a formula $\phi(x, y_1, \ldots, y_n)$ in $L$ and $\sigma_1, \ldots, \sigma_n \in M$. Then we can define

$$D := \{b \in B : \exists \tau \in M \quad \textit{such that} \quad b \leq [\![\phi(\tau, \sigma_1, \ldots, \sigma_n)]\!]\}.$$

We notice immediately that $D$ is dense below $[\![\exists x\phi(x, \sigma_1, \ldots, \sigma_n)]\!]$: if $c \leq [\![\exists x\phi(x, \sigma_1, \ldots, \sigma_n)]\!]$, this means that $c \leq \bigvee_{\tau \in M}[\![\phi(\tau, \sigma_1, \ldots, \sigma_n)]\!]$. This implies that there exists $\eta \in M$ such that $b := c \wedge [\![\phi(\eta)]\!] > 0$. In particular, $b \in D$ and, since $b \leq c$, we have that $D$ is dense below $\exists x\phi(x, \sigma_1, \ldots, \sigma_n)$.
Let $A \subset D$ be a maximal antichain in $D$. Clearly, $\bigvee A \leq \bigvee D$. Conversely, by contradiction assume that $\bigvee A < \bigvee D$, and let $c := \bigvee D \wedge \neg \bigvee A > 0$. Since $D$ is dense, there exists $b \in D$ such that $b \leq c$ but $b \wedge \bigvee A = 0$, that is $b \wedge a = 0$ for every $a \in A$. This means that $A \cup \{b\}$ is an antichain, against the maximality of $A$ in $D$. We conclude that $\bigvee A = \bigvee D$.
In particular, we have that $\bigvee A = [\![\exists x\phi(x, \sigma_1, \ldots, \sigma_n)]\!]$, which means that, for every $a \in A$, there exists $\tau_a \in M$ such that $a \leq [\![\phi(\tau_a, \sigma_1, \ldots, \sigma_n)]\!]$. Since $\mathcal{M}$ satisfies the mixing property, let $\tau \in M$ be such that $a \leq [\![\tau = \tau_a]\!]$ for every $a \in A$. We obtain that

$$[\![\phi(\tau, \sigma_1, \ldots, \sigma_n)]\!] \geq [\![\tau = \tau_a]\!] \wedge [\![\phi(\tau_a, \sigma_1, \ldots, \sigma_n)]\!] \geq a.$$

We conclude that $[\![\phi(\tau, \sigma_1, \ldots, \sigma_n)]\!] \geq \bigvee A = [\![\exists x\phi(x, \sigma_1, \ldots, \sigma_n)]\!]$. Hence $\mathcal{M}$ is full. □

*Remark* 2.1.8. An important fact we have to observe is that satisfying the mixing property does not depend on the language we are considering. In particular, let $\mathcal{M}$ be a $\mathsf{B}$-valued model for a language $L$ and suppose having proved that $\mathcal{M}$ satisfies the mixing property. If $\mathcal{M}$ is a $\mathsf{B}$-valued model for any other language $L'$ and, in this language, the interpretation of the equality symbol $=$ remain the same, then $\mathcal{M}$ is a full $\mathsf{B}$-valued model for $L'$.

**Lemma 2.1.9.** *Let $\langle \mathcal{M}_i : i \in I \rangle$ be a family of first order structures for the language L. Then, the $\mathcal{P}(I)$-valued model $\prod_{i \in I} M_i$ satisfies the mixing property.*

*Proof.* Let $A \subset \mathcal{P}(I)$ be an antichain and let $\{\tau_a : a \in A\} \subseteq \prod_{i \in I} M_i$. We observe that $A$ is a family of pairwise disjoint subsets of $I$. Then we can define a $\tau \in \prod_{i \in I} M_i$ in the following way: for every $i \in I$, if there exists an (unique) $a \in A$ such that $i \in a$, then we define $\tau(i) := \tau_a(i)$. Otherwise, we set $\tau(i) := 0$. We check that this $\tau$ is the element that satisfies the mixing property: for every $a \in A$,

$$[\![\tau = \tau_a]\!] = \{i \in I : \tau(i) = \tau_a(i)\} \supseteq a,$$

which means that $[\![\tau = \tau_a]\!] \geq a$. □

The main motivation for introducing the notion of full boolean valued model arises from the following result.

**Theorem 2.1.10** (Łoś Theorem)**.** *Let* $\mathsf{B}$ *be a (complete) boolean algebra. Assume $\mathcal{M}$ to be a full* $\mathsf{B}$*-valued model. For any $U \in \mathrm{St}(\mathsf{B})$ $f_1, \ldots, f_n \in M$, and for all formulae $\varphi(x_1, \ldots, x_n)$*

$$\mathcal{M}/_U \vDash \varphi([f_1]_U, \ldots, [f_n]_U) \quad \textit{if and only if} \quad [\![\varphi(f_1, \ldots, f_n)]\!]^{\mathcal{M}}_{\mathsf{B}} \in U.$$

This theorem is important because there is no reason to believe that, if a formula which is not quantifier free has boolean value 1 in a $\mathsf{B}$-valued model $\mathcal{M}$, then it is true in $\mathcal{M}/_U$ for every $U \in \mathrm{St}(\mathsf{B})$. An example of a boolean valued model $\mathcal{M}$ (clearly not full) that admits a formula true

in $\mathcal{M}$ but no longer true in $\mathcal{M}/_U$ for some ultrafilter $U$ can be found in [19].
Moreover, Lemma 2.1.9 allows us to notice that this Łoś Theorem is the generalization of its first order version, namely Theorem 1.2.12.

**Definition 2.1.11.** Let $\mathcal{M}_1$ be a $\mathsf{B}_1$-valued model and $\mathcal{M}_2$ be a $\mathsf{B}_2$-valued model, where $\mathsf{B}_1$ and $\mathsf{B}_2$ are complete boolean algebras. Let $i : \mathsf{B}_1 \to \mathsf{B}_2$ be a complete morphism of complete boolean algebras. Then a map $\Phi : M_1 \to M_2$ is an injective $i$-*morphism* if for every $n$-ary relational symbol $R$ in the language and for every $a_1, \dots, a_n \in M_1$

$$[\![R(\Phi(a_1), \dots, \Phi(a_n))]\!]^{\mathcal{M}_2}_{\mathsf{B}_2} = i([\![R(a_1, \dots, a_n)]\!]^{\mathcal{M}_1}_{\mathsf{B}_1})$$

$$[\![\Phi(a_1) = \Phi(a_2)]\!]^{\mathcal{M}_2}_{\mathsf{B}_2} = i([\![a_1 = a_2]\!]^{\mathcal{M}_1}_{\mathsf{B}_1}).$$

Clearly, if $\mathsf{B}_1 = \mathsf{B}_2 = 2$, we have the usual definition of morphism between two different Tarski models.

**Proposition 2.1.12.** *Let $\mathcal{M}_1$ be a $\mathsf{B}_1$-valued model and $\mathcal{M}_2$ be a $\mathsf{B}_2$-valued model, where $\mathsf{B}_1$ and $\mathsf{B}_2$ are complete boolean algebras. Suppose that $i : \mathsf{B}_1 \to \mathsf{B}_2$ is a complete morphism of complete boolean algebras. If $\Phi : M_1 \to M_2$ is an injective $i$-morphism and $U$ is an ultrafilter on $\mathsf{B}_2$, then is well defined the map $\Phi^U : \mathcal{M}_1/W \to \mathcal{M}_2/U$, where we have that $W := i^{-1}[U]$ and*

$$\Phi^U([a]_{\mathcal{M}_1}) := [\Phi(a)]_{\mathcal{M}_2}. \tag{2.6}$$

*Moreover, $\Phi^U$ is an embeddding of 2-valued models.*

*Proof.* First of all if $U$ is an ultrafilter on $\mathsf{B}_2$, then $i^{-1}[U]$ is an ultrafilter on $\mathsf{B}_1$.
Let us now show that $\Phi^U$ is well defined, i.e. for every $a \in M_1$, $\Phi\big[[a]_{\mathcal{M}_1}\big] \subseteq [\Phi(a)]_{\mathcal{M}_2}$. This fact holds true since:

$$\begin{aligned}
\Phi\big[[a]_{\mathcal{M}_1}\big] &= \{\Phi(b) : [\![b = a]\!] \in i^{-1}[U]\} \\
&= \{\Phi(b) : i([\![b = a]\!]) \in U\} \\
&= \{\Phi(b) : [\![\Phi(b) = \Phi(a)]\!] \in U\} \\
&\subseteq \{c \in M_2 : [\![c = \Phi(a)]\!] \in U\} = [\Phi(a)]_{\mathcal{M}_2}.
\end{aligned}$$

Finally, we prove that $\Phi^U$ is a morphism of 2-valued models. We have to show that, for every $a_1, \dots, a_n \in M_1$ and for every $R$ $n$-ary relational symbol in the language, $R([a_1]_{\mathcal{M}_1}, \dots, [a_n]_{\mathcal{M}_1})$ if and only of $R(\Phi^U([a_1]_{\mathcal{M}_1}, \dots, \Phi^U([a_n]_{\mathcal{M}_1}))$. We prove it as follows:

$$\begin{aligned}
R([a_1]_{\mathcal{M}_1}, \dots, [a_n]_{\mathcal{M}_1}) &\iff [\![R(a_1, \dots, a_n)]\!] \in i^{-1}[U] \\
&\iff i([\![R(a_1, \dots, a_n)]\!] \in U \\
&\iff [\![R(\Phi(a_1), \dots, \Phi(a_n)]\!] \in U \\
&\iff R([\Phi(a_1)]_{\mathcal{M}_2}, \dots, [\Phi(a_n)]_{\mathcal{M}_2}) \\
&\iff R(\Phi^U([a_1]_{\mathcal{M}_1}), \dots, \Phi^U([a_n]_{\mathcal{M}_1})).
\end{aligned}$$

□

## 2.2 Boolean ultrapowers

We will now introduce the construction due to Mansfield of the boolean ultrapower of a first order structure. In the next chapter, we will use it to obtain saturated elementary extensions of a structure $\mathcal{M}$.
Let $\mathsf{B}$ be a complete boolean algebra.

**Definition 2.2.1.** Let $A \subset \mathsf{B}$ be a maximal antichain. A subset $W \subseteq \mathsf{B}$ is a *refinement* of $A$ if it is a maximal antichain such that, for every $w \in W$, there exists an $a \in A$ such that $w \leq a$.

Clearly, for every $w \in W$ the $a \in A$ such that $w \leq a$ is unique. Moreover, every finite family of maximal antichains of $\mathsf{B}$ admits a common refinement.
Now suppose $A \subset \mathsf{B}$ is a maximal antichain and let $f : A \to X$ be a function. For every refinement $W$ of $A$ the *reduction* of $f$ to $W$ is the map

$$f \downarrow W : W \to X$$

such that

$$w \mapsto f(a),$$

where $a \in A$ is the unique such that $w \leq a$.

**Definition 2.2.2.** Let $L$ be a relational language, $\mathcal{M}$ be an $L$-structure, and $\mathsf{B}$ be a complete boolean algebra. We define the $\mathsf{B}$-*power* of $\mathcal{M}$ as the $\mathsf{B}$-valued model $\mathcal{M}^{\downarrow \mathsf{B}}$ such that:

1. Its domain is the set $\mathcal{M}^{\downarrow \mathsf{B}} := \{\sigma : A \to M : A \subseteq \mathsf{B}$ is a maximal antichain $\}$.
2. If $\sigma, \tau \in \mathcal{M}^{\downarrow \mathsf{B}}$ and we fix a common refinement $W$ of $\mathrm{dom}(\sigma)$ and $\mathrm{dom}(\tau)$, then we define
$$[\![\sigma = \tau]\!] := \bigvee \{w \in W : (\sigma \downarrow W)(w) = (\tau \downarrow W)(w)\}.$$
3. the definition of the interpretation of symbols in $L$ is the following:
    - If $R$ is a $n$-ary relational symbol and $\sigma_1, \ldots, \sigma_n \in \mathcal{M}^{\downarrow \mathsf{B}}$, then we fix a common refinement $W$ of $\mathrm{dom}(\sigma_1), \ldots, \mathrm{dom}(\sigma_n)$ and we define
    $$[\![R(\sigma_1, \ldots \sigma_n)]\!] := \bigvee \{w \in W : \mathcal{M} \vDash R((\sigma_1 \downarrow W)(w), \ldots, (\sigma_n \downarrow W)(w))\};$$
    - if $c$ is a constant symbol, $c^{\mathcal{M}^{\downarrow \mathsf{B}}}$ is the map $\{1\} \to M$ such that $1 \mapsto c^{\mathcal{M}}$.

It is easy to check that these interpretations are well-defined, and also that $\mathcal{M}^{\downarrow \mathsf{B}}$ satisfies the definition of $\mathsf{B}$- valued model.
Observe also that, given a valuation taking values in $\mathcal{M}^{\downarrow \mathsf{B}}$, it is defined the boolean value of truth for every formula in the language $L$. More precisely,

**Proposition 2.2.3.** *Let $\varphi(x_1, \ldots, x_n)$ an $L$-formula and let $\sigma_1, \ldots, \sigma_n \in \mathcal{M}^{\downarrow \mathsf{B}}$. Let $W$ be a common refinement of* $\mathrm{dom}(\sigma_1), \ldots, \mathrm{dom}(\sigma_n)$*, then:*

$$[\![\varphi(\sigma_1, \ldots, \sigma_n)]\!] = \bigvee \{w \in W : \mathcal{M} \vDash \varphi((\sigma_1 \downarrow W)(w), \ldots, (\sigma_n \downarrow W)(w))\}. \tag{2.7}$$

*Proof.* We prove it by induction on the complexity of the formula $\varphi$. We can suppose $\sigma_1, \ldots, \sigma_n$ already reduced to $W$. If $\varphi$ is an atomic formula, it holds by definition.
If $\varphi(x_1, \ldots, x_n) = \neg\psi(x_1, \ldots, x_n)$, then, since $W$ is a maximal antichain,

$$\begin{aligned}
[\![\phi(\sigma_1, \ldots, \sigma_n)]\!] &= \neg[\![\psi(\sigma_1, \ldots, \sigma_n)]\!] \\
&= \neg \bigvee \{w \in W : \mathcal{M} \vDash \psi(\sigma_1(w), \ldots, \sigma_n(w))\} \\
&= \bigwedge \{\neg w : w \in W \quad \textit{and} \quad \mathcal{M} \vDash \psi(\sigma_1(w), \ldots, \sigma_n(w))\} \\
&= \bigvee \{v \in W : \mathcal{M} \vDash \neg\psi(\sigma_1(v), \ldots, \sigma_n(v))\}.
\end{aligned}$$

If $\varphi(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_n) \wedge \chi(x_1, \ldots, x_n)$, then:

$$\begin{aligned}
&[\![\varphi(\sigma_1, \ldots, \sigma_n)]\!] = [\![\psi(\sigma_1, \ldots, \sigma_n)]\!] \wedge [\![\chi(\sigma_1, \ldots, \sigma_n)]\!] \\
&= \bigvee \{w \in W : \mathcal{M} \vDash \psi(\sigma_1(w), \ldots, \sigma_n(w))\} \wedge \wedge \bigvee \{w \in W : \mathcal{M} \vDash \chi(\sigma_1(w), \ldots, \sigma_n(w))\} \\
&= \bigvee \{w \in W : \mathcal{M} \vDash \psi(\sigma_1(w), \ldots, \sigma_n(w)) \textit{ and } \mathcal{M} \vDash \chi(\sigma_1(w), \ldots, \sigma_n(w))\} \\
&= \bigvee \{w \in W : \mathcal{M} \vDash (\psi(\sigma_1(w), \ldots, \sigma_n(w)) \wedge \chi(\sigma_1(w), \ldots, \sigma_n(w)))\}.
\end{aligned}$$

Finally, if $\varphi(x_1, \ldots, x_n) = \exists y \psi(y, x_1, \ldots, x_n)$, then:

$$\begin{aligned}
[\![\exists y \psi(\sigma_1, \ldots, \sigma_n)]\!] &= \bigvee_{\tau \in \mathcal{M}^{\downarrow \mathsf{B}}} [\![\psi(\tau, \sigma_1, \ldots, \sigma_n)]\!] \\
&= \bigvee_{\tau \in \mathcal{M}^{\downarrow \mathsf{B}}} \bigvee \{w \in W : \mathcal{M} \vDash \psi(\tau(w), \sigma_1(w), \ldots, \sigma_n(w))\} \\
&= \bigvee \bigcup_{\tau \in \mathcal{M}^{\downarrow \mathsf{B}}} \{w \in W : \mathcal{M} \vDash \psi(\tau(w), \sigma_1(w), \ldots, \sigma_n(w))\} \\
&= \bigvee \{w \in W : \textit{ exists } \tau \in \mathcal{M}^{\downarrow \mathsf{B}} \textit{ such that } \mathcal{M} \vDash \psi(\tau(w), \sigma_1(w), \ldots, \sigma_n(w))\} \\
&= \bigvee \{w \in W : \mathcal{M} \vDash \exists y \psi(y, \sigma_1(w), \ldots, \sigma_n(w))\}.
\end{aligned} \tag{2.8}$$

□

**Proposition 2.2.4.** *The* $\mathsf{B}$*-valued model* $\mathcal{M}^{\downarrow \mathsf{B}}$ *satisfies the mixing property and so, in particular, it is full.*

*Proof.* We show that $\mathcal{M}^{\downarrow \mathsf{B}}$ satisfies the mixing property. By Proposition 2.1.7, it is also full.
Let $A$ be an antichain in $\mathsf{B}$, and let $\{\sigma_a : a \in A\} \subseteq \mathcal{M}^{\downarrow \mathsf{B}}$. By Zorn's Lemma, we can assume $A$ to be maximal. For $a \in A$, we define

$$D_a := \{b \wedge a : b \in \mathrm{dom}(\sigma_a)\}.$$

We note that, if $a_1 \neq a_2$, then $D_{a_1} \cap D_{a_2}$ is empty. We define $\sigma \in \mathcal{M}^{\downarrow \mathsf{B}}$ as follows: the domain is $\mathrm{dom}(\sigma) := \bigcup_{a \in A} D_a$ and, if $d \in D_a$, then $\sigma(d) := \sigma_a(b)$, where $d \leq b$.

We want to prove that, for every $a \in A$, $a \leq [\![\sigma = \sigma_a]\!]$. We can always assume that $\mathrm{dom}(\sigma)$ is a refinement of $\mathrm{dom}(\sigma_a)$ and we will write $\sigma_a$ for $\sigma_a \downarrow \mathrm{dom}(\sigma)$. Then:

$$\begin{aligned} a \wedge [\![\sigma = \sigma_a]\!] &= a \wedge \bigvee\{d \in \mathrm{dom}(\sigma) : \sigma(d) = \sigma_a(d)\} \\ &= \bigvee\{a \wedge d : d \in \mathrm{dom}(\sigma), \sigma(d) = \sigma_a(d)\} \\ &\geq \bigvee\{a \wedge d : d \in D_a, \sigma(d) = \sigma_a(d)\} = \bigvee\{a \wedge d : d \in D_a\} = a, \end{aligned}$$

as we wanted. □

**Definition 2.2.5.** Let $\mathcal{M}$ be an $L$-structure, $\mathsf{B}$ a complete boolean algebra and $U \subset \mathsf{B}$ an ultrafilter. We call the $\mathsf{B}$-*ultrapower* of $\mathcal{M}$ by $U$ the quotient $\mathcal{M}^{\downarrow \mathsf{B}}/_U$.

**Theorem 2.2.6.** *The map*

$$\begin{aligned} j :& \mathcal{M} \to \mathcal{M}^{\downarrow \mathsf{B}}/_U, \\ & x \mapsto [c_x]_U \end{aligned}$$

*where*

$$\begin{aligned} c_x :& \{1\} \to M, \\ & 1 \mapsto x \end{aligned}$$

*is an elementary embedding.*

*Proof.* Let $\varphi(x_1, \ldots, x_n)$ be a formula in $L$ and let $a_1, \ldots, a_n \in M$. Then, $\mathcal{M}^{\downarrow \mathsf{B}}/_U \vDash \varphi(j(a_1), \ldots, j(a_n))$ if and only if $[\![\phi(c_{a_1}, \ldots, c_{a_n})]\!] \in U$. Using (2.7), this is exactly the same of writing that

$$\bigvee\{w \in \{1\} : \mathcal{M} \vDash \phi(c_{a_1}(w), \ldots, c_{a_n}(w))\} \in U,$$

but this means that $\mathcal{M} \vDash \phi(a_1, \ldots, a_n)$. □

## 2.3 The boolean valued models $M^{\mathsf{B}}$ and $\check{M}^{\mathsf{B}}$

The method of forcing was introduced in [7] by Cohen to prove indipendence results, and today it is still the core instrument for researches in modern set theory. The approach to forcing via boolean valued models can be summarized in two main steps. We start having a model $V$ (called the *ground model*) for the ZFC axiomatization of set theory, and we fix a (complete) boolean algebra $\mathsf{B}$ in $V$. The main goal of the method of forcing is to find a new model of ZFC in which a given sentence $\phi$ is true. First of all, a $\mathsf{B}$-valued model $V^{\mathsf{B}}$ for ZFC is constructed. Then, by an accurate choice of an ultrafilter $U$ in $\mathsf{B}$, one construct the first order structure $V[U]$, that is a model of ZFC in which $\phi$ holds. This section is not devoted to the presentation of how $V[U]$ can be built from $V^{\mathsf{B}}$, for our aims it is sufficent to say that if $U$ is a $V$-generic filter for $\mathsf{B}$, $V[U] \cong V^{\mathsf{B}}/_U$ (see [19, Theorem 5.2.3] for the proof). Moreover, for a complete description of the forcing method using boolean valued models, we address the reader to [9] or [19].
Let $\mathcal{L} = \{\in, \subseteq\}$ be the language of set theory and let $V$ be a model for ZFC in this language. Fix a complete boolean algebra $\mathsf{B} \in V$. Let $M \subseteq V$ be defined by an $\mathcal{L}$ formula $\varphi_M(x)$. The collection

of $\mathsf{B}$-names for elements of $M$ is the family $M^{\mathsf{B}}$ of elements $\tau \in V^B$ such that $[\![\varphi_M(\tau)]\!] = 1$. Moreover, $V$ can be embedded in $V^{\mathsf{B}}$ and so the elements of $M \subseteq V$ are represented by elements of $V^{\mathsf{B}}$. The collection of these representatives is denoted with $\check{M}^{\mathsf{B}}$. We will now characterize $\check{M}^{\mathsf{B}}$ using a boolean ultrapower. In principle the classes $M^{\mathsf{B}}$ and $\check{M}^{\mathsf{B}}$ are unrelated, but if $M$ has a sufficiently simple definition we have at least that $M^{\mathsf{B}} \subseteq \check{M}^{\mathsf{B}}$ even if the inclusion may be strict. We will see a specific case of this phenomenon analyzing the case $M = 2^\omega$.

**Definition 2.3.1.** We define the *class of* $\mathsf{B}$*-names* $V^{\mathsf{B}}$ by induction on $\mathrm{Ord}$:

1. $V_0^{\mathsf{B}} := \emptyset$
2. $V_{\alpha+1}^{\mathsf{B}} := \{f : X \to \mathsf{B} : X \subseteq V_\alpha^{\mathsf{B}}\}$;
3. $V_\alpha^{\mathsf{B}} := \bigcup_{\beta<\alpha} V_\beta^{\mathsf{B}}$ if $\alpha$ is a limit ordinal;
4. $V^{\mathsf{B}} := \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha^{\mathsf{B}}$.

For every $x \in V^{\mathsf{B}}$, the *rank* $\rho(x)$ of $x$ is the least $\alpha \in \mathrm{Ord}$ such that $x \in V_{\alpha+1}\mathsf{B}$. We define the boolean value of the two relational symbols $\in$ and $\subseteq$. Since we have to consider only binary relational symbols, we define the boolean value by induction on $\langle \rho(x), \rho(y)\rangle$, ordered with the canonical square well-order of $\mathrm{Ord} \times \mathrm{Ord}$. We will write $a \to b$ for $\neg a \vee b$.

**Definition 2.3.2.** The boolean value of $=$, $\in$ and $\subseteq$ in $V^{\mathsf{B}}$ is:

$$[\![x \in y]\!] := \bigvee_{t\in\mathrm{dom}(y)} ([\![x = t]\!] \wedge y(t));$$
$$[\![x \subseteq y]\!] := \bigwedge_{t\in\mathrm{dom}(x)} (x(t) \to [\![t \in y]\!]);$$
$$[\![x = y]\!] := [\![x \subseteq y]\!] \wedge [\![y \subseteq x]\!].$$

**Theorem 2.3.3.** *$V^{\mathsf{B}}$ satisfies the mixing property, hence it is a full $\mathsf{B}$-valued model for $\mathcal{L}$. Moreover, if $\varphi$ is an axiom of* $\mathsf{ZFC}$*, then* $[\![\varphi]\!]^{V^{\mathsf{B}}} = 1$.

*Proof.* See, for example, [9, Chapter 14] or [19]. □

For every set $x \in V$ there exists a canonical $\mathsf{B}$-name for $x$, defined by induction on $\in$ in $V$:

$$\check{x} := \{\langle \check{y}, 1\rangle : y \in x\}.$$

**Definition 2.3.4.** Let $M$ be any class and let $\varphi_M(x)$ a formula in the language $\mathcal{L}$ such that

$$a \in M \textit{ if and only if } V \vDash \varphi_M(a).$$

The *set* $M^{\mathsf{B}}$ *of* $\mathsf{B}$*-names for elements of* $M$ is the set of $\tau \in V^{\mathsf{B}}$ such that

$$[\![\varphi_M(\tau)]\!] = 1,$$

modulo the equivalence relation

$$\sigma \sim \tau \quad \textit{if and only if} \quad [\![\sigma = \tau]\!] = 1.$$

Moreover, the *set of* $\mathsf{B}$*-names for elements of* $M$ *in* $V$ is the set

$$\check{M}^{\mathsf{B}} := \{\tau \in V^{\mathsf{B}} : \bigvee_{x\in M} [\![\tau = \check{x}]\!] = 1\}.$$

Notice that, if $\varphi_M$ is $\Sigma_1$, $\check{M}^{\mathsf{B}} \subseteq M^{\mathsf{B}}$. Indeed, if $\tau \in (\check{M})^{\mathsf{B}}$,

$$1 = \bigvee_{x \in M} [\![\tau = \check{x}]\!] = \bigvee_{x \in M} ([\![\tau = \check{x}]\!] \wedge [\![\varphi_M(\check{x})]\!]) \leq [\![\varphi_M(\tau)]\!]$$

since $[\![\varphi_M(\check{x})]\!] = 1$ for every $x \in M$, because[1] $\phi_M$ is $\Sigma_1$. Therefore $\tau \in M^{\mathsf{B}}$. Furthermore, $M^B$ and $\check{M}^{\mathsf{B}}$ are $\mathsf{B}$-valued model for $\mathcal{L}$ with the same definition of the boolean relations $\in^{\mathsf{B}}$, $\subseteq^{\mathsf{B}}$, $=^{\mathsf{B}}$.

**Lemma 2.3.5.** *$M^{\mathsf{B}}$ satisfies the mixing property.*

*Proof.* Let $A$ be an antichain in $\mathsf{B}$ and let, for every $a \in A$, $\tau_a \in M^{\mathsf{B}}$. By further extending $A$, we can assume $A$ to be maximal.
By Theorem 2.3.3, $V^{\mathsf{B}}$ satisfies the mixing property: there exists $\tau \in V^{\mathsf{B}}$ such that

$$[\![\tau = \tau_a]\!] \geq a$$

for every $a \in A$. We only have to check that $\tau \in M^{\mathsf{B}}$. Since, for every $a \in A$, $\tau_a \in M^{\mathsf{B}}$, we have that $[\![\varphi_M(\tau_a)]\!] = 1$. Therefore

$$a \leq [\![\tau = \tau_a]\!] = [\![\tau = \tau_a]\!] \wedge [\![\varphi_M(\tau_a)]\!] \leq [\![\varphi_M(\tau)]\!].$$

Then,

$$[\![\varphi_M(\tau)]\!] \geq \bigvee A = 1$$

by maximality of $A$. $\square$

The proof of the following results can be found in [16, Theorem 2.5.3, Proposition 2.5.4], else see Theorem 2.3.9 below.

**Proposition 2.3.6.** *The $\mathsf{B}$-valued model $\check{M}^B$ has the mixing property.*

**Theorem 2.3.7.** *If $M$ is a class, $\varphi(x_1, \ldots, x_n)$ is an $L$-formula and $a_1, \ldots, a_n \in M$, then*

$$\langle M, \in, \subseteq \rangle \vDash \varphi(a_1, \ldots, a_n) \quad \textit{if and only if} \quad [\![\varphi(\check{a_1}, \ldots, \check{a_n})]\!]^{\check{M}^B} = 1.$$

An immediate consequence is:

**Corollary 2.3.8.** *If $U$ is an ultrafilter on $\mathsf{B}$, the map*

$$\begin{aligned} i : &M \to \check{M}^B/U \\ &x \mapsto [\check{x}]_U \end{aligned}$$

*is an elementary embedding.*

The following result improves [16, Theorem 2.5.6] (which is now Corollary 2.3.10).

**Theorem 2.3.9.** *$\mathcal{M}^{\downarrow\mathsf{B}}$ and $\check{M}^{\mathsf{B}}$ are isomorphic $\mathsf{B}$-valued models for the language $\{=\}$.*

[1] If for example $\phi(x) = \forall y \psi(x, y)$ is $\Pi_1$ with $\psi(x, y)$ $\Delta_0$ it could be the case that $[\![\exists y \psi(\check{x}, y)]\!] = 1$ while $V \models \forall y \neg\psi(x, y)$.

*Proof.* Since $\check{M}^B$ has the mixing property, for every antichain $A \subset \mathsf{B}$ and every function $f : A \to M$ in $\mathcal{M}^{\downarrow\mathsf{B}}$, there exists $\tau_f \in \check{M}^B$ such that

$$a \leq [\![\tau_f = f\check{(a)}]\!]^{\check{M}^B}$$

for every $a \in A$.
If $W$ is a refinement of $A$, then $[\![\tau_f = \tau_{(f\downarrow W)}]\!]^{\check{M}^B} = 1$: if $w \in W$ and $a$ is the unique element of $A$ such that $w \leq a$, then

$$w \leq a \leq [\![\tau_f = f\check{(a)}]\!]^{\check{M}^B} = [\![\tau_f = (f \downarrow \check{W})(w)]\!]^{\check{M}^B},$$

therefore

$$w \leq [\![\tau_f = (f \downarrow \check{W})(w)]\!]^{\check{M}^B} \wedge [\![\tau_{(f\downarrow W)} = (f \downarrow \check{W})(w)]\!]^{\check{M}^B} \leq [\![\tau_f = \tau_{(f\downarrow W)}]\!]^{\check{M}^B},$$

hence $1 = \bigvee W \leq [\![\tau_f = \tau_{(f\downarrow W)}]\!]^{\check{M}^B}$, as we claimed.
We can now define the map

$$\begin{aligned} \theta : \mathcal{M}^{\downarrow\mathsf{B}} &\to \check{M}^{\mathsf{B}}. \\ f &\mapsto \tau_f \end{aligned} \tag{2.9}$$

**$\theta$ is surjective.** Let $\tau \in \check{M}^{\mathsf{B}}$. By definition, then, $\bigvee_{x\in M} [\![\tau = \check{x}]\!] = 1$. Let $\{x_i : i \in I\}$ be an enumeration of $M$. Then

$$A := \{a_i := [\![\check{x_i} = \tau]\!] : i \in I\}$$

is an antichain, since $[\![\check{x_1} = \check{x_2}]\!] = 0$ for every $x_1 \neq x_2$ is $M$. Moreover, it is maximal since $\bigvee A = \bigvee_{x\in M} [\![\tau = \check{x}]\!] = 1$. Let

$$\begin{aligned} f :&A \to M. \\ &a_i \mapsto x_i \end{aligned}$$

It is immediate to see that $[\![\tau_f = \tau]\!] \geq \bigvee_{i\in I} [\![\check{x_i} = \tau]\!] = \bigvee A = 1$.

**$\theta$ is injective.** Let $f, g$ be two distinct elements in $\mathcal{M}^{\downarrow\mathsf{B}}$. For sake of easiness, assume $\mathrm{dom}(f) = \mathrm{dom}(g) = A$. Since $f \neq g$, there exists $a \in A$ such that $f(a) \neq g(a)$. Moreover, we have by construction that

$$a \leq \left[\!\!\left[\tau_f = f\check{(a)}\right]\!\!\right], \left[\!\!\left[\tau_g = g\check{(a)}\right]\!\!\right].$$

By contraddiction, assume that $\tau_f = \tau_g$, so that $[\![\tau_f = \tau_g]\!] = 1$. Then we have:

$$a \leq [\![\tau_f = \tau_g]\!] \wedge \left[\!\!\left[\tau_f = f\check{(a)}\right]\!\!\right] \leq \left[\!\!\left[\tau_g = f\check{(a)}\right]\!\!\right],$$

which is false since

$$0 < a \leq \left[\!\!\left[\tau_g = g\check{(a)}\right]\!\!\right] \wedge \left[\!\!\left[\tau_g = g\check{(a)}\right]\!\!\right] \leq \left[\!\!\left[f\check{(a)} = g\check{(a)}\right]\!\!\right] = 0.$$

**$\theta$ preserves the interpretation of $=$.** Let $w \in W$ be such that $(f \downarrow W)(w) = (g \downarrow W)(w)$. Then

$$\begin{aligned} w &\leq [\![\tau_{(f\downarrow W)} = (f \downarrow \check{W})(w)]\!]^{\check{M}^B} \wedge [\![\tau_{(g\downarrow W)} = (g \downarrow \check{W})(w)]\!]^{\check{M}^B} \\ &\leq [\![\tau_{(f\downarrow W)} = \tau_{(g\downarrow W)}]\!]^{\check{M}^B} = [\![\tau_f = \tau_g]\!]^{\check{M}^{\mathsf{B}}}, \end{aligned}$$

and so we have $[\![f = g]\!]^{\mathcal{M}^{\downarrow \mathsf{B}}} \leq [\![\tau_f = \tau_g]\!]^{\check{M}^{\mathsf{B}}}$.

Conversely, with the same steps we obtain $[\![f \neq g]\!]^{\mathcal{M}^{\downarrow \mathsf{B}}} \leq [\![\tau_f \neq \tau_g]\!]^{\check{M}^{\mathsf{B}}}$.

Then $[\![f = g]\!]^{\mathcal{M}^{\downarrow \mathsf{B}}} = [\![\tau_f = \tau_g]\!]^{\check{M}^{\mathsf{B}}}$.

□

**Corollary 2.3.10.** *Let $U$ be an ultrafilter on $\mathsf{B}$. Let $j : M \to \mathcal{M}^{\downarrow \mathsf{B}}/U$ be the canonical elementary embedding. Then there exists an isomorphism $\pi : \mathcal{M}^{\downarrow \mathsf{B}}/U \to \check{M}^B/U$ such that, for every $x \in M$, $i(x) = \pi(j(x))$.*

*Proof.* The proof is straightforward: the isomorphism $\theta$ passes to the quotient. In particular, surjectivity is trivially granted, and injectivity comes from the fact that, by Łoś Theorem 2.1.10, the interpretation of the equality is preserved by the quotient. □

## 2.4 $\mathsf{B}$-names for the Cantor space

Let us fix throughout this section a complete boolean algebra $\mathsf{B}$ and consider the *Cantor space* $2^\omega$ given by the set of infinite binary strings endowed with the product topology, where $2 = \{0, 1\}$ has the discrete topology. The Cantor space is a well known example of compact Polish space, that is, a compact, second countable, completely metrizable, topological space.
Our first goal is to establish an isomorphism of $\mathsf{B}$-valued models between $(2^\omega)^{\mathsf{B}}$ and $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$. We will also describe the image of $(\check{2^\omega})^{\mathsf{B}}$ under this isomorphism. Finally, we will discuss which kind or relations this isomorphism preserves. The material is taken and expands from [18].
We assume that the family $V$ of all sets is such that $(V, \in, \subseteq) \models \mathsf{ZFC}$ for the segnature $\mathcal{L} = \{=, \in, \subseteq\}$.
We will use basic facts about the Cantor space (the standard reference will be [11]), in particular that $2^\omega$ is compact, Hausdorff and zero-dimensional.

**Definition 2.4.1.** The *space of continuous functions* from $\mathrm{St}(\mathsf{B})$ to $2^\omega$ is the set

$$\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega) := \{f : \mathrm{St}(\mathsf{B}) \to 2^\omega : f \textit{ is continuous}\}$$

equipped with the topology induced by the distance

$$d_\infty(f, g) := \sup \{d_{2^\omega}(f(G), g(G)) : G \in \mathrm{St}(\mathsf{B})\}.$$

The *space of locally constant continuous functions* from $\mathrm{St}(\mathsf{B})$ to $2^\omega$ is denoted by

$$\mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega) := \{f : \mathrm{St}(\mathsf{B}) \to 2^\omega : \bigcup_{y \in 2^\omega} \mathrm{Reg}\left(f^{-1}[\{y\}]\right) \textit{ is a dense open subset of } \mathrm{St}(\mathsf{B})\},$$

with the induced topology of subspace of $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$.

Since we assumed $\mathsf{B} \cong \mathrm{CLOP}(\mathrm{St}(\mathsf{B}))$ to be complete, by Corollary 1.1.11 $\mathsf{B}$ is isomorphic to $\mathsf{RO}(\mathrm{St}(\mathsf{B})) = \mathrm{CLOP}(\mathrm{St}(\mathsf{B}))$. Thus we will feel free to use any of the latter or the former representations of $\mathsf{B}$.

**Proposition 2.4.2.** *Loc*$(\mathrm{St}(\mathsf{B}), 2^\omega)$ *is a dense subset of* $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$.

*Proof.* Since we have defined a metric topology on $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$, the proof consists in showing that, for every $f \in \mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$ and for every $\varepsilon > 0$, there exists $g \in \mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$ such that

$$d_\infty(f, g) < \varepsilon.$$

Now, fix $f \in \mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$ and $\varepsilon > 0$ and let $Q$ be a countable dense subset of $2^\omega$. Then, the following inclusion holds true

$$f[\mathrm{St}(\mathsf{B})] \subseteq \bigcup \{B(q, \varepsilon) : q \in Q \cap f[\mathrm{St}(\mathsf{B})]\}.$$

By further refining each $B(q_i, \varepsilon)$ we may without loss of generality assume that they are all clopen subsets of $2^\omega$.
Since $\mathrm{St}(\mathsf{B})$ is compact and $f$ is continuous, $f[\mathrm{St}(\mathsf{B})]$ is compact, which implies that there exist $q_1, \ldots, q_n \in Q$ such that

$$f[\mathrm{St}(\mathsf{B})] \subseteq B(q_1, \varepsilon) \cup \cdots \cup B(q_n, \varepsilon);$$

If we define $U_i := f^{-1}[B(q_i, \varepsilon)]$ for $i = 1, \ldots, n$, then $\{U_1, \ldots, U_n\}$ is a finite open cover of $\mathrm{St}(\mathsf{B})$. By letting $V_i = U_i \cap (\mathrm{St}(\mathsf{B}) \setminus \bigcup_{j<i} U_j)$, we get that $\{V_1, \ldots, V_n\}$ is a clopen cover of $\mathrm{St}(\mathsf{B})$ made by disjoint sets. Moreover, they are such that, for each $i$ and $G \in V_i$, $|f(G) - q_i| < \varepsilon$.
Let us define

$$\begin{aligned} g : \mathrm{St}(\mathsf{B}) &\to 2^\omega, \\ V_i \ni G &\mapsto q_i. \end{aligned}$$

Then $g \in \mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$ is such that

$$d_\infty(f, g) = \sup_{x \in \mathrm{St}(\mathsf{B})} d_{2^\omega}(f(x), g(x)) < \varepsilon.$$

□

We endow $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$ with the structure of $\mathsf{B}$-valued model for the language $\{=\}$, letting for $f, g \in \mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$

$$[\![f = g]\!]^{\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)} := \mathrm{Reg}\left(\{G \in \mathrm{St}(\mathsf{B}) : f(G) = g(G)\}\right). \tag{2.10}$$

Clearly, in $\mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$ the interpretation of $=$ is the same.
We have to ensure that this interpretation is well-defined.
Let $f, g \in \mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$, and consider

$$W := \{U \in \mathrm{St}(\mathsf{B}) : f(U) = g(U)\}.$$

Since $f, g$ are continuous functions so is $(f \times g) : U \mapsto (f(U), g(U))$, therefore we have that $W = (f \times g)^{-1}[\{(r, r) : r \in 2^\omega\}]$ is a closed subset of $\mathrm{St}(\mathsf{B})$. By Proposition 1.3.3, $W$ has the Baire property. By [8, Chapter 29, Lemma 2], $\mathrm{Reg}\left(\{G \in \mathrm{St}(\mathsf{B}) : f(G) = g(G)\}\right)$ is the unique regular set with meager difference with $W$. Secondarily, we have to check that our definition satisfies the axioms for equality in a boolean valued model. Let $f, g, h \in \mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$. Then:

- $[\![f = f]\!] = \{U \in \mathrm{St}(\mathsf{B}) : f(U) = f(U)\} = \mathrm{St}(\mathsf{B}) = 1;$

- $[\![f = g]\!] = \{U \in \mathrm{St}(\mathsf{B}) : f(U) = g(U)\} = [\![g = f]\!]$;
- $[\![f = g]\!] \wedge [\![g = h]\!] \leq [\![f = h]\!]$. Observe that:

$$\begin{aligned}
[\![f = g]\!] \wedge [\![g = h]\!] &= \mathrm{Reg}\left(\{U \in \mathrm{St}(\mathsf{B}) : f(U) = g(U)\}\right) \wedge \mathrm{Reg}\left(\{U \in \mathrm{St}(\mathsf{B}) : g(U) = h(U)\}\right) = \\
&= \mathrm{Reg}\left(\{U \in \mathrm{St}(\mathsf{B}) : f(U) = g(U) = h(U)\}\right) \subseteq \\
&\subseteq \{U \in \mathrm{St}(\mathsf{B}) : f(U) = h(U)\} = \\
&= [\![f = h]\!].
\end{aligned}$$

**Lemma 2.4.3.** *Under* (2.10), $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$ *satisfies the mixing property, hence it is a full* $\mathsf{B}$*-valued model.*

*Proof.* Let $A$ be an antichain in $\mathsf{B}$ and let $\{f_a : a \in A\}$ be a subset of $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$. Set $b := \neg \bigvee A$ and define $f : \mathrm{St}(\mathsf{B}) \to 2^\omega$ to be the null sequence on $N_b$ and $f \restriction N_a := f_a \restriction N_a$ for all $a \in A$. Then $f$ is defined on the open dense set $W := N_b \cup \bigcup_{a \in A} N_a$. Since $\mathrm{St}(\mathsf{B})$ is extremally disconnected, by Lemma 1.3.12 $f$ can be uniquely extended to a continuous function $f$ on $\mathrm{St}(\mathsf{B})$.
We conclude that for all $a \in A$

$$a = N_a \subseteq [\![f = f_a]\!],$$

as was to be shown. □

It can be proved that $\mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$ satisfies the mixing property as well, with a similar proof, otherwise see Corollary 2.4.5 below.

**Theorem 2.4.4.** *The spaces* $(2^\omega)^{\mathsf{B}}$ *and* $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$ *are isomorphic* $\mathsf{B}$*-valued models for the language* $\{=\}$.

*Proof.* Consider the map

$$\begin{aligned}
\Psi : (2^\omega)^{\mathsf{B}} &\to \mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega), \\
\tau &\mapsto f_\tau
\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
f_\tau : \mathrm{St}(\mathsf{B}) &\to 2^\omega. \\
U &\mapsto \left\{(n, i_n) : [\![\tau(\check{n}) = \check{i_n}]\!] \in U\right\}_{n \in \omega}
\end{aligned}$$

We prove that $\Psi$ is an isomorphism.

**$\Psi$ is well-defined:**

- We first check that for every $\tau \in (2^\omega)^{\mathsf{B}}$ and for every $U \in \mathrm{St}(\mathsf{B})$ $f_\tau(U)$ is a function $\omega \to 2$. This means that we have to show that for every $\tau \in (2^\omega)^{\mathsf{B}}, U \in \mathrm{St}(\mathsf{B})$ and $n \in \omega$ exactly one among $[\![\tau(\check{n}) = \check{0}]\!]$ or $[\![\tau(\check{n}) = \check{1}]\!]$ is in $U$.
  Now notice that for all $n \in \omega$

  $$1 = [\![\tau(\check{n}) \in \check{2}]\!] = [\![\tau(\check{n}) = \check{0}]\!] \vee [\![\tau(\check{n}) = \check{1}]\!],$$

  while

  $$[\![\tau(\check{n}) = \check{0}]\!] \wedge [\![\tau(\check{n}) = \check{1}]\!] \leq [\![\check{0} = \check{1}]\!] = 0.$$

- We now prove that $f_\tau$ is continuous. A base for the topology of $2^\omega$ is the family of clopen sets

$$A_{a_0,\dots,a_k} = \{s : \omega \to 2 : s(0) = a_0, \dots, s(k) = a_k\},$$

for some $k \in \omega$ and $a_0, \dots, a_k \in 2$. If $A = A_{a_0,\dots,a_k}$ is such a clopen set, then

$$\begin{aligned} f_\tau^{-1}[A] &= \left\{U \in \mathrm{St}(\mathsf{B}) : \llbracket \tau(\check{0}) = \check{a_0} \rrbracket \wedge \cdots \wedge \llbracket \tau(\check{k}) = \check{a_k} \rrbracket \in U\right\} \\ &= N_{\left\llbracket \left(\tau(\check{0})=\check{a_0}\right) \wedge \cdots \wedge \left(\tau(\check{k})=\check{a_k}\right) \right\rrbracket}, \end{aligned} \tag{2.12}$$

which is a basic clopen set in $\mathrm{St}(\mathsf{B})$. Hence $f_\tau$ is continuous.

**$\Psi$ is injective:** Assume $\tau_1 \neq \tau_2$. This implies that there exists $n \in \omega$ such that

$$\llbracket \tau_1(\check{n}) \neq \tau_2(\check{n}) \rrbracket > 0.$$

Let $U \in \mathrm{St}(\mathsf{B})$ be such that $\llbracket \tau_1(\check{n}) \neq \tau_2(\check{n}) \rrbracket \in U$ and assume, for instance, that $\llbracket \tau_1(\check{n}) = \check{1} \rrbracket \in U$ and $\llbracket \tau_2(\check{n}) = \check{0} \rrbracket \in U$. Then,

$$f_{\tau_1}(U)(n) = 1 \quad \textit{and} \quad f_{\tau_2}(U)(n) = 0,$$

meaning $f_{\tau_1} \neq f_{\tau_2}$.

**$\Psi$ is surjective:** First of all, let us consider the following clopen cover of $2^\omega$:

$$\left\{A_{(n,i_n)} := \{s \in 2^\omega : s(n) = i_n\} : n \in \omega, i_n \in 2\right\}.$$

Given $f \in \mathcal{C}(\mathrm{St}(\mathsf{B}, 2^\omega)$, define the following $\mathsf{B}$-name:

$$\tau := \left\{\langle (n,\check{i_n}), f^{-1}[A_{(n,i_n)}] \rangle, n \in \omega, i_n \in 2\right\},$$

well-defined since $f^{-1}[A_{(n,i_n)}]$ is a basic clopen set in $\mathrm{St}(\mathsf{B})$. We want to show that $\Psi(\tau) = f$, i.e. $f_\tau = f$. In order to prove it, we need the following fact.

**Claim 1.** *Let $U$ be a point in* $\mathrm{St}(\mathsf{B})$. *Then* $\llbracket \tau(\check{n}) = \check{i_n} \rrbracket \in U$ *if and only if* $U \in f^{-1}[A_{(n,i_n)}]$.

*Proof.* First of all, notice that

$$\llbracket \tau(\check{n}) = \check{i_n} \rrbracket = \left\llbracket (n,\check{i_n}) \in \tau \right\rrbracket.$$

Then, by definition we have

$$\left\llbracket (n,\check{i_n}) \in \tau \right\rrbracket = \bigvee_{\sigma \in \mathrm{dom}(\tau)} \tau(\sigma) \wedge \left\llbracket (n,\check{i_n}) = \sigma \right\rrbracket.$$

By taking $\sigma_0 := (n,\check{i_n})$, we have that $\tau(\sigma_0) = f^{-1}[N_{(n,i_n)}]$ and $\left\llbracket (n,\check{i_n}) = \sigma_0 \right\rrbracket = 1$. Consequently,

$$\left\llbracket (n,\check{i_n}) \in \tau \right\rrbracket = \bigvee_{\sigma \in \mathrm{dom}(\tau)} \tau(\sigma) \wedge \left\llbracket (n,\check{i_n}) = \sigma \right\rrbracket \geq f^{-1}[A_{(n,i_n)}].$$

In particular, this implies that, if $U \in f^{-1}[A_{(n,i_n)}]$, then $\left[\!\left[ (n, i_n)\check{} \in \tau \right]\!\right] \in U$.
Conversely, with the same proof we obtain that

$$\left[\!\left[ (n, 1 - i_n)\check{} \in \tau \right]\!\right] \geq f^{-1}[A_{(n,1-i_n)}] = \mathrm{St}(\mathsf{B}) \setminus f^{-1}[A_{(n,i_n)}].$$

Being $f^{-1}[A_{(n,i_n)}]$ and $f^{-1}[A_{(n,1-i_n)}]$ disjoint, we conclude that the following inclusions are equalities:

$$f^{-1}[A_{(n,i_n)}] \subseteq N_{[\![(n,i_n)\check{}\in\tau]\!]}, \quad f^{-1}[A_{(n,1-i_n)}] \subseteq N_{[\![(n,1-i_n)\check{}\in\tau]\!]}.$$

□

We have shown that $f_\tau(U)(n) = i_n$ if and only if $[\![\tau(\check{n}) = \check{i_n}]\!] \in U$, which is equivalent to say that

$$f_\tau(U)(n) = i_n \textit{ if and only if } f(U) \in A_{(n,i_n)} = \{s \in 2^\omega : s(n) = i_n\}.$$

**$\Psi$ preserves the interpretation of the equality symbol:** We have to prove that, for every $\tau_1, \tau_2$,

$$[\![\tau_1 = \tau_2]\!] = \mathrm{Reg}\left((f_{\tau_1} \times f_{\tau_2})^{-1}[\{(x, x) : x \in 2^\omega\}]\right).$$

Assume $[\![\tau_1 = \tau_2]\!] \in U$. Then, for all $n \in \omega$, $[\![\tau_1(\check{n}) = \tau_2(\check{n})]\!]$. Therefore $f_{\tau_1}(U) = f_{\tau_2}(U)$ by definition of $f_\tau$. We conclude that $N_{[\![\tau_1=\tau_2]\!]} \subseteq \{U \in \mathrm{St}(\mathsf{B}) : f_{\tau_1}(U) = f_{\tau_2}(U)\}$ which gives also that

$$N_{[\![\tau_1=\tau_2]\!]} \leq \mathrm{Reg}\left(\{U \in \mathrm{St}(\mathsf{B}) : f_{\tau_1}(U) = f_{\tau_2}(U)\}\right).$$

Conversely assume $[\![\tau_1 \neq \tau_2]\!] \in U$. Then, for some $\sigma \in V^{\mathsf{B}}$, $b = [\![\sigma \in \check{\omega}]\!] \wedge [\![\tau_1(\sigma) \neq \tau_2(\sigma)]\!] \in U$. Since $\check{\omega} = \{\langle \check{n}, 1_{\mathsf{B}} \rangle : n \in \omega\}$,

$$[\![\sigma \in \check{\omega}]\!] = \bigvee \{[\![\sigma = \check{n}]\!] \wedge \check{\omega}(\check{n}) : n \in \omega\} = \bigvee \{[\![\sigma = \check{n}]\!] : n \in \omega\}.$$

This yields that if

$$a_n = [\![\sigma = \check{n}]\!] \wedge [\![\tau_1(\sigma) \neq \tau_2(\sigma)]\!] = [\![\sigma = \check{n}]\!] \wedge b,$$

then $N_b = \mathrm{Reg}\,(W)$, where

$$W := \bigcup \{N_{a_n} : n \in \omega\}.$$

Now for any $H \in W$, if $H \in N_{a_n}$, $[\![\tau_1(\check{n}) \neq \tau_2(\check{n})]\!] \in H$ since

$$H \ni a_n = [\![\sigma = \check{n}]\!] \wedge [\![\tau_1(\sigma) \neq \tau_2(\sigma)]\!] \leq [\![\tau_1(\check{n}) \neq \tau_2(\check{n})]\!].$$

This gives that, for any $H \in W$, $f_{\tau_1}(H) \neq f_{\tau_2}(H)$, yielding that

$$U \in N_b = \mathrm{Reg}\,(W) \leq \mathrm{Reg}\left(\{H \in \mathrm{St}(\mathsf{B}) : f_{\tau_1}(H) \neq f_{\tau_2}(H)\}\right) = [\![f_{\tau_1} \neq f_{\tau_2}]\!].$$

The proof is concluded. □

**Corollary 2.4.5.** $(\check{2^\omega})^{\mathsf{B}}$ *and* $Loc(\mathrm{St}(\mathsf{B}), 2^\omega)$ *are isomorphic* $\mathsf{B}$*-valued models for the language* $\{=\}$.

*Proof.* Let $\Psi$ be the map defined by (2.11). We only have to prove that $\Psi$ maps the entire space $(\check{2^\omega})^{\mathsf{B}}$ to $\mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$, i.e.

$$\Psi\left[(\check{2^\omega})^{\mathsf{B}}\right] = \mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega).$$

Let $\tau \in (\check{2^\omega})^{\mathsf{B}}$; by definition, $\bigvee_{x \in 2^\omega} [\![\tau = \check{x}]\!] = 1$. As in the proof of Theorem 2.3.9, fix an enumeration $\{x_i : i \in I\}$ of $2^\omega$ and, following the same steps, find a maximal antichain $A = \{a_i : i \in J\}$ for some $J \subseteq I$ such that, for every $i \in J$,

$$[\![\tau = \check{x_i}]\!] \geq a_i.$$

This implies that $\bigcup_{i \in J} N_{a_i}$ is a dense subset of $\mathrm{St}(\mathsf{B})$, and also that for every $i \in J$, if $U \in N_{a_i}$, then $f_\tau(U) = x_i$, yelding that $f_\tau$ is locally constant on a dense subset of $\mathrm{St}(\mathsf{B})$. $f_\tau$ can be uniquely extended to continuous function on $\mathrm{St}(\mathsf{B})$, hence it uniquely identifies a locally constant continuous function. This shows that $\Psi\left[(\check{2^\omega})^{\mathsf{B}}\right] \subseteq \mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$.
Now pick $f \in \mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$. For every $r \in 2^\omega$, define

$$a_r := \mathrm{Reg}\left(f^{-1}[\{r\}]\right).$$

Then $\bigcup_{r \in 2^\omega} N_{a_r}$ is dense open in $\mathrm{St}(\mathsf{B})$, since $f \in \mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$. This shows that $A = \{a_r : r \in 2^\omega, a_r > 0\}$ is a maximal antichain.
We can now define (using the mixing property) $\tau \in (2^\omega)^{\mathsf{B}}$ requiring that

$$[\![\tau = \check{r}]\!] \geq a_r$$

for any $a_r \in A$. Consequently, we have that, for any $U \in \bigcup_{a_r \in A} N_{a_r}$, $f(U) = r$ if and only if $a_r \in U$ and so

$$[\![\tau = \check{r}]\!] \geq N_{a_r} = \mathrm{Reg}\left(\{U : f(U) = r\}\right).$$

Hence $\{G : \Psi(\tau)(U) = f(U)\} \supseteq \bigcup_{a_r \in A} N_{a_r}$ which is a dense open subset of $\mathrm{St}(\mathsf{B})$. Then $\Psi(\tau)$ and $f$ are two continuous functions which coincide on a dense set, hence are equal. $\square$

# Chapter 3

# Saturation via boolean valued models and good ultrafilters

This chapter presents a method to construct saturated structures of a first order theory by means of boolean valued models. This method is intertwined with forcing; this will become apparent as we proceed presenting the main results.
We start defining in the first section the notion of $\kappa$-good ultrafilter $U$ for an arbitrary boolean algebra $\mathsf{B}$ and proving that the quotient $\mathcal{M}/_U$ of a $\mathsf{B}$-valued model $\mathcal{M}$ with the mixing property by a $\kappa$-good ultrafilter $U$ is a $\kappa$-saturated 2-valued structure.
In the second section we isolate sufficient conditions for a boolean algebra $\mathsf{B}$ granting that $\mathrm{St}(\mathsf{B})$ has (densely many) $\kappa$-good ultrafilters.
Almost all the results of the first and second sections expand on Parente's Master thesis [16] and elaborate on the work of Mansfield [15] and Balcar and Franek's [1]. However, we rephrase the key theorems in a more general setting, yielding results which can be applied to a wider class of examples than those presented in [16] or in [15]. In particular, our effort is to isolate the optimal hypothesis required to perform this method for constructing saturated structures.
The third section investigates a specific example of boolean valued models that comes from the field of non-standard analysis. In 2012, Benci introduced in [2] the notion of space of ultrafunctions for a functional space $V(\Omega)$. Roughly speaking, a space of ultrafunctions $V_\Lambda(\Omega)$ for $V(\Omega)$ is an extension of $V(\Omega)$ in which every net in $V(\Omega)$ has a limit point (even nets which do not converge even in the space of distributions). Our main goal is to show that the construction of a space of ultrafunctions for $V(\Omega)$ presented by Benci and Luperi Baglini turns out to be an ultraproduct of all the finite-dimensional subspaces of $V(\Omega)$. This observation leads us to some considerations about the saturation of a space of ultrafunctions, relating it to the converence of nets with arbitrary values in $V_\Lambda(\Omega)$.

## 3.1 Good ultrafilters and saturated quotients of boolean valued models

**Definition 3.1.1.** Let $\mathsf{B}$ be a boolean algebra.

- A function $f : \mathcal{P}_\omega(X) \to \mathsf{B}$ is:
  - *multiplicative* if $f(S \cup T) = f(S) \wedge f(T)$ for all $S, T \in \mathcal{P}_\omega(X)$;

– *monotonically decreasing* if for every $S, T \in \mathcal{P}_\omega(X)$, $S \subseteq T$ implies[1] $f(T) \leq F(S)$.

- Let $U \subseteq \mathsf{B}$ be an ultrafilter, and $\kappa$ be a cardinal number.
  – $U$ is *$\kappa$-good* if for every monotonically decreasing function $f : \mathcal{P}_\omega(\kappa) \to U$, there exists a multiplicative function $g : \mathcal{P}_\omega(\kappa) \to U$ refining $f$.
  – $U$ is *$|\mathsf{B}|$-good* if it is $\lambda$-good for all $\lambda < \mathfrak{d}(\mathsf{B})$, the density of $\mathsf{B}$.

It is simple to check that for all $\lambda < \kappa$ a $\kappa$-good ultrafilter is also $\lambda$-good.

**Theorem 3.1.2.** *Let $\mathsf{B}$ be a $\kappa$-complete boolean algebra and $\mathcal{M}$ a full $\mathsf{B}$-valued model in the language $L$ satisfying the $\leq \kappa$-mixing property for some cardinal $\kappa$ such that $\aleph_0 + |L| < \kappa$. Assume $U \in \mathrm{St}(\mathsf{B})$ is $\aleph_1$-incomplete and $\kappa$-good. Then $\mathcal{M}/_U$ is $\kappa$-saturated.*

*Proof.* Let $A \subseteq M/_U$ be a subset of size $\lambda < \kappa$ and fix a complete 1-type $p(x)$ over $A$ which is finitely satisfiable in $\mathcal{M}/_U$. It suffices to prove that $p(x)$ is satisfied in $\mathcal{M}/_U$. Our assumptions grant that $|p(x)| = \lambda + |L| = \lambda$. Therefore we can fix an enumeration $p(x) = \{\varphi_\alpha(x) : \alpha < \lambda\}$. Since $p(x)$ is finitely satisfied in $\mathcal{M}/_U$, for every $S \in \mathcal{P}_\omega(\lambda)$,

$$\left[\!\!\left[ \exists x \bigwedge_{\alpha \in S} \phi_\alpha(x) \right]\!\!\right] \in U.$$

Now, by the $\aleph_1$-incompleteness of $U$, there exists $\{a_n : n < \omega\} \subseteq U$ such that $\bigwedge_{n<\omega} a_n = b \notin U$, refining each $a_n$ to $\neg b \wedge \bigwedge_{i \leq n} a_i$, we may further assume that $\bigwedge_{n<\omega} a_n = 0$ and $a_i \geq a_j$ if $i \leq j$. Define the monotonically decreasing map

$$\begin{aligned} f :& \mathcal{P}_\omega(\lambda) \to U \\ & S \mapsto a_{|S|} \wedge \left[\!\!\left[ \exists x \bigwedge_{\alpha \in S} \phi_\alpha(x) \right]\!\!\right]. \end{aligned}$$

By assumption $U$ is $\kappa$-good, hence there exists a multiplicative refinement $g : \mathcal{P}_\omega(\lambda) \to U$ of $f$. Consider the map

$$\begin{aligned} h :& \mathcal{P}_\omega(\lambda) \to B \\ & S \mapsto h(S) := g(S) \wedge \bigwedge \{\neg g(T) : |T| > |S|\}. \end{aligned}$$

We will prove later that $h$ is not the constant map $S \mapsto 0_\mathsf{B}$.
The following observation is crucial in what follows:

**Claim 2.** *For all $S, T \in \mathcal{P}_\omega(\lambda)$, if $g(S) \wedge h(T) > 0_\mathsf{B}$, then $S \subseteq T$.*

*Proof.* Suppose not. Then $|T| < |T \cup S|$; hence (since $g$ is multiplicative)

$$g(S) \wedge h(T) \leq g(S) \wedge g(T) \wedge \neg g(S \cup T) = g(S) \wedge g(T) \wedge \neg(g(S) \wedge g(T)) = 0,$$

against our assumption. □

We get the following:

[1] Notice that being multiplicative implies being monotonically decreasing.

**Claim 3.** $\operatorname{ran}(h) \setminus \{0\}$ *is an antichain.*

*Proof.* Assume that for some $S, T \in \mathcal{P}_\omega(\lambda)$ we have $h(S) \wedge h(T) > 0_{\mathsf{B}}$. We must show that $S = T$. By definition of $h$, we immediately observe that $h(S) \wedge g(T) > 0$ and $g(S) \wedge h(T) > 0$. Now apply the previous claim to conclude that $S = T$. $\square$

By the fullness of $\mathcal{M}$, we can find a subset $\{\sigma_S : S \in \mathcal{P}_\omega(\lambda)\} \subseteq M$ such that

$$\left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \phi_\alpha(x)\right]\!\!\right] = \left[\!\!\left[\bigwedge_{\alpha \in S} \phi_\alpha(\sigma_S)\right]\!\!\right]$$

for every $S \in \mathcal{P}_\omega(\lambda)$. By the mixing property for $\mathcal{M}$, using that $\operatorname{ran}(h)$ is an antichain, we can find $\tau \in M$ such that

$$h(S) \leq [\![\tau = \sigma_S]\!]$$

for every $S \in \mathcal{P}_\omega(\lambda)$. This means that, for every fixed $S \in \mathcal{P}_\omega(\lambda)$ and $\mathcal{P}_\omega(\lambda) \ni T \supseteq S$, we have

$$\left[\!\!\left[\bigwedge_{\alpha \in S} \phi_\alpha(\tau)\right]\!\!\right] \geq \left[\!\!\left[\bigwedge_{\alpha \in T} \phi_\alpha(\tau)\right]\!\!\right] \geq [\![\tau = \sigma_T]\!] \wedge \left[\!\!\left[\bigwedge_{\alpha \in T} \phi_\alpha(\sigma_T)\right]\!\!\right] \geq h(T). \tag{3.1}$$

We conclude that

$$\left[\!\!\left[\bigwedge_{\alpha \in S} \phi_\alpha(\tau)\right]\!\!\right] \geq \bigvee \{h(T) : T \supseteq S\}. \tag{3.2}$$

If we can prove that, for every $S \in \mathcal{P}_\omega(\lambda)$ the second member of (3.2) is in $U$, then also the first one is in $U$; this means that $[\tau]_U$ realizes the type $p(x)$ in $\mathcal{M}/U$, as desired.

**Claim 4.** $\bigvee\{h(T) : \mathcal{P}_\omega(\lambda) \ni T \supseteq S\} \in U$ *for every* $S \in \mathcal{P}_\omega(\lambda)$. *In particular,* $h$ *is not identically* $0_{\mathsf{B}}$.

*Proof.* Fix $S$ and let

$$b := g(S) \wedge \bigwedge\{\neg h(T) : T \supseteq S\}.$$

We notice that

$$\begin{aligned} b \vee \bigvee\{h(T) : T \supseteq S\} &= \Big(g(S) \wedge \bigwedge\{\neg h(T) : T \supseteq S\}\Big) \vee \bigvee\{h(T) : T \supseteq S\} = \\ &= \Big(g(S) \vee \bigvee\{h(T) : T \supseteq S\}\Big) \wedge \Big(\bigwedge\{\neg h(T) : T \supseteq S\} \vee \bigvee\{h(T) : T \supseteq S\}\Big) \geq \\ &\geq g(S) \wedge 1 = g(S) \in U. \end{aligned}$$

Since $U$ is an ultrafilter, either $b \in U$ or $\bigvee\{h(T) : T \supseteq S\} \in U$. Aiming for a contradiction, suppose that $b \in U$. For $n < \omega$, let us define

$$c_n := \bigvee\{g(T) : |T| = n\}.$$

Clearly, since $g$ is monotonically decreasing, for every $n < \omega$ $c_{n+1} \leq c_n$ and $b \leq c_{|S|}$. Since $g(T) \leq a_{|T|}$ for every $T$, we get that $c_n \leq a_n$ for all $n \in \omega$. This gives that

$$b \not\leq \bigwedge_{n<\omega} c_n,$$

since $b \in U$, while $\bigwedge_{n<\omega} c_n \leq \bigwedge_{n<\omega} a_n = 0 \notin U$.
Hence there exists $m < \omega$ such that $b \wedge c_m \wedge \neg c_{m+1} > 0_{\mathsf{B}}$. Notice that $m \geq |S|$, since $b \leq g(S) \leq c_{[S]}$. This gives that for some $R$ of cardinality $m \geq |S|$ (since $b \leq c_{|S|}$ and $c_m = \bigvee\{g(T) : |T| = m\}$)

$$0_{\mathsf{B}} < b \wedge g(R) \wedge \neg c_{m+1}.$$

Now observe that $g(R) \wedge \neg c_{m+1} = h(R)$, since

$$\begin{aligned} h(R) :&= g(R) \wedge \bigwedge\{\neg g(T) : |T| > |R|\} \\ &= g(R) \wedge \neg \bigvee\{g(T) : |T| > |R|\} = g(R) \wedge \neg \bigvee_{n>|R|} c_n, \end{aligned}$$

and, since $|R| = m$ and $c_{n+1} \leq c_n$ for every $n < \omega$, we obtain that $\bigvee_{n>|R|} c_n = c_{|R|+1} = c_{m+1}$.
We conclude that $0_{\mathsf{B}} < b \wedge h(R)$. Since $b \leq g(S)$ we can apply again Claim 2, to get that $R \supseteq S$. Therefore

$$0_{\mathsf{B}} < b \wedge h(R) = g(S) \wedge \bigwedge\{\neg h(T) : T \supseteq S\} \wedge h(R) \leq \neg h(R) \wedge h(R) = 0_{\mathsf{B}},$$

a contradiction. □

This completes the proof. □

## 3.2 Constructing good ultrafilters

The purpose of this section is to isolate minimal conditions on a boolean algebra $\mathsf{B}$ in order to guarantee the existence of $\kappa$-good ultrafilters on $\mathsf{B}$.

**Lemma 3.2.1.** *Let $\mathsf{B}$ be a $< \kappa$-disjointable, $< \kappa$-complete boolean algebra, and $D \subset \mathsf{B}$ a prefilter of cardinality less than $\kappa$.*
*Assume $f : \mathcal{P}_\omega(\lambda) \to D$ to be a monotonically decreasing function for some $\lambda < \kappa$. Then there exists a prefilter $D' \supseteq D$ of cardinality less than $\kappa$ and a multiplicative function $g : \mathcal{P}_\omega(\lambda) \to D'$ refining $f$.*

*Proof.* Without loss of generality (by further extending $D$ if necessary without increasing its size), we may assume that $D$ is closed under finite conjunctions (i.e. if $d_1, \ldots, d_n \in D$ also $d_1 \wedge \cdots \wedge d_n \in D$). Now let

$$\begin{aligned} l :&\mathcal{P}_\omega(\lambda) \times D \to \mathsf{B}^+ && (3.3) \\ &(S, d) \mapsto f(S) \wedge d. && (3.4) \end{aligned}$$

Since $\mathsf{B}$ is $< \kappa$-disjointable, there exists a disjoint $h : \mathcal{P}_\omega(\lambda) \times D \to \mathsf{B}^+$ refining $l$ i.e. with the property that $h(S, d) \leq f(S) \wedge d$ for every $(S, d) \in \mathcal{P}_\omega(\lambda) \times D$, $\operatorname{ran}(h)$ is an antichain, and $h$ is injective. Let

$$\begin{aligned} g :&\mathcal{P}_\omega(\lambda) \to \mathsf{B}^+ \\ &S \mapsto \bigvee\{h(T, d) : T \supseteq S, d \in D\}. \end{aligned}$$

**Claim 5.** *$g$ is multiplicative, refines $f$, and $\operatorname{ran}(g) \cup D$ is a prefilter.*

*Proof.* First of all $g$ refines $f$: For every $S \in \mathcal{P}_\omega(\lambda)$, $T \supseteq S$, and $d \in D$,

$$h(T, d) \leq f(T) \wedge d \leq f(T) \leq f(S).$$

Therefore

$$g(S) = \bigvee\{h(T, d) : T \supseteq S, d \in D\} \leq f(S).$$

Also, $g(S) \geq h(S, d) > 0_{\mathsf{B}}$ is positive for all $S \in \mathcal{P}_\omega(\lambda)$.
We now prove that $g$ is multiplicative: since $\operatorname{ran}(h)$ is an antichain, if $S_1, S_2 \in \mathcal{P}_\omega(\lambda)$,

$$\begin{aligned} g(S_1) \wedge g(S_2) &= \bigvee\{h(T_1, d_1) : T_1 \supseteq S_1, d_1 \in D\} \wedge \bigvee\{h(T_2, d_2) : T_2 \supseteq S_2, d_2 \in D\} \\ &= \bigvee\{h(T_1, d_1) \wedge h(T_2, d_2) : T_1 \supseteq S_1, T_2 \supseteq S_2, d_1 \in D, d_2 \in D\}. \end{aligned}$$

Since $\operatorname{ran}(h)$ is an antichain and $h$ is injective, $h(T_1, d_1) \wedge h(T_2, d_2) > 0_{\mathsf{B}}$ if and only if $T_1 = T_2$ and $d_1 = d_2$; therefore $h(T_1, d_1) \wedge h(T_2, d_2) > 0_{\mathsf{B}}$ with $T_1 \supseteq S_1$ and $T_2 \supseteq S_2$ if and only if $d_1 = d_2$ and $T_1 = T_2 \supseteq S_1 \cup S_2$. We conclude that

$$g(S_1) \wedge g(S_2) = \bigvee\{h(T, d) : T \supseteq S_1 \cup S_2, d \in D\} = g(S_1 \cup S_2).$$

Finally we show that $\operatorname{ran}(g) \cup D$ is a prefilter: Fix $d_1, \ldots, d_n \in D$ and $S_1, \ldots, S_m \in \mathcal{P}_\omega(\lambda)$. Then

$$g(S_1) \wedge \cdots \wedge g(S_m) \wedge d_1 \cdots \wedge d_m = g(S) \wedge d \geq h(S, d) > 0_{\mathsf{B}},$$

where $S = S_1 \cup \cdots \cup S_m$ and $d = d_1 \cdots \wedge d_m$ (we are crucially using that $g$ is multiplicative and $D$ is closed under finite conjunctions). $\square$

The prefilter

$$D' := \operatorname{ran}(g) \cup D$$

satisfies the conclusion of the Lemma. The proof of the Lemma is completed. $\square$

We can now prove the following.

**Theorem 3.2.2.** *Assume $\kappa$ is a regular cardinal such that $|\kappa^{<\kappa}| = \kappa$. Let $\mathsf{B}$ be a $< \kappa$-disjointable $< \kappa$-complete boolean algebra of cardinality $\kappa$.*
*Then every filter $H$ on $\mathsf{B}$ of size less than $\kappa$ can be extended to a $\kappa$-good ultrafilter $U \supseteq H$.*

*Proof.* First of all, we fix an enumeration $\{b_\alpha : \alpha < \kappa\}$ of $\mathsf{B}$. Since $|\kappa^{<\kappa}| = \kappa$, we can also fix an enumeration $\{f_\alpha : \alpha < \kappa\}$ of all the partial monotonically decreasing functions $\mathcal{P}_\omega(\lambda) \to \mathsf{B}^+$, where $\lambda$ is any ordinal less then $\kappa$.
By induction, we want to obtain a sequence $\{D_\alpha : \alpha \leq \kappa\}$ of prefilters on $\mathsf{B}^+$ each of them closed under finite conjuctions and also satisfying the following properties:

- $D_0 = H$;
- $|D_\alpha| < \kappa$;
- For all $\alpha < \kappa$ there exists a multiplicative function $g : \mathcal{P}_\omega(\lambda) \to D_\alpha$ refining $f_\alpha$;
- For all $\alpha < \kappa$ either $b_\alpha \in D_{\alpha+1}$ or $\neg b_\alpha \in D_{\alpha+1}$.

We proceed by induction on $\alpha > 0$ according to the following rules:

- If $\alpha > 0$ is limit, we let $D_\alpha = \bigcup_{\alpha>\beta} D_\beta$.
- If $\alpha = \beta + 2n$ with $\beta$ limit and $n < \omega$, we let $\xi$ be the least ordinal such that neither $b_\xi$ nor $\neg b_\xi$ belong to $D_\alpha$ and we let $D_{\alpha+1}$ be some prefilter of size $|D_\alpha| + \aleph_0$, closed under finite conjunctions, and containing $D_\alpha$ and exactly one among $b_\xi$ or $\neg b_\xi$.
- If $\alpha = \beta + 2n + 1$ with $\beta$ limit and $n < \omega$, we let $\xi$ be the least ordinal such that $f_\xi : \mathcal{P}_\omega(\lambda) \to D_\alpha$ is a partial monotonically decreasing function with the property that no multiplicative $g$ with range contained in $D_\alpha$ refines $f_\xi$. Then we let $D_{\alpha+1}$ be a prefilter of size $|D_\alpha| + \lambda$, closed under finite conjunctions, containing $D_\alpha$, with the property that some multiplicative $g : \mathcal{P}_\omega(\lambda) \to D_{\alpha+1}$ refines $f_\xi$ (by the previous Lemma $D_{\alpha+1}$ can be defined).

Now, we consider
$$U := \bigcup_{\alpha<\kappa} D_\alpha.$$
$U$ is an ultrafilter by construction (we included at most one among $b_\xi$ and $\neg b_\xi$ at each even non-limit stage of the construction). Also $U$ is $\kappa$-good: Assume $\xi$ is the least ordinal such that $f_\xi : \mathcal{P}_\omega(\lambda) \to U$ is monotonically decreasing for some $\lambda < \kappa$, but no $g : \mathcal{P}_\omega(\lambda) \to U$ refines $f_\xi$. For each $\eta < \xi$, let $g_\eta : \mathcal{P}_\omega(\lambda) \to U$ be a multiplicative refinement of $f_\eta$. By regularity of $\kappa$ there exists a least limit $\beta$ such that $\mathrm{ran}(g_\eta) \subseteq D_\beta$ for all $\eta < \xi$. Then at stage $\beta + 1$, $f_\xi$ must be chosen to define $D_{\beta+1}$, which gives that some multiplicative $g : \mathcal{P}_\omega(\lambda) \to D_{\beta+1} \subseteq U$ refines $f_\xi$, a contradiction. $\square$

By Remark 1.1.15 the following result (appearing in [1]) summarizes the optimal conditions to have good ultrafilters:

**Corollary 3.2.3.** *Let $\kappa$ be a regular cardinal number such that $|\kappa^{<\kappa}| = \kappa$ and assume that $\mathsf{B}$ is a $< \kappa$-complete boolean algebra of size $\kappa$ with the property that for each $b \in \mathsf{B}^+$ and $\alpha < \kappa$ there is an antichain $\{c_\xi : \xi < \alpha\}$ with $\bigvee_{\xi<\alpha} c_\xi \leq b$. Then every filter $H$ on $\mathsf{B}$ of size less than $\kappa$ can be extended to a $\kappa$-good ultrafilter $U \supseteq H$.*

The following is also a straightforward consequence of the above theorem in combination with Theorem 2.2.6:

**Corollary 3.2.4.** *Let $\mathcal{M}$ be any first order structure. Let $\kappa$ be a regular cardinal such that $|\kappa^{<\kappa}| = \kappa$ and let $\mathsf{B}$ be a $< \kappa$-disjointable $< \kappa$-complete boolean algebra of cardinality $\kappa$. Then $\mathcal{M}^{\downarrow\mathsf{B}}/_U$ is a saturated elementary extension of $\mathcal{M}$ for densely many $U \in \mathrm{St}(\mathsf{B})$.*

We now address the degree of goodness that an ultrafilter in a powerset can have.

**Theorem 3.2.5.** *Let $X$ be a set of cardinality $\eta$, where we assume that both $\eta$ and $2^\eta$ are regular. Let $E \subseteq \mathcal{P}(X)$ be any subset of size less than $2^\eta$ satisfying the finite intersection property, and such that each element in $E$ has size $\eta$. Then there exists a $\eta^+$-good ultrafilter on $\mathcal{P}(X)$ extending the set $E$.*
*In particular, taking $E = \emptyset$, there exist $\eta^+$-good ultrafilters in $\mathcal{P}(X)$.*

*Proof.* Let us consider the set
$$F := \{Y \subseteq X : |X \setminus Y| < \eta\}.$$

It is immediate to see that $F \cup E$ is a prefilter in $\mathcal{P}(X)$ that satisfies the finite intersection property. Indeed, $F$ is a filter; moreover $|Y \cap Z| = \eta$ for any $Y \in F$ and $Z \in E$: on the one hand $|(X \setminus Y) \cap Z| \leq |X \setminus Y| < \eta$, on the other hand, since $|Z| = \eta$, it must be that $|Y \cap Z| = \eta$.
We want to show that $F \cup E$ can be extended to a $\eta^+$-good ultrafilter. We can do this repeating exactly the same argument used in the proof of Theorem 3.2.2 (since $\mathcal{P}(X)$ is a complete boolean algebra). We only have to pay attention to the indexing. We start showing that $\mathcal{P}(X)$ satisfies Lemma 3.2.1.

**Claim 6.** *Let $D \supset F$ be a prefilter and let $f : \mathcal{P}_\omega(\gamma) \to D$ be a partial monotonically decreasing function for some cardinal $\gamma < \eta^+$. Then $f$ can be disjoint.*

*Proof.* Let $\{x_\alpha : \alpha < \gamma\}$ enumerate the domain of $f$. Let $c_\alpha = f(x_\alpha)$. Define $a_\alpha := c_\alpha \setminus \bigcup_{\beta<\alpha} c_\beta = c_\alpha \setminus \bigcup_{\beta<\alpha} a_\beta$. Let $I$ be the set of $\alpha < \eta$ such that $a_\alpha$ has size $\eta$. Then $\{a_\alpha : \alpha \in I\}$ is an antichain of size $\delta$ for some $\delta \leq \gamma$. The regularity of $\eta$ allows us to split each $a_\alpha$ with $\alpha \in I$ in $\eta$ sets of size $\eta$: $\{a_\alpha^\beta : \beta < \eta\}$.
Finally:

- if $a_\alpha = c_\alpha \setminus \bigcup_{\beta<\alpha} c_\beta = c_\alpha \setminus \bigcup_{\beta<\alpha} a_\beta$ has size $\eta$, define $b_\alpha := a_\alpha^\alpha$;
- otherwise $\alpha \notin I$, hence there must be $\beta(\alpha)$ least $\beta < \alpha$ such that $|c_\alpha \cap a_\beta| = \eta$; in this case define $b_\alpha := a_{\beta(\alpha)}^\alpha$.

The map $x_\alpha \mapsto b_\alpha$ is disjoint. □

To use Theorem 3.2.2 for our set-up, we only need to prove that the set of all the partial monotonically decreasing functions $\mathcal{P}_\omega(\gamma) \to \mathcal{P}(X) \setminus \{0\}$ for $\gamma \leq \eta$ can be enumerated in type of order $2^\eta$ i.e. the same size of $\mathcal{P}(X)$. If this is the case the same inductive construction given in the proof of 3.2.2 can be carried over.
It suffices to check that

$$(2^\eta)^{<\eta^+} = 2^\eta.$$

By a simple computation:

$$\begin{aligned}(2^\eta)^{<\eta^+} &= \bigcup \{(2^\eta)^\alpha : \alpha < \eta^+\} = \eta^+ \cdot (2^\eta)^\eta \\ &= (2^\eta)^\eta = 2^{\eta \cdot \eta} = 2^\eta.\end{aligned}$$

□

The following result will be used later.

**Proposition 3.2.6.** *Let $\kappa$ be such that $\kappa^{<\kappa} = \kappa$ and $\mathsf{B}_1$ and $\mathsf{B}_2$ be boolean algebras, with $\mathsf{B}_2$ of cardinality $\kappa > |\mathsf{B}_1|$. Suppose that $B_2$ is $< \kappa$-disjointable and $< \kappa$-complete. Assume $m : \mathsf{B}_1 \to \mathsf{B}_2$ is an injective complete morphism of boolean algebras.*
*Then every ultrafilter $G$ on $\mathsf{B}_1$ can be extended to a $\kappa$-good ultrafilter $U$ on $\mathsf{B}_2$ such that $G = m^{-1}[U]$.*

*Proof.* Apply Theorem 3.2.2 to the prefilter $D_0 := m[G]$. □

### 3.2.1 The Lévy collapse

In this section we construct explicitly an example of boolean algebra that admits good ultrafilters. Let $\kappa$ be an inaccessible cardinal. Let $\mathrm{Coll}(\omega, <\kappa)$ consist of all functions $p : \kappa \times \omega \to \kappa$ with finite domain and such that $p(\alpha, n) \in \alpha$ for all $(\alpha, n) \in \mathrm{dom}(p)$. Order $\mathrm{Coll}(\omega, <\kappa)$ by reverse inclusion. We define

$$\begin{aligned} \pi_\lambda : \mathrm{Coll}(\omega, <\kappa) &\to \mathrm{Coll}(\omega, \lambda) = \{q : \omega \to \lambda : \ q \text{ is finite}\} \\ p &\mapsto \{(n, \alpha) : \ p(\lambda, n) = \alpha\} \end{aligned}$$

where $\mathrm{Coll}(\omega, \lambda)$ is also ordered by reverse inclusion.

**Lemma 3.2.7.** *Let $C \subset \mathrm{Coll}(\omega, <\kappa)$ be a filter, i.e. such that:*

1. *if $p, q \in C$, then $p$ and $q$ are compatible;*
2. *if $p \in C$, $q \in \mathrm{Coll}(\omega, <\kappa)$ and $p \leq q$, then $q \in \mathrm{Coll}(\omega, <\kappa)$.*

*Then for every infinite cardinal $\lambda < \kappa$, there is a maximal antichain $A \subset \mathrm{Coll}(\omega, <\kappa)$ such that $|A| = \lambda$ and $A \cap C = \emptyset$.*

*Proof.* Given such a filter $C$ and $\lambda$, let $C_\lambda := \pi_\lambda[C]$. $C_\lambda$ is also a filter on $\mathrm{Coll}(\omega, \lambda)$, since it still satisfies 1 and 2 (now for $\mathrm{Coll}(\omega, \lambda)$). Notice that $\mathrm{Coll}(\omega, \lambda) \setminus C_\lambda$ is a dense subset of $\mathrm{Coll}(\omega, \lambda)$: if $p \in P_\lambda$, find $q, r \leq p$ with $r$ and $q$ incompatible (for example $q = p \cup \{(n, 0)\}$, $r = p \cup \{(n, 1)\}$ for some $n \notin \mathrm{dom}(p)$), then at least one of them cannot stay in $C_\lambda$ since $C_\lambda$ is a filter.
Now given $q \in \mathrm{Coll}(\omega, \lambda) \setminus C_\lambda$ and $n \notin \mathrm{dom}(q)$ let $A = \{q \cup \{(n, \alpha)\} : \alpha \in \lambda\}$ This is an antichain of cardinality $\lambda$ in $\mathrm{Coll}(\omega, \lambda)$.
Notice that $A^* = \{\{(\langle \alpha, j\rangle, \beta) : q_\xi(j) = \beta\} : \xi < \lambda\} \subseteq \mathrm{Coll}(\omega, <\kappa)$ is disjoint from $C$ since $\pi_\lambda[A^*] = A$ and $q \in C \cap A^*$ entails that $\pi_\lambda(q) \in A \cap C_\lambda$, which is not possible. Finally notice that $\pi_\lambda \restriction A^*$ is injective. $\square$

By Corollary 1.1.21, we know that there exist an unique complete boolean algebra $\mathsf{B}_\kappa$ (called the *Lévy collapse*) and a map $e : \mathrm{Coll}(\omega, <\kappa) \to \mathsf{B}_\kappa$ such that:

1. if $p \leq q$, then $e(p) \leq e(q)$;
2. $p$ and $q$ are incompatible in $P$ if and only if $e(p) \wedge e(q) = 0$;
3. $e[P]$ is dense in $\mathsf{B}_\kappa^+$.

**Proposition 3.2.8.** *Let $F$ be a filter in $\mathsf{B}_\kappa$. Then, for every $\lambda < \kappa$ there exists a maximal antichain $A$ of $\mathsf{B}_\kappa$ of cardinality $\lambda$ such that $A \cap F = \emptyset$.*
*In particular, every ultrafilter $U \subset \mathsf{B}_\kappa$ is $\aleph_1$-incomplete.*

*Proof.* If we consider $C := e^{-1}[F] \subseteq P$, then $C$ satisfies the hypothesis of Lemma 3.2.7 and so there exists an antichain $W \subset P$ of cardinality $\lambda$ such that $W \cap e^{-1}[F] = \emptyset$. Now we only have to define $A := e[W]$ to obtain the required antichain. $\square$

**Theorem 3.2.9.** *$\mathsf{B}_\kappa$ satisfies the $<\kappa$-chain condition, and so $|\mathsf{B}_\kappa| = \kappa$.*

*Proof.* We will prove that $\mathrm{Coll}(\omega, < \kappa)$ satisfies the $< \kappa$-chain condition. The thesis will follow by density of $e[P]$ in $\mathsf{B}_\kappa$.
For all $p \in \mathrm{Coll}(\omega, < \kappa)$ let

$$\mathrm{supp}(p) = \{\alpha < \kappa : \exists n(\alpha, n) \in \mathrm{dom}(p)\}.$$

Now let $W \subseteq \mathrm{Coll}(\omega, < \kappa)$ be an antichain. We construct two increasing sequences $\langle A_n : n < \omega\rangle$ and $\langle W_n : n < \omega\rangle$ such that $A_n \subseteq A_{n+1} \subseteq \kappa$ and $W_n \subseteq W_{n+1} \subseteq W$. We start from $A_0 := \emptyset =: W_0$ and, supposing we have defined $A_n$ and $W_n$, we construct $A_{n+1}$ and $W_{n+1}$ in the following way: For every $p \in \mathrm{Coll}(\omega, < \kappa)$ with $\mathrm{supp}(p) \subseteq A_n$, we choose $q_p \in W$ such that $q_p \restriction A_n \times \omega = p$ every time that $A_n \supseteq \mathrm{supp}(p)$. Then we set

$$W_{n+1} := W_n \cup \{q_p : p \in P, \mathrm{supp}(p) \subseteq A_n\}$$

and

$$A_{n+1} := \bigcup\{\mathrm{supp}(q) : q \in W_{n+1}\}.$$

We define also $A := \bigcup_{n<\omega} A_n$, and we want to prove that $W = \bigcup_{n<\omega} W_n$. Let $q \in W$. By finiteness of $\mathrm{supp}(q)$, we can find $n \in \omega$ such that $\mathrm{supp}(q) \cap A = \mathrm{supp}(q) \cap A_n$. Then, by construction, let $q' \in W_{n+1}$ be such that

$$q' \restriction A_n \times \omega = q \restriction A_n \times \omega.$$

Since $\mathrm{supp}(q') \subseteq A$, we have that

$$\mathrm{supp}(q) \cap \mathrm{supp}(q') = \mathrm{supp}(q) \cap \mathrm{supp}(q') \cap A = \mathrm{supp}(q) \cap \mathrm{supp}(q') \cap A_n \subseteq A_n.$$

This means that $q$ and $q'$ are compatible, since

$$q' \restriction \mathrm{supp}(q) \cap \mathrm{supp}(q') \times \omega = q \restriction \mathrm{supp}(q) \cap \mathrm{supp}(q') \times \omega.$$

Since $W$ is an antichain $q = q' \in W_{n+1}$.

It remains to prove that each $W_n$ has cardinality less then $\kappa$. Then, by regularity of $\kappa$, we conclude that also $W = \bigcup_{n\in\omega} W_n$ has size less then $\kappa$.
We prove it by induction on $n \in \omega$: For $n = 0$ the thesis is trivial. Assume that $|W_n| < \kappa$. Since we have that

$$|A_n| = |\bigcup\{\mathrm{supp}(q) : q \in W_n\}| \leq \aleph_0 \dot{|} W_n| < \kappa,$$

we obtain that $|\{p \in P : \mathrm{supp}(p) \subseteq A_n\}| < \kappa$ and so $|W_{n+1}| < \kappa$.
Finally, by Proposition 1.1.22, we conclude that $|\mathsf{B}_\kappa| \leq |\mathrm{Coll}(\omega, < \kappa)^{<\kappa}| = |\kappa^{<\kappa}| = \kappa$, since $|\mathrm{Coll}(\omega, < \kappa)| = \kappa$ and $\mathrm{Coll}(\omega, < \kappa)$ is completely embedded as a dense subset of $\mathsf{B}\kappa$. □

We can now easily prove the following:

**Proposition 3.2.10.** *There exists a $\kappa$-good ultrafilter on $\mathsf{B}_\kappa$.*

*Proof.* We want to show that $\mathsf{B}_\kappa$ is $< \kappa$-disjointable. Since it is complete, by Remark 1.1.15, we only have to prove that for every $b \in \mathrm{Coll}(\aleph_0, < \kappa)$ and for every $\alpha < \kappa$ there exists an antichain $\{c_\xi : \xi < \alpha\}$ such that $\bigvee_{\xi<\alpha} c_\xi \leq b$. Since $\mathrm{Coll}(\omega, < \kappa)$ can be identified with a dense subset of $\mathsf{B}_\kappa$, let $p \in P$ be such that $p \leq b$. Then $\mathrm{supp}(p)$ is finite. Let $\gamma < \kappa$ be such that $\mathrm{supp}(p) \subset \gamma$.

Since $\kappa$ is a cardinal, we can always find an ordinal $\delta < \kappa$ such that $\alpha < \delta$ and $\gamma < \delta$. Finally, define, for every $\xi < \alpha$,

$$p_\xi = p \cup \{(\langle \delta, 0\rangle, \xi)\}$$

It is clear that $\{p_\xi : \xi < \alpha\}$ is an antichain below $p$ and so below $q$.
We have obtained that $\mathsf{B}_\kappa$ is a $< \kappa$-disjointable $< \kappa$-complete boolean algebra of size $\kappa$. By Theorem 3.2.2, there exists a $\kappa$-good ultrafilter in $\mathsf{B}_\kappa$. $\square$

**Corollary 3.2.11.** *Let $\mathcal{M}$ be an $L$-structure and $\kappa$ be an inaccessible cardinal such that $|L|^+ + |M| \leq \kappa$. Then there exists a boolean algebra $\mathsf{B}$ and an ultrafilter $U \subset \mathsf{B}$ such that $\mathcal{M}^{\downarrow \mathsf{B}}/_U$ is a saturated structure of cardinality less or equal than $\kappa$.*

*Proof.* Let $\mathsf{B} := \mathsf{B}_\kappa$, and $U$ a $\kappa$-good ultrafilter on $\mathsf{B}$. Then $U$ is $\aleph_1$-incomplete by Proposition 3.2.8, and so, by Theorem 3.1.2, $\mathcal{M}^{\downarrow \mathsf{B}}/_U$ is $\kappa$-saturated.
Finally, we have to consider the cardinality of $\mathcal{M}^{\downarrow \mathsf{B}}$:

$$|M^{\downarrow \mathsf{B}}| = |\{\sigma : A \to M : A \subseteq \mathsf{B} \textit{ is a maximal antichain }\}| \leq |\bigcup \{M^A : A \subseteq B, |A| < \kappa\}|,$$

using Theorem 3.2.9. Now, since $\kappa$ is inaccessible, if $|A| < \kappa$, then $|M^A| \leq \kappa$ and so

$$|M^{\downarrow \mathsf{B}}| \leq \kappa \cdot \kappa^{<\kappa} = \kappa \cdot \kappa = \kappa.$$

$\square$

## 3.3 Spaces of ultrafunctions

In non-standard analysis, ultraproducts are of common use. An example can be found in [10], where the costruction of a non-archimedean field extension of $\mathbb{R}$ via an ultrapower of the real field is presented. Here we use ultraproducts as quotients of boolean valued models, and we investigate their degree of saturation. We rephrase, using ultraproducts, the notion of space of ultrafunctions introducted by Benci in [2] and developed in several other works with Luperi Baglini ([3] and [4], for instance).
For a general introduction to non-standard analysis, we refer to [12].

### 3.3.1 Construction of $\Lambda$-limits

Let $\Omega$ be an open subset of $\mathbb{R}^n$ and let $V(\Omega)$ be some functional space such that

$$\mathcal{D}(\Omega) \subseteq V(\Omega) \subseteq \mathcal{C}(\bar{\Omega}) \cap L^1(\Omega) \cap L^2(\Omega),$$

where:

- $\mathcal{D}(\Omega)$ is the space of infinitely differentiable functions $\Omega \to \mathbb{R}$ having compact support;
- $\mathcal{C}(\bar{\Omega})$ is the space of real-valued continuous functions defined on the closure of $\Omega$;
- $L^1(\Omega)$ is the space of functions $\Omega \to \mathbb{R}$ whose absolute value is Lebesgue integrable;
- $L^2(\Omega)$ is the space of functions $\Omega \to \mathbb{R}$ whose square is Lebesgue integrable.

We choose this kind of functional spaces to follow closely the construction of Benci and Luperi Baglini in [4].
We want to extend this space of real-valued functions to a new space of functions, denoted by $V_\Lambda(\Omega)$. More precisely, we are interested in finding a space in which we can embed $V(\Omega)$, large enough so that certain types of nets - taking values in $V(\Omega)$ - converge.
In the second part of this section we will discuss which saturation/completeness properties such an extension can have. From here on, we will denote with $\Lambda$ the set $\mathcal{P}_\omega(\Omega)$ of finite subsets of $\Omega$ ordered with the natural upward directed partial order structure on this set given by $\subseteq$.

**Definition 3.3.1** (Benci - Luperi Baglini)**.** A real vector space $W$ extending $V(\Omega)$ is called a *space of ultrafunctions* for $V(\Omega)$ if for every net $\mathcal{N} : \Lambda \to V(\Omega)$ there exists a unique $f \in W$ called the $\Lambda$-*limit* for $\mathcal{N}$ and denoted with $\lim_{\lambda\uparrow\Lambda} \mathcal{N}(\lambda)$, satisfying the following conditions:

1. if $\mathcal{N}$ is eventually constant, i.e. if there exist $\lambda_0 \in \Lambda$ and $g \in V(\Omega)$ such that for every $\lambda > \lambda_0$ $\mathcal{N}(\lambda) = g$, then
$$\lim_{\lambda\uparrow\Lambda} \mathcal{N}(\lambda) = g;$$
2. if $\mathcal{M} : \Lambda \to V(\Omega)$ is another net and if $a, b \in \mathbb{R}$, then
$$a \lim_{\lambda\uparrow\Lambda} \mathcal{N}(\lambda) + b \lim_{\lambda\uparrow\Lambda} \mathcal{M}(\lambda) = \lim_{\lambda\uparrow\Lambda} [a\mathcal{N}(\lambda) + b\mathcal{M}(\lambda)].$$

Let us now build the space of ultrafunctions.
Let us fix a Hamel base $\{e_{\alpha\in\Omega}\}$ of $V(\Omega)$ with the labels given by the points of $\Omega$, since we can suppose it has the continuum cardinality.
Now, for every $\lambda \in \Lambda$, we define
$$V_\lambda(\Omega) := \text{span}(\{e_\alpha\}_{\alpha\in\lambda}).$$
Since, for every $\lambda \in \Lambda$, $V_\lambda(\Omega)$ is a subspace of $V(\Omega)$, we can consider the canonical projection $\pi_\lambda : V(\Omega) \to V_\lambda(\Omega)$.
We define also
$$\bar{V}(\Omega) := \prod_{\lambda\in\Lambda} V_\lambda(\Omega).$$
A suitable quotient of $\bar{V}(\Omega$ will give the space of ultrafunctions. To reach this extent, let us first describe the element $f$ in $\bar{V}(\Omega)$ that in the quotient will be the $\Lambda$-limit for a fixed net $\mathcal{N}$.
Let $\mathcal{N} : \Lambda \to V(\Omega)$ be a net, and write $g_i := \mathcal{N}(i)$ for every $i \in \Lambda$. Since we can embed $V(\Omega)$ in $\bar{V}(\Omega)$ with the map $\iota : h \mapsto \langle \pi_\lambda(h) \rangle_\lambda$, we can associate to $\mathcal{N}$ the net
$$\begin{aligned} \mathcal{N} :&\Lambda \to \bar{V}(\Omega). \\ &i \mapsto \langle \pi_\lambda(g_i) \rangle_\lambda \end{aligned}$$
We define $f = (f_\lambda)_\lambda \in \bar{V}(\Omega)$ as
$$f_\lambda := \pi_\lambda(g_\lambda).$$
Clearly, $\bar{V}(\Omega)$ is a real vector with the usual operations, and satisfies condition 2 of Definition 3.3.1. However, in general property 1 is not satisfied. Indeed, let $\mathcal{N}$ be an eventually constant net

and let $\lambda_0$ be such that $\mathcal{N}(\lambda) = g$ for every $\lambda > \lambda_0$. If we assume that $\mathcal{N}(\lambda_0) = f_{\lambda_0} \neq g$ and that $\pi_{\lambda_0}(f_{\lambda_0}) \neq \pi_{\lambda_0}(g)$, in our setting we obtain

$$\big(\lim_{\lambda \uparrow \Lambda} \mathcal{N}\big)_{\lambda_0} = \pi_{\lambda_0}(f_{\lambda_0}) \neq \pi_{\lambda_0}(g) = (\iota(g))_{\lambda_0}.$$

For this reason we will quotient $\bar{V}(\Omega)$.
We want to quotient it with an ultrafilter on $\mathcal{P}(\Lambda)$, in order to preserve good properties of $V(\Omega)$. Let us consider, for each $\lambda \in \Lambda$,

$$X_\lambda := \{\mu \in \Lambda : \lambda \subseteq \mu\}$$

and define

$$E := \{X_\lambda : \lambda \in \Lambda\}. \tag{3.5}$$

**Definition 3.3.2.** Let $X$ be an infinite set. We say that an ultrafilter $U$ on $\mathcal{P}(X)$ is *regular* if there exists a subset $F \subseteq U$ such that

1. $|F| = |X|$;
2. each $x \in X$ is contained only in finitely many elements of $F$.

**Lemma 3.3.3.** *The family $E$ can be extended to a regular ultrafilter on $\Lambda$. Moreover, each ultrafilter on $\Lambda$ extending the family $E$ is regular.*

*Proof.* First of all, we notice that the family $E$ has the finite intersection property. To prove it, let us fix $\lambda, \eta \in \Lambda$. Then,

$$\begin{aligned} X_\lambda \cap X_\eta &= \{\mu \in \Lambda : \lambda \subseteq \mu \textit{ and } \eta \subseteq \mu\} \\ &= \{\mu \in \Lambda : \lambda \cup \eta \subseteq \mu\} \\ &= X_{\lambda \cup \eta}. \end{aligned}$$

Applying Zorn's Lemma, we can then extend this family to an ultrafilter $U$, which is regular, by setting $F \subseteq U$ as the family $E$. Then, we notice that $\mu \in X_\lambda$ if and only if $\lambda \subseteq \mu$. Thus, since each $\mu \in \Lambda$ is finite, there are only finitely many $\lambda \in \Lambda$ such that $\mu \in X_\lambda$, as required. Finally, it is trivial to see that $E$ has the same cardinality of $\Lambda$ because the map $\lambda \mapsto X_\lambda$ is injective. □

**Theorem 3.3.4.** *There exists an ultrafilter $U$ on $\Lambda$ such that $V_\Lambda(\Omega) := \bar{V}(\Omega)/_U$ is a space of ultrafunctions for $V(\Omega)$.*

*Proof.* Since linear conditions are preserved by the quotient, we only have to check condition 1 of Definition 3.3.1. We have to prove that there exists an ultrafilter $U$ such that, for every eventually constant net $\mathcal{N}$,

$$\{\lambda \in \Lambda : \pi_\lambda(\mathcal{N}(\lambda)) = \pi_\lambda(g)\} \in U, \tag{3.6}$$

where $g$ is the limit point of $\mathcal{N}$. Now, let $U$ be an ultrafilter extending the family $\{X_\lambda : \lambda \in \Lambda\}$ (such an ultrafilter exists because of Lemma 3.3.3). Let $\mathcal{N}$ be a net in $V(\Omega)$ eventually constant and let $\lambda_0 \in \Lambda$ and $g \in V(\Omega)$ be such that $\mathcal{N}(\lambda) = g$ for every $\lambda \supseteq \lambda_0$. Then condition (3.6) is satisfied, since

$$\{\lambda \in \Lambda : \pi_\lambda(\mathcal{N}(\lambda) = \pi_\lambda(g)\} \supseteq X_{\lambda_0}$$

and each $X_\lambda$ is in $U$ for every $\lambda \in \Lambda$. □

### 3.3.2 Saturating a space of ultrafunctions

We want to consider $V(\Omega)$ as a first order structure for a language $L$. From here on, let us fix a language $L$ whose symbols have an interpretation in $V(\Omega)$. We require also that each linear subspace of $\bar{V}(\Omega)$ is an $L$-substructure of $V(\Omega)$. Without loss of generality, we can consider any language only with relational symbols: if $f$ is an $n$-ary functional symbol, we just have to define a new $n+1$-ary relational symbol $R_f$ letting $R_f(x_1,\ldots,x_n,y)$ if and only if $f(x_1,\ldots,x_n)=y$. Also the presence of constant symbols is not relevant. We can assume $L=\{R_i\}_{i\in I}$.
We want to consider in this setting the construction of the boolean ultrapower. To this extent, the first order structure $\mathcal{M}$ will be the space $V(\Omega)$ and the complete boolean algebra will be the boolean completion $\mathsf{B}_\kappa$ of the Lévy collapse $\mathrm{Coll}(\omega,<\kappa)$. We will see that not only the space $V(\Omega)$ can be embedded in $V(\Omega)^{\downarrow\mathsf{B}_\kappa)}/_U$, but also the space of ultrafunctions $V_\Lambda(\Omega)$ as an $L$-structure can be thought as an $L$-substructure of $V(\Omega)^{\downarrow\mathsf{B}_\kappa}/_U$.
We have already seen that $V(\Omega)^{\downarrow\mathsf{B}_\kappa}$ is a $\mathsf{B}_\kappa$-valued model. We naturally endow $\bar{V}(\Omega)=\prod_{\lambda\in\Lambda}V_\lambda(\Omega)$ of the structure of a $\mathsf{C}$-valued model for $L$, where $\mathsf{C}:=\mathcal{P}(\Lambda)$. Indeed, if $f_1,\ldots,f_n\in\bar{V}(\Omega)$ and $R\in L$ is a relational symbol, the standard definition for a product of first order structures is

$$[\![R(f_1,\ldots,f_n)]\!]_{\mathcal{P}(\Lambda)}:=\{\lambda\in\Lambda : V_\lambda(\Omega)\vDash R^{V_\lambda(\Omega)}(f_1(\lambda),\ldots,f_n(\lambda))\}.$$

Moreover, Lemma 2.1.9 ensure the fact that the $\mathsf{C}$-valued model $\bar{V}(\Omega)$ satisfies the mixing property. Another observation we can do is the following: each $V_\lambda(\Omega)$ is a substructure of $V(\Omega)$, since $R^{V(\Omega)}\subseteq V(\Omega)^k$ for some $k$, $R^{V_\lambda(\Omega)}$ is simply $V_\lambda(\Omega)^k\cap R^{V(\Omega)}$. For this reason, we have that

$$V_\lambda(\Omega)\vDash R^{V_\lambda(\Omega)}(f_1(\lambda),\ldots,f_n(\lambda))\quad \textit{if and only if}\quad V(\Omega)\vDash R^{V(\Omega)}(f_1(\lambda),\ldots,f_n(\lambda)).$$

Thus, we can write:

$$[\![R(f_1,\ldots,f_n)]\!]_{\mathcal{P}(\Lambda)}=\{\lambda\in\Lambda : V(\Omega)\vDash R(f_1(\lambda),\ldots,f_n(\lambda))\}.$$

We want now to embed $\bar{V}(\Omega)$ in $V(\Omega)^{\downarrow\mathsf{B}_\kappa}$. First of all, we define a morphism of complete boolean algebras $m:\mathsf{C}\to\mathsf{B}_\kappa$. Remember that $|\Lambda|=|\mathcal{P}_\omega(\Omega)|=|\Omega|=2^{\aleph_0}$. We note that the set $\{\{\lambda\}:\lambda\in\Lambda\}$ is a maximal antichain in $\mathsf{C}$ of length $2^{\aleph_0}$. If we take an antichain $A$ in $\mathsf{B}_\kappa$ of cardinality $2^{\aleph_0}<\kappa$, we can define $m$ as a bijection between these two antichains and then we can extend it in an unique way to a complete, injective morphism of complete boolean algebras $m:\mathsf{C}\to\mathsf{B}_\kappa$.
Define

$$\begin{aligned}\Phi :&\bar{V}(\Omega)\to V(\Omega)^{\downarrow\mathsf{B}_\kappa}\\ &[f:\Lambda\to\bigcup_{\lambda\in\Lambda}V_\lambda(\Omega)]\mapsto\Phi(f),\end{aligned}$$

where

$$\begin{aligned}\Phi(f) :&A\to V(\Omega)\\ &m(\{\lambda\})\mapsto f(\lambda).\end{aligned}\tag{3.7}$$

$\Phi$ is well defined since $m:\Lambda\to A$ is a bijection.
Clearly, $\Phi$ is an injective map.

We can now check that the pair $(\Phi, i)$ is a morphism of boolean valued models by fixing a relational symbol $R$ and $f_1, \dots, f_n \in \bar{V}(\Omega)$. Then

$$\begin{aligned} m(\llbracket R(f_1, \dots, f_n) \rrbracket_{\mathcal{P}(\Omega)}) &= m(\bigcup_{\lambda \in \Lambda} \{\{\lambda\} : V(\Omega) \vDash R(f_1(\lambda), \dots, f_n(\lambda))\} = \\ &= \bigvee_{\lambda \in \Lambda} \{m(\{\lambda\}) : V(\Omega) \vDash R(f_1(\lambda), \dots, f_n(\lambda))\} = \\ &= \bigvee \{a \in A : V(\Omega) \vDash R(\Phi(f_1)(a), \dots, \Phi(f_n)(a))\} = \\ &= \llbracket R(\Phi(f_1), \dots, \Phi(f_n)) \rrbracket_{\mathsf{B}_\kappa}, \end{aligned} \tag{3.8}$$

using that $i$ is a complete morphism of complete boolean algebras. We conclude, applying Proposition 2.1.12, that the following result holds true:

**Proposition 3.3.5.** *The pair $(\Phi, m)$ with $\Phi$ defined by* (3.7) *is an embedding of boolean valued models.*
*Moreover fix $U \in \mathrm{St}(\mathsf{B}_\kappa)$ such that $X_\lambda \in m^{-1}[U] = G$ for all $\lambda \in \Lambda$. Then the space of ultrafunction $V_\Lambda(\Omega)$ induced by $G$ embeds in the 2-valued structure $V(\Omega)^{\downarrow \mathsf{B}_\kappa}/_U$ via the quotient map associated to the pair $(\Phi, i)$.*

**Proposition 3.3.6.** *The canonical embedding $i : V(\Omega) \to V_\Lambda(\Omega)$ is $\Sigma_1$-elementary.*

*Proof.* We can consider the embedding $\iota : V(\Omega) \to \bar{V}(\Omega)$ we already defined, so that $i = p \circ \iota$, where $p$ is the projection to the quotient.
Assume that $V_\Lambda(\Omega) \vDash \exists x \varphi(x, i(f_1), \dots, i(f_n))$. We must show that there exists $g \in V(\Omega)$ such that $V_\Lambda(\Omega) \vDash \varphi(i(g), i(f_1), \dots, i(f_n))$. Our hypotesis is:

$$T := \{\lambda \in \Lambda : V_\lambda(\Omega) \vDash \exists x \varphi(x, \pi_\lambda(f_1), \dots, \pi_\lambda(f_n))\} \in G.$$

Let $\mu \in \Lambda$ be such that $f_1, \dots, f_n \in V_\mu(\Omega)$. Since $X_\mu \in G$ and $G$ is an ultrafilter, then $X_\mu \cap T \neq \emptyset$. So we can find $\eta \supseteq \mu$ such that $V_\eta(\Omega) \vDash \exists x \varphi(x, f_1, \dots, f_n)$. Let $g \in V_\eta(\Omega) \subseteq V(\Omega)$ be such that

$$V_\eta(\Omega) \vDash \varphi(g, f_1, \dots, f_n).$$

Then, since $\varphi$ is a quantifier-free formula, for every $\lambda \in X_\eta$,

$$V_\lambda(\Omega) \vDash \varphi(g, f_1, \dots, f_n) = \varphi(\pi_\lambda(g), \pi_\lambda(f_1), \dots, \pi_\lambda(f_n)).$$

This allows us to conclude that

$$\{\lambda \in \Lambda : V_\lambda(\Omega) \vDash \varphi(\pi_\lambda(g), \phi_\lambda(f_1), \dots, \pi_\lambda(f_n))\} \supseteq X_\eta \in G,$$

as we claimed. $\square$

Which degree of saturation can we obtain on a space of ultrafunctions $V_\Lambda(\Omega)$? Observing that $\bar{V}(\Omega)$ is a $\mathcal{P}(\Lambda)$-valued model, we get:

**Theorem 3.3.7.** *Suppose that $2^{\aleph_0}$ and $2^{2^{\aleph_0}}$ are regular. Then there exists an ultrafilter $G$ on $\mathcal{P}(\Lambda)$ such that $V_\Lambda(\Omega) := \bar{V}(\Omega)/_G$ is a $(2^{\aleph_0})^+$-saturated space of ultrafunctions.*
*Moreover, if $2^{2^{\aleph_0}} = (2^{\aleph_0})^+$ holds, $V_\Lambda(\Omega)$ is a saturated space of ultrafunctions.*

*Proof.* By Theorem 3.2.5 there exists a $(2^{\aleph_0})^+$-good ultrafilter $G$ on $\mathcal{P}(\Lambda)$ extending the set $E$ defined by (3.5). In particular, $V_\Lambda(\Omega) := \bar{V}(\Omega)/_G$ is a space of ultrafunctions. To conclude that it is saturated, we use Theorem 3.1.2. By Lemma 2.1.9, $\bar{V}(\Omega)$ satisfies the mixing property. Moreover, the $(2^{\aleph_0})^+$-good ultrafilter is $\aleph_1$ incomplete since, by Lemma 3.3.3, it is regular. Then, by Theorem 3.1.2, $V_\Lambda(\Omega)$ is $(2^{\aleph_0})^+$-saturated. $\square$

Our results yield that for any ultrafilter $G$ in $\mathcal{P}(\Lambda)$ we can find a $\kappa$-good ultrafilter on $\mathsf{B}_\kappa$ whose preimage via the inclusion $i$ is exactly $G$. We can summarize it in the following

**Fact 3.3.8.** *Using the notation of Theorem 3.3.5, assume that $2^{\aleph_0}$ and $2^{2^{\aleph_0}}$ are regular. Then there exists a $\kappa$-good ultrafilter $U$ on $\mathsf{B}_\kappa$ such that $G := m^{-1}[U]$ is a $(2^{\aleph_0})^+$-good ultrafilter on $\mathcal{P}(\Lambda)$ and the map*

$$\Phi/_U : V_\Lambda(\Omega) := \bar{V}(\Omega)/_G \to V(\Omega)^{\downarrow \mathsf{B}_\kappa}/_U$$

*induced by $\Phi$ is an embedding of a $(2^{\aleph_0})^+$-saturated space of ultrafunctions in a saturated structure of inaccessible cardinality.*

# Chapter 4

# Sheaves and boolean valued models

Sheaves are at the very heart of category theory and algebraic geometry. Our goal in this last chapter is to use them to characterize the boolean valued models satisfying the mixing property. A standard reference for our approach to sheaves is [14].
Roughly speaking, a sheaf structure on a topological space $X$ allows to patch together objects defined locally (one for each open subset of $X$) which overlap coherently in their common domain, yielding a global object defined on the whole of $X$. In the field of boolean valued models, the mixing property plays the same role. Our main result is the formalization of this connection. It is not transparent, though, whether in the language of sheaves one can characterize the fullness property for boolean valued models.
In the final part of the chapter we reexamine the principal boolean valued models introduced troughout the dissertation. Since they all satisfy the mixing property, we can associate to each of them an appropriate sheaf, which also reflects the boolean $L$-structure of the model. In certain cases this is possible only if we impose certain restrictions on the boolean interpretation of the relation symbols in $L$. A byproduct of our results brings that some saturated extensions of $2^\omega$ (the ones obtained using the methods of Chapter 3) can be represented as the stalks of a sheaf of continuous functions.

**Definition 4.0.1.** A *category* $\mathcal{C}$ consists of:

1. a class $C = \mathrm{Ob}_{\mathcal{C}}$ whose elements are called *objects*;
2. a class $\mathrm{Arw}_{\mathcal{C}}$ whose elements are called *arrows* or *morphisms*;
3. a function $\mathrm{dom}_{\mathcal{C}} : \mathrm{Arw}_{\mathcal{C}} \to C$ assigning to each arrow its *domain*;
4. a function $\mathrm{cod}_{\mathcal{C}} : \mathrm{Arw}_{\mathcal{C}} \to C$ assigning to each arrow its *codomain*;
5. a function $\mathrm{Id}_{\mathcal{C}} : C \to \mathrm{Arw}_{\mathcal{C}}$ attaching to each object $c$ its *identity arrow* $\mathrm{Id}_c$;
6. a function $\circ_{\mathcal{C}} : E \to \mathrm{Arw}_{\mathcal{C}}$ where $E = \big\{(f,g) \in \mathrm{Arw}_{\mathcal{C}}^2 : \mathrm{cod}_{\mathcal{C}}(g) = \mathrm{dom}_{\mathcal{C}}(f)\big\}$.

We require that:

- $\mathrm{dom}_{\mathcal{C}}(\mathrm{Id}_c) = \mathrm{cod}_{\mathcal{C}}(\mathrm{Id}_c) = c$ for every $c \in \mathcal{C}$;
- if $\mathrm{dom}_{\mathcal{C}}(f) = c = \mathrm{cod}_{\mathcal{C}}(g)$ then $f \circ_{\mathcal{C}} \mathrm{Id}_c = f$ and $\mathrm{Id}_c \circ_{\mathcal{C}} g = g$;

- $\mathrm{dom}_{\mathcal{C}}(g \circ_{\mathcal{C}} f) = \mathrm{dom}_{\mathcal{C}}(f)$ and $\mathrm{cod}_{\mathcal{C}}(g \circ_{\mathcal{C}} f) = \mathrm{cod}_{\mathcal{C}}(g)$:
- $\circ_{\mathcal{C}}$ is associative.

Given a category $\mathcal{C}$ and $x, y \in \mathrm{Ob}_{\mathcal{C}}$, the collection of the arrows from $x$ to $y$ is denoted by $\mathrm{Hom}_{\mathcal{C}}(x, y)$. Moreover, the *opposite category* of $\mathcal{C}$ is the category $\mathcal{C}^{\mathrm{op}}$ such that $\mathrm{Ob}_{\mathcal{C}^{\mathrm{op}}} := \mathrm{Ob}_{\mathcal{C}}$ and, if $x, y \in \mathrm{Ob}_{\mathcal{C}^{\mathrm{op}}}$, then $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) := \mathrm{Hom}_{\mathcal{C}}(y, x)$.
An arrow $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ is an *isomorphism* if there exists an arrow $g \in \mathrm{Hom}_{\mathcal{C}}(y, x)$ such that $f \circ_{\mathcal{C}} g = \mathrm{Id}_y$ and $g \circ_{\mathcal{C}} f = \mathrm{Id}_x$. In this case, it is easily proved that $g$ is unique, and it is called the *inverse* of $f$.

For instance, if $(X, \tau)$ is a topological space, we can define the category $\mathcal{O}(X)$ of all the open sets of $X$ where

- the class of objects is $\tau$;
- for every $U, V \in \tau$, we say that there is an arrow from $U$ to $V$ if and only if $U \subseteq V$.

**Definition 4.0.2.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, a *(covariant) functor* $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ is a function $\mathcal{F} : \mathrm{Ob}_{\mathcal{C}} \cup \mathrm{Arw}_{\mathcal{C}} \to \mathrm{Ob}_{\mathcal{D}} \cup \mathrm{Arw}_{\mathcal{D}}$ such that

- $\mathcal{F}(x) \in \mathrm{Ob}_{\mathcal{D}}$ for every $x \in \mathrm{Ob}_{\mathcal{C}}$;
- $\mathcal{F}[\mathrm{Hom}_{\mathcal{C}}(x, y)] \subseteq \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(x), \mathcal{F}(y))$ for every $x, y \in \mathrm{Ob}_{\mathcal{C}}$;
- $\mathcal{F}(\mathrm{Id}_x) = \mathrm{Id}_{\mathcal{F}(x)}$ for every $x \in \mathrm{Ob}_{\mathcal{C}}$;
- $\mathcal{F}(f \circ_{\mathcal{C}} g) = \mathcal{F}(f) \circ_{\mathcal{D}} \mathcal{F}(g)$ for every composable arrows $f, g \in \mathrm{Arw}_{\mathcal{C}}$.

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from $\mathcal{C}$ to $\mathcal{D}^{\mathrm{op}}$.

Let us now consider $L$ to be a signature and define the category $\mathcal{C}_L^{Bool}$ in the following way:

- objects are pairs $(\mathcal{M}, \mathsf{B})$, where $\mathsf{B}$ is a complete boolean algebra and $\mathcal{M}$ is a $\mathsf{B}$-valued model for $L$;
- if $(\mathcal{M}, \mathsf{B})$ and $(\mathcal{N}, \mathsf{C})$ are objects, a morphism between them is a pair $(\Phi, i)$, where $i : \mathsf{B} \to \mathsf{C}$ is a complete morphism of complete boolean algebras and $\Phi : \mathcal{M} \to \mathcal{N}$ is an $i$-morphism;
- the composition of morphisms is the composition of morphisms of boolean valued models.

**Definition 4.0.3.** Let $(X, \tau)$ be a topological space. A *presheaf* of boolean valued models for $L$ on $X$ is a contravariant functor $\mathcal{F}$ from $\mathcal{O}(X)$ to $\mathcal{C}_L^{Bool}$.
Let $U$ be an open set and let $\{U_i : i \in I\}$ be an open covering of $U$. A presheaf $\mathcal{F}$ is called a *sheaf* if it satisfies the following conditions:

1. if $f, g \in \mathcal{F}(U)$ are such that
$$\mathcal{F}(U_i \subseteq U)(f) = \mathcal{F}(U_i \subseteq U)(g) \quad \textit{for every } i \in I,$$
then $f = g$;

2. if, for each $i \in I$, there exists $f_i \in \mathcal{F}(U_i)$ such that, for $i \neq j$,

$$\mathcal{F}(U_i \cap U_j \subseteq U_i)(f_i) = \mathcal{F}(U_i \cap U_j \subseteq U_j)(f_j),$$

then there exists $f \in \mathcal{F}(U)$ such that

$$\mathcal{F}(U_i \subseteq U)(f) = f_i \quad \textit{for every } i \in I.$$

**Definition 4.0.4.** Let $\mathcal{F}, \mathcal{G} : \mathcal{O}(X) \to \mathcal{C}_L^{Bool}$ be two sheaves. A *morphism of sheaves* from $\mathcal{F}$ to $\mathcal{G}$ is a family $\{\varphi_U : U \in \mathcal{O}(X)\}$ of morphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that, for every $V \subseteq U$ is $\mathcal{O}(X)$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathcal{F}(V\subseteq U)"'] & \mathcal{G}(U) \arrow[d, "\mathcal{G}(V\subseteq U)"] \\
\mathcal{F}(V) \arrow[r, "\varphi_V"'] & \mathcal{G}(V)
\end{tikzcd}$$

It can be checked that, with this definition of morphisms, the family of $\mathcal{C}_L^{Bool}$-valued sheaves on $X$ is a category. In particular, an *isomorphism of sheaves* is an isomorphism in this category.

**Definition 4.0.5.** Let $\mathcal{F} : \mathcal{O}(X) \to \mathcal{C}_L^{Bool}$ be a sheaf and let $x \in X$. The *stalk* of $\mathcal{F}$ at $x$ is

$$\mathcal{F}_x := \Big( \bigsqcup_{x \in U \in \mathcal{O}(X)} \mathcal{F}(U) \Big) /_{\sim},$$

where $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ are equivalent ($f \sim g$) if there exists $W \in \mathcal{O}(X)$ such that $x \in W \subseteq U \cap V$ and such that $\mathcal{F}(W \subseteq U)(f) = \mathcal{F}(W \subseteq V)(g)$.

## 4.1 A characterization of the mixing property using sheaves

Now we want to associate a sheaf structure to any boolean valued model satisfying the mixing property. To this extent, let $\mathsf{B}$ be a complete boolean algebra and let $\mathcal{M}$ be a $\mathsf{B}$-valued model. The topological space on which we will construct our sheaf is $\mathrm{St}(\mathsf{B})$.
First of all, we have to define a contravariant functor $\mathcal{F}$ from $\mathcal{O}(\mathrm{St}(\mathsf{B}))$ to $\mathcal{C}_L^{Bool}$. To do so, we define $\mathcal{F}(N_b)$ for every basic open set $N_b$, in order to later extend $\mathcal{F}$ on every open set in a consistent way. For every $b \in B$, let $F_b$ be the filter generated by $b$. For every $b \in B$, define

$$\mathcal{F}(N_b) := \mathcal{M}/_{F_b},$$

that is a $\mathsf{B}/_{F_b}$-valued model. Since we have assumed that $\mathsf{B}$ is complete, we have defined $\mathcal{F}(U)$ for every regular open set. For an arbitrary $U \in \mathcal{O}(\mathrm{St}(\mathsf{B}))$, we set

$$\mathcal{F}(U) := \mathcal{F}(\mathrm{Reg}\,(U)).$$

Finally, we have to say what the images of morphisms are. Now, our morphisms in $\mathcal{O}(\mathrm{St}(\mathsf{B}))$ are the inclusions $U \subseteq V$. However, we can restrict ourselves to consider only the image of inclusions of basic open sets. Let us now assume that $N_b \subseteq N_c$, so that $b \leq c$. In particular let us take into account the case $b < c$, since the equality represents a trivial case. Being $\mathcal{F}$ contravariant, we need to have a morphism $\mathcal{M}/_{F_c} \to \mathcal{M}/_{F_b}$. Now, notice that, if we quotient a boolean algebra $\mathsf{C}$ by a filter $F$ and we call $p$ the projection to the quotient, the filters of $\mathsf{C}/_F$ are exactly the sets $p[G]$, for $G$ a filter of $\mathsf{C}$ extending $F$. In our setting, if $p : \mathsf{B} \to \mathsf{C} := \mathsf{B}/_{F_c}$ is the projection to the quotient, $p[F_b]$ is a non-trivial filter since $b < c$. Moreover, if $p' : \mathsf{B} \to \mathsf{B}/_{F_b}$ and $q : \mathsf{C} \to \mathsf{C}/_{p[F_b]}$ are the two projections, we have that $p' = q \circ p$. In conclusion, we can define

$$\begin{aligned} \mathcal{F}(N_b \subseteq N_c) : \mathcal{M}/_{F_c} &\to \mathcal{M}/_{F_b}, \\ [x]_{F_c} &\mapsto [x]_{F_b} \end{aligned}$$

which is well-defined by our previous observations. Let us stress the fact that until now we have built nothing else than a presheaf.

**Theorem 4.1.1.** *In this setting, $\mathcal{M}$ satisfies the mixing property if and only if the presheaf $\mathcal{F} : \mathcal{O}(\mathrm{St}(\mathsf{B})) \to \mathcal{C}_L^{Bool}$ defined above is a sheaf.*

*Proof.* Assume that $\mathcal{M}$ satisfies the mixing property. Let $U$ be an open set and let $\{U_i : i \in I\}$ be an open cover of $U$ and $\{f_i : i \in I\}$. Without loss of generality, we can assume each $U_i$ to be a regular open set and so let us write $U_i = N_{b_i}$. In particular, let us assume that $U = \mathrm{Reg}\left(\bigcup_{i\in I} U_i\right) = N_b$, where $b = \bigvee_{i\in I} b_i$. From now on, fix a well-order $\leq$ on $I$.
First of all, let $f, g \in F(U) = \mathcal{M}/_{F_b}$ be such that $\mathcal{F}(U_i \subseteq U)(f) = \mathcal{F}(U_i \subseteq U)(g)$ for every $i \in I$. Let $x, y \in \mathcal{M}$ be such that $[x]_{F_b} = f$ and $[y]_{F_b} = g$. Then our hypothesis implies that $[\![x = y]\!] \geq b_i$ for every $i \in I$. Since $b = \bigvee_{i\in I} b_i$, it is clear that $[\![x = y]\!] \geq b$, and thus $f = g$.
Now, let $f_i \in F(U_i) = \mathcal{M}/_{F_{b_i}}$ for every $i \in I$ and suppose that, if $i \neq j$, then $\mathcal{F}(U_i \cap U_j \subseteq U_i)(f_i) = \mathcal{F}(U_i \cap U_j \subseteq U_j)(f_j)$. In particular, we can assume that, for every $i \in I$, $b_i \wedge \neg \bigvee_{j<i} b_j \neq 0$, otherwise we may omit $b_i$. Let us choose, for every $i \in I$, an element $x_i \in \mathcal{M}$ such that $[x_i]_{f_{b_i}} = f_i$. We can always refine the family $\{b_i : i \in I\}$ to an antichain $A$: consider

$$a_{\min I} := b_{\min I}$$

and, for $i > \min I$,

$$a_i := b_i \wedge \neg \bigvee_{j<i} b_j.$$

Then $A := \{a_i : i \in I\}$ is an antichain in $\mathsf{B}$ and, for every $i \in I$, $a_i \leq b_i$. Let

$$g_i := \mathcal{F}(N_{a_i} \subseteq N_{b_i})(f_i).$$

In particular, $g_i = [x_i]_{F_{a_i}}$. Since $\mathcal{M}$ satisfies the mixing property, there exists $y \in \mathcal{M}$ such that

$$[\![y = x_i]\!] \geq a_i \quad \text{for every } i \in I.$$

By induction on the well order of $I$, $[\![y = x_i]\!] \geq b_i$. Indeed, $[\![y = x_{\min I}]\!] \geq a_{\min I} = b_{\min I}$ and, if we assume that $[\![y = x_j]\!] \geq b_j$ for all $j < i$,

$$\begin{aligned} [\![y = x_i]\!] &\geq a_i \vee \bigvee_{j<i} \left([\![x_i = x_j]\!] \wedge [\![y = x_j]\!]\right) \geq a_i \vee \bigvee_{j<i} ((b_i \wedge b_j) \wedge b_j) \\ &= \big(b_i \wedge \neg \bigvee_{j<i} b_j\big) \vee \bigvee_{j<i} (b_j \wedge b_i) = \big(b_i \wedge \neg \bigvee_{j<i} b_j\big) \vee \big(b_i \wedge \bigvee_{j<i} b_j\big) \\ &= b_i \wedge \big(\bigvee_{j<i} b_j \vee \neg \bigvee_{j<i} b_j\big) = b_i. \end{aligned}$$

This means that $\mathcal{F}(U_i \subseteq U)([y]_{f_b}) = f_i$ for every $i \in I$.
Conversely, suppose $\mathcal{F}$ is a sheaf. Let $A$ be an antichain in $\mathsf{B}$ and let $x_a \in \mathcal{M}$ for every $a \in A$. In particular, if $a \neq a'$, since $A$ is an antichain, $N_a \cap N_{a'} = \emptyset = N_0$, and so it is clear that $\mathcal{F}(N_a \cap N_{a'} \subseteq N_a)([x_a]_{F_a}) = \mathcal{F}(N_a \cap N_{a'} \subseteq N_{a'})([x_{a'}]_{F_{a'}})$. Let $c := \bigvee A$. Being $F$ a sheaf, there exists $y \in \mathcal{M}$ such that $\mathcal{F}(N_a \subseteq N_c)([y]_{F_c}) = [x_a]_{F_a}$. This is equivalent to say that $[\![y = x_a]\!] \geq a$ for every $a \in A$. Hence $\mathcal{M}$ satisfies the mixing property. $\square$

Finally, we want to describe the stalk of $\mathcal{F}$ at $U \in \mathrm{St}(\mathsf{B})$. Translating the definition of stalk in our setting, we have that

$$\mathcal{F}_U := \Big(\bigsqcup_{b \in U} \mathcal{M}/_{F_b}\Big)/_{\sim},$$

where

$$[x]_{F_b} \sim [y]_{F_c} \quad \textit{if and only if} \quad [x]_{F_{b\wedge c}} = [y]_{F_{b\wedge c}}.$$

This means that, for each $x \in \mathcal{M}$,

$$\begin{aligned}[x]_{\sim} &= \{y \in \mathcal{M} : [x]_{F_b} = [y]_{F_b} \textit{ for some } b \in U\} = \{y \in \mathcal{M} : [\![x = y]\!] \geq b \textit{ for some } b \in U\} \\ &= \{y \in \mathcal{M} : [\![x = y]\!] \in U\} = [x]_U.\end{aligned}$$

We have obtained that the stalk $\mathcal{F}_U$ of $\mathcal{F}$ at $U$ is exactly the first order structure $\mathcal{M}/_U$.

## 4.2 Some examples

We now consider the sets $M^{\mathsf{B}}$, $\check{M}^{\mathsf{B}}$, $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$ and $\mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$ and we discuss which $\mathsf{B}$-valued structures can be defined on them. We will take advantage of the isomorphisms defined in Theorem 2.3.9, Theorem 2.4.4 and Corollary 2.4.5, showing that they are isomorphisms even with respect to to certain $\mathsf{B}$-valued structures on these objects.
Observe that, if we translate these isomorphisms in the setting of Section 4.1, we are essentially describing three isomorphisms of sheaves.
We now fix a complete boolean algebra $\mathsf{B}$.
Given $L = \{R_i :\in I\}$ a relational language and recall that, whenever $\mathcal{M} = \langle M, R : i^{\mathcal{M}} : i \in I\rangle$ is an $L$-structure, then $\mathcal{M}^{\downarrow\mathsf{B}}$ is a $\mathsf{B}$-valued model for $L$. Since we have shown in Theorem 2.3.9 that $\mathcal{M}^{\downarrow\mathsf{B}} \cong \check{M}^{\mathsf{B}}$ for the language $\{=\}$, we can endow $\check{M}^{\mathsf{B}}$ of the structure of $\mathsf{B}$-valued model for the language $L$. To this extent, fix an enumeration $\{x_\xi : \xi \in \gamma\}$ of $M$. Given $R$ an $n$-ary symbol of relation in $L$, fix $\tau_1, \ldots, \tau_n \in \check{M}^{\mathsf{B}}$. Set for $\xi_1, \ldots, \xi_n \in \gamma$

$$a^{\tau_1,\ldots,\tau_n}_{\xi_1,\ldots,\xi_n} := \bigwedge_{j=1}^{n} [\![\tau_j = \check{x_{\xi_j}}]\!] .$$

Define

$$[\![R(\tau_1, \ldots, \tau_n)]\!]^{\check{M}^{\mathsf{B}}}_{\mathsf{B}} := \bigvee \Big\{a^{\tau_1,\ldots,\tau_n}_{\xi_1,\ldots,\xi_n} : \xi_1, \ldots, \xi_n \in \gamma,\, \mathcal{M} \vDash R(x_{\xi_1}, \ldots, x_{\xi_n})\Big\}. \tag{4.1}$$

If $\langle \check{M}^{\mathsf{B}}, R_i^{\check{M}^{\mathsf{B}}} : i \in I\rangle$ is a $\mathsf{B}$-valued model, it satisfies the mixing property by Proposition 2.3.6, since fulfilling the mixing property does not depend on the language. In particular $\langle \check{M}^{\mathsf{B}}, R_i^{\check{M}^{\mathsf{B}}} : i \in I\rangle$ is a full $\mathsf{B}$-valued model (if it is a $\mathsf{B}$-valued model).

**Corollary 4.2.1.** *Let $L = \{R_i : i \in I\}$ be any relational language, and $\mathcal{M} = \langle M, R^{\mathcal{M}} : i \in I\rangle$ an $L$-structure.*
*Then $\langle \check{M}^{\mathsf{B}}, R_i^{\check{M}^{\mathsf{B}}} : i \in I\rangle$ and $\langle \mathcal{M}^{\downarrow\mathsf{B}}, R_i^{\mathcal{M}^{\downarrow\mathsf{B}}} : i \in I\rangle$ are isomorphic* $\mathsf{B}$*-valued models for $L$.*

*Proof.* We have simply to show that the map $\theta$ defined in (2.9) preserves the interpretation of symbols in $L$.
To simplify notation we assume the family $\{R_i : i \in I\}$ is closed under complements, i.e. if $R$ is a binary relational symbols, so it is $\neg R$, with the natural interpretation, we just deal with case of a binary relation symbol.
Now assume $R$ is a binary relational symbol and let $f, g \in \mathcal{M}^{\downarrow\mathsf{B}}$. Fix a refinement $W$ of $\mathrm{dom}(f), \mathrm{dom}(g)$: by (2.3) we recall that

$$w \leq \left[\!\!\left[\tau_f = (f \downarrow \check{W})(w)\right]\!\!\right]^{\check{M}^{\mathsf{B}}}$$

holds for every $w \in W$. Therefore

$$\begin{aligned}
[\![R(f,g)]\!]^{\mathcal{M}^{\downarrow\mathsf{B}}} :&= \bigvee \left\{w \in W : \mathcal{M} \vDash R^{\mathcal{M}}((f \downarrow W)(w), (g \downarrow W)(w))\right\} \\
&\leq \bigvee \left\{ \left[\!\!\left[\tau_f = (f \downarrow \check{W})(w)\right]\!\!\right] \wedge \left[\!\!\left[\tau_g = (g \downarrow \check{W})(w)\right]\!\!\right] : \right. \\
&\qquad\qquad\qquad \left. \mathcal{M} \vDash R^{\mathcal{M}}((f \downarrow W)(w), (g \downarrow W)(w)) \right\} \\
&\leq \bigvee_{t,s \in T} \left\{[\![\tau_f = \check{x_t}]\!] \wedge [\![\tau_g = \check{x_s}]\!] : \mathcal{M} \vDash R^{\mathcal{M}}(x_t, x_s)\right\} = [\![\tau_f, \tau_g]\!]^{\check{M}^{\mathsf{B}}}.
\end{aligned}$$

Being $\neg R$ a relational symbol itself, we obtain as well

$$\neg [\![R(f,g)]\!]^{\mathcal{M}^{\downarrow\mathsf{B}}} = [\![\neg R(f,g)]\!]^{\mathcal{M}^{\downarrow\mathsf{B}}} \leq [\![\neg R(\tau_f, \tau_g)]\!]^{\check{M}^{\mathsf{B}}} = \neg [\![R(\tau_f, \tau_g)]\!]^{\check{M}^{\mathsf{B}}}.$$

We conclude that

$$[\![R(f,g)]\!]^{\mathcal{M}^{\downarrow\mathsf{B}}} = [\![R(\tau_f, \tau_g)]\!]^{\check{M}^{\mathsf{B}}}.$$

□

**Corollary 4.2.2.** *Let $U$ be an ultrafilter on* $\mathsf{B}$*. Let $j : M \to \mathcal{M}^{\downarrow\mathsf{B}}/_U$ and $i : M \to \check{M}^{\mathsf{B}}/_U$ be the two canonical embeddings. Then they are elementary with respect to the language $L$ and the map*

$$\begin{aligned}
\pi : M^{\downarrow\mathsf{B}}/_U &\to \check{M}^{\mathsf{B}}/_U \\
[f]_U &\mapsto [\tau_f]_U
\end{aligned}$$

*is an $L$-isomorphism. Moreover, $\pi$ is such that $i(x) = \pi(j(x))$ for every $x \in M$.*

*Proof.* The elementarity of $j$ comes from Theorem 2.2.6. Moreover, $\pi$ is the map $\theta$ passed to the quotient and since $\theta$ is an isomorphism, so is $\pi$. The fact that $i = \pi \circ j$ comes from Corollary 2.3.10 and, due to the elementarity of $j$ and the fact that $\pi$ is an isomorphism, implies that also $i$ is elementary. □

### 4.2.1 The case of the Cantor space

Let us now focus on the two sets introduced in Section 2.4.
Let $L = \{R_i : i \in I\}$ be a relational language. Assume $2^\omega$ is an $L$-structure in which each $R_i$ is interpreted by an appropriate relation of the correct arity in $2^\omega$. Let $R$ be any $n$-ary relational symbol in $L$.
The interpretation of the relational symbols in $\mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)$ is the following:

$$[\![R(f_1, \dots, f_n)]\!]^{\mathrm{Loc}(\mathrm{St}(\mathsf{B}),2^\omega)} := \mathrm{Reg}\left(\{G \in \mathrm{St}(\mathsf{B}) : 2^\omega \vDash R^{2^\omega}(f_1(G), \dots, f_n(G))\}\right). \quad (4.2)$$

**Lemma 4.2.3.** *The interpretation of relational symbols in Loc*$(\mathrm{St}(\mathsf{B}), 2^\omega)$ *given by* (4.2) *is well-defined.*

*Proof.* To simplify notation, assume $R$ is an unary relational symbol. First of all, we have to prove that

$$[\![R(f)]\!] \wedge [\![\neg R(f)]\!] = 0.$$

Defining $A := R^{2^\omega}$ and $B := (\neg R)^{2^\omega}$, we have that $A \cap B = \emptyset$ and $A \cup B = 2^\omega$. Then $f^{-1}[A] \cap f^{-1}[B] = \emptyset$, and $f^1[A] \cup f^{-1}[B] = \mathrm{St}(\mathsf{B})$. Moreover, if we define for $r \in 2^\omega$

$$N_r := \mathrm{Reg}\left(f^{-1}[\{r\}]\right),$$

the set

$$\bigcup_{r \in A} N_r \cup \bigcup_{s \in B} N_s$$

is dense in $\mathrm{St}(\mathsf{B})$ since $f$ is locally constant.

**Claim 7.** $f^{-1}[B] \cap \bigcup_{r \in A} N_r$ *is meager.*

*Proof.* Otherwise, there would be a basic non-empty open set $W$ such that $W \subseteq f^{-1}[B]$ and $W \subseteq \bigcup_{r \in A} N_r$. In particular,

$$W = \bigcup_{r \in A} (W \cap N_r).$$

Being $W$ non-empty, there exists $r \in A$ such that $W \cap N_r \neq \emptyset$. This implies that

$$W \cap N_r \subseteq W \subseteq f^{-1}[B],$$

and so $f^{-1}[B] \cap N_r$ is non-meager. Since $f^{-1}[\{r\}]$ has meager difference with $N_r$, we conclude that $f^{-1}[\{r\}] \cap f^{-1}[B]$ is non-empty, which is a contradiction. □

With the same argument we have that $f^{-1}[A]$ has meager intersection with $\bigcup_{b \in B} N_b$. In conclusion, $f^{-1}[A]$ differs from $\bigcup_{a \in A} N_a$ on a meager set, and the same result holds for $f^{-1}[B]$.
We can now prove our thesis. Towards a contradiction, assume that there exists a basic open set $W$ such that it is non-empty and

$$W \subseteq \mathrm{Reg}\left(f^{-1}[A]\right) \cap \mathrm{Reg}\left(f^{-1}[B]\right).$$

Then, $W$ has meager difference with $f^{-1}[A] \cap f^{-1}[B]$, and so it has also meager difference with $\left(\bigcup_{r \in A} N_r\right) \cap \left(\bigcup_{s \in B} N_s\right)$. Thus we can suppose that

$$W \subseteq \left(\bigcup_{r \in A} N_r\right) \cap \left(\bigcup_{s \in B} N_s\right).$$

In particular, there exist $r \in A$ and $s \in B$ such that

$$N_r \cap N_s \neq \emptyset.$$

This implies that $f^{-1}[\{r\}] \cap f^{-1}[\{s\}] \neq \emptyset$, which is a contraddiction.
Finally, we have to prove that this interpretation of $R$ satisfies axioms for a $\mathsf{B}$-valued model. To this extent, let $f, g \in \mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$. We can observe that

$$\begin{aligned}\llbracket f = g \rrbracket \wedge \llbracket R(f) \rrbracket &= \text{Reg}\left(\{U \in \mathrm{St}(\mathsf{B}) : f(U) = g(U)\}\right) \cap \text{Reg}\left(\{U \in \mathrm{St}(\mathsf{B}) : 2^\omega \models R(f(U))\}\right) \subseteq \\ &\subseteq \text{Reg}\left(\{U \in \mathrm{St}(\mathsf{B}) : 2^\omega \models R(g(U))\}\right) = \llbracket R(g) \rrbracket .\end{aligned}$$

□

Notice that we could have defined the interpretation of $R$ in $\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$ in the same way i.e. $R(f) = \text{Reg}\left(\{U : 2^\omega \models R(f(U))\}\right)$. However, if we consider $\neg R$ as a relational symbol itself, with its interpretation in $2^\omega$ given by the complement of $R$, we can not ensure that

$$\neg\text{Reg}\left(\{G \in \mathrm{St}(\mathsf{B}) : R^{2^\omega}(f(G))\}\right) = \text{Reg}\left(\{G \in \mathrm{St}(\mathsf{B}) : (\neg R)^{2^\omega}(f(G))\}\right)$$

for any $f \in \mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)$ (the problem is that if the relation $R \subseteq 2^\omega$ has no regularity property, its preimage may not have meager difference with a regular open set). However, if $R$ is a relational symbol whose interpretation in $2^\omega$ is a Borel set, then our definition works: Indeed, if $R^{2^\omega} \subseteq (2^\omega)^n$ is a Borel subset, then

$$W := \{U \in \mathrm{St}(\mathsf{B}) : 2^\omega \vDash R(f_1(U), \ldots, f_n(U))\} = (f_1 \times \cdots \times f_n)^{-1}\left[R^{2^\omega}\right]$$

is the continuous preimage of a Borel set, hence it is Borel himself. Then

$$\text{Reg}\left(\{G \in \mathrm{St}(\mathsf{B}) : 2^\omega \vDash R^{2^\omega}(f_1(G), \ldots, f_n(G))\}\right)$$

is the unique regular open set with meager difference with $W$, similarly to the case of $=^{\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)}$.
We refer to [18] for futher details.

**Notation 4.2.4.** Given a Borel set $R \subseteq (2^\omega)^n$

$$\llbracket R(f_1, \ldots, f_n) \rrbracket^{\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)} := \text{Reg}\left(\{G \in \mathrm{St}(\mathsf{B}) : 2^\omega \vDash R^{2^\omega}(f_1(G), \ldots, f_n(G))\}\right).$$

From now on, we assume that all the relational symbols in $L = \{R_i : i \in I\}$ have a Borel interpretation in $2^\omega$.
Since having the mixing property does not depend on the language, by Lemma 2.4.3 both $\langle \mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega), R_i^{\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)} : i \in I \rangle$ and $\langle \mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega), R_i^{\mathrm{Loc}(\mathrm{St}(\mathsf{B}), 2^\omega)} : i \in I \rangle$ are full $\mathsf{B}$-valued models for the language $L$.

Let us now address the problem of interpreting the relational symbols in $(2^\omega)^{\mathsf{B}}$.
We will use the following fact (see e. g. [18]):

**Fact 4.2.5.** *For every Borel subset $R$ of $(2^\omega)^n$ there exist an $r \subseteq \omega$ and a (ZFC provably) $\Delta_1$-property $P_R(x_1, \ldots, x_n, y)$ in the language $\mathcal{L}$ such that*

$$(a_1, \ldots, a_n) \in R \quad \textit{if and only if} \quad V \vDash_{\mathcal{L}} P_R(a_1, \ldots, a_n, r).$$

*In particular, there exists a $\Delta_1$-property $\varphi_{2^\omega}(x, y)$ and a $r \subseteq \omega$ such that $a \in 2^\omega$ if and only if $V \vDash_{\mathcal{L}} \varphi_{2^\omega}(a, r)$.*

**Notation 4.2.6.** Given $R \in L$ with interpretation $R \subseteq (2^\omega)^n$, let $\varphi_R(x_1, \ldots, x_n, y)$ be the $\Delta_1$-definible formula in $\mathcal{L}$ such that

$$(a_1, \ldots, a_n) \in R \quad \textit{if and only if} \quad V \vDash_{\mathcal{L}} \varphi_R(a_1, \ldots, a_n, r).$$

For every $\tau_1, \ldots, \tau_n \in (2^\omega)^{\mathsf{B}}$, define

$$[\![R(\tau_1, \ldots, \tau_n)]\!]_{\mathsf{B}}^{(2^\omega)^{\mathsf{B}}} := [\![\varphi_R(\tau_1, \ldots, \tau_n, \check{r})]\!]_{\mathsf{B}}^{V^{\mathsf{B}}}.$$

Notice that, since the property $\varphi_{2^\omega}$ is define in particular by a $\Sigma_1$-formula, $(\check{2^\omega})^{\mathsf{B}} \subseteq (2^\omega)^{\mathsf{B}}$ and the interpretation of the relational symbols in $(2^\omega)^{\mathsf{B}}$ restricted to $(\check{2^\omega})^{\mathsf{B}}$ is the same of the one defined by (4.1). Indeed, let (to simplify notation) $R$ be an unary realtional symbol. Then, for every $\tau \in (\check{2^\omega})^{\mathsf{B}}$ and for any enumeration $\{x_t : t \in T\}$ of $2^\omega$,

$$\begin{aligned}
[\![R(\tau)]\!]^{(\check{2^\omega})^{\mathsf{B}}} &= \bigvee \{[\![\tau = \check{x}_t]\!] : t \in T \textit{ such that } R(x_t)\} \\
&= \bigvee \{[\![\tau = \check{x}_t]\!] : t \in T \textit{ such that } [\![\varphi_R(\check{x}_t)]\!] = 1\} \\
&\leq \bigvee \{[\![\tau = \check{x}_t]\!] \wedge [\![\varphi_R(\check{x}_t)]\!] : t \in T \textit{ s. t. } [\![\varphi_R(\check{x}_t)]\!] = 1\} \\
&\leq \bigvee_{t \in T} \left([\![\tau = \check{x}_t]\!] \wedge [\![\varphi_R(\check{x}_t)]\!]\right) \\
&\leq \bigvee_{t \in T} [\![\varphi_R(\tau)]\!] = [\![R(\tau)]\!]^{(2^\omega)^{\mathsf{B}}}.
\end{aligned} \tag{4.3}$$

Considering (4.3) for $\neg R$, we conclude that

$$[\![R(\tau_1, \ldots, \tau_n)]\!]^{(\check{2^\omega})^{\mathsf{B}}} = [\![R(\tau_1, \ldots, \tau_n)]\!]^{(2^\omega)^{\mathsf{B}}}.$$

This fact holds because $\neg R$ is defined by $\neg\varphi_R(x, y)$, which is also a $\Delta_1$-property.

**Theorem 4.2.7.** *In this setting,* $\langle \mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega), R_i^{\mathcal{C}(\mathrm{St}(\mathsf{B}), 2^\omega)} : i \in I\rangle$ *and* $\langle (2^\omega)^{\mathsf{B}}, R_i^{(2^\omega)^{\mathsf{B}}} : i \in I\rangle$ *are isomorphic* $\mathsf{B}$*-valued models for the language* $L$*.*

*Proof.* The main step is to show that the map defined by (2.11) preserves the interpretation of relational symbols. Let us fix an $m$-ary relational symbol $R \in L$ and consider $r \subseteq \omega$ and a $\Delta_1$-formula $\varphi_R(x_1, \ldots, x_m, y)$ such that

$$(a_1, \ldots, a_m) \in R^{2^\omega} \quad \textit{if and only if} \quad V \vDash_{\mathcal{L}} \varphi_R(a_1, \ldots, a_m, r).$$

Our goal is to prove that

$$[\![R(\tau_1, \ldots, \tau_m)]\!] = [\![R(f_{\tau_1}, \ldots, f_{\tau_m})]\!]$$

for every $\tau_1, \ldots, \tau_m \in (2^\omega)^{\mathsf{B}}$. To simplify notation, assume $R$ is a unary relational symbol. We have to show that

$$[\![R(\tau)]\!] = \mathrm{Reg}\left(\{U \in \mathrm{St}(\mathsf{B}) : R(f_\tau(U)\}\right) = \mathrm{Reg}\left(f_\tau^{-1}[R]\right).$$

Since we have assumed $R \subseteq 2^\omega$ to be a Borel set, we can prove our thesis by induction on the Borel complexity of $R$, i.e. by induction on the Borel sets $\Sigma^0_\alpha$, $\Pi^0_\alpha$, for $\alpha$ a countable ordinal. First of all, suppose $R$ to be a basic clopen set:

$$\{s \in 2^\omega : \varphi(s, r)\} = R^{2^\omega} = A_{a_0, \ldots, a_k} = \{s : \omega \to 2 : s(0) = a_0, \ldots, s(k) = a_k\}.$$

We know, by equation (2.12), that

$$f_\tau^{-1}[A_{a_0,\dots,a_k}] = N_{\left[\!\left[\left(\tau(\check{0})=\check{a_0}\right)\wedge\cdots\wedge\left(\tau(\check{k})=\check{a_k}\right)\right]\!\right]}.$$

This implies that $U \in f_\tau^{-1}[R^{2\omega}]$ if and only if $\left[\!\left[\left(\tau(\check{0}) = \check{a_0}\right) \wedge \cdots \wedge \left(\tau(\check{k}) = \check{a_k}\right)\right]\!\right] \in U$. Now observe that

$$\left[\!\left[\left(\tau(\check{0}) = \check{a_0}\right) \wedge \cdots \wedge \left(\tau(\check{k}) = \check{a_k}\right)\right]\!\right] = \bigvee \left\{[\![\tau = \check{s}]\!] : s \in A_{a_0,\dots,a_k}\right\}.$$

Since $[\![\varphi(\check{s}, \check{r})]\!] = 1$ for every $s \in A_{a_0,\dots,a_k}$, we can conclude that $U \in f_\tau^{-1}[R^{2\omega}]$ if and only if

$$[\![\varphi(\tau, \check{r})]\!] \geq \bigvee \left\{[\![\tau = \check{s}]\!] \wedge [\![\varphi(\check{s}.\check{r})]\!] : s \in A_{a_0,\dots,a_k}\right\} \in U,$$

meaning that $U \in f_\tau^{-1}[R^{2\omega}]$ only if $U \in N_{[\![\varphi(\tau,\check{r})]\!]}$. Then, our thesis follows carrying out the same argument for $\neg R^{2^\omega}$.
Assume the thesis holds for basic clopen sets, and let

$$R = \bigcup_i U_i$$

be an arbitrary open set, with $U_i$ basic clopen sets, for $i \in \omega$. Then, we have that

$$f^{-1}[R^{2^\omega}] = \bigcup_i f^{-1}[U_i],$$

whose difference with

$$\bigvee_i [\![U_i(\tau)]\!] = \bigvee \left[\!\left[\tau \in \check{U_i}\right]\!\right] = \left[\!\left[\tau \in \check{R}\right]\!\right] = [\![R(\tau)]\!]$$

is a countable union of meager sets, hence meager. This allows us to conclude that

$$[\![R(\tau)]\!] = \text{Reg}\left(f_\tau^{-1}[R^{2^\omega}]\right).$$

If we have proved that, for any $\Sigma^0_\alpha$ relation $R$, $f_\tau^{-1}[R]$ has meager difference with $[\![R(\tau)]\!]$, then the same situation holds for $\Pi^0_\alpha$ Borel sets, just taking the complement.
The case of $\Sigma^0_{\alpha+1}$ Borel sets from $\Pi^0_\alpha$ is handled exactly in the same way we handled the case of open sets starting from basic clopen sets.
The case of $\alpha$ limit is similar. □

We stress the fact that the regularity assumption on the interpretation of relational symbols in $2^\omega$ can be dropped in the case of $\text{Loc}(\text{St}(\mathsf{B}), 2^\omega)$.

**Corollary 4.2.8.** *Let $L$ be any language for which $2^\omega$ is an $L$-structure. Then, the $\mathsf{B}$-valued models $(2^\omega)^{\downarrow\mathsf{B}}$, $(\check{2^\omega})^{\mathsf{B}}$ and Loc$(\text{St}(\mathsf{B}), 2^\omega)$ are all isomorphic.*

*Proof.* We have already shown (see Corollary 4.2.1) that $(2^\omega)^{\downarrow\mathsf{B}} \cong (\check{2^\omega})^{\mathsf{B}}$. With a similar proof of 4.2.7, we obtain that the bijection $\Psi : (\check{2^\omega})^{\mathsf{B}} \to \text{Loc}(\text{St}(\mathsf{B}), 2^\omega)$ preserves the interpretation of relational symbols. Thus, $(\check{2^\omega})^{\mathsf{B}} \cong \text{Loc}(\text{St}(\mathsf{B}), 2^\omega)$. □

**Corollary 4.2.9.** *Let $U$ be an ultrafilter in $\mathsf{B}$. Then, Loc$(\text{St}(\mathsf{B}), 2^\omega)/_U$ is an elementary extension of $2^\omega$.*

In other terms, we have constructed an elementary extension of $2^\omega$ as the stalk of a sheaf of continuous functions on $\mathrm{St}(\mathsf{B})$. All these statements hold for any compact Polish space $Y$, not only for the Cantor space, even though the proofs are slightly more intricate, since in our arguments we have used heavily the zero-dimensionality of $2^\omega$. If $Y$ is a non-compact Polish space, it is not anymore true that $\mathcal{C}(\mathrm{St}(\mathsf{B}), Y) \cong Y^{\mathsf{B}}$. Indeed to obtain the isomorphism the idea is that the space $\mathcal{C}(\mathrm{St}(\mathsf{B}), Y)$ needs to be enlarged. To define this extension, remember that any Polish space $Y$ is homeomorphic to a $G_\delta$ subset (i.e. a countable intersection of open sets) of the Hilbert cube $\mathcal{H} := [0,1]^\omega$ (for a proof of this fact, we refer to [11]). Then, if we define

$$\mathcal{C}^+(\mathrm{St}(\mathsf{B}), Y) := \left\{ f : \mathrm{St}(\mathsf{B}) \to \mathcal{H} : f \textit{ is continuous and } f^{-1}[\mathcal{H} \setminus Y] \textit{ is meager in } \mathrm{St}(\mathsf{B}) \right\},$$

it can be shown that

$$\mathcal{C}^+(\mathrm{St}(\mathsf{B}), Y) \cong Y^{\mathsf{B}}.$$

A complete overview of the case where a generic Polish space is treated is given in [18].

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