The solution to CH (at least for me)

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Which is the “natural” signature for set theory?

The signature $\in$ is **not** the right signature for set theory.

Consider the sentence

x is the ordered pair with first component y and second component z.

On the board we write

$$x = \langle y, z \rangle,$$

in the $\in$-signature (using Kuratowski’s trick) we write:

$$\exists t \exists u \, [\forall w \, (w \in x \leftrightarrow w = t \vee w = u) \wedge \forall v \, (v \in t \leftrightarrow v = y) \wedge \forall v \, (v \in u \leftrightarrow v = y \vee v = z)].$$

More complex concepts which we still consider basic such as: being a function, being the domain of a relation, being a transitive set, being an ordinal, etc.... are incredibly complicate to formulate in the $\in$-signature.

The solutions adopted in the standard set theory textbooks such as Kunen’s and Jech’s are to consider *definable* extensions of ZFC, where many of the concepts we consider basic are introduced as new symbols of the language together with axioms forcing their interpretation to be the natural meaning of the concept they should name.

The signature τ_{ST}

τ_{ST} extends the signature $\{\in\}$ by adjoining:

- predicate symbols R_ϕ of arity n for any Δ_0-formula $\phi(x_1, \ldots, x_n)$,
- function symbols f_θ of arity k for any Δ_0-formula $\theta(y, x_1, \ldots, x_k)$,
- constant symbols for ω and $\emptyset$.

T_{ST} is the τ_{ST}-theory given by the axioms

$$\forall \vec{x}\, (R_{\forall x \in y \phi}(y, \vec{x}) \leftrightarrow \forall x (x \in y \rightarrow R_\phi(y, x, \vec{x}))$$

$$\forall \vec{x}\, [R_{\phi \wedge \psi}(\vec{x}) \leftrightarrow (R_\phi(\vec{x}) \wedge R_\psi(\vec{x}))]$$

$$\forall \vec{x}\, [R_{\neg\phi}(\vec{x}) \leftrightarrow \neg R_\phi(\vec{x})]$$

$$\forall \vec{x}[(\exists! y\, R_\phi(y, \vec{x})) \rightarrow R_\phi(f_\phi(\vec{x}), \vec{x})]$$

for all Δ_0-formulae $\phi(\vec{x})$, together with the Δ_0-sentences

$$\forall x \in \emptyset\, \neg(x = x),$$

$$\omega \text{ is the first infinite ordinal}$$

(the former is an atomic τ_{ST}-sentence, the latter is expressible as the Π_1-sentence for τ_{ST} stating that ω is a non-empty limit ordinal contained in any other non-empty limit ordinal).

The theory $\mathrm{ZFC}_{\mathrm{ST}}$ and second order arithmetic.

- $\mathrm{ZFC}_{\mathrm{ST}}$ is the τ_{ST}-theory $\mathrm{ZFC} + T_{\mathrm{ST}}$.
- $\mathrm{ZFC}^-_{\mathrm{ST}}$ is the τ_{ST}-theory $\mathrm{ZFC}^- + T_{\mathrm{ST}}$ (ZFC^- is ZFC–*power-set axiom*).

$\mathrm{ZFC}^-_{\mathrm{ST}}$ is a theory in which second order arithmetic can be formalized quite naturally:

Assume $(V, \in)$ models ZFC (or just ZFC^-). Then it has a unique extension to a $\mathrm{ZFC}_{\mathrm{ST}}$ (or $\mathrm{ZFC}^-_{\mathrm{ST}}$) τ_{ST}-structure $(V, \tau^V_{\mathrm{ST}})$.

$\mathcal{M} = (M, \tau^{\mathcal{M}})$ is a shorthand for $(M, R^{\mathcal{M}} : R \in \tau)$.
Given $(V, \in)$ model of ZFC^-, consider the τ_{ST}-structures

$$(\mathcal{P}(\omega)^V, \tau^V_{\mathrm{ST}}), \qquad (H^V_{\omega_1}, \tau^V_{\mathrm{ST}}).$$

Almost all interesting results of second order arithmetic (such as projective determinacy, almost all of Kechris' book, etc.) can be naturally formalized in these two structures.

First key observation:

Theorem (Levy absoluteness)

Let $(V, \in)$ *be a model of* ZFC. *Then*

$$(H_{\omega_1}^V, \tau_{\mathrm{ST}}^V, A : A \subseteq \mathcal{P}(\omega)^k, k \in \mathbb{N}) \prec_1 (V, \tau_{\mathrm{ST}}^V, A : A \subseteq \mathcal{P}(\omega)^k, k \in \mathbb{N})$$

Note that here we allow arbitrary subsets of $\mathcal{P}(\omega)^k$ as new predicate symbols for our formulae.

The standard textbook formulation of Levy's absoluteness is

$$(H_{\omega_1}^V, \tau_{\mathrm{ST}}^V) \prec_1 (V, \tau_{\mathrm{ST}}^V),$$

but minor variations of its proof allow to prove the enhanced version.

For the structure $(\mathcal{P}(\omega)^V, \tau_{\mathrm{ST}}^V)$ we can just say $(\mathcal{P}(\omega)^V, \tau_{\mathrm{ST}}^V) \sqsubseteq (V, \tau_{\mathrm{ST}}^V)$. This is one of the reasons why it is convenient (at least in set theory) to formalize second order numer theory as the first order theory of $(H_{\omega_1}^V, \tau_{\mathrm{ST}}^V)$ rather than that of $(\mathcal{P}(\omega)^V, \tau_{\mathrm{ST}}^V)$.

Forcing enters the picture

A key property of the signature τ_{ST} is the following corollary of Shoenfield's and Levy's absoluteness:

Theorem

Assume $(V, \in)$ *models* ZFC. *Let* G *be* V*-generic for some forcing* $P \in V$. *Then*

$$V \prec_1 V[G]$$

In particular the Π_1*-fragment of the* τ_{ST}*-theory of* V *is* **invariant** *across all forcing extensions of* V*.*

Proof

$(H_{\omega_1}^V, \tau_{\mathrm{ST}}^V) \prec_1 (V, \tau_{\mathrm{ST}}^V)$ and
$(H_{\omega_1}^{V[G]}, \tau_{\mathrm{ST}}^{V[G]}) \prec_1 (V[G], \tau_{\mathrm{ST}}^{V[G]})$
by Levy's absoluteness.

$(H_{\omega_1}^V, \tau_{\mathrm{ST}}^V) \prec_1 (H_{\omega_1}^{V[G]}, \tau_{\mathrm{ST}}^{V[G]})$
by Shoenfield's absoluteness for Σ_2^1-properties
(since Σ_2^1-properties code Σ_1-properties of H_{ω_1} via the Π_1^1-set of countable well-founded extensional graphs).

Model theory enters the picture

The structure $(H_{\omega_1}^V, \tau_{ST}^V, \mathcal{P}(\mathcal{P}(\omega))^V)$ is not the right one where to formalize second order number theory:

- The language has cardinality $2^{(2^{\aleph_0})}$ (the size of $\mathcal{P}(\mathcal{P}(\omega))$); this is not a problem if we are platonists, but if we are formalists we should have a *recursive* signature. HOWEVER:
- Wild sets which are not part of second order arithmetic such as non-measurable sets, etc are among the predicates considered in the above structure. This is a problem also for a platonist These sets should not be definable in the first order axiomatization of second order artithmetic.

SOLUTIONS: the first two OK for formalists and platonists, the third OK just for platonists

- **Minimal solution:** just consider τ_{ST}.
- **Good model-theoretic solution:** consider τ_{ST} enriched with predicate symbols for all *lightface definable projective* subsets of $\mathcal{P}(\omega)$.
- **Platonist solution:** consider τ_{ST} enriched with predicate symbols for all *universally Baire* subsets of $\mathcal{P}(\omega)$.

Model companionship for second order number theory

Theorem (Viale, Venturi)

No τ_{ST}-theory $T \supseteq \mathrm{ZFC}_{\mathrm{ST}}$ has a model companion.

σ_ω is the signature τ_{ST} enriched by a new predicate symbol S_ϕ of arity n for any τ_{ST}-formula ϕ with n-many free variables.

T_ω is the σ_ω-theory given by the axioms

$$\forall x_1 \dots x_n \left[S_\psi(x_1, \dots, x_n) \leftrightarrow \left(\bigwedge_{i=1}^{n} x_i \subseteq \omega \wedge \psi^{\mathcal{P}(\omega)}(x_1, \dots, x_n)\right)\right]$$

as ψ ranges over the τ_{ST}-formulae.

ZFC^*_ω is the *definable extension of* ZFC given by $\mathrm{ZFC}_{\mathrm{ST}} + T_\omega$ (similarly for ZFC^{*-}_ω).

In ZFC^*_ω we have quantifier elimination for formulae whose quantifiers range just over $\mathcal{P}(\omega)$. In particular ZFC^*_ω considers the projective sets as elementary properties of set theory.

Projective determinacy is expressible by an axiom schema of atomic sentence in ZFC^*_ω, while it is expressible by sentences with prenex normal form of arbitrarily high complexity in $\mathrm{ZFC}_{\mathrm{ST}}$.

Model companionship for second order number theory

For a τ-theory T, $T_\forall$ is the family of Π_1-sentences for τ which are T-provable (accordingly we define $T_\exists$, $T_{\forall\exists}$).

Theorem (Viale, and for a weaker version also Venturi)

Any σ_ω-theory $T \supseteq \mathrm{ZFC}^*_\omega$ has as model companion the theory

$$T_\forall + \mathrm{ZFC}^{*-}_\omega + \forall x \exists f\ (f : \omega \to x \text{ is surjective}).$$

An immediate corollary of (the proof of) the above result is the following:

Corollary

TFAE for any σ_ω-theory $T \supseteq \mathrm{ZFC}^*_\omega$ and any Π_2-sentence ψ for σ_ω:

1. $T \vdash \psi^{H_{\omega_1}}$;
2. *For all complete theory $S \supseteq T$, $S_\forall + \psi$ is consistent.*

If T is complete (2) becomes $T_\forall + \psi$ is consistent.

A brief digression on model companionship

The model companion T^* of a τ-theory T maximizes the family of Π_2-sentences which are compatible with the universal fragment of T for τ.

Characterizations of model companionship for all **complete** theories:

The model companion T^* of a *complete* τ-theory T (if it exists) is characterized by the following three properties:

1. $T^*_\forall = T_\forall$;
2. T^* is axiomatized by its Π_2-fragment $T^*_{\forall\exists}$ for τ;
3. For all Π_2-sentences ψ for τ:

$$\psi \in T^* \qquad \text{if and only if} \qquad T_\forall + \psi \text{ is consistent.}$$

Moreover for $\mathcal{M}$ a τ-model of T^*, $\mathcal{M} \prec_1 \mathcal{N}$ whenever $\mathcal{M} \sqsubseteq \mathcal{N}$ and $\mathcal{N} \models T$.

The characterization of model companionship for non complete theories is more delicate.

Also large cardinals enter the picture

Definition

Let (X, τ) be a Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

From now on UB denotes the family of universally Baire sets.

Universal Baireness describes the absolutely regular sets of reals:

Consider 2^ω as a closed subspace of $[0;1]$. It is meager.
Now take a subset P of 2^ω which does not have the Baire property in 2^ω.
Seen as a subset of $[0;1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of 2^ω inside $[0;1]$.
This map is continuous, and the preimage of P does not have the Baire property in 2^ω.
Hence $P \subseteq [0;1]$ is not universally Baire, even if it has the Baire property.

Theorem (Woodin)

Assume $(V, \in)$ is a model of ZFC+*there are class many Woodin cardinals and* UB *be the family of universally Baire subsets of $\mathcal{P}(\omega)$ in V.*
Then for all forcing notions $P \in V$, there are canonical $\dot{P}$-names $\dot{A}$ for all $A \in$ UB such that for all G V-generic for P:

$$(H_{\omega_1}^V, \tau_{\mathrm{ST}}^V, \mathrm{UB}) \prec (H_{\omega_1}^{V[G]}, \tau_{\mathrm{ST}}^{V[G]}, \dot{A}_G : A \in \mathrm{UB})$$

In particular the Π_1-fragment of the $\tau_{\mathrm{ST}} \cup$ UB-theory of V is **invariant** *across all forcing extensions of V.*

Theorem (Woodin?, Steel?, Martin?)

Let $(V, \in)$ be a model of ZFC+*there are class many Woodin cardinals, and $A \subseteq \mathcal{P}(\omega)^k$ be a set definable by $\phi^{\mathcal{P}(\omega)}(\vec{x})$ for $\phi(\vec{x})$ a σ_ω-formula.*
Then A is universally Baire in V.

Putting everything together we get:

Theorem (Viale, and for weaker versions also Venturi)

Let T be any σ_ω-theory extending ZFC^*_ω+there are class many Woodin cardinals. Then T has a model companion T^*, and TFAE for any Π_2-sentence ψ:

1. $\psi \in T^*$;
2. $T \vdash \psi^{H_{\omega_1}}$;
3. *T proves that*

$$\exists P\ [(P \text{ is a partial order}) \wedge \Vdash_P \psi^{H_{\omega_1}}];$$

4. *for all complete σ_ω-theory $S \supseteq T$, $S_\forall + \psi$ is consistent.*

In particular **forcibility, consistency, provability** overlap for second order arithmetic if we assume large cardinals.

From $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega_1)$

- The theory ZFC^*_ω is a definable extension of ZFC, i.e. any σ_ω-formula is ZFC^*_ω-equivalent to some $\in$-formula; hence using ZFC^*_ω we can prove exactly the same theorems we can prove from ZFC.
- The result holds for many families of universally Baire sets $\mathcal{A}$ such that

$$\sigma_\omega \subseteq \tau_{\mathsf{ST}} \cup \mathcal{A} \subseteq \tau_{\mathsf{ST}} \cup \mathsf{UB}$$

  (together with the appropriate axioms giving the correct interpretations of the symbols in $\mathcal{A}$); for example it holds (assuming appropriate large cardinals) for

  - $\mathcal{A}$ being the family of $L(\mathbb{R})$-definable subsets of $\mathcal{P}(\omega)$,
  - $\mathcal{A}$ being the family of $L(\mathsf{UB})$-definable subsets of $\mathcal{P}(\omega)$,
  - $\dots$

What happens if we add to τ_{ST} new predicate symbols which are not talking about universally Baire sets?

It is a natural move, but brings our focus away from $\mathcal{P}(\omega)$ and second order number theory, towards more complicated fragments of the universe of sets.

If V models a τ-theory T, which τ-structures can be models of its model companion T^*?

For all models $(V, \in)$ of ZFC and all cardinals $\lambda \in V$ and all signatures

$$\tau_{\mathrm{ST}} \cup \{\lambda\} \subseteq \tau \subseteq \tau_{\mathrm{ST}} \cup \{\lambda\} \cup \mathcal{P}(\mathcal{P}(\lambda))$$

1. $(H^V_{\lambda^+}, \tau^V) \prec_1 (V, \tau^V)$, therefore $H^V_{\lambda^+}$ and V share the same Π_1-theory for τ;
2. $H^V_{\lambda^+}$ is the unique transitive substructure of V containing $\mathcal{P}(\lambda)$ which models ZFC^- and the Π_2-sentence ψ_λ for $\tau_{\mathrm{ST}} \cup \{\lambda\}$

$$\forall X \exists f\, (f : \lambda \to X \text{ is surjective}).$$

Hence if a τ-theory $T \supseteq$ ZFC has a model companion T^*, ψ_λ should be in T^*, hence T^* should be the τ-theory common to the τ-structures $H^{\mathcal{M}}_{\lambda^+}$ as $\mathcal{M}$ vary in a certain class of models of ZFC $+\ T_\forall$.

Generic absoluteness detects the model companions of set theory

PROGRAM: For each cardinal λ, look for signatures τ such that

- $\tau_{\mathrm{ST}} \cup \{\lambda\} \cup \mathcal{P}(\mathcal{P}(\lambda)) \supseteq \tau \supseteq \tau_{\mathrm{ST}} \cup \{\lambda\}$.
- For $(V, \in)$ a model of ZFC+*large cardinals*, the Π_1-theory of V in the signature τ is invariant across forcing extensions of V.
- Understand whether ZFC+*large cardinals* as formulated in the signature τ admits a model companion, which *should* be equivalently given by:
  - the τ-theory T^* of $H_{\lambda^+}^{\mathcal{M}}$ as $\mathcal{M}$ varies among a class C_λ given by *certain* models of set theory.
  - the Π_2-sentences ψ for τ such that $\psi^{H_{\lambda^+}}$ is *provably forcible* from T.

THIS PROGRAM IS SUCCESSFUL FOR:

- $\tau = \sigma_\omega$;
- $\lambda = \omega$;
- $C_\lambda =$ *all* models of ZFC+*there are class many Woodin*.

Signatures for third order number theory

Suppose now we want to talk about the properties of $\mathcal{P}(\omega_1)$. Then we should at least consider a language adding to τ_{ST} a constant symbol for ω_1.

Since the non-stationary ideal plays a central role in our analysis of $\mathcal{P}(\omega_1)$, it is natural to add also a predicate symbol $\mathbf{NS}_{\omega_1}$ for the non-stationary ideal. Let:

- $\tau_{\omega_1} = \tau_{\mathrm{ST}} \cup \{\omega_1\}$,
- $\tau_{\mathbf{NS}_{\omega_1}} = \tau_{\omega_1} \cup \{\mathbf{NS}_{\omega_1}\}$.
- T_{ω_1} is the τ_{ω_1}-theory given by T_{ST} together with the axiom

$$\omega_1 \text{ is the first uncountable cardinal.}$$

- $T_{\mathbf{NS}_{\omega_1}}$ is the $\tau_{\mathbf{NS}_{\omega_1}}$-theory given by T_{ω_1} together with the axiom

$$\forall x\, [(x \subseteq \omega_1 \text{ is non-stationary}) \leftrightarrow \mathbf{NS}_{\omega_1}(x)].$$

Why CH is false and the continuum is $\aleph_2$.

Remark (Viale, Venturi)

Assume $\sigma \supseteq \tau_{\omega_1}$ is some signature and $T \supseteq \mathrm{ZFC} + T_{\omega_1}$ is a σ-theory such that:

(A) *T admits a model companion T^*.*

(B) *If $\mathcal{M} \models T$, $P \in \mathcal{M}$ is a forcing notion, and ϕ is a Π_1-sentence for σ:*

$$\mathcal{M} \models \phi \qquad \Longleftrightarrow \qquad \mathcal{M} \models [\Vdash_P \phi].$$

Then

1. *$\neg\mathrm{CH}$ is in T^*.*
2. *Moreover if for some model $(V, \in)$ of ZFC, $(H^V_{\lambda^+}, \sigma^V)$ models T^*, we have that $V \models 2^{\aleph_0} = \aleph_2$.*

$\neg$CH is formalized by the Π_2-sentence for τ_{ω_1}

$$\forall f\,[(f \text{ is a function} \wedge \mathrm{dom}(f) = \omega_1) \to \exists r\,(r \subseteq \omega \wedge r \notin \mathrm{ran}(f))]$$

Why CH is false (if condition (B) holds).

Proof of the remark:

To simplify matters assume T is complete. By Condition (A), its model companion T^* exists.
Since T is complete, T^* is axiomatized by the Π_2-sentences ψ consistent with $T_\forall$.
By Condition (B), $\neg$CH $+ T_\forall$ holds in some forcing extension of a model of T, hence is consistent.
Therefore $\neg$CH is in T^*.

It is slightly more delicate to argue as above in case T is not Π_1-complete.

If $\mathcal{M} \models$ ZFC+*large cardinals* and $H_{\lambda^+}^{\mathcal{M}} \models T^*$, $\mathcal{M} \models 2^\omega = \omega_2$.

Use the Π_2-sentence θ:

$$\forall C \text{ ladder system on } \omega_1 \, \forall r \subseteq \omega \, \exists \alpha \, \exists f \, [(f : \omega_1 \to \alpha \text{ is surjective}) \wedge \psi(C, r, \alpha)]$$

where $\psi(x, y, z)$ is a Σ_1-formula for $\tau_{ST} \cup \{\omega_1\}$ which can be used to define for each ladder system x an injective map $\mathcal{P}(\omega) \to \omega_2$ with assignment $y \mapsto z$ of the real y to a corresponding ordinal z. ($\psi(x, y, z)$ exists and θ is forcible by a result of Caicedo and Veličkovič).

For which signatures σ does condition (B) holds?.

Condition (B) says that forcing can be used only to change the truth value of complicated σ-sentences; the basic properties of models of T (i.e. the universal part of their theory) are preserved through forcing.

Condition (B) holds for:

- any τ_{ST}-theory $T \supseteq \mathrm{ZFC}_{\mathrm{ST}}$;
- any σ_ω-theory $T \supseteq \mathrm{ZFC}^*_\omega$+*there are class many Woodin cardinals*;
- the $\tau_{\mathrm{ST}} \cup \mathrm{UB}^V$-theory of V assuming $(V, \in) \models \mathrm{ZFC}$+*there are class many Woodin cardinals*.

Key issue:

Is condition (B) void of content for signatures σ extending $\tau_{\mathrm{ST}} \cup \{\omega_1\}$?

Boban remarked that condition (B) fails if σ contains a constant naming ω_2 (see the last slides), BUT:

Condition (B) is realized by $\tau_{\mathbf{NS}_{\omega_1}} \cup \mathrm{UB}$.

Theorem (Viale)

Assume $(V, \in)$ models that there are class many Woodin cardinals. Then the Π_1-theory of V for the language $\tau_{\mathbf{NS}_{\omega_1}} \cup \mathrm{UB}$ is invariant under set sized forcings.

Theorem (Levy absoluteness for $\mathcal{P}(\omega_1)$)

Let $(V, \in)$ be a model of ZFC$+$*there are class many Woodin cardinals. Then*

$$(H_\lambda^V, \tau_{\mathbf{NS}_{\omega_1}}^V, \mathrm{UB}^V) \prec_1 (V, \tau_{\mathbf{NS}_{\omega_1}}^V, \mathrm{UB}^V)$$

for any $\lambda > \omega_1$.

Why Woodin's axiom $(*)$ should be true

Definition

MAX(UB): There are class many Woodin cardinals in V, and for all G V-generic for some forcing notion $P \in V$:

1. Any subset of $(2^\omega)^{V[G]}$ definable in $(H_{\omega_1}^{V[G]} \cup \mathrm{UB}^{V[G]}, \in)$ is universally Baire in $V[G]$.
2. Let H be $V[G]$-generic for some forcing notion $Q \in V[G]$. Then:

$$(H_{\omega_1}^{V[G]} \cup \mathrm{UB}^{V[G]}, \in) \prec (H_{\omega_1}^{V[G][H]} \cup \mathrm{UB}^{V[G][H]}, \in)$$

   via the map which is the identity on $H_{\omega_1}^{V[G]}$ and maps A to $\dot{A}_H$ for A universally Baire set in $V[G]$ and $\dot{A} \in V[G]^Q$ its corresponding canonical Q-name.

MAX(UB) holds in every forcing extension of V collapsing some δ supercompact in V to become countable. It is a way to express the existence of a $\sharp$ for UB.

Why Woodin's axiom $(*)$ should be true

See Larson's handbook chapter for a definition of $\mathbb{P}_{\max}$.
$L(\mathrm{UB})$ denotes the smallest transitive model of ZF which contains UB.

Definition

$(*)$-UB holds if there are class many Woodin cardinals, $\mathbf{NS}_{\omega_1}$ is saturated, and there exists an $L(\mathrm{UB})$-generic filter G for $\mathbb{P}_{\max}$.

Theorem (Asperò, Schindler)

$\mathrm{MM}^{++} + \mathbf{MAX}(\mathrm{UB})+$*there are class many Woodins implies* $(*)$-UB.

Therefore $(*)$-UB *is forcible over any model of* ZFC$+$*there exist two supercompact cardinals*$+$*there are class many Woodin cardinals.*

Why Woodin's axiom $(*)$ should be true for a Platonist

Theorem (Viale)

Let $\mathcal{V} = (V, \in)$ *be a model of*

$$\mathrm{ZFC} + \mathbf{MAX}(\mathrm{UB}) + \mathbf{NS}_{\omega_1} \textit{ is precipitous}+$$

$$+\textit{there are class many supercompact cardinals},$$

TFAE:

1. $(V, \in)$ *models* $(*)$-UB;
2. *Let* T *be the* $\tau_{\mathbf{NS}_{\omega_1}} \cup \mathrm{UB}$-*theory of* V *and* T^* *be the* $\tau_{\mathbf{NS}_{\omega_1}} \cup \mathrm{UB}$-*theory of* H_{ω_2}. *Then* T *has* T^* *as its model companion and* V *models* $\mathbf{NS}_{\omega_1}$ *is precipitous.*

$\mathbf{NS}_{\omega_1}$ is precipitous is independent of CH.

On the basis *just* of large cardinal axioms (and **MAX**(UB)) one has a *natural canonical* theory for H_{ω_2}. This is the theory of H_{ω_2} assuming forcing axioms.

PROBLEM:

This result is not satisfactory for a formalist, because we are dealing with a non-recursive signature and a non-recursive theory of sets.

Why Woodin's axiom $(*)$ should be true for a formalist

- Let $\sigma_{\omega,\mathbf{NS}_{\omega_1}}$ be $\tau_{\mathbf{NS}_{\omega_1}}$ enriched with a predicate symbol S_ϕ for any τ_{ST}-formula ϕ.
- $T_{\text{l-UB}}$ is the $\sigma_{\omega,\mathbf{NS}_{\omega_1}}$-theory given by the axioms

$$\forall x_1 \dots x_n \, [S_\psi(x_1,\dots,x_n) \leftrightarrow (\bigwedge_{i=1}^{n} x_i \subseteq \omega \wedge \psi^{L(\mathrm{UB})}(x_1,\dots,x_n))]$$

  as ψ ranges over the $\in$-formulae.
- $\mathrm{ZFC}^{*-}_{\text{l-UB},\mathbf{NS}_{\omega_1}}$ is the $\sigma_{\omega,\mathbf{NS}_{\omega_1}}$-theory

$$\mathrm{ZFC}^{-}_{\mathrm{ST}} \cup T_{\text{l-UB}} \cup T_{\mathbf{NS}_{\omega_1}};$$

- Accordingly we define $\mathrm{ZFC}^{*}_{\text{l-UB},\mathbf{NS}_{\omega_1}}$.

$\sigma_{\omega,\mathbf{NS}_{\omega_1}}$ is a *recursive* signature and $\mathrm{ZFC}^{*}_{\text{l-UB},\mathbf{NS}_{\omega_1}}$ is a *recursive* first order theory which is a *definable* extension of ZFC.

Why Woodin's axiom $(*)$ should be true for a formalist

Theorem (Viale)

Let T be any $\sigma_{\omega,\mathbf{NS}_{\omega_1}}$-theory extending

$$T_0 = \mathrm{ZFC}^*_{\mathrm{1\text{-}UB},\mathbf{NS}_{\omega_1}} + \mathbf{MAX}(\mathrm{UB}) + \textit{there are class many supercompacts.}$$

Then T has a model companion T^*, and TFAE for any Π_2-sentence ψ for $\sigma_{\omega,\mathbf{NS}_{\omega_1}}$:

1. $\psi \in T^*$
2. $T_\forall + T_0 + (*)\text{-UB} \vdash \psi^{H_{\omega_2}}$ *(equivalently one could write* MM^{++} *in the place of* $(*)$-UB*).*
3. *T proves*
$$\exists P\ [(P \text{ is a partial order}) \wedge \Vdash_P \psi^{H_{\omega_2}}];$$
4. *for all complete $\sigma_{\omega,\mathbf{NS}_{\omega_1}}$-theory $S \supseteq T$, $S_\forall + \psi$ is consistent.*

Is **MAX**(UB) really necessary?

Theorem (Viale)

Let T be any $\tau_{\mathbf{NS}_{\omega_1}}$-theory extending

$$\mathrm{ZFC} + T_{\mathrm{ST}} + T_{\mathbf{NS}_{\omega_1}} + \textit{there are class many supercompacts.}$$

*Then T may not have a model companion T^*, **BUT** TFAE for any Π_2-sentence ψ for $\tau_{\mathbf{NS}_{\omega_1}}$:*

1. $T_\forall + T_0 + (*)\text{-UB} \vdash \psi^{H_{\omega_2}}$ *(equivalently one could write* MM^{++} *in the place of* $(*)$-UB*).*
2. *T proves*
$$\exists P\ [(P \text{ is a partial order}) \wedge \Vdash_P \psi^{H_{\omega_2}}];$$
3. *for all complete $\tau_{\mathbf{NS}_{\omega_1}}$-theory $S \supseteq T$, $S_\forall + \psi$ is consistent.*

$2^{\aleph_0} = \aleph_2$ follows by a Π_2-sentence for $\tau_{\mathbf{NS}_{\omega_1}}$ holding in H_{ω_2}.

What about the theory of $\mathcal{P}(\lambda)$ for $\lambda > \omega_1$?

Veličkovič remarked the following:

$\square_{\omega_2}$ is a Σ_1-statement for $\tau_{\omega_2} = \tau_{\mathrm{ST}} \cup \{\omega_1\} \cup \{\omega_2\}$:

$$
\begin{aligned}
\exists \{C_\alpha : \alpha < \omega_2\}[& \\
& \forall \alpha \in \omega_2 \, (C_\alpha \text{ is a club subset of } \alpha) \wedge \\
& \wedge \forall \alpha \in \beta \in \omega_2 \, (\alpha \in \lim(C_\beta) \to C_\alpha = C_\beta \cap \alpha) \wedge \\
& \wedge \forall \alpha \in \omega_2 \, (\mathrm{otp}(C_\alpha) \leq \omega_1) \\
&].
\end{aligned}
$$

$\square_{\omega_2}$ is forcible by very nice forcings (countably directed and $< \omega_1$-strategically closed), and its negation is forcible by $\mathrm{Coll}(\omega_1, < \delta)$ whenever δ is Mahlo.

In particular the Π_1-theory for τ_{ω_2} of any forcing extension $V[G]$ of V can be destroyed in a further forcing extension $V[G][H]$ assuming mild large cardinals.

CONCLUSION: Condition (B) fails badly for any $\sigma \supseteq \tau_{\omega_2}$ assuming mild large cardinals!

How to unveil the theory of $\mathcal{P}(\omega_2)$?

Boban's remark shows that the Π_1-theory of $V[G]$ in any signature $\sigma \supseteq \tau_{ST} \cup \{\omega_1, \omega_2\}$ can be destroyed in a further forcing extension $V[G][H]$.

But:

- There are many concepts expressible by atomic sentences in τ_{ω_2} and even more in $\tau_{\mathbf{NS}_{\omega_1}} \cup \mathrm{UB} \cup \{\omega_2\}$ in models of ZFC. It is not clear whether for this family of atomic sentences invariance under *all* forcings fails.
- It is also not hard to check that the property $x \in H_{\omega_2}$ is a Δ_1-property in the signature τ_{ω_2} assuming $(*)$-UB. In particular the Π_2-theory of H_{ω_2} in models of $(*)$-UB is axiomatized by Π_1-sentences in the signature $\tau_{\mathbf{NS}_{\omega_1}} \cup \mathrm{UB} \cup \{\omega_2\}$.
- The good strategy could now be that of maximizing the number of Π_2-sentences for $\tau_{\mathbf{NS}_{\omega_1}} \cup \mathrm{UB} \cup \{\omega_2\}$ which are compatible with $(*)$-UB $+$ **MAX**(UB), by checking which of these sentences can be forced together with $(*)$-UB. For example $\square_{\omega_2}$ is incompatible with $(*)$-UB. (Condition (B) is now weakened requiring preservation only through forcing extensions which maintain $(*)$-UB)

However for now these are just mere speculations.....

In particular the proposed solution to CH still has many aspects that need to be clarified, because it is based on arguments which work perfectly for $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega_1)$, but (as of our current understanding) need to be completely revised for any $\mathcal{P}(\lambda)$ with $\lambda > \omega_1$.

Definition of model companionship

For the sake of completeness....

Definition

Let τ be a signature and T be a τ-theory.
A τ-theory T^* is the *model companion* of T if:

1. $T_\forall = T^*_\forall$;
2. every *existential* τ-formula $\phi(\vec{x})$ is T^*-equivalent to a *universal* τ-formula $\psi(\vec{x})$.

1. is equivalent to

   Every τ-model of T is a substructure of a τ-model of T^* and conversely.

2. is equivalent to

   If a τ-model of T^* is a substructure of a τ-model of $T_\forall$, then it is a Σ_1-elementary substructure.

THANKS FOR YOUR ATTENTION!