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The ranks of $Ext^1(A, \mathbb{Z})$ and Whitehead's Problem

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Abstract

In this thesis we analyse Whitehead's conjecture, stating that a commutative group A is free if and only if the associated group $Ext^1(A, \mathbb{Z})$ is trivial. First we introduce the necessary background in homological algebra and logic needed to properly formulate and tackle the conjecture. Then we present a characterization of the ranks of $Ext^1(A, \mathbb{Z})$ for countable abelian groups A due to Chase (1963), which improves the classical theorem of Stein (1951), and solves positively Whitehead's conjecture for the case of countable abelian groups. Finally we discuss the undecidability of the conjecture with respect to the standard axioms of mathematics, focusing on its instantiation for abelian groups of cardinality $\aleph_1$.

Contents

Introduction **1**

1 Background **3**

1.1 Algebraic Prerequisites . . . 3

1.1.1 Rings and modules . . . 3

1.1.2 Homological Algebra . . . 6

1.2 Logical Prerequisites . . . 13

1.2.1 Basic set theoretic notions . . . 13

1.2.2 Diamond . . . 16

1.2.3 Martin's axiom . . . 19

2 ZFC results **23**

2.1 $Ext^1(A, \mathbb{Z})$ for torsion groups A . . . 24

2.2 The ranks of a group . . . 24

2.3 Classification of divisible groups . . . 28

2.4 The torsion-free rank of $Ext^1(A, \mathbb{Z})$ . . . 29

2.5 The p-rank of $Ext^1(A, \mathbb{Z})$ . . . 36

3 Whitehead's problem **39**

3.1 W-groups . . . 39

3.1.1 Chase($\aleph_1$)-condition and $\Gamma(A)$ . . . 40

3.2 Diamond Principle implies Whitehead's conjecture . . . 43

3.3 Martin's Axiom implies Whitehead's conjecture is false . . . 46

Bibliography **55**

Introduction

In this thesis we analyse Whitehead's conjecture, which states that a commutative group A is free if and only if its associated group $Ext^1(A, \mathbb{Z})$ is trivial; groups satisfying the latter property are called Whitehead's groups, from here onwards W-groups. Whitehead's Problem appeared at the beginning of the fifties of the past century. In 1951 it was proved by Stein that every countable W-group is free. For uncountable W-groups only partial results have been obtained in ZFC until the early seventies. In an unexpected turn of events in 1974 Saharon Shelah proved that, at least for cardinality $\aleph_1$, Whitehead's Problem is undecidable on the basis of ZFC set theory, by showing that there are distinct models of ZFC in one of which every W-group of size $\aleph_1$ is free, while in the other there are non-free W-groups of size $\aleph_1$. It follows that either the affirmative or the negative answer to Whitehead's Problem are both consistent with ZFC. We will retrace the history of the problem, presenting the progress made over the years. We will follow an approach which was successive to the first solutions, and which massively uses homological algebraic tools. The thesis is organized as follows:

- In the first Chapter we introduce the necessary background in homological algebra and logic needed to properly formulate and tackle the conjecture. We also introduce basic set theoretic concepts, as well as the Diamond principle and Martin's axiom, two combinatorial assumptions which will be used in the third chapter to establish the undecidability with respect to ZFC of Whitehead's problem.

- In the second Chapter we reduce Whitehead's problem to the case of torsion-free groups A having a divisible $Ext^1(A, \mathbb{Z})$. This result brings us to the introduction of a set of invariants for divisible abelian groups: the torsion-free rank, and the p-torsion rank, as p ranges over the primes. Making use of the Pontryagin's Criterion, we present a characterization of the ranks of $Ext^1(A, \mathbb{Z})$ for a countable abelian group A due to Chase (1963), which improves the classical theorem of Stein (1951), and solves positively Whitehead's conjecture for the case of countable abelian groups.

- In the third Chapter we discuss the undecidability of Whitehead conjecture with respect to the standard axioms of mathematics given by ZFC, focusing on its instantiation for abelian groups of cardinality $\aleph_1$. First we provide a generalization of the Pontryagin's Criterion to κ-free abelian groups of cardinality κ, where κ is a regular uncountable cardinal, introducing the Chase(κ)-condition for κ-free $\mathbb{Z}$-modules. Then we prove that the Diamond Principle implies that each W-group of cardinality $\aleph_1$ satisfies Chase($\aleph_1$)-condition, and showing (again by means of the Diamond principle) that this assertion brings an affirmative answer to Whitehead's Problem for abelian groups of cardinality $\aleph_1$. In the last section, we prove that Martin's Axiom entails that there exists W-groups of size $\aleph_1$ which are not free.

The first two chapters can be read and followed by any reader with a basic acquaintance with the notions of algebra taught in undergraduate courses. The third chapter requires the reader to have some familiarity with basic properties of cardinals and ordinal numbers, and with the basic results regarding stationary sets and Cubs.

1

Background

In this Chapter, we introduce the definitions and theorems needed to understand the work presented in this thesis. In the following sections we recall basic definitions as well as deep theorems. If proofs are omitted, proper references are given. Our main reference text for what concerns homological algebra is the book *An Introduction to Homological Algebra* of Joseph Rotman and for what concerns set theory is *Set Theory* of Thomas Jech.

1.1 Algebraic Prerequisites

1.1.1 Rings and modules

Definition 1.1.1. Let $(R, +_R, \cdot_R, 0_R, 1_R)$ be a unitary ring. A *left R-module* $(M, +_M, 0_M)$ is an abelian group, together with an operation $\cdot : R \times M \to M$ such that for all $a, b \in R$ and $x, y \in M$ we have

1. $a \cdot (x +_M y) = a \cdot x +_M a \cdot y$;
2. $(a +_R b) \cdot x = a \cdot x +_M b \cdot x$;
3. $(a \cdot_R b) \cdot x = a \cdot (b \cdot x)$;
4. $1_R \cdot x = x$.

Similarly, one defines a *right R-module.* If R is commutative, then left R-modules are the same as right R-modules and we will refer to them as *R-module.* Unless otherwise specified in what follows"ring" will be a shorthand for "commutative ring with identity" and "group" will be a shorthand for "commutative group".

Remark 1. Any commutative group $(A, +)$ can be endowed of the structure of a $\mathbb{Z}$-module posing $n \cdot a = \underbrace{a + \cdots + a}_{n}$.

Definition 1.1.2. Given an R-module M, and a submodule N of M, the *quotient R-module* is the set of all equivalence classes $[m] = \{m + n : n \in N\}$ defined by the equivalence relation $m \sim m'$ if and only if $m' +_M (-m) \in N$ for any $m, m' \in M$.

The sum of two equivalence classes $[m]$ and $[m']$ is the equivalence class $[m+m']$ and the multiplication by $r \in R$ of $[m]$ is defined as $[r \cdot m]$. In this way the set becomes itself a module over R.

Definition 1.1.3. Let R be a ring, and $\{M_i : i \in I\}$ a family of R-modules indexed by the set I. The *direct sum* of $\{M_i\}_{i \in I}$ is then defined to be the set of all sequences $(\alpha_i)_{i \in I}$ where $\alpha_i \in M_i$ for all $i \in I$ and $\alpha_i = 0$ for cofinitely many indices i. The *direct product* is analogous but the indices do not need to cofinitely vanish. In both cases, the set inherits the R-module structure via component-wise addition and scalar multiplication. They are respectively denoted by $\bigoplus_{i \in I} M_i$ and $\prod_{i \in I} M_i$.

Definition 1.1.4. A *free* R-module is the direct sum of modules isomorphic to R. If these modules are generated by elements x_i $(i \in I)$, the free R-module F is denoted by

$$F = \bigoplus_{i \in I} \langle x_i \rangle.$$

The generators $\{x_i\}_{i \in I}$ form a *free set of generators* or a *basis* of F.

Remark 2. In case R is a PID, a domain in which every ideal is principal, following the notation used in the previous definition, we remark that F is, up to isomorphism, uniquely determined by the cardinality of the index set I, which is called the *dimension* of F: given a prime element $p \in R$, observe that F/pF is a vector space over the field R/pR, whose R/pR-dimension is exactly the cardinality of I.

We will make use of the following:

Theorem 1.1.5. *[10, Thm. 7.1] Let R be a commutative PID, M a free R-module and N a submodule of M. Then N is free and its dimension is less than or equal to the dimension of M.*

Definition 1.1.6. Given an R-module M, $a \in M$ is a *torsion element* of M if $n \cdot a = 0$ for some $n \in \mathbb{Z}$. In this case, the least $n \in \mathbb{N}$ such that $n \cdot a = 0$ is called the *order* of a, ord(a).

The set given by the torsion elements of M is a submodule of M, the *torsion part* $T(M)$ of M. If $T(M)$ is trivial, M is a *torsion-free* module.

Proposition 1.1.7. *For any R-module M, $M/T(M)$ is torsion-free.*

Proposition 1.1.8. *[5, Thm. 15.5] A finitely generated torsion-free group is free.*

Definition 1.1.9. A *torsion group* is a group where each element has finite order and, similarly, a *p-group* is a group in which the order of each element is a some power of a prime number $p \in \mathbb{N}$. Given a group $(A, \cdot)$, for each prime number $p \in \mathbb{N}$

$$A_p = \{a \in A : \exists k \in \mathbb{N} \ s.t. \ ord(a) = p^k\}.$$

A_p is a subgroup of A and a p-group, it is called the *p-component* of A.

Theorem 1.1.10. *[5, Thm. 8.4] An abelian torsion group $(A, +)$ is representable as $\mathbb{Z}$-module as the direct sum of its p-components:*

$$A = \bigoplus_p A_p.$$

Moreover if A is isomorphic to $\bigoplus_p B_p$ with $B_p \subseteq A$, a p-group, then $B_p = A_p$ for all primes p.

Definition 1.1.11. Let p a prime number. The *Prüfer p-group* or the group of type p^∞, here denoted by $\mathbb{Z}(p^\infty)$, is the (unique up to isomorphism) group whose set of generators are non-null elements $\{c_n\}_{n \in \mathbb{N}}$ uniquely characterized by the property that (viewing $\mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module)

$$p \cdot c_1 = 0 \quad \text{and} \quad p \cdot c_{n+1} = c_n \quad \text{for all } n \in \mathbb{N}.$$

Remark 3. Equivalently, $\mathbb{Z}(p^\infty)$ may be defined as the p-component of the quotient group $\mathbb{Q}/\mathbb{Z}$ (using addition of rational numbers as group operation). Alternatively it could be identified with the subgroup of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ consisting of all p^n-th roots of unity for all $n \in \mathbb{N}$ (the group operation on $\mathbb{C}^*$ being the multiplication of complex numbers).

Definition 1.1.12. Let R be a ring. We say that an R-module $(M, +, \cdot)$ is *divisible* if $M = r \cdot M = \{r \cdot m : m \in M\}$ for all $r \in R \setminus \{0\}$.

Example. *Both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible $\mathbb{Z}$-modules. Also the groups $\mathbb{Z}(p^\infty)$ (as p ranges over the primes) are other examples of divisible $\mathbb{Z}$-modules.*

Lastly we introduce a group we will make use of in Chapter 2.

Definition 1.1.13. A *p-adic integer* is a formal series $\alpha = \sum_{n \in \mathbb{N}} a_i p^i$ with $0 \leq a_i < p$, i.e. a sequence in $\{0, \dots, p-1\}^{\mathbb{N}}$. The set of p-adic integers is denoted by $\widehat{\mathbb{Z}}_p$.

$\widehat{\mathbb{Z}}_p$ can be endowed of the structure of a commutative group as follows: Assume $\alpha = (a_i)_{i \in \mathbb{N}}$ and $\beta = (b_i)_{i \in \mathbb{N}}$ are two p-adic integers, we define their sum according to the following procedure: by induction we define a sequence $(c_i)_{i \in \mathbb{N}}$ with $0 \leq c_i < p$ and a sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ of elements of $\{0, 1\}$ as follows: $\varepsilon_0 = 0$ and

$$c_i = \begin{cases} a_i + b_i + \varepsilon_i & \text{if } 0 \leq a_i + b_i + \varepsilon_i < p \\ a_i + b_i + \varepsilon_i - p & \text{if } 0 \leq a_i + b_i + \varepsilon_i - p < p. \end{cases}$$

In the former case, $\varepsilon_{i+1} = 0$ and in the latter, $\varepsilon_{i+1} = 1$. We let $\alpha + \beta = (c_i)_{i\in\mathbb{N}}$[1].
If we cut an element $\alpha \in \widehat{\mathbb{Z}}_p$ at its i-th term $\alpha_i = a_0 + a_1 p + ... + a_{i-1}p^{i-1}$ we get a well defined element of $\mathbb{Z}/p^i\mathbb{Z}$. This yields to a map from $\widehat{\mathbb{Z}}_p$ to $\mathbb{Z}/p^i\mathbb{Z}$.
A sequence $\{\alpha_i\}_{i>0}$ such that $\alpha_i \equiv \alpha_j \mod (p^j)$ for all $j < i$ defines a unique p-adic integer $\alpha \in \widehat{\mathbb{Z}}_p$. This is a bijection which define an isomorphism between $\widehat{\mathbb{Z}}_p$ and the *inverse limit* of $\{\mathbb{Z}/p^i\mathbb{Z}\}_{i\in\mathbb{N}^*}$ which is

$$\varprojlim \mathbb{Z}/p^i\mathbb{Z} := \{(x_i)_{i\in\mathbb{N}^*} \in \prod_{i\in\mathbb{N}^*} \mathbb{Z}/p^i\mathbb{Z} : x_i \equiv x_j \mod (p^j)\ \forall j < i\}.$$

Proposition 1.1.14. *$\widehat{\mathbb{Z}}_p$ is torsion-free.*

Proof. We can identify $\widehat{\mathbb{Z}}_p$ as a subgroup of $\prod_{i\in\mathbb{N}} \mathbb{Z}/p^i\mathbb{Z}$ appealing to the isomorphism between $\widehat{\mathbb{Z}}_p$ and $\varprojlim \mathbb{Z}/p^i\mathbb{Z}$. Now given a non-zero element $(x_i)_{i\in\mathbb{N}}$, suppose there exists $n \in \mathbb{Z}$ such that $n \cdot x_i \equiv 0 \mod (p^i)$ for all $i \in \mathbb{N}$. Let x_j be the least coordinate which is not zero: since $x_i \equiv x_j \mod (p^j)$ for all $i \geq j$ the maximum power of p that divides x_i is at most p^{j-1}. However $n \cdot x_i \equiv 0$ for all $i \in \mathbb{N}$ and this means p^{i-j+1} divides n for all $i \in \mathbb{N}$, a contradiction. □

Theorem 1.1.15. *[14, Prop. 5.26 - Ex. 5.20] The group of endomorphisms of $\mathbb{Z}(p^\infty)$ is isomorphic to the group of p-adic integers, i.e. $End_{\mathbb{Z}}(\mathbb{Z}(p^\infty)) \cong \widehat{\mathbb{Z}}_p$.*

1.1.2 Homological Algebra

Definition 1.1.16. Let R be a ring and suppose M, N and P are R-modules: the sequence of homomorphisms

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} P$$

is *exact* at N if $\mathrm{Im}(\alpha) = \ker(\beta)$. A longer sequence

$$\cdots \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+2} \longrightarrow \cdots$$

is *exact* if it is exact at each term. An exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ is a *short exact sequence.*

Remark 4. By this definition, $\alpha : M \longrightarrow N$ is injective if and only if $0 \longrightarrow M \xrightarrow{\alpha} N$ is exact and is surjective if and only if $M \xrightarrow{\alpha} N \longrightarrow 0$ is exact.

[1]Note that we can identify $\mathbb{N}$ with the series $\sum_{i\in\mathbb{N}} a_i$ whose terms a_i are eventually null, by mapping $(a_0, \ldots, a_n, 0, 0, 0, \ldots)$ to the natural number $\sum_{i=0}^n a_i \cdot p^i$. Observe that for such series the above map is an homomorphism, i.e. $(a_0, \ldots, a_n, 0, 0, 0, \ldots) + (b_0, \ldots, b_n, 0, 0, 0, \ldots)$ is mapped to $\sum_{i=0}^{\max\{m,n\}} (a_i + b_i)p^i$. Restricted to these p-adic numbers, the rules described above are exactly the rules used for adding natural numbers represented in base p.

Fact 1.1.17. *Assume*

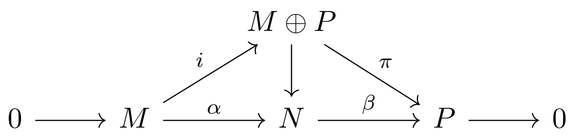
$$0 \longrightarrow G' \longrightarrow G \longrightarrow G/G' \longrightarrow 0$$

is an exact sequence of groups such that G' and G/G' are torsion-free. Then G is torsion-free as well.

Proof. Suppose that there exists $g \in G$ such that $ng = 0$ for some $n > 0$; then consider $\overline{g} \in G/G'$; we have $n\overline{g} = \overline{0}$ in G/G', which means that $g \in G'$, a contradiction. □

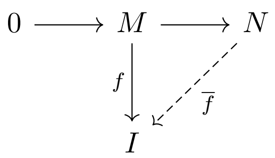
Proposition 1.1.18. *[2, Question 1] Let R be a ring and $0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} P \longrightarrow 0$ be a short exact sequence of R-modules. The following are equivalent:*

1. *there exists a* section *of β, that is, a map $s : P \longrightarrow N$ such that $\beta \circ S = id_P$;*
2. *there exists a* retraction *of α, that is, a map $r : N \longrightarrow M$ such that $r \circ \alpha = id_M$;*
3. *there exists an isomorphism $N \cong M \oplus P$ such that the following diagram commutes.*



If one of the above conditions is true we say that the sequence *splits*.

Definition 1.1.19. Let R be a ring. I is an *injective* R-module if to each diagram of R-modules with exact row



there exists $\overline{f} : N \longrightarrow I$ making the diagram commute.

This means that every homomorphism from a submodule M of N into I can always be extended to a homomorphism from the module N into I.

The definition of *projective* module is the dual notion obtained by reversing the direction of all the arrows:

Definition 1.1.20. Let R be a ring. P is a *projective* R-module if to each diagram of R-modules with exact row

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & 0 \\ & \overline{f} \nwarrow & \uparrow f & & \\ & & P & & \end{array}$$

there exists $\overline{f} : P \longrightarrow M$ making the diagram commute.

Proposition 1.1.21. *[10, Ch. XX, §4 - Ch. III, §4] Let R be a ring and suppose M and N are R-modules. I is an injective R-module if and only if each short exact sequence*

$$0 \longrightarrow I \longrightarrow M \longrightarrow N \longrightarrow 0$$

splits. Dually, P is a projective R-module if and only if each short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

splits.

Proposition 1.1.22. *[14, Thm. 3.5 - Lemma 3.33] Let R be a commutative PID and suppose I and P are R-modules. Then P is projective if and only if it is free and I is injective if and only if it is divisible.*

Remark 5. A free R-module is always projective, even if R is not a PID. Instead, assuming again that R is not a PID, there are R-modules which are divisible but not injective.[2] Moreover, there are no relations between injective modules and free modules. $\mathbb{Z}$ is free but not injective and $\mathbb{Q}$ is injective but not free.

Proposition 1.1.23. *[14, Cor. 3.35] Assume R is a commutative PID. Then any quotient of a divisible R-module is divisible.*

Definition 1.1.24. Let M be a $\mathbb{Z}$-module and let I^0 and I^1 be injective $\mathbb{Z}$-modules. Assume the following sequence

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{d^0} I^1 \longrightarrow 0$$

is exact. Then the (non-exact) sequence

$$0 \longrightarrow I^0 \xrightarrow{d^0} I^1 \longrightarrow 0$$

is an *injective resolution* of M.

Definition 1.1.25. Let M be a $\mathbb{Z}$-module and let P_0 and P_{-1} be projective $\mathbb{Z}$-modules. If the following sequence

$$0 \longrightarrow P_{-1} \xrightarrow{d_{-1}} P_0 \longrightarrow M \longrightarrow 0$$

[2] For example consider $R = \mathbb{Z}[t]$, the ring of polynomials over the integers $\mathbb{Z}$ and its field of fractions K (the smallest field in which it can be embedded): then K/R is divisible, seen as R-module, but it is not injective.

is exact, then the (non-exact) sequence

$$0 \longrightarrow P_{-1} \xrightarrow{d_{-1}} P_0 \longrightarrow 0$$

is called a *projective resolution* of M.

Theorem 1.1.26. *Each $\mathbb{Z}$-module M is both a quotient of a projective $\mathbb{Z}$-module and a submodule of an injective $\mathbb{Z}$-module.*

Proof. Let M^* be the set $M \setminus \{0\}$. Consider the free $\mathbb{Z}$-module $\bigoplus_{m \in M^*} \langle e_m \rangle$, where $\langle e_m \rangle \cong \mathbb{Z}$, and define

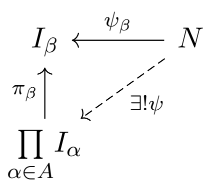
$$\varphi : \bigoplus_{m \in M^*} \langle e_m \rangle \longrightarrow M \quad \text{as} \quad n_1 e_{m_1} + ... + n_k e_{m_k} \mapsto n_1 m_1 + ... + n_k m_k,$$

where $n_1, ..., n_k \in \mathbb{Z}$ and $m_1, ..., m_k \in M$. This is a group homomorphism onto M, and therefore $M \cong \left(\bigoplus_{m \in M^*} \langle e_m \rangle \right) / \ker(\varphi)$. Since free modules are projective, M is a quotient of a projective $\mathbb{Z}$-module.

To prove that each $\mathbb{Z}$-module is a subgroup of an injective one, we need a lemma.

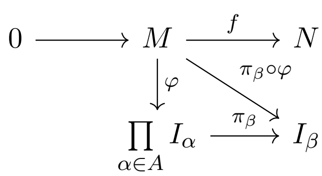
Lemma 1.1.27. *Let $\{I_\alpha\}_{\alpha \in A}$ be a family of injective $\mathbb{Z}$-modules. Then $\prod_{\alpha \in A} I_\alpha$ is injective too.*

Proof. We recall that the *direct product* is characterized by the following universal property: for each group N and each indexed family of group homomorphisms $\{\psi_\alpha : N \longrightarrow I_\alpha\}_{\alpha \in A}$ there exists a unique group homomorphism $\psi : N \longrightarrow \prod_{\alpha \in A} I_\alpha$ that makes the following diagram commute for all $\beta \in A$



where $\pi_\beta : \prod_{\alpha \in A} I_\alpha \longrightarrow I_\beta$ is the β-th coordinate projection.

Let M and N be two groups and $f : M \longrightarrow N$ be an injective group homomorphism and consider the following diagram.



Since I_β is injective for all $\beta \in A$, there exists $\varphi_\beta : N \longrightarrow I_\beta$ which extends $\pi_\beta \circ \varphi$. Thus we have a family of group homomorphisms $\{\varphi_\alpha\}_{\alpha\in A}$ and therefore there exists $\overline{\varphi} : N \longrightarrow \prod_{\alpha\in A} I_\alpha$ such that $\pi_\beta \circ \overline{\varphi} = \varphi_\beta$ for all $\beta \in A$. Moreover, $\pi_\beta \circ \overline{\varphi} \circ f = \varphi_\beta \circ f = \pi_\beta \circ \varphi$ for all $\beta \in A$. Thus, for universal property of the direct product, $\overline{\varphi} \circ f = \varphi$ and so $\prod_{\alpha\in A} I_\alpha$ is injective.

□

Given $m \in M^*$, if m is a torsion element, then $\langle m\rangle \cong \mathbb{Z}/ord(m)\mathbb{Z}$. Hence, defining $\psi_m(m) = [\frac{1}{ord(m)}]$, $\psi_m : \langle m\rangle \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a group homomorphism such that $\varphi_m(m) \neq [0]$. Otherwise, $\langle m\rangle \cong \mathbb{Z}$ and, posing $\varphi_m(m)$ different from $[0]$, one gets a group homomorphism such that $\varphi_m(m) \neq [0]$. In both cases, by injectivity of $\mathbb{Q}/\mathbb{Z}$ the inclusion map $\langle m\rangle \subseteq M$ induces an extension of ψ_m to $\varphi_m : M \longrightarrow \mathbb{Q}/\mathbb{Z}$: in particular $\varphi_m(m) \neq [0]$ for all $m \in M^*$.

Consider the family $\{\varphi_m\}_{m\in M^*}$. By universal property of the direct product, there exists a unique group homomorphism

$$\varphi : M \longrightarrow \prod_{m\in M^*} \mathbb{Q}/\mathbb{Z} \quad \text{such that} \quad \varphi(x) = (\varphi_m(x))_{m\in M^*} \text{ for all } x \in M.$$

Moreover, φ is injective, since $\varphi_m(m) \neq 0$ for all $m \in M^*$. By the previous lemma, $\prod_{m\in M^*} \mathbb{Q}/\mathbb{Z}$ is injective. Therefore M is a subgroup of an injective $\mathbb{Z}$-module. □

Corollary 1.1.28. *Every $\mathbb{Z}$-module M admits injective and projective resolutions.*

Proof. Since there exists an injective homomorphism between M and I, an injective group, we have the exact short sequence

$$0 \longrightarrow M \xrightarrow{i} I \longrightarrow I/i(M) \longrightarrow 0.$$

By Proposition 1.1.23,

$$0 \longrightarrow I \longrightarrow I/i(M) \longrightarrow 0$$

is an injective resolution of M. Similarly, we have an exact sequence

$$0 \longrightarrow \ker(\varphi) \longrightarrow P \xrightarrow{\varphi} M \longrightarrow 0,$$

where P is projective, thus free; hence also $\ker(\varphi)$ is free by Theorem 1.1.5. Therefore

$$0 \longrightarrow \ker(\varphi) \longrightarrow P \longrightarrow 0$$

is a projective resolution for M. □

Definition 1.1.29. Fix $\mathbb{Z}$-modules M and N. Using the notation of the previous proof, let $0 \longrightarrow I \xrightarrow{\pi} I/i(N) \longrightarrow 0$ be an injective resolution for N. One defines

$$Ext^1(M,N) \cong \frac{Hom_{\mathbb{Z}}(M, I/i(N))}{\pi_*(Hom_{\mathbb{Z}}(M,I))},$$

where $\pi_* : Hom_{\mathbb{Z}}(M, I) \longrightarrow Hom_{\mathbb{Z}}(M, I/i(N))$ is defined as $\pi_*(f) = \pi \circ f$ with $f \in Hom_{\mathbb{Z}}(M, I)$.

The above definition is well posed, i.e. it is independent of the injective resolution of N one chooses:

Theorem 1.1.30. *[14, Prop. 6.40] Let M, N be $\mathbb{Z}$-modules. Fix*

$$0 \longrightarrow N \xrightarrow{i_0} I_0 \xrightarrow{\pi_0} I_0/i_0(N) \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N \xrightarrow{i_1} I_1 \xrightarrow{\pi_1} I_1/i_1(N) \longrightarrow 0$$

short exact sequences such that I_0, I_1 are injective, and let

$$0 \longrightarrow I_0 \xrightarrow{\pi_0} I_0/i_0(N) \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow I_1 \xrightarrow{\pi_1} I_1/i_1(N) \longrightarrow 0$$

be the respectively injective resolutions for N. Then

$$\frac{Hom_{\mathbb{Z}}(M, I_0/i_0(N))}{\pi_{0*}(Hom_{\mathbb{Z}}(M, I_0))} \cong \frac{Hom_{\mathbb{Z}}(M, I_1/i_1(N))}{\pi_{1*}(Hom_{\mathbb{Z}}(M, I_1))}.$$

Definition 1.1.31. Fix $\mathbb{Z}$-modules M and N. With the notation used in the proof of the Corollary 1.1.28, let $0 \longrightarrow \ker(\varphi) \xrightarrow{i} P \longrightarrow 0$ be a projective resolution for M. One defines

$$\underline{Ext}^1(M, N) \cong \frac{Hom_{\mathbb{Z}}(\ker(\varphi), N)}{i^*(Hom_{\mathbb{Z}}(P, N))},$$

where $i^* : Hom_{\mathbb{Z}}(P, N) \longrightarrow Hom_{\mathbb{Z}}(\ker(\varphi), N)$ is defined as $i^*(f) = f \circ i$ with $f \in Hom_{\mathbb{Z}}(P, N)$.

As before, $\underline{Ext}^1(M, N)$ does not depend on the projective resolution for M chosen. Indeed:

Theorem 1.1.32. *[14, Prop. 6.20] Let M, N be $\mathbb{Z}$-modules. Fix*

$$0 \longrightarrow \ker(\varphi_0) \xrightarrow{i_0} P_0 \xrightarrow{\varphi_0} M \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \ker(\varphi_1) \xrightarrow{i_1} P_1 \xrightarrow{\varphi_1} M \longrightarrow 0$$

short exact sequences such that P_0, P_1 are projective, and let

$$0 \longrightarrow \ker(\varphi_0) \xrightarrow{i_0} P_0 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \ker(\varphi_1) \xrightarrow{i_1} P_1 \longrightarrow 0$$

be the respectively projective resolutions for M. Then

$$\frac{Hom_{\mathbb{Z}}(\ker(\varphi_0), N)}{i_0^*(Hom_{\mathbb{Z}}(P_0, N))} \cong \frac{Hom_{\mathbb{Z}}(\ker(\varphi_1), N)}{i_1^*(Hom_{\mathbb{Z}}(P_1, N))}.$$

We now present a result which is a particular case of a deep theorem in Homological Algebra, which will be frequently used in this thesis: in the most general formulation the setting is that of Abelian Categories, but the simplified assertion is all we need in this thesis. For a detailed discussion, see Chapter 6 of *An Introduction of Homological Algebra*, book of Joseph J. Rotman.

Theorem 1.1.33. *[14, Thm. 6.27 and 6.43] Let* $0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$ *be an exact sequence of* $\mathbb{Z}$*-modules,and let* M, N *be* $\mathbb{Z}$*-modules. There is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow Hom_{\mathbb{Z}}(Z,N) \xrightarrow{\beta^*} Hom_{\mathbb{Z}}(Y,N) \xrightarrow{\alpha^*} Hom_{\mathbb{Z}}(X,N) \longrightarrow \\ \longrightarrow \underline{Ext}^1(Z,N) \xrightarrow{\overline{\beta^*}} \underline{Ext}^1(Y,N) \xrightarrow{\overline{\alpha^*}} \underline{Ext}^1(X,N) \longrightarrow 0, \end{aligned}$$

where the maps β^* *and* α^* *are defined as the right composition by respectively* β $(- \circ \beta)$ *and* α $(- \circ \alpha)$ *and the maps* $\overline{\beta^*}$ *and* $\overline{\alpha^*}$ *are always defined as the right composition by respectively* β *and* α*, but passed to the quotient. Moreover, there is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow Hom_{\mathbb{Z}}(M,X) \xrightarrow{\alpha_*} Hom_{\mathbb{Z}}(M,Y) \xrightarrow{\beta_*} Hom_{\mathbb{Z}}(M,Z) \longrightarrow \\ \longrightarrow Ext^1(M,X) \longrightarrow Ext^1(M,Y) \longrightarrow Ext^1(M,Z) \longrightarrow 0, \end{aligned}$$

where the maps $\beta_*, \alpha_*, \overline{\beta_*}$ *and* $\overline{\alpha_*}$ *are defined as before, but this time as the left composition.*

Theorem 1.1.34. (Balance for *Ext*) *[10, Cor XX.8.5] Let* M, N *be* $\mathbb{Z}$*-modules. Then*

$$Ext^1(M,N) \cong \underline{Ext}^1(M,N).$$

Example. Let us consider $M = \mathbb{Z}/m\mathbb{Z}$ and $N = \mathbb{Z}/n\mathbb{Z}$, where m, n and positive integers. A projective resolution for $\mathbb{Z}_m$ could be obtained by the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0,$$

where the first map is the multiplication by m. Therefore

$$\underline{Ext}^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \frac{Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})}{(\cdot m)^* \left(Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})\right)} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

Taking instead an injective resolution from the exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \xrightarrow{\cdot n} \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \\ & 1 & \longmapsto & \frac{1}{n} & \end{array}$$

we obtain $Ext^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \dfrac{Hom_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}/\mathbb{Z})}{(\cdot n)_* \left(Hom_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}/\mathbb{Z})\right)} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$. As expected, we have $\underline{Ext}^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong Ext^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$.

Keeping in mind the previous isomorphism, from here onwards we will identify the two groups and write Ext^1 instead of $\underline{Ext}^1$. The *Balance for Ext* allows us to use injective resolution for N as well as projective resolution for M to compute $Ext^1(M,N)$.

Lastly, we remind the behaviour of the group of homomorphisms $Hom_{\mathbb{Z}}(A,B)$ toward direct sums and products and, consequently, $Ext^1(A,B)$.

Theorem 1.1.35. *[5, Thm.s 43.1 and 43.2] Let* $A, B, \{A_i\}_{i\in I}, \{B_i\}_{i\in I}$ *be all* $\mathbb{Z}$-*modules. Then*

$$Hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} A_i, B\Big) \cong \prod_{i\in I} Hom_{\mathbb{Z}}(A_i, B) \tag{1.1}$$

$$Hom_{\mathbb{Z}}\Big(A, \prod_{i\in I} B_i\Big) \cong \prod_{i\in I} Hom_{\mathbb{Z}}(A, B_i) \tag{1.2}$$

Proof. Let $\varphi : \bigoplus_{i\in I} A_i \longrightarrow B$ be a homomorphism and consider the map $\varphi \mapsto (\varphi|_{A_i})_{i\in I}$ of $Hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} A_i, B\Big)$ into $\prod_{i\in I} Hom_{\mathbb{Z}}(A_i, B)$. It is left to the reader to check that the map is an isomorphism.
Similarly, if $\pi_j : \prod_{i\in I} B_i \longrightarrow B_j$ is the j-th coordinate projection, given a homomorphism $\psi : A \longrightarrow \prod_{i\in I} B_i$, one conclude that $\psi \mapsto (\pi_i \circ \psi)_{i\in I}$ is an isomorphism too. □

Corollary 1.1.36. *[5, Cor. 43.4] Let* $T(A)$ *be the torsion subgroup of a group* A.

$$Hom_{\mathbb{Z}}(T(A), \mathbb{Q}/\mathbb{Z}) \cong \prod_p Hom_{\mathbb{Z}}(T_p(A), \mathbb{Z}(p^\infty))$$

where $T_p(A)$ *and* $\mathbb{Z}(p^\infty)$ *are respectively the* p-*components of* $T(A)$ *and* $\mathbb{Q}/\mathbb{Z}$ *as* p *ranges over the primes.*

1.2 Logical Prerequisites

1.2.1 Basic set theoretic notions

Like any other mathematical theory, Set Theory has a *universe* and the axioms and theorems have to hold within the universe. Informally, the objects in the universe of Set Theory are called *sets*, but we talk also about collections of elements from that universe. More formally, any collection of the form $\{x : \phi(x)\}$, where ϕ is a formula, is a *class*. We allow ϕ to have free variables other then x, which are thought of as parameters upon which the class depends.
A *set* is a class which belongs to another class. A class is not necessarily a set and the class of all sets $V = \{x : x = x\}$, called the *universal class* or the *universe*, is an example.

Definition 1.2.1. A total order $<$ of a set P is a *well-ordering* if every non-empty subset of P has a least element.

Definition 1.2.2. A set T is *transitive* if $x \subseteq T$ for each element $x \in T$. A set is an *ordinal number* (simply an *ordinal*) if it is transitive and well-ordered by $\in$.

The class of all ordinal numbers is denoted by *Ord*. Unless otherwise specified, when we talk about ordinals by $\beta < \alpha$ we will mean $\beta \in \alpha$.

By Axiom of Union, for any set X there exists a set $Y = \bigcup X$, the union of all elements of X. In symbols, $Y = \{x : x \in y \text{ for some } y \in X\}$.

Definition 1.2.3. Given a partially ordered class $(C, \leq)$ and X a non-empty subset of C, then $c \in C$ is a *upper bound* of X if for all $x \in X$ we have $x \leq c$. c is the *supremum* of X if c is the least upper bound of X. The supremum of X (if it exists) is denoted by $\sup X$.

Lemma 1.2.4. *[7, Lemma 2.3] If X is a non-empty set of ordinals, then $\bigcup X$ is an ordinal, and $\bigcup X = \sup X$.*

Definition 1.2.5. A *transfinite sequence* is a function whose domain is an ordinal α, and it is denoted by $\langle a_\xi : \xi \in \alpha \rangle$. It is also called a *sequence of lenght* α. Given a sequence $\langle \beta_\xi \rangle_{\xi \in \alpha}$ of ordinals indexed by α, we define the $\sup\limits_{\xi \in \alpha} \beta_\xi$ as the supremum of the set $\{\beta_\xi : \xi \in \alpha\}$, which is an ordinal by Lemma 1.2.4.

We define $\alpha + 1 = \alpha \cup \{\alpha\}$ (the *successor* of α). If $\alpha = \beta + 1$, then α is a *successor ordinal*. Otherwise $\alpha = \sup\{\beta : \beta \in \alpha\} = \bigcup \alpha$ and α is called a *limit ordinal*.
We denote the least non-zero limit ordinal ω (which is $\mathbb{N}$): the ordinals less than ω are called *finite ordinals*, or *natural numbers*.

We recall that between linearly ordered set an *isomorphism* is a one-to-one order preserving function.

Proposition 1.2.6. *[7, Thm. 2 and 15] Every set can be well-ordered and each well-ordered set is* isomorphic *to a unique ordinal number.*

The Induction Principle and the Recursion Theorem are common tools for proving theorems about natural numbers: now we present their generalizations for ordinal numbers that we will use in the third Chapter.

Theorem 1.2.7 (Transfinite Induction)**.** *[1, Thm. 40.1] Let $P(x)$ be a property. Assume that, for all ordinal numbers α,*

$$if\ P(\beta)\ holds\ for\ all\ \beta \in \alpha,\ then\ P(\alpha).$$

Then $P(\alpha)$ holds for all ordinals α.

Theorem 1.2.8 (Transfinite Recursion, Definition by Transfinite Induction)**.** *[1, Thm. 30.2] Let $H : Ord \times V \longrightarrow V$ be a class function (a* class function *is a* *class which is a relation with the property that if both* (x, y) *and* (x, y') *belong to it then* $y = y'$*.). Then there exists a unique function* $F : Ord \longrightarrow V$ *such that* $F(\alpha) = H(\alpha, F|_{\beta:\beta\in\alpha})$ *for all ordinals* α*.*

Definition 1.2.9. An ordinal α is called a *cardinal number*, or a *cardinal*, if $|\alpha| \neq |\beta|$ for all $\beta \in \alpha$.

If X is a well-ordered set, by 1.2.6 there exists an ordinal α such that $|X| = |\alpha|$. Thus we let $|X|$ be the least ordinal such that $|X| = |\alpha|$, which is obviously a cardinal. For every $\alpha \in Ord$, we let α^+ denote the least cardinal greater than α (the *cardinal successor* of $|\alpha|$). We will use $\aleph_0$ to denote $|\omega|$ and $\aleph_1$ for $|\omega|^+$ when we want to underline the cardinality of those sets.

We briefly report the arithmetic operations on cardinals: it is defined as follows:

$\kappa + \lambda = |A \bigcup B|$ where $|A| = \kappa$, $|B| = \lambda$ and A, B are disjoint,

$\kappa \cdot \lambda = |A \times B|$ where $|A| = \kappa$, $|B| = \lambda$,

$\kappa^\lambda = |{}^B A|$ where $|A| = \kappa$, $|B| = \lambda$ and ${}^B A$ is the set of all function from A to B.

Naturally, the above definition is meaningful only if it does not depend on the choice of the sets A and B. However, addition and multiplication of infinite cardinal numbers is a trivial matter.

Theorem 1.2.10. *Let* κ *and* λ *be infinite cardinals. Then we have* $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$*.*

For the exponentiation of cardinals we have the following theorem.

Theorem 1.2.11. *[7, Thm. 6] (Cantor's Theorem) For every cardinal* κ*,* $\kappa < 2^\kappa$*.*

Definition 1.2.12. Let α and β be two ordinals. A function $f : \alpha \longrightarrow \beta$ is *cofinal* in β if its range is not limited, i.e. $\sup\limits_{\xi\in\alpha} f(\xi) = \beta$. The *cofinality* of β, denoted by $cf(\beta)$, is the least ordinal α such that there exists a cofinal function from α into β. An ordinal β is *regular* if $cf(\beta) = \beta$.

Unless otherwise specified, κ will be a regular uncountable cardinal.

Definition 1.2.13. We say that a set $C \subseteq \kappa$ is *closed unbounded* (*Cub*) in κ if

1. for every increasing sequence $\{\alpha_\delta\}_{\delta\in\gamma\in\kappa}$, $\sup\limits_{\delta\in\gamma} \alpha_\delta \in C$ (*closed*);
2. for every $\alpha \in \kappa$, there is a $\beta > \alpha$ such that $\beta \in C$ (*unbounded*).

Lemma 1.2.14. *[7, Lemma 7.3] If C and D are closed unbounded, then $C \cap D$ is closed unbounded.*

Definition 1.2.15. A function $f : \kappa \longrightarrow \kappa$ is *normal* if it is increasing and continuous or, equivalently, if $f(\alpha) = \sup_{\delta \in \alpha} f(\delta)$ for each limit ordinal $\alpha \in \kappa$.

Remark 6. A range of a normal function on κ is a Cub on κ. Conversely, if C is a Cub there is a unique normal function that enumerates C (i.e. such that its range is exactly C).

Definition 1.2.16. A set $S \subseteq \kappa$ is said *stationary* if $S \cap C \neq \varnothing$ for every Cub $C \subseteq \kappa$.

Definition 1.2.17. Let A be a set of cardinality $\leq \kappa$. A *κ-filtration* of A is an ascending chain

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots$$

where α belongs to κ such that:

1. $|A_\alpha| < \kappa$ for each $\alpha \in \kappa$;
2. $A_\lambda = \bigcup_{\alpha \in \lambda} A_\alpha$ for each limit ordinal $\lambda \in \kappa$ (*continuity*);
3. $A = \bigcup_{\alpha \in \kappa} A_\alpha$.

The κ-filtration is said *strictly increasing* if the containments are all strict. When A is a group for a *κ-filtration* of A we mean a κ-filtration of A as a set and such that A_α is a group for all $\alpha \in \kappa$.

The statements and theorems of Chapter 2 are framed in the standard context of ZFC set theory. However the problem we will discuss is connected with questions of provability and consistency: so now we introduce the additional axioms which we will make use in Chapter 3.

1.2.2 Diamond

A principle we will see is a combinatorial consequence of the *Axiom of Constructibility* (denoted by V=L). Kurt Gödel introduced the Axiom of Constructibility in connection with his prof that the Axiom of Choice and the Continuum Hypothesis (i.e. $\aleph_1 = 2^{\aleph_0}$) are relatively consistent with ZF ([7, Thm. 32 and 33]).
As mentioned, in the second section of Chapter 3 we will work in the framework of ZFC plus the additional axiom $V = L$ but actually we will make use only of the following "prevision" property.

Definition 1.2.18. Let E be a stationary subset of a regular uncountable ordinal κ: $\diamondsuit_\kappa(E)$ (the *Diamond Principle* for E) holds if there exists a sequence of sets

$\{X_\alpha\}_{\alpha \in E}$ such that

1. $X_\alpha \subseteq \alpha$ for all $\alpha \in E$;
2. the set $\{\alpha \in E : X_\alpha = X \cap \alpha\}$ is stationary in κ for all $X \subseteq \kappa$.

REMARK 7. Remark that if $E \subseteq F$ are stationary sets and $\diamondsuit_\kappa(E)$ holds, $\diamondsuit_\kappa(F)$ holds as well.

$\diamondsuit_\kappa$ holds if $\diamondsuit_\kappa(E)$ holds for all stationary sets E. The consistence of this property follows from the next theorem proved by Jensen.

Theorem 1.2.19. *[8, Lemma 6.5] ZFC + $V = L$ implies that $\diamondsuit_\kappa$ holds for all regular uncountable cardinal κ. Hence $\diamondsuit_\kappa$ is consistent with ZFC.*

Now we present some convenient consequences of the Diamond Principles for κ-filtrations.

Theorem 1.2.20. *Let A be a set of cardinality κ and $\{A_\alpha\}_{\alpha \in \kappa}$ a κ-filtration strictly increasing of A. Let E be a stationary subset of κ. If $\diamondsuit_\kappa(E)$ holds then:*

1. *there exists a family $\{Y_\alpha\}_{\alpha \in E}$ such that for each $\alpha \in E$, $Y_\alpha \subseteq A_\alpha$, and for all $X \subseteq A$ the set $\{\alpha \in E : Y_\alpha = X \cap A_\alpha\}$ is stationary in κ;*
2. *let B be a set of cardinality $\leq \kappa$ and let $\{B_\alpha\}_{\alpha \in \kappa}$ a κ-filtration of B: there is a family $\{g_\alpha\}_{\alpha \in E}$ such that for each $\alpha \in E$, g_α is a function between A_α and B_α, and for every function $f : A \longrightarrow B$ the set $\{\alpha \in E : f|_{A_\alpha} = g_\alpha\}$ is stationary in κ.*

First of all, we need a lemma.

Lemma 1.2.21. *Given $\{A_\alpha\}_{\alpha \in \kappa}$ as above there exists a Cub $C \subseteq \kappa$ such that for all $\beta \in C$, $|A_{\beta^+} \setminus A_\beta| = |\beta^+ \setminus \beta|$, where $\beta^+ = \inf\{\gamma \in C : \beta \in \gamma\}$.*

Proof. Since the κ-filtration is strictly increasing $|A_\gamma \setminus A_\beta| \geq |\gamma \setminus \beta|$ for all $\beta \in \gamma \in \kappa$. Fixed β, we have

$$|\gamma_0 \setminus \beta| \leq |A_{\gamma_0} \setminus A_\beta| \leq |A_{\gamma_0}| < \kappa,$$

thus there exists a $\gamma_1 \in \kappa$ such that $|A_{\gamma_0}| \leq |\gamma_1 \setminus \beta|$ (for istance, one can define $\gamma_1 = \beta + |A_{\gamma_0}| \in \kappa$). In this way, by induction, one can build $\{\gamma_n\}_{n \in \omega} \subseteq \kappa$ so that:

$$\cdots \leq |\gamma_n \setminus \beta| \leq |A_{\gamma_n} \setminus A_\beta| \leq |\gamma_{n+1} \setminus \beta| \leq \cdots .$$

If $\gamma(\beta) = \sup_{n \in \omega} \gamma_n$ (which still belongs to κ, since κ is regular and uncountable), then

$$|A_{\gamma(\beta)} \setminus A_\beta| = |\gamma(\beta) \setminus \beta|.$$

Therefore given $\beta \in \kappa$, set $\gamma(\beta)$ equal to an ordinal such that $|A_{\gamma(\beta)} \setminus A_\beta| = |\gamma(\beta) \setminus \beta|$, and define $\gamma^\mu(\beta)$ by transfinite induction as follows:

If $\mu = \delta + 1$, then set $\gamma^{\mu}(\beta) = \gamma(\gamma^{\delta}(\beta)) \in \kappa$.
If μ is a limit ordinal, then set $\gamma^{\mu}(\beta) = \sup_{\delta \in \mu} \gamma^{\delta}(\beta)$ which belongs to κ always by regularity of κ.
This construction gives rise to a function $f : \kappa \longrightarrow \kappa$ which maps μ into $\gamma^{\mu}(\beta)$. This is a normal function by how we have defined $\gamma^{\mu}(\beta)$. Therefore $f[\kappa] = C$ is a Cub and it has the required properties. □

Proof of 1. Now consider the Cub C foreseen by Lemma 1.2.21: since $|A_{\beta^+} \setminus A_{\beta}| = |\beta^+ \setminus \beta|$ for all $\beta \in C$, then one could define by transfinite induction a bijective function $F : \kappa \longrightarrow A$ such that $F[\beta] = A_{\beta}$ for all $\beta \in C$. Consider the sequence of sets $\{X_{\alpha}\}_{\alpha \in E}$, whose existence is guaranteed by $\diamondsuit_{\kappa}(E)$, and define $Y_{\alpha} = F[X_{\alpha}]$ if $\alpha \in E \cap C$ and $Y_{\alpha} = \varnothing$ if $\alpha \notin E \cap C$. First of all we notice that $Y_{\alpha} = F[X_{\alpha}] \subseteq F[\alpha] = A_{\alpha}$ for all $\alpha \in E \cap C$, since $X_{\alpha} \subseteq \alpha$. On the other hand, $\varnothing \subseteq A_{\alpha}$ always, and so $Y_{\alpha} \subseteq A_{\alpha}$ for each $\alpha \in E$. Now consider $X \subseteq A$. We have the following set relations:

$$\{\alpha \in E : Y_{\alpha} = X \cap A_{\alpha}\} \supseteq \{\alpha \in E \cap C : Y_{\alpha} = X \cap A_{\alpha}\} = \{\alpha \in E \cap C : F[X_{\alpha}] = X \cap F[\alpha]\}.$$

Define $\overline{X} = F^{-1}(X)$. Then the set $\{\alpha \in E : X_{\alpha} = \overline{X} \cap \alpha\}$ is stationary in κ. However, if S is stationary and C is a Cub, $S \cap C$ is stationary. Thus the set

$$\{\alpha \in E : X_{\alpha} = \overline{X} \cap \alpha\} \cap C = \{\alpha \in E \cap C : X_{\alpha} = \overline{X} \cap \alpha\}$$

is stationary. If $X_{\alpha} = \overline{X} \cap \alpha$, then $F[X_{\alpha}] = F[\overline{X} \cap \alpha] = F[\overline{X}] \cap F[\alpha] = X \cap F[\alpha]$ since F is a bijection. Hence

$$\{\alpha \in E \cap C : X_{\alpha} = \overline{X} \cap \alpha\} \subseteq \{\alpha \in E : Y_{\alpha} = X \cap A_{\alpha}\}$$

and the set $\{\alpha \in E : Y_{\alpha} = X \cap A_{\alpha}\}$ is stationary too.

□

Proof of 2. Define Z as $A \times B$ and $\{Z_{\alpha}\}_{\alpha \in \kappa}$ as $\{A_{\alpha} \times B_{\alpha}\}_{\alpha \in \kappa}$. Observe that Z has still cardinality equal to κ and $\{Z_{\alpha}\}_{\alpha \in \kappa}$ is an increasing κ-filtration of Z. By the first part of this theorem, there exists a family $\{Y_{\alpha}\}_{\alpha \in E}$ such that for each $\alpha \in E$, $Y_{\alpha} \subseteq A_{\alpha} \times B_{\alpha}$, and for all $X \subseteq A \times B$, the set $\{\alpha \in E : Y_{\alpha} = X \cap (A_{\alpha} \times B_{\alpha})\}$ is stationary in κ. For each α define $g_{\alpha} = Y_{\alpha}$ if Y_{α} is a function, and g_{α} being an arbitrary function otherwise.

Pick a function $f \subseteq A \times B$; then $S = \{\alpha \in E : Y_{\alpha} = f \cap (A_{\alpha} \times B_{\alpha})\}$ is stationary. Now define $\overline{B}_{\alpha} = B_{\alpha} \cap f[A]$. $\{\overline{B}_{\alpha}\}_{\alpha \in \kappa}$ is a κ-filtration for $f[A]$. Finally consider $\overline{B'}_{\alpha} = f[A_{\alpha}]$; since $f[\bigcup_{i \in I} A_i] = \bigcup_{i \in I} f[A_i]$, then also $\{\overline{B'}_{\alpha}\}_{\alpha \in \kappa}$ is a κ-filtration of $f[A]$. By Lemma 3.1.1 we have that the set $C = \{\alpha \in \kappa : B_{\alpha} \cap f[A] = f[A_{\alpha}]\}$ is a Cub. If $B_{\alpha} \cap f[A] = f[A_{\alpha}]$, then $f[A_{\alpha}] \subseteq B_{\alpha}$. Hence we have the following relations:

$$S \cap C \subseteq \{\alpha \in E : Y_{\alpha} = f \cap (A_{\alpha} \times B_{\alpha})\} \bigcap \{\alpha \in \kappa : f[A_{\alpha}] \subseteq B_{\alpha}\}.$$

The right-side set is equal to $\{\alpha \in E : Y_{\alpha} = f|_{A_{\alpha}}\}$. It follows that for all $\alpha \in S \cap C$, $Y_{\alpha} = f|_{A_{\alpha}}$, a function and so $g_{\alpha} = f|_{A_{\alpha}}$. Since $S \cap C$ is stationary, the family $\{g_{\alpha}\}_{\alpha \in E}$ is the desired one. □

1.2.3 Martin's axiom

Let $(P, <)$ be a partially ordered set (also called a *poset*).

Definition 1.2.22. A set $D \subseteq P$ is *dense* in P if for every $p \in P$ there exists $q \in D$ such that $q \leq p$.

Definition 1.2.23. A set $F \subseteq P$ is a *filter* on P if it satisfies the following requests:

1. F is non-empty;
2. if $p \leq q$ and $p \in F$, then $q \in F$;
3. if $p, q \in F$, there exists an $r \in F$ such that $r \leq p$ and $r \leq q$.

Definition 1.2.24. Consider a set $\mathcal{D} \subseteq \mathcal{P}(P)$. A subset $G \subseteq P$ is a *$\mathcal{D}$-generic filter* on P if:

1. G is a filter;
2. $G \cap D \neq \emptyset$ for each dense subset of P contained in $\mathcal{D}$.

Definition 1.2.25. Given a poset $(P, <)$ two elements p_1 and p_2 are *compatible* ($p_1 \parallel p_2$), if there exists $q \in P$ such that both $q \leq p_1$ and $q \leq p_2$. Otherwise they are *incompatible* ($p_1 \perp p_2$).

Definition 1.2.26. A poset $(P, <)$ satisfies the *countable chain condition* (*ccc*) if every collection of pairwise incompatible elements of P is at most countable, i.e. for each subset $\{p_\alpha\}_{\alpha \in \aleph_1}$ there exist $\gamma \in \beta \in \aleph_1$ such that $p_\gamma \parallel p_\beta$.

Lemma 1.2.27. *If $(P, <)$ is a poset and $\mathcal{D}$ is a countable collection of dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.*

Proof. Let $\{D_n\}_{n \in \omega}$ be an enumeration of the sets in $\mathcal{D}$. Define by induction a family $\{d_n\}_{n \in \omega}$ as follows: let d_0 be a generic element of D_0. Let $d_n \in D_n$ be such that $d_n \leq d_{n-1}$ (a such defined d_n exists since D_n is dense). Then the set $G = \{p \in P :$ there is $n \in \omega$ such that $d_n \leq p\}$ is a filter on P, and $G \cap D_n \neq \varnothing$. Thus G is a $\mathcal{D}$-generic filter on P. □

Remark 8. If $\mathcal{D}$ is an uncountable collection of dense subsets of $(P, <)$ it is not always true that there exists a $\mathcal{D}$-generic filter on P.

Proof. Consider the set $P = \{f : F \longrightarrow \omega_1 : F \subseteq \omega \text{ and } |F| < \aleph_0\}$ ordered by reversed inclusion. For each $\alpha \in \aleph_1$ define $D_\alpha = \{f \in P : \alpha \in \operatorname{ran}(f)\}$, where $\operatorname{ran}(f)$ is the range of f. If $g : G \longrightarrow \omega_1$ is an element of $P \setminus D_\alpha$, we choose an integer which does not belong to the domain of g and define $\widetilde{g} : G \cup \{n\} \longrightarrow \omega_1$ as the map which extends g letting $\widetilde{g}(n) = \alpha$. It follows that $\widetilde{g} \in D_\alpha$, and so D_α is dense for each $\alpha \in \omega_1$. Thus the set $\mathcal{D} = \{D_\alpha : \alpha \in \aleph_1\}$ is an uncountable collection of dense subsets of P.

Suppose by way of contradiction that there exists a $\mathcal{D}$-generic filter $\mathcal{G}$ on $(P, <)$. If f and g belong to $\mathcal{G}$ then they have to coincide over the intersection of their domains, since there exists a function which extends both of them. It follows that, taking only the functions in $\mathcal{G}$, each non-negative integer can be mapped at most in one element of ω_1. Thus the union of the ranges of the functions in $\mathcal{G}$ is a function g with countable domain (being a subset of $\mathbb{N}$). However the condition for $\mathcal{G}$ to be $\mathcal{D}$-generic grants that the range of g is the whole ω_1 which is uncountable. This is a contradiction.

□

Martin's Axiom springs from a generalization of Lemma 1.2.27.

Definition 1.2.28 (Martin's Axiom, MA)**.** If $(P, <)$ is a poset that satisfies the ccc and $\mathcal{D}$ is a collection of less than $2^{\aleph_0}$ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.

The consistency of MA with ZFC follows from the following theorem:

Theorem 1.2.29 (Solovay, Tennenbaum)**.** *[7, Thm. 51] Martin's Axiom plus the negation of the Continuum Hypothesis* $(\aleph_1 < 2^{\aleph_0})$ *is consistent with ZFC.*

The particular consequence of Martin's axiom we will use is given by the following:

Lemma 1.2.30. *Assume MA. Let A and B be two sets of cardinality $< 2^{\aleph_0}$, and let P be a family of functions with the following properties:*

1. *if $f \in P$, then f is a function $f : A' \longrightarrow B$ where $A' \subseteq A$;*
2. *for every $a \in A$ and every $f \in P$, there exists $g \in P$ such that $f \subseteq g$ and $a \in \mathrm{dom}(g)$;*
3. *for each uncountable $P' \subseteq P$, there exists $f_1, f_2 \in P'$ and $f \in P$ such that $f_1 \neq f_2$ and $f_1 \subseteq f$ and $f_2 \subseteq f$.*

Then there exists a function $g : A \longrightarrow B$ defined on all of A such that for each finite $F \subseteq A$ there exists $f \in P$ such that $F \subseteq \mathrm{dom}(f)$ and $f\restriction_F = g\restriction_F$.

Proof. Consider P ordered by reversed inclusion, i.e. $f_1 \leq f_2$ if and only if f_1 extends f_2. For every $F \subseteq A$ of finite cardinality, let D_F be the set $\{f \in P : F \subseteq \mathrm{dom}(f)\}$, where $\mathrm{dom}(f)$ is the domain of f, and let $\mathcal{D}$ be the collection of the D_F. The cardinality of $\mathcal{D}$ is less than $2^{\aleph_0}$ since

$$|\mathcal{D}| \leq |\{F \subseteq A : |F| < \aleph_0\}| \leq \sum_{n\in\omega} |A|^n = \aleph_0 \cdot |A| < 2^{\aleph_0}.$$

By the second property of P each D_F is dense. The third property is exactly the ccc for the order defined on P. Therefore by Martin's Axiom, it follows that there exists a $\mathcal{D}$-generic filter $\mathcal{G}$ on P.

For each $F \subseteq A$ of finite cardinality let g_F be an element of $\mathcal{G} \cap D_F$. Define $g : A \longrightarrow B$ as $\bigcup \mathcal{G}$. Since $\mathcal{G}$ is a filter, g is a function. Clearly for any F finite subset of A, $g \restriction F = g_F$. In particular since $\mathcal{G} \cap D_F$ is dense for any finite subset of A, for any $a \in A$ $g(a) = g_F(a)$ for some $F \ni a$ with $g_F \in \mathcal{G}$.

$\square$

2

ZFC results

In this Chapter we give a variety of results on the structure of $Ext^1(A, \mathbb{Z})$ which can be proved in ZFC. First of all we prove that if A is torsion free $Ext^1(A, \mathbb{Z})$ is divisible and draw some elementary but very useful consequences from this fact.

Lemma 2.0.1. *Assume A is a $\mathbb{Z}-$module such that $T(A) = 0$, where $T(A)$ denote the torsion subgroup of A. Then $Ext^1(A, M)$ is injective for any $\mathbb{Z}-$module M.*

Proof. Consider an injective resolution of M

$$0 \longrightarrow M \xrightarrow{i} I \xrightarrow{\pi} I/i(M) \longrightarrow 0.$$

Then we have

$$Ext^1(A, M) = \frac{Hom_{\mathbb{Z}}(A, I/i(M))}{\pi_*(Hom_{\mathbb{Z}}(A, I))}.$$

By Remark 1.1.23, in order to see that $Ext^1(A, M)$ is divisible it is enough to prove this for $Hom_{\mathbb{Z}}(A, I/i(M))$. Consider the map $A \xrightarrow{\cdot n} A$ given by the multiplication by a non-zero integer n: A torsion-free implies $\cdot n$ in injective. Since $I/i(M)$ is injective any map between A and $I/i(M)$ factorizes through $\cdot n$.

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\cdot n} & A \\ & & {\scriptstyle\phi}\downarrow & \swarrow{\scriptstyle\psi} & \\ & & I/i(M) & & \end{array}$$

Or rather $Hom_{\mathbb{Z}}(A, I/i(M))$ is divisible, and so injective. □

Let A be a group: consider the short exact sequence

$$0 \longrightarrow T(A) \longrightarrow A \longrightarrow A/T(A) \longrightarrow 0.$$

Applying $Hom_{\mathbb{Z}}(-, \mathbb{Z})$, by Theorem 1.1.33, we obtain the long exact sequence

$$0 \longrightarrow Hom_{\mathbb{Z}}(A/T(A), \mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(A, \mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(T(A), \mathbb{Z}) \longrightarrow$$
$$\longrightarrow Ext^1(A/T(A), \mathbb{Z}) \longrightarrow Ext^1(A, \mathbb{Z}) \longrightarrow Ext^1(T(A), \mathbb{Z}) \longrightarrow 0.$$

Since $Hom_{\mathbb{Z}}(T(A), \mathbb{Z}) = 0$ (given that the unique element of finite order of $\mathbb{Z}$ is 0) we get the following short exact sequence:

$$0 \longrightarrow Ext^1(A/T(A), \mathbb{Z}) \longrightarrow Ext^1(A, \mathbb{Z}) \longrightarrow Ext^1(T(A), \mathbb{Z}) \longrightarrow 0. \tag{2.1}$$

Since $A/T(A)$ has no torsion part, $Ext^1(A/T(A), \mathbb{Z})$ is injective and so the sequence 2.1 splits by Proposition 1.1.21 and Lemma 2.0.1. Hence we obtain:

Proposition 2.0.2. *Let A be any commutative group. Then:*

$$Ext^1(A, \mathbb{Z}) \cong Ext^1(T(A), \mathbb{Z}) \bigoplus Ext^1(A/T(A), \mathbb{Z}).$$

2.1 $Ext^1(A, \mathbb{Z})$ for torsion groups A

We briefly describe how to characterize $Ext^1(A, \mathbb{Z})$ for torsion groups A. Take the injective resolution of $\mathbb{Z}$ given by

$$0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Since $Hom_{\mathbb{Z}}(T(A), \mathbb{Q})$ is trivial, we get that

$$Ext^1(T(A), \mathbb{Z}) \cong \frac{Hom_{\mathbb{Z}}(T(A), \mathbb{Q}/\mathbb{Z})}{\pi_*(Hom_{\mathbb{Z}}(T(A), \mathbb{Q}))} \cong Hom_{\mathbb{Z}}(T(A), \mathbb{Q}/\mathbb{Z}).$$

By Corollary 1.1.36, $Hom_{\mathbb{Z}}(T(A), \mathbb{Q}/\mathbb{Z}) \cong \prod_p Hom_{\mathbb{Z}}(T_p(A), (\mathbb{Q}/\mathbb{Z})_p)$, where $T_p(A)$ and $(\mathbb{Q}/\mathbb{Z})_p \cong \mathbb{Z}(p^\infty)$ are respectively the p-components of $T(A)$ and $\mathbb{Q}/\mathbb{Z}$, as p ranges over the primes. For a torsion group A a complete description of the homomorphism groups $Hom_{\mathbb{Z}}(A, \mathbb{Z}(p^\infty))$ and its structure is given in [5, Thm. 47.1]. In the next sections of this Chapter we give a thorough and detailed analysis of the structure of $Ext^1(A, \mathbb{Z})$ for A torsion free and countable. Instead in the next Chapter we will discuss the possible structure of $Ext^1(A, \mathbb{Z})$ for uncountable groups A.

2.2 The ranks of a group

Every abelian group A contains subgroups that are direct sums of cyclic groups. We will use those which are, in a certain sense, maximal among them to define cardinal numbers depending only on A. This leads to the definition of *ranks* of A, which extend to groups the notion of dimension for vector spaces.

Definition 2.2.1. A system $\{a_1, ..., a_k\}$ of non-zero elements of a group A is called independent if

$$\sum_{i=1}^{k} n_i a_i = 0 \quad (n_i \in \mathbb{Z}) \quad \Longrightarrow \quad n_i a_i = 0 \quad \forall i \in \{1, \dots, k\}.$$

This means that $n_i = 0$ if $ord(a_i) \notin \mathbb{N}$ and $ord(a_i) | n_i$ otherwise.

Definition 2.2.2. A system $L = \{a_i\}_{i \in I}$ of non-zero elements of A is *independent* if each finite subsystem of L is independent (remark that I could be infinite). An independent system M of A is *maximal* if there is no independent system in A containing M properly.

An independent system cannot contain equal elements, hence it is a set.

Definition 2.2.3. Given an independent system L, an element $g \in A$ is *dependent* on L if there exist $m, n_1, \dots, n_k \in \mathbb{Z}$ and $a_1, \dots, a_k \in L$ such that

$$mg = \sum_{i=1}^{k} n_i a_i \neq 0.$$

Definition 2.2.4. Given a group A, let M_0 be an independent system of A containing only elements of infinite order maximal with respect to this property[1]. The *torsion-free rank* of A, denoted by $r_0(A)$, is the cardinality of M_0.
Analogously, for p ranging over the prime numbers, we define the *p-rank* $r_p(A)$ of A as the cardinality of an independent system M_p which contains only elements whose orders are powers of p rather than infinite.

We will argue that for any group A the cardinals $r_0(A)$ and $r_p(A)$ are independent respectively of the maximal systems M_0 and M_p chosen to compute them.

Theorem 2.2.5. *For any group A $r_0(A)$ and $r_p(A)$ do not depend on the chosen independent systems M_0 and M_p, for any p, hence it gives well defined notions of torsion-free rank and p-ranks for A.*

We start with the torsion-free rank.

Proof. We first prove the theorem assuming A is torsion-free.

Let M be an independent maximal system for A. Since A is torsion-free, every element of M has infinite order. For $g \in A \setminus \{0\}$, $\{M, g\}$ is no longer independent, which means that there exist $n, n_1, \dots, n_k \in \mathbb{Z}$ and $a_1, \dots, a_k \in M$ such that

$$ng = \sum_{i=1}^{k} n_i a_i.$$

Assume $ng' = ng$, then $n(g' - g) = 0$, giving that $g' = g$ (since A is torsion-free). So one can injectively associate a tuple $\{n, n_1, \dots, n_k, a_1, \dots, a_k\}$ to each element of A. It follows that

$$|A| \leq |\bigsqcup_{k \in \mathbb{N}} \mathbb{Z}^{k+1} \times M^k| = \sum_{k \in \mathbb{N}} |M| \cdot \aleph_0 = \max\{|M|, \aleph_0\}.$$

[1] Zorn's Lemma ensures its existence.

Since $|M| \leq |A|$ always, if $|M| \geq \aleph_0$, then $|A| = |M|$. Hence in this case the rank is well defined. Suppose now that $M = \{a_1, \dots, a_n\}$ is finite. Assume $\{b_1, \dots, b_m\}$ is an independent system then

$$\langle b_i \rangle \bigcap \sum_{j \neq i} \langle b_j \rangle = \{0\} \qquad \Longrightarrow \qquad \langle b_1, \dots, b_m \rangle = \bigoplus_{i=1}^{m} b_i \cong \mathbb{Z}^m.$$

By maximality of M, for all $j \in \{1, \dots, m\}$ there exists $m_j \in \mathbb{Z}$ such that $0 \neq m_j b_j \in \langle a_1, \dots, a_n \rangle$. Observe that if $\{b_1, \dots, b_m\}$ is an independent system then $\{m_1 b_1, \dots, m_m b_m\}$ is also independent. Thus $\bigoplus_{i=1}^{m} m_i b_i \subseteq \bigoplus_{j=1}^{n} a_j$. Since $\mathbb{Z}$ is a PID, by Theorem 1.1.5 we obtain $m \leq n$. We conclude that every maximal independent system in A has the same cardinality also in case M is finite.

Now we assume A has torsion and we fix M independent system for A maximal with respect to the property of containing only elements of infinite order. We will show that $|M| = r_0(A/T(A))$, yielding that $r_0(A) = |M| = r_0(A/T(A))$ is well defined.

Assume $a \in M$ then $\overline{a} \neq 0$ in $A/T(A)$, since $a \notin T(A)$. Now pick a subset $\{a_1, \dots, a_m\} \subseteq M$:

$$n_1 \overline{a_1} + \dots + n_m \overline{a_m} = \overline{0} \qquad \Longrightarrow \qquad n_1 a_1 + \dots + n_m a_m = b \in T(A).$$

If $b = 0$, then $n_i = 0$ for all $i = 1, \dots, m$ (since M is an independent system of torsion-free elements $\langle b_1, \dots, b_m \rangle = \bigoplus_{i=1}^{m} b_i \cong \mathbb{Z}^m$).

Otherwise, multiplying by the order of b, we obtain

$$ord(b) n_1 \cdot a_1 + \dots + ord(b) n_m \cdot a_m = 0,$$

which (by the same argument) holds only if $ord(b) n_i = 0$ for all $i = 1, \dots, m$, whence $n_i = 0$ for all $i = 1, \dots, m$.

Vice versa, if $\{\overline{a_1}, \dots, \overline{a_m}\} \subseteq A/T(A)$ is an independent system for $A/T(A)$ and $\overline{a_i} = b_i + T(A)$ for all i, we get that

$$n_1 b_1 + \dots + n_m b_m = 0 \qquad \Longrightarrow \qquad n_1 \overline{a_1} + \dots + n_m \overline{a_m} = \overline{0},$$

giving once again that $n_i = 0$ for all $i \in \{1, \dots, m\}$. This proves that $r_0(A) = r_0(A/T(A))$, and it is well defined also in this case.

□

We now prove the theorem for p an arbitrary prime.

Definition 2.2.6. We recall that the *socle* $S(A)$ of a group A is the subgroup consisting of all $a \in A$ whose order is a square-free integer.

Proof. The proof is an immediate corollary of the following remark.

REMARK 9. Let A be a group. $M = \{a_i\}_{i\in I}$ is an independent system of A, containing only elements whose orders are powers of a fixed prime p, if and only if $M' = \{p^{o(a_i)-1}a_i\}_{i\in I}$ is an indepedent system of $S(A_p)$, where $ord(a_i) = p^{o(a_i)}$ for $i \in I$.

Proof. Let $\{a_1, \dots, a_k\}$ be an independent system. If there exists a_j whose order is greater that p, it is clear that $\{a_1, \dots, p \cdot a_j, \dots, a_k\}$ is independent too.
Vice versa, if $\{p \cdot a_1, a_2, \dots, a_k\}$ is independent, then from $n_1a_1 + \dots + n_ka_k = 0$ follows that $p \cdot n_1a_1 + \dots + p \cdot n_ka_k = 0$ or $p \cdot n_ia_i = 0$ for all $i \in \{1, \dots, k\}$. In particular $ord(a_1)|p \cdot n_1$ and, since $ord(a_1) \geq p^2$, $p|n_1$. By the independence of $\{p \cdot a_1, a_2, \dots, a_k\}$ we can conclude that $n_ia_i = 0$ for all $i \in \{1, \dots, k\}$, which is exactly the independence of $\{a_1, \dots, a_k\}$.
□

Therefore for any group A and any independent system M for A containing only elements of order a power of p and maximal with this property we get that M has the same cardinality of $M' = \{p^{o(a_i)-1}a_i\}_{i\in I}$, a maximal independent system for $S(A_p)$. Hence $r_p(A) = r_p(S(A_p))$. In this case we note that $M' \subseteq S(A_p) = \{a \in A_p : ord(a) = p\}$, and $S(A_p)$ can be naturally seen as a vector space over the field $\mathbb{F}_p$. By its very definition M' is $\mathbb{F}_p$-linearly independent and maximal with this property, and so $|M'| = \dim_{\mathbb{F}_p}(S(A_p))$. Hence $r_p(A)$ is equal to the $\mathbb{F}_p$-dimension of the vector space $S(A_p)$, which means that the p-rank of A is well defined for any p. □

Corollary 2.2.7. *If A is torsion-free then $|A| \leq \max\{r_0(A), \aleph_0\}$.*

Proof. It is a direct consequence of the previous proof. □

Lemma 2.2.8. *Let A and B be two groups. If there exists $f : A \longrightarrow B$ surjective map or $g : B \longrightarrow A$ injective map, then $r_0(A) \geq r_0(B)$.*

Proof. Let $f : A \longrightarrow B$ be a surjective map and let $M = \{b_j\}_{i\in J}$ be an independent system of B, maximal with respect to the property of containing only elements of infinite order. By taking the preimages of the elements we obtain a system $M' = \{a_j\}_{j\in J}$ such that $b_j = f(a_j)$ for all $j \in J$. If $n_1a_{j_1} + \dots + n_ka_{j_k} = 0$, then $n_1b_{j_1} + \dots + n_kb_{j_k} = 0$ and, by the independence of M, we have that $n_hb_{j_h} = 0$, giving that $n_h = 0$ for all $h = 1, \dots, k$, since $ord(b_{j_h})$ is not finite. Hence M' is an independent system containing only elements of infinite order. The proof of the second part of the assumption is analogous. □

Theorem 2.2.9. *(Pontryagin's Criterion) A countable torsion-free group A is free if and only if every finite rank subgroup is free.*

Proof. A subgroup of a free group is always free. Therefore the necessity is clear. Let us prove the sufficiency. List the element of A, $\{a_n\}_{n\in\mathbb{N}}$, and define

$$A_n = \{a \in A \setminus \{0\} : \ a \text{ depends on } \{a_0, \dots, a_n\}\} \cup \{0\}.$$

Since $T(A) = 0$, A_n is a subgroup: indeed, if $a, b \in A_n \setminus \{0\}$ then there exist $n_a, n_b \in \mathbb{Z} \setminus \{0\}$ such that $\{n_a a, n_b b\} \subseteq \langle a_0, \dots, a_n \rangle \setminus \{0\}$. If $a + b \neq 0$ then $n_a n_b \cdot (a + b) \in \langle a_0, \dots, a_n \rangle \setminus \{0\}$. The torsion-free rank of A_n is not greater than $n + 1$, considering that $\{a_0, \dots, a_n, g\}$ is a dependent system for all $g \in A_n$. By well definition of the rank, $r_0(A_n) \leq n + 1$. Clearly $r_0(A_{n+1}) \leq r_0(A_n) + 1$: because $\bigcup_{n \in \mathbb{N}} A_n = A$, and so either A has finite rank, in which case the assertion is obvious, or for each n there is a subgroup between the A_m whose rank is exactly $n + 1$. Let B_n be a subsequence of A_n such that $r_0(B_n) = n + 1$.
Notice that we still have $A = \bigcup_{n \in \mathbb{N}} B_n$. Now consider B_{n+1}/B_n. If B_{n+1}/B_n has a torsion part then there exists $a \in B_{n+1} \setminus B_n$ such that $na \in B_n$, but if na is dependent on $\{a_0, \dots, a_k\}$, then also a depends on them. Thus B_{n+1}/B_n is torsion-free of rank 1; therefore $B_{n+1}/B_n = \langle b_{n+1} \rangle \cong \mathbb{Z}$ and is projective. Hence $B_{n+1} \cong B_n \oplus \langle b_{n+1} \rangle$ and we obtain $A \cong \bigoplus_{n \in \mathbb{N}} b_n$. □

2.3 Classification of divisible groups

We now provide a complete classification of divisible groups, in terms of the torsion-free rank and p-ranks. We will distinguish our analysis in two cases, according to whether the divisible group has torsion or not.

Theorem 2.3.1. *Any divisible group A is isomorphic to*

$$\bigoplus_p \mathbb{Z}(p^\infty)^{\gamma_p} \bigoplus \mathbb{Q}^\delta \tag{2.2}$$

where p runs over the primes and γ_p and δ are cardinals; moreover, these cardinals define a complete and independent system of invariants for A.

Proof. First of all one observes that $T(A)$ is divisible. Indeed, for all $a \in A$ and for all $n \in \mathbb{Z}$ there exists $x \in A$ such that $nx = a$. If $a \in T(A)$ then $ord(x) | n \cdot ord(a)$, thus it is finite. Therefore $x \in T(A)$. Similarly, one proves that $A/T(A)$ is divisible too.
Owing to the divisibility (and thus the injectivity) of $T(A)$, the following exact sequence

$$0 \longrightarrow T(A) \longrightarrow A \longrightarrow A/T(A) \longrightarrow 0$$

splits. If $T_p(A)$ denotes the p-component of $T(A)$, by Theorem 1.1.10 we have

$$A = \bigoplus_p T_p(A) \oplus A/T(A).$$

Hence it is enough to prove the theorem for p-groups and torsion-free groups.

If A is a p-group, pick a maximal independent system $\{a_i\}_{i \in I}$ of $S(A)$, the socle of A. For each $i \in I$ consider a countable set $\{a_{i,n}\}_{n \in \mathbb{N}^*}$, where $a_{i,1} = a_i$ and $a_{i,n+1}$ is such that $p \cdot a_{i,n+1} = a_{i,n}$.

By the divisibility of A, the previous sets are well-defined. Furthermore $A_i = \langle a_{i,1}, \dots, a_{i,n}, \dots\rangle$ is isomorphic to $\mathbb{Z}(p^\infty)$. Now observe that $A_i \cap (A_{j_1} + \dots + A_{j_k}) = 0$, since each element of A_j is a multiple of a_{j,m_j} for m_j great enough and $\{a_i, a_{j_1}, \dots, a_{j_k}\}$ is independent if and only if $\{a_{i,m_i}, a_{j_1,m_{j_1}}, \dots, a_{j_k,m_{j_k}}\}$ is independent.
Define $B = \bigoplus_{i\in I} A_i \subseteq A$: it is divisible, hence $A = B \oplus A/B$. At the same time $S(A) = S(B)$, thus $A = \bigoplus_{i\in I} A_i \cong \bigoplus_{i\in I} \mathbb{Z}(p^\infty)$ where $|I| = r_p(A)$.

If A is torsion-free, choose a maximal independent system $\{a_i\}_{i\in I}$ of A. A is divisible, thus for every $n \in N^*$ there is $x \in A$ such that $nx = a_i$. Actually if $nx = ny$, then $(x-y)$ has finite order, which means $x = y$. Hence there exists exactly one $x \in A$ that satisfies $nx = a_i$, which means that every a_i can be embedded in a subgroup $A_i \cong \mathbb{Q}$ of A. Similarly to the previous case, the A_i are in direct sum. The direct sum $\bigoplus_{i\in I} A_i$ is a direct summand of A containing a maximal independent system of A and thus we have

$$A = \bigoplus_{i\in I} A_i \cong \bigoplus_{i\in I} \mathbb{Q} \qquad \text{where } |I| = r_0(A).$$

Consequently, for a divisible group A there is the decomposition

$$A \cong \bigoplus_p \mathbb{Z}(p^\infty)^{r_p(A)} \bigoplus \mathbb{Q}^{r_0(A)}.$$

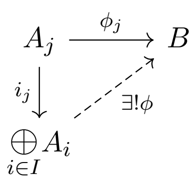
By Theorem 2.2.5, the cardinal numbers of the sets of components $\mathbb{Z}(p^\infty)$ and $\mathbb{Q}$ are uniquely determined by A. Thus they form a complete and independent system of invariants for A. □

2.4 The torsion-free rank of $Ext^1(A, \mathbb{Z})$

Theorem 2.4.1. *Let $\{A_i\}_{i\in I}, B$ be all $\mathbb{Z}-$modules. Then*

$$Ext^1\left(\bigoplus_{i\in I} A_i, B\right) \cong \prod_{i\in I} Ext^1(A_i, B).$$

Proof. We recall that the *direct sum* is characterized by the following universal property: for each group B and each indexed family of group homomorphism $\{\phi_i : A_i \longrightarrow B\}_{i\in I}$ there exists a unique group homomorphism $\phi : \bigoplus_{i\in I} A_i \longrightarrow B$ making the following diagram commute for all $j \in I$



where $i_j : A_j \longrightarrow \bigoplus_{i\in I} A_i$ is the j-th coordinate immersion.

Take an injective resolution for B, $0 \longrightarrow J \xrightarrow{\pi} J/B \longrightarrow 0$. Then we have

$$Ext^1\Big(\bigoplus_{i\in I} A_i, B\Big) \cong \frac{Hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} A_i, J/B\Big)}{\pi_*\Big(Hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} A_i, J\Big)\Big)} \quad \text{and} \quad Ext^1(A_i, B) \cong \frac{Hom_{\mathbb{Z}}(A_i, J/B)}{\pi_*(Hom_{\mathbb{Z}}(A_i, J))}.$$

By Theorem 1.1.35,

$$Hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} A_i, J/B\Big) \cong \prod_{i\in I} Hom_{\mathbb{Z}}(A_i, J/B).$$

Hence we can define the map $\varphi \mapsto (\varphi|_{A_i})_{i\in I} \mapsto (\overline{\varphi|_{A_i}})_{i\in I}$ of $Hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} A_i, J/B\Big)$ onto $\prod_{i\in I} \frac{Hom_{\mathbb{Z}}(A_i, J/B)}{\pi_*(Hom_{\mathbb{Z}}(A_i, J))}$ obtained by composing the above-mentioned isomorphism componentwise with the quotient maps. If $(\overline{\varphi|_{A_i}})_{i\in I}$ is the zero element of the product $\prod_{i\in I} \frac{Hom_{\mathbb{Z}}(A_i, J/B)}{\pi_*(Hom_{\mathbb{Z}}(A_i, J))}$, then for all $i \in I$ there exists a homomorphism $\phi_i : A_i \longrightarrow J$ such that $\varphi|_{A_i} = \pi \circ \phi_i$. By the universal property of the direct sum, there exists a homomorphism $\phi : \bigoplus_{i\in I} A_i \longrightarrow J$ such that $\phi_j = \phi \circ i_j$ for all $j \in I$. Composing ϕ with the projection π we obtain

$$\varphi \circ i_j = \varphi|_{A_j} = \pi \circ \phi_j = \pi \circ \phi \circ i_j \text{ for all } j \in I.$$

Hence, due to the uniqueness of the homomorphism that makes the following diagram commute

$$\begin{array}{ccc} A_j & \xrightarrow{\varphi|_{A_j}} & J/B \\ {\scriptstyle i_j}\downarrow & \nearrow & \\ \bigoplus_{i\in I} A_i & & \end{array}$$

we obtain $\varphi = \pi \circ \phi \in \pi_*\Big(Hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} A_i, J\Big)\Big)$. Since for a homomorphism $\psi \in Hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} A_i, J\Big)$ we have $\overline{\pi \circ \psi \circ i_j} = \overline{0}$ for all $j \in I$, we conclude that the kernel of the map $\varphi \mapsto (\overline{\varphi|_{A_i}})_{i\in I}$ is exactly $\pi_*\Big(Hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} A_i, J\Big)\Big)$. Then we obtain, as desired,

$$Ext^1\Big(\bigoplus_{i\in I} A_i, B\Big) \cong \frac{Hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} A_i, J/B\Big)}{\pi_*\Big(Hom_{\mathbb{Z}}\Big(\bigoplus_{i\in I} A_i, J\Big)\Big)} \cong \prod_{i\in I} \frac{Hom_{\mathbb{Z}}(A_i, J/B)}{\pi_*(Hom_{\mathbb{Z}}(A_i, J))} \cong \prod_{i\in I} Ext^1(A_i, B).$$

□

Now we are ready to study the torsion-free rank of $Ext^1(A,\mathbb{Z})$.

Theorem 2.4.2. *Let A be a countable torsion-free group. If A is free, then $Ext^1(A,\mathbb{Z})$ is trivial. Otherwise $r_0(Ext^1(A,\mathbb{Z})) = 2^{\aleph_0}$.*

Proof. If A is free, then it is is projective and so a projective resolution for A is

$$0 \longrightarrow A \xrightarrow{id} A \longrightarrow 0.$$

This means that $Ext^1(A,\mathbb{Z})$ is trivial by *Balance of Ext*. Thus assume A is not free. Observe that

$$Ext^1(A,\mathbb{Z}) \cong \frac{Hom_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})}{\pi_*(Hom_{\mathbb{Z}}(A,\mathbb{Q}))} \quad \Longrightarrow \quad |Ext^1(A,\mathbb{Z})| \leq |Hom_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})| \leq {\aleph_0}^{\aleph_0}.$$

and it follows that $r_0(Ext^1(A,\mathbb{Z})) \leq 2^{\aleph_0}$. Thus we have to prove the converse inequality. Suppose for the moment that the theorem holds for groups of rank 1.

If A is not free and of finite rank, proceed by induction. Let A be with torsion-free rank equal to n. By Theorem 2.4.1 we can assume that A is indecomposable (recall that an indecomposable group is a non-trivial group that cannot be expressed as direct sum of two subgroups). Let $M = \{a_1, \dots, a_n\}$ be a maximal independent system for A. Define the subgroup

$$B = \{a \in A \setminus \{0\} : a \text{ depends on } \{a_1, \dots, a_{n-1}\}\} \cup \{0\},$$

which is a countable torsion-free group of rank $n-1$.

Since

$$A = \{a \in A \setminus \{0\} : a \text{ depends on } \{a_1, \dots, a_n\}\} \cup \{0\},$$

A/B is a countable torsion-free group of rank 1. It cannot be free, else it would be projective, yielding that $A \cong B \bigoplus A/B$ is free contrary to our assumptions. Hence A/B is not free and $r_0(Ext^1(A/B,\mathbb{Z})) = 2^{\aleph_0}$.

Now consider the short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \to 0$$

and apply $Hom_{\mathbb{Z}}(-,\mathbb{Z})$:

$$\begin{aligned} 0 \longrightarrow Hom_{\mathbb{Z}}(A/B,\mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(A,\mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(B,\mathbb{Z}) \xrightarrow{\delta} \\ \xrightarrow{\delta} Ext^1(A/B,\mathbb{Z}) \longrightarrow Ext^1(A,\mathbb{Z}) \longrightarrow Ext^1(B,\mathbb{Z}) \longrightarrow 0. \end{aligned}$$

By exactness we have

$$0 \longrightarrow \frac{Ext^1(A/B,\mathbb{Z})}{\delta(Hom_{\mathbb{Z}}(B,\mathbb{Z}))} \longrightarrow Ext^1(A,\mathbb{Z}).$$

Since the rank of B is finite, $Hom_{\mathbb{Z}}(B,\mathbb{Z})$ is at most countable and thus $|\delta(Hom_{\mathbb{Z}}(B,\mathbb{Z}))| \leq \aleph_0$.

Lemma 2.4.3. *Let A be a group whose torsion-free rank is an infinite cardinal number and let B be a subgroup such that $r_0(B) < r_0(A)$: then $r_0(A/B) = r_0(A)$.*

Proof. Pick a maximal independent system $\{a_j\}_{j \in r_0(B)}$ of element of B of infinite order and extend it to a maximal independent system $\{a_i\}_{i \in r_0(A)}$ of element of A of infinite order. Since $r_0(B) < r_0(A)$ there exists a subset S of $r_0(A) \setminus r_0(B)$ of cardinality equal to $r_0(A)$ such that $\overline{a_i} \neq \overline{a_j}$ in A/B for all $i, j \in S$. Let $\{i_1, \dots, i_k\}$ be indices in S and let $n_1, \dots, n_k$ be integer numbers.

$$n_1 \cdot \overline{a_{i_1}} + \dots + n_k \cdot \overline{a_{i_k}} = \overline{0} \qquad \Longrightarrow \qquad n_1 \cdot a_{i_1} + \dots + n_k \cdot a_{i_k} \in B.$$

This means that the element $n_1 \cdot a_{i_1} + \dots + n_k \cdot a_{i_k}$ depends on $\{a_j\}_{j \in r_0(B)}$. Since $\{a_{i_1}, \dots, a_{i_k}\} \cup \{a_j\}_{j \in r_0(B)}$ is an independent system, it follows that $n_1 \cdot a_{i_1} + \dots + n_k \cdot a_{i_k} = 0$. Hence the integers n_h are trivial for all $h \in \{1, \dots, k\}$ and so $\{\overline{a_i}\}_{i \in S}$ is an independent system of element of infinite order for A/B. By Lemma 2.2.8, we have $r_0(A/B) \leq r_0(A)$ and this implies that $r_0(A) = |S| \leq r_0(A/B) \leq r_0(A)$.

□

By Cantor's Theorem $\aleph_0 < 2^{\aleph_0}$ and so the torsion-free rank of the quotient $\dfrac{Ext^1(A/B, \mathbb{Z})}{\delta(Hom_{\mathbb{Z}}(B, \mathbb{Z}))}$ remains $2^{\aleph_0}$. By Lemma 2.2.8, $r_0(Ext^1(A, \mathbb{Z})) = 2^{\aleph_0}$.

If A is not free and of infinite rank, by Pontryagin's criterion there is a subgroup $B \subseteq A$ of finite rank which is not free. Now consider the short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0$$

and apply $Hom_{\mathbb{Z}}(-, \mathbb{Z})$. The result is:

$$\begin{aligned} 0 \longrightarrow Hom_{\mathbb{Z}}(A/B, \mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(A, \mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(B, \mathbb{Z}) \longrightarrow \\ \longrightarrow Ext^1(A/B, \mathbb{Z}) \longrightarrow Ext^1(A, \mathbb{Z}) \longrightarrow Ext^1(B, \mathbb{Z}) \longrightarrow 0. \end{aligned}$$

By Lemma 2.2.8 $r_0(Ext^1(A, \mathbb{Z})) \geq r_0(Ext^1(B, \mathbb{Z})) = 2^{\aleph_0}$ and therefore

$$r_0(Ext^1(A, \mathbb{Z})) = 2^{\aleph_0}.$$

Now we discuss the case when A has torsion-free rank equal to 1. First of all we prove that it is not a loss of generality to suppose that A is a subgroup of $\mathbb{Q}$.

Lemma 2.4.4. *If A is a torsion-free group of rank 1 then it is isomorphic to a subgroup of $\mathbb{Q}$.*

Proof. Fix a maximal independent system, $\{a\}$: then for all $g \in A \setminus \{0\}$ there exists a least $n \in \mathbb{N}^*$ such that $ng = n_g a$, with $n_g \in \mathbb{N}^*$ (observe that n_g is unique). Consider the function $f : A \longrightarrow \mathbb{Q}$ that maps a into 1, 0 into 0 and g into $\frac{n_g}{n}$: obviously f is injective, and we claim that it is also a homomorphism: If g and h are respectively mapped into $\frac{n_g}{n}$ and $\frac{m_h}{m}$ then

$$(n_g\, m + m_h\, n) \cdot a = mn_g \cdot a + nm_h \cdot a = mn \cdot g + nm \cdot h = mn \cdot (g+h).$$

If $g+h$ is mapped into $\frac{k}{l}$ then $l(g+h) = ka$ and, multiplying by mn,

$$kmn \cdot a = lmn \cdot (g+h) = l(n_g\, m + m_h\, n) \cdot a \qquad \Longrightarrow \qquad l(n_g\, m + m_h\, n) = kmn$$

since A is torsion-free. Thus

$$\frac{n_g}{n} + \frac{m_h}{m} = \frac{(n_g\, m + m_h\, n)}{mn} = \frac{k}{l},$$

as was to be shown. □

From now on and for the rest of this proof we suppose that a is equal to 1 and A is a subgroup of $\mathbb{Q}$. Observe that, for relatively prime numbers m and n, $\frac{m}{n} \in A$ if and only if $\frac{1}{n} \in A$. Indeed if m and n are relatively primes then there exists s and t in $\mathbb{Z}$ such that $sm + tn = 1$ and so

$$\frac{1}{n} = s\frac{m}{n} + t \in A.$$

It follows that, for relatively prime numbers m and n, $\frac{1}{mn} \in A$ if and only if both $\frac{1}{m}$ and $\frac{1}{n}$ belong to A: indeed if $\frac{1}{m}$, $\frac{1}{n} \in A$ then

$$\frac{1}{m} + \frac{1}{n} = \frac{m+n}{mn} \in A$$

which gives also that $\frac{1}{mn} \in A$, since $m+n$ and mn are relatively primes.

To proceed in our analysis we need the following:

Claim. *For the group A under consideration (which is now identified with a subgroup of $\mathbb{Q}$) there exists a prime number p such that $1/p^k \in A$ for infinitely many k or there are infinitely many primes q such that $1/q \in A$.*

Proof. Suppose that for each prime p there exists a maximum power of p, p^n, such that $1/p^n \in A$ and that there are finitely many primes $\{p_1, \dots, p_k\}$ such that $1/p_j \in A$ for any $j = 1, \dots, k$. Let n_j be the maximum exponent of p_j such that $1/p_j^{n_j} \in A$ for each $j = 1, \dots, k$. Then $y = \frac{1}{p_1^{n_1} \cdot \dots \cdot p_k^{n_k}} \in A$, since $p_j^{n_j}, p_i^{n_i}$ are relatively prime for each $i \neq j$, and we have already seen that $1/mn \in A$ if m, n are relatively prime and $1/n, 1/m \in A$. Since $\langle y \rangle \subsetneq A$ (otherwise A would be free) there is an element $h \in A \setminus \langle y \rangle$. Assume $h = s/r$ with s, r relatively prime. By what we already know we also get that $1/r \in A$ and $1/r \notin \langle y \rangle$ else also $h \in \langle y \rangle$. Let $r = q_1^{m_1} \cdot \ldots \cdot q_s^{m_s}$ with each q_i a prime number. This gives that $1/q_i^{m_i} = \left(\prod_{l \neq i} q_l^{m_l} \right) / r \in A$ for all $i = 1, \ldots, s$. Since $\{p_1, \ldots, p_k\}$ exhausts the set of primes p such that $1/p \in A$ we have that $q_i \in \{p_1, \ldots, p_k\}$ for all $i = 1, \ldots, s$. Thus (modulo a rearrangement) $q_i = p_i$ for all $i = 1, \ldots, s$ and so $m_i \leq n_i$ for all $i = 1, \ldots, s$. Therefore $1/r = \left(\prod_{i=1}^{s} p_i^{n_i - m_i} \right) y \in \langle y \rangle$, which is a contradiction. □

Let us now focus on $T = A/\mathbb{Z}$. T is a torsion group since for all $g \in A \setminus \{0\}$ there exists (n, n_g) such that $ng = n_g$, which gives that $n[g] = [0]$. Thus by the Factorization Theorem 1.1.10 we have

$$Ext^1(T, \mathbb{Z}) \cong Ext^1\Big(\bigoplus_p T_p, \mathbb{Z}\Big) \cong Hom_{\mathbb{Z}}\Big(\bigoplus_p T_p, \mathbb{Q}/\mathbb{Z}\Big).$$

We want to prove that $r_0(Ext^1(T, \mathbb{Z})) \geq 2^{\aleph_0}$. By the Claim there are two cases we have to handle.

First Case: *there is a prime number p such that $1/p^k \in A$ for infinitely many k.* This yields that $1/p^k \in A$ for all $k \in \mathbb{N}^*$. Hence there exists a subgroup of T_p which is isomorphic to the Prüfer p-group. The Prüfer p-group is divisible, therefore

$$\begin{aligned} &0 \longrightarrow \mathbb{Z}(p^\infty) \longrightarrow T_p \ \text{ exact } \Longrightarrow \\ &\Longrightarrow Hom_{\mathbb{Z}}(T_p, \mathbb{Z}(p^\infty)) \longrightarrow End_{\mathbb{Z}}(\mathbb{Z}(p^\infty)) \longrightarrow 0 \ \text{ exact,} \end{aligned}$$

and (by Lemma 2.2.8) $r_0(Hom_{\mathbb{Z}}(T_p, \mathbb{Z}(p^\infty))) \geq r_0(End_{\mathbb{Z}}(\mathbb{Z}(p^\infty)))$. Let us focus on $End_{\mathbb{Z}}(\mathbb{Z}(p^\infty))$. By Theorem 1.1.15 $End_{\mathbb{Z}}(\mathbb{Z}(p^\infty))$ is isomorphic to the additive group $\widehat{\mathbb{Z}}_p$ of p-adic integers. Since this group is torsion-free by Proposition 1.1.14, we have that $r_0(\widehat{\mathbb{Z}}_p) = 2^{\aleph_0}$. Thus $r_0(\, Hom_{\mathbb{Z}}(T_p, \mathbb{Z}(p^\infty))\,) = 2^{\aleph_0}$. By Corollary 1.1.36 we have

$$Hom_{\mathbb{Z}}\Big(\bigoplus_p T_p, \mathbb{Q}/\mathbb{Z}\Big) \cong \prod_p Hom_{\mathbb{Z}}(T_p, \mathbb{Z}(p^\infty)).$$

Therefore the torsion-free rank of $Hom_{\mathbb{Z}}\Big(\bigoplus_p T_p, \mathbb{Q}/\mathbb{Z}\Big)$ is $\geq 2^{\aleph_0}$.

Second Case: *there are infinitely many primes p such that $1/p \in A$.* Enumerate them as $\{p_n\}_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$ $\frac{1}{p_n}$ belongs to A, hence there is a subgroup isomorphic to $\mathbb{Z}/p_n\mathbb{Z}$ in T_{p_n} for each n. We obtain

$$\begin{aligned} &0 \longrightarrow \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z} \longrightarrow \bigoplus_{n \in \mathbb{N}} T_{p_n} \quad \text{exact } \Longrightarrow \\ &\Longrightarrow \quad Hom_{\mathbb{Z}}\Big(\bigoplus_{n \in \mathbb{N}} T_{p_n}, \mathbb{Q}/\mathbb{Z}\Big) \longrightarrow Hom_{\mathbb{Z}}\Big(\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}\Big) \longrightarrow 0 \ \text{ exact.} \end{aligned}$$

Therefore, by Lemma 2.2.8, we have the following inequalities for what concerns the torsion-free ranks:

$$r_0(\, Hom_{\mathbb{Z}}\Big(\bigoplus_{n\in\mathbb{N}} T_{p_n}, \mathbb{Q}/\mathbb{Z}\Big)\,) \geq r_0(\, Hom_{\mathbb{Z}}\Big(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}\Big)\,).$$

Similarly considering the following exact sequence

$$0 \longrightarrow \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z} \longrightarrow \bigoplus_{p} (\mathbb{Q}/\mathbb{Z})_p \cong \mathbb{Q}/\mathbb{Z},$$

and applying $Hom_{\mathbb{Z}}\Big(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}, -\Big)$ we obtain the exact sequence

$$0 \longrightarrow Hom_{\mathbb{Z}}\Big(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}\Big) \longrightarrow Hom_{\mathbb{Z}}\Big(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}\Big).$$

This gives the inequality

$$r_0(\, Hom_{\mathbb{Z}}\Big(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}\Big)\,) \geq r_0(\, Hom_{\mathbb{Z}}\Big(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}\Big)\,).$$

Thus, in order to obtain a lower bound for the torsion-free rank of $Ext^1(T,\mathbb{Z})$, it is enough to study $Hom_{\mathbb{Z}}\Big(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}\Big)$, which is isomorphic to $\prod_{n\in\mathbb{N}} Hom_{\mathbb{Z}}\Big(\mathbb{Z}/p_n\mathbb{Z}, \bigoplus_{j\in\mathbb{N}} \mathbb{Z}/p_j\mathbb{Z}\Big)$ by Theorem 1.1.35. $Hom_{\mathbb{Z}}\Big(\mathbb{Z}/p_n\mathbb{Z}, \mathbb{Z}/p_j\mathbb{Z}\Big)$ is trivial for all $j \neq n$, and $Hom_{\mathbb{Z}}\Big(\mathbb{Z}/p_n\mathbb{Z}, \mathbb{Z}/p_n\mathbb{Z}\Big)$ is isomorphic to $\mathbb{Z}/p_n\mathbb{Z}$, we get that $\prod_{n\in\mathbb{N}} Hom_{\mathbb{Z}}\Big(\mathbb{Z}/p_n\mathbb{Z}, \bigoplus_{j\in\mathbb{N}} \mathbb{Z}/p_j\mathbb{Z}\Big)$ is actually isomorphic to $\prod_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}$.

Now partition $\mathbb{N}$ into $\aleph_0$ sets $\{I_n\}_{n\in\aleph_0}$, each of them of cardinality $\aleph_0$: hence we have $\prod_{n\in\aleph_0}\Big(\prod_{p\in I_n} \mathbb{Z}/p\mathbb{Z}\Big)$ and each $\prod_{p\in I_n} \mathbb{Z}/p\mathbb{Z}$ has at least an element a_n of infinite order. Therefore the subgroup $\prod_{p\in I_n} \langle a_n \rangle$ is a torsion-free group with at least $\aleph_0$-many independent elements. By Corollary 2.2.7 its torsion-free rank is equal to its cardinality which is $2^{\aleph_0}$. This yields that the torsion free rank of $\prod_{n\in\mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}$ is also $2^{\aleph_0}$. Therefore also in this second case $r_0(Ext^1(T,\mathbb{Z})) = 2^{\aleph_0}$.

Now consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow A \longrightarrow T \longrightarrow 0;$$

applying $Hom_{\mathbb{Z}}(-,\mathbb{Z})$ we obtain the exact sequence

$$\ldots \longrightarrow Hom_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \xrightarrow{\delta} Ext^1(T,\mathbb{Z}) \longrightarrow Ext^1(A,\mathbb{Z}) \longrightarrow 0 = Ext^1(\mathbb{Z},\mathbb{Z}).$$

Therefore $Ext^1(A,\mathbb{Z}) \cong \dfrac{Ext^1(T,\mathbb{Z})}{\delta(End_{\mathbb{Z}}(\mathbb{Z}))}$. Since $End_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}$, $\delta(End_{\mathbb{Z}}(\mathbb{Z}))$ is a subgroup of $Ext^1(T,\mathbb{Z})$ isomorphic to a quotient of $\mathbb{Z}$, hence its torsion-free rank is at most 1. By Lemma 2.4.3 we conclude that the torsion-free rank of $\dfrac{Ext^1(T,\mathbb{Z})}{\delta(End_{\mathbb{Z}}(\mathbb{Z}))}$ remains $2^{\aleph_0}$.

□

The solution for Whitehead's problem for countable groups is an easy corollary:

Corollary 2.4.5. *(Stein's Theorem) Let A be a countable group. Then $Ext^1(A,\mathbb{Z}) = 0$ if and only if A is free.*

2.5 The p-rank of $Ext^1(A,\mathbb{Z})$

We turn to the analysis of the p-ranks of $Ext^1(A,\mathbb{Z})$ for a countable A.

Lemma 2.5.1. *(Stein's Lemma) Any countable group A can be written as $A = N \oplus F$, where F is free and N is a subgroup such that $Hom_{\mathbb{Z}}(N,\mathbb{Z}) = 0$.*

Proof. Define A^* as the set $Hom_{\mathbb{Z}}(A,\mathbb{Z})$. Let N be the intersection of the kernels of all homomorphisms $\varphi : A \longrightarrow \mathbb{Z}$, i.e. $N = \bigcap_{\varphi \in A^*} \ker(\varphi)$. Then A/N is isomorphic to a subgroup of $\prod_{\varphi \in A^*} A/\ker(\varphi)$ and every $A/\ker(\varphi)$ is isomorphic to a subgroup of $\mathbb{Z}$, and so it is either trivial or isomorphic to $\mathbb{Z}$.

Write $P = \prod_{\varphi \in A^*} A/\ker(\varphi) = \prod_{i \in I} \langle a_i \rangle$ where I is an at most infinite set and $\langle a_i \rangle \cong \mathbb{Z}$ for each i. If I is finite then P is free, otherwise we claim that it is $\aleph_1$-free. In order to show this we prove that all finite torsion-free rank subgroups of P are free, which (by Pontriagin's Criterion) also yields that all countable subgroups of P are free.

Towards this aim it suffices to prove the following:

Claim. *Every finite subset $\{b_1, \dots, b_k\} \subseteq P$ can be embedded in a finitely generated direct summand of P itself.*

Proof. We proceed by induction on k.

$k = 1$: Let $b = (n_i a_i)_{i \in I}$ be an element of P with $n_i \in \mathbb{Z}$ and define m as the least positive integer among all the $|n_i|$ with $n_i \neq 0$. Proceed by induction on m. If $m = 1$ then there is a $j \in I$ such that $|n_j| = 1$, and so $P = \langle b \rangle \oplus B_j$ where B_j is the subgroup of all elements with j-th coordinate 0. Otherwise set $n_i = q_i m + r_i$ with $0 \le r_i < m$ and define $c = (q_i a_i)_{i \in I}$ and $d = (r_i a_i)_{i \in I}$ so that $b = cm + d$. By definition there is $j \in I$ such that $|q_j| = 1$ and $r_j = 0$: as before $P = \langle c \rangle \oplus B_j$ where B_j has the same definition above. It happens that $d \in B_j$ with coefficient $|r_i| < m$, and so, by induction on m applied to d and B_j, B_j has a finitely generated direct summand B'_j containing d. Hence $\langle c \rangle \oplus B'_j$ is a direct summand of P which contain b.

$k > 1$: Given a subset $\{b_1, \dots, b_k\}$, by induction we can assume there exists a finitely generated direct summand B such that $\{b_1, \dots, b_{k-1}\} \subseteq B$. We may in addition suppose that, in the direct sum $A = B \oplus C$, C is the direct product of almost all $\langle a_i \rangle$ (it follows from our choice of B_j at each step according to the procedure described in the previous case). Decomposing $b_k = b + c$ where $b \in B$ and $c \in C$ we can find a finitely generated direct summand of C which c belongs to (applying the procedure described in the previous case to c, C).

Hence the Claim is proved. □

Therefore, since A/N is countable, it is free (hence projective), therefore $A \cong N \oplus A/N$ (by Proposition 1.1.21, given that

$$0 \longrightarrow N \longrightarrow A \longrightarrow A/N \longrightarrow 0$$

is exact with A/N projective). Lastly observe that each homomorphism $\psi : N \longrightarrow \mathbb{Z}$ can be extended to a homomorphism $\overline{\psi} : N \oplus A/N \longrightarrow \mathbb{Z}$. By its very definition, $N \subseteq \ker(\overline{\psi})$, i.e. $\psi = 0$. Thus $Hom_{\mathbb{Z}}(N, \mathbb{Z})$ is trivial. □

Theorem 2.5.2. *Let A be a countable torsion-free group. For each prime number p, $r_p(Ext^1(A,\mathbb{Z}))$ is either finite or $2^{\aleph_0}$.*

Proof. Consider the subgroups F and N of A such that $A = N \oplus F$ with F free and N such that $Hom_{\mathbb{Z}}(N, \mathbb{Z}) = 0$, whose existence is ensured by Stein's Lemma. Since

$$Ext^1(A,\mathbb{Z}) = Ext^1(N \oplus F, \mathbb{Z}) \cong Ext^1(N,\mathbb{Z}) \bigoplus Ext^1(F,\mathbb{Z}) \cong Ext^1(N,\mathbb{Z}),$$

(where the second equality holds by Theorem 2.4.1 and the latter by Corollary 2.4.5), without loss of generality we may assume that A has the property that $Hom_{\mathbb{Z}}(A,\mathbb{Z}) = 0$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0.$$

Due to the fact that $Hom_{\mathbb{Z}}(A,\mathbb{Z}) = 0$, applying $Hom_{\mathbb{Z}}(A,-)$ we have (by Theorem 1.1.33) the exact sequence:

$$0 \longrightarrow Hom_{\mathbb{Z}}(A,\mathbb{Z}/p\mathbb{Z}) \longrightarrow Ext^1(A,\mathbb{Z}) \xrightarrow{p_*} Ext^1(A,\mathbb{Z}) \longrightarrow Ext^1(A,\mathbb{Z}/p\mathbb{Z}) \longrightarrow 0,$$

where p_* is the multiplication by p, by the Theorem 1.1.33. By definition of the socle $S(Ext^1(A,\mathbb{Z})_p) = \{x \in Ext^1(A,\mathbb{Z}) : px = 0\}$ coincides with the kernel of p^*. Therefore (by Remark 9) $\ker(p^*)$ is exactly the subgroup to consider in order to compute $r_p(Ext^1(A,\mathbb{Z}))$. By exactness, $ker(p^*) \cong Hom_{\mathbb{Z}}(A,\mathbb{Z}/p\mathbb{Z})$, hence $r_p(Ext^1(A,\mathbb{Z})) = r_p(Hom_{\mathbb{Z}}(A,\mathbb{Z}/p\mathbb{Z}))$. Since $pA \subseteq ker(\phi)$ for all $\phi \in Hom_{\mathbb{Z}}(A,\mathbb{Z}/p\mathbb{Z})$, it follows that

$$Hom_{\mathbb{Z}}(A,\mathbb{Z}/p\mathbb{Z}) \cong Hom_{\mathbb{Z}}(A/pA,\mathbb{Z}/p\mathbb{Z}).$$

A/pA is endowed of the structure of a $\mathbb{Z}/p\mathbb{Z}$-vector space, hence $Hom_{\mathbb{Z}}(A/pA,\mathbb{Z}/p\mathbb{Z})$ is the dual $\mathbb{Z}/p\mathbb{Z}$-vector space $(A/pA)^*$. This gives that $r_p(Hom_{\mathbb{Z}}(A/pA,\mathbb{Z}/p\mathbb{Z}))$ is the cardinality of the basis of the $\mathbb{Z}/p\mathbb{Z}$-vector space $(A/pA)^*$. If the dimension of A/pA is finite, this is also the dimention of $Hom_{\mathbb{Z}}(A/pA, \mathbb{Z}/p\mathbb{Z})$. Otherwise, if the dimension of $A/pA \cong \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p\mathbb{Z}$ is countable, then $\dim_{\mathbb{Z}/p\mathbb{Z}}((A/pA)^*) = 2^{\aleph_0}$, since the isomorphism 1.1 gives that

$$Hom_{\mathbb{Z}}(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \prod_{n\in\mathbb{N}} End_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}) \cong \prod_{n\in\mathbb{N}} \mathbb{Z}/p\mathbb{Z},$$

which has $\mathbb{Z}/p\mathbb{Z}$-dimension equal to $2^{\aleph_0}$.

□

3

Whitehead's problem

In this Chapter we prove results about $Ext^1(A, \mathbb{Z})$ which are consistent with ZFC but not provable in it. We will study the effects certain principles (the *Diamond Principle* and *Martin's Axiom* — which have been proved to be consistent with ZFC) have on the structure of $Ext^1(A, \mathbb{Z})$ for uncountable groups A.

Definition 3.0.1. A is a *Whitehead group* (*W-group*) if $Ext^1(A, \mathbb{Z})$ is trivial.

We will prove the following two statements.

Theorem 3.0.2.

1. *The Diamond principle $\diamondsuit_{\omega_1}$ entails that each W-group of cardinality $\aleph_1$ is free.*
2. *Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ implies that there exists a non-free W-group of cardinality $\aleph_1$.*

Since the two axioms yield contradictory answers to the problem, the equivalence between being a W-group and being free is undecidable in ZFC for a group of cardinality $\aleph_1$.

3.1 W-groups

We start giving an equivalent characterization of the notion of W-group.

Lemma 3.1.1. *A is a W-group if and only if each exact sequence of group homomorphisms*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} B \longrightarrow A \longrightarrow 0,$$

splits.

Proof. The sequence splits if and only if there exists a retraction of α. If the latter exists, then the map $\alpha^* : Hom_{\mathbb{Z}}(B, \mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ is surjective, hence the sequence

$$0 \longrightarrow Hom_{\mathbb{Z}}(A, \mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(B, \mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0$$

is exact, giving that $Ext^1(A, \mathbb{Z}) = 0$ (by Thm. 1.1.33). Vice versa if $Ext^1(A, \mathbb{Z}) = 0$, the sequence

$$0 \longrightarrow Hom_{\mathbb{Z}}(A, \mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(B, \mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0$$

is exact (again by Thm. 1.1.33), therefore α^* is surjective, and each element of $(\alpha^*)^{-1}(\{id_{\mathbb{Z}}\})$ is a retraction of α. □

We have already seen that a free group A is also a W-group and also that a countable W-group A is free. *Whitehead's Problem* asks whether being a W-group corresponds to being free. Saharon Shelah proved that Whitehead's problem is undecidable within ZFC theory, by showing that there are distinct models of ZFC in one of which every W-group of size $\aleph_1$ is free, while in the other there are non-free W-groups of size $\aleph_1$.
Throughout this Chapter κ will denote an uncountable regular cardinal.

Definition 3.1.2. A group A is called *κ-free* if every subgroup of cardinality $< \kappa$ is free.

Remark 10. For instance a W-group is always $\aleph_1$-free, since a subgroup of a W-group is itself a W-group and each countable W-group is free.

For groups of cardinality κ there is another characterization of being κ-free.

Lemma 3.1.3. *A group A of cardinality κ is κ-free if and only if it has a κ-*free filtration*, or rather a κ-filtration $\{A_\alpha\}_{\alpha\in\kappa}$ such that each A_α is free.*

Proof. Suppose A is κ-free. Then enumerate its elements $\{a_\beta\}_{\beta\in\kappa}$ and for each $\alpha \in \kappa$ define $A_\alpha = \langle a_\beta : \beta \in \alpha\rangle$: the cardinality of A_α is less than κ (κ is regular), therefore A_α is free for each $\alpha \in \kappa$. Thus $\{A_\alpha\}_{\alpha\in\kappa}$ is a κ-filtration of free subgroups.

Conversely since κ is regular if A has a κ-filtration $\{A_\alpha\}_{\alpha\in\kappa}$ of free subgroups, then for each subgroup $B \subseteq A$ of cardinality $< \kappa$ there exists an A_α which contains B. Thus B is a subgroup of a free group, hence B is free too. □

3.1.1 Chase($\aleph_1$)-condition and $\Gamma(A)$

In the second Chapter, we saw the Pontryagin's criterion for countable torsion-free groups. We now provide a generalization of this principle to κ-free groups of cardinality κ.

Definition 3.1.4. Let A be a κ-free group. A subgroup $B \subseteq A$ is *κ-pure* if A/B is κ-free.

Definition 3.1.5. We say that a group A satisfies the *Chase(κ)-condition* if it is κ-free and each of its subgroups of cardinality $< \kappa$ is contained in a κ-pure subgroup of cardinality $< \kappa$.

If we restrict ourselves to groups of cardinality κ we have an equivalent description of the Chase(κ)-condition through the filtrations.

Lemma 3.1.6. *Let A be a group of cardinality κ. A satisfies the Chase(κ)-condition if and only if it has a κ-free filtration such that $A_0 = 0$ and $A_{\alpha+1}$ is κ-pure for each $\alpha \in \kappa$.*

Proof. Suppose that A satisfies the Chase(κ)-condition and proceed by transfinite induction to define the requested κ-filtration as follows:

Let $\{a_\alpha\}_{\alpha\in\kappa}$ be an enumeration of the elements of A. For $\alpha = 0$, $A_0 = 0$. Now suppose that A_β has been defined for all $\beta \in \alpha$. If $\alpha = \delta + 1$, let $A_{\delta+1}$ be one of the κ-pure subgroups of cardinality $< \kappa$, which contain $A_\delta \cup \{a_\delta\}$. Such an $A_{\delta+1}$ exists by the Chase(κ)-condition, since $|A_\delta \cup \{a_\delta\}| < \kappa$. Otherwise, if α is a limit ordinal set $A_\alpha = \bigcup_{\beta\in\alpha} A_\beta$. This union is free since its cardinality is less than κ.
Since $a_\alpha \in A_{\alpha+1}$ for each $\alpha \in \kappa$, $A \subseteq \bigcup_{\alpha\in\kappa} A_\alpha$. Therefore $\{A_\alpha\}_{\alpha\in\kappa}$ is the filtration that we were looking for.

Conversely, by Lemma 3.1.3 we have that A is κ-free. Now let $B \subseteq A$ be a subgroup of cardinality κ: by regularity of κ there exists $\alpha \in \kappa$ such that $B \subseteq A_\alpha \subseteq A_{\alpha+1}$; the latter is a κ-pure subgroup of cardinality $< \kappa$. □

We now give a necessary and sufficient condition to grant freeness for a group of cardinality κ satisfying the Chase(κ)-condition. Towards this aim, we need to introduce an equivalence relation on the filtrations of a group.

Definition 3.1.7. Given E, F in $\mathcal{P}(\kappa)$, $E \sim F$ if and only if there exists a Cub C such that $E \cap C = F \cap C$.

Since the intersection of two Cub is itself a Cub, $\sim$ is an equivalence relation on $\mathcal{P}(\kappa)$. We denote by $[E]$ the equivalence class of E.

Remark 11. Notice that for $E \subseteq \kappa$ being stationary is equivalent to $[E] \neq [\emptyset]$.

Definition 3.1.8. Given a group A satisfying the Chase(κ)-condition, let $\mathcal{F} = \{A_\alpha : \alpha < \kappa\}$ be a κ-free filtration such that $A_0 = 0$ and $A_{\alpha+1}$ is κ-pure for each $\alpha \in \kappa$ (which exists by the previous Lemma). Define

$$E_{\mathcal{F}} \coloneqq \{\alpha \in \kappa : A_\alpha \text{ is } \textit{not } \kappa\text{-pure}\} \qquad \text{and} \qquad \Gamma(A) = [E_{\mathcal{F}}].$$

We will show that while $E_{\mathcal{F}}$ is a set of limit ordinals which depends on the filtration $\mathcal{F}$ we choose on A, its equivalence class $\Gamma(A) = [E_{\mathcal{F}}]$ does not, hence:

Lemma 3.1.9. *For any group A satisfying the Chase(κ)-condition the map $A \mapsto \Gamma(A)$ is well defined.*

Proof. We need the following:

Claim. *Let A be a set of cardinality κ and let $\{A_\alpha\}_{\alpha\in\kappa}$ and $\{A'_\alpha\}_{\alpha\in\kappa}$ be two κ-filtrations of A. Then the set $C = \{\alpha \in \kappa : \; A'_\alpha = A_\alpha\}$ is a Cub.*

Proof. Given $\alpha_0 \in \kappa$, consider A_{α_0}. Since it has size less than κ and κ is regular, there exists $\beta_0 \in \kappa$ with $\alpha_0 \in \beta_0$, and such that $A_{\alpha_0} \subseteq A'_{\beta_0}$. Analogously one can find $\alpha_1 \in \kappa$, with $\beta_0 \in \alpha_1$, and such that $A'_{\beta_0} \subseteq A_{\alpha_1}$. Inductively, we can define the chain

$$A_{\alpha_0} \subseteq A'_{\beta_0} \subseteq \cdots \subseteq A_{\alpha_n} \subseteq A'_{\beta_n} \subseteq A_{\alpha_{n+1}} \subseteq \cdots$$

We conclude that $\alpha = \sup\{\alpha_n : n \in \omega\} = \sup\{\beta_n : n \in \omega\}$, which, by the continuity of the two filtrations, yields

$$A_\alpha = \bigcup_{n<\omega} A_{\alpha_n} = \bigcup_{n<\omega} A'_{\beta_n} = A'_\beta.$$

Thus C is unbounded. It is also clear that C is closed (again by the continuity of the filtrations). □

Given two distinct κ-filtrations $\mathcal{F}$, $\mathcal{F}'$ of A, let C be the Cub given by the previous Claim, then $E_{\mathcal{F}} \cap C = E_{\mathcal{F}'} \cap C$. It follows that $\Gamma(A)$ is well defined and does not depend on the chosen filtration of A. □

We also need the following Lemma:

Lemma 3.1.10. *Let $\{A_\alpha\}_{\alpha\in\kappa}$ be a κ-filtration of the group A such that A_0 is free and $A_{\alpha+1}/A_\alpha$ is free for each $\alpha \in \kappa$. Then A is free.*

Proof. Firstly we prove that for every $\alpha \in \kappa$, each A_α is free by transfinite induction, finding a basis for it.
For $\alpha = 0$, A_0 is free by assumption. Let B_0 be a basis of A_0. Suppose now that A_β is free for all $\beta \in \alpha$ and let B_β be a basis of A_β such that $B_\beta \cap A_\gamma = B_\gamma$ for all $\gamma < \beta$. If α is a limit ordinal, then let B_α be the union of the previous bases $\bigcup_{\lambda\in\alpha} B_\lambda$. This is a basis of A_α: since $A_\alpha = \bigcup_{\beta\in\alpha} A_\beta$, pick $a \in A_\alpha$, then $a \in A_\beta$ giving that $a \in \langle B_\beta \rangle$. Hence B_α generates A_α. Since B_α is also an independent set of elements of A_α, A_α free.
If $\alpha = \delta + 1$, consider the exact sequence

$$0 \longrightarrow A_\delta \longrightarrow A_{\delta+1} \longrightarrow A_{\delta+1}/A_\delta \longrightarrow 0.$$

Since $A_{\delta+1}/A_\delta$ is free, the sequence splits and we obtain

$$A_{\delta+1} \cong A_\delta \bigoplus A_{\delta+1}/A_\delta.$$

Therefore we can let $B_\alpha = B_\delta \cup B$, where B is a basis for the copy of $A_{\delta+1}/A_\delta$ inside A_α which is in direct sum with A_δ. Then B_α is a basis for A_α which extends B_δ. If we set $B = \bigcup_{\alpha\in\kappa} B_\alpha$, then we will have a basis for A, which therefore is free. □

Theorem 3.1.11. *Let A be a group of cardinality κ satisfying the Chase(κ)-condition. Then A is free if and only if $\Gamma(A) = [\emptyset]$.*

Proof. If $\Gamma(A) = [\emptyset]$, then for each $E_{\mathcal{F}}$ there exists a Cub C which does not intersect $E_{\mathcal{F}}$. Since C is a Cub and κ regular, $|C| = \kappa$ and $A = \bigcup_{\alpha\in C} A_\alpha$. For all $\alpha \in C$, $A_{\alpha+1}/A_\alpha$ has cardinality $< \kappa$ in A/A_α, which is κ-free. Thus $A_{\alpha+1}/A_\alpha$ is free. Considering the κ-filtration of A induced by C and using the Lemma 3.1.10, we obtain that A is free.

Vice versa if A is free, then $A = \bigoplus_{\beta\in\kappa} \langle a_\beta \rangle$. Defining $A_\alpha = \bigoplus_{\beta\in\alpha} \langle a_\beta \rangle$ it follows that $\mathcal{F} = \{A_\alpha\}_{\alpha\in\kappa}$ is a κ-filtration such that $A/A_\alpha \cong \bigoplus_{\alpha\in\beta\in\kappa} \langle a_\beta \rangle$ is free (and in particular κ-free). Thus $E_{\mathcal{F}}$ is empty. □

3.2 Diamond Principle implies Whitehead's conjecture

In this section we prove the former part of Theorem 3.0.2. The crucial algebraic tool in our proof will be the following lemma.

Lemma 3.2.1. *Let $A_0 < A_1$ be countable free groups such that A_1/A_0 is not free. Given a short exact sequence of type*

$$0 \longrightarrow \mathbb{Z} \longrightarrow B_0 \xrightarrow{\pi} A_0 \longrightarrow 0,$$

and a section ρ of π (which exists since A_0 is free), there exists a group B_1 which is an extension of B_0 and a map $\overline{\pi} : B_1 \to A_1$ which extends π such that:

1. *$\overline{\pi}[B_1 \setminus B_0] = A_1 \setminus A_0$;*
2. *the sequence $0 \longrightarrow \mathbb{Z} \longrightarrow B_1 \xrightarrow{\overline{\pi}} A_1 \longrightarrow 0$ is exact;*
3. *ρ cannot be extended to a section of $\overline{\pi}$.*

Proof. Due to the existence of a section of π, the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow B_0 \xrightarrow{\pi} A_0 \longrightarrow 0,$$

splits. Therefore we can assume that $B_0 = A_0 \oplus \mathbb{Z}$, that π is exactly the projection on the first component, and that the map $a \mapsto (a, 0)$ is the section ρ.
Now consider the following exact sequence:

$$0 \longrightarrow A_0 \xrightarrow{i} A_1 \longrightarrow A_1/A_0 \longrightarrow 0,$$

where i is the inclusion map. By applying $Hom_{\mathbb{Z}}(-, \mathbb{Z})$, we obtain the long exact sequence

$$0 \longrightarrow Hom_{\mathbb{Z}}(A_1/A_0, \mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(A_1, \mathbb{Z}) \xrightarrow{i^*} Hom_{\mathbb{Z}}(A_0, \mathbb{Z}) \longrightarrow$$
$$\longrightarrow Ext^1(A_1/A_0, \mathbb{Z}) \longrightarrow Ext^1(A_1, \mathbb{Z}) \longrightarrow Ext^1(A_0, \mathbb{Z}) \longrightarrow 0.$$

Since A_1 and A_0 are free $Ext^1(A_1, \mathbb{Z}) = Ext^1(A_0, \mathbb{Z}) = 0$. Therefore we have

$$Ext^1(A_1/A_0, \mathbb{Z}) \cong \frac{Hom_{\mathbb{Z}}(A_0, \mathbb{Z})}{i^*(Hom_{\mathbb{Z}}(A_1, \mathbb{Z}))} \neq 0$$

since A_1/A_0 is a countable non-free group, hence not a W-group.
This means that there exists a homomorphism $\phi : A_0 \longrightarrow \mathbb{Z}$ which does not factorize through the inclusion $i : A_0 \longrightarrow A_1$.

Define $\widehat{B}_1 = A_1 \oplus \mathbb{Z}$. Let $f : \widehat{B}_1 \longrightarrow A_1 \times \mathbb{Z}$ be defined by

$$f(a,n) = \begin{cases} (a,n) & \text{if } \ a \notin A_0, \\ (a, n - \phi(a)) & \text{if } \ a \in A_0. \end{cases}$$

Clearly f is injective. Furthermore, if $(a,m) \in A_0 \times \mathbb{Z}$, $f(a, m + \phi(a)) = (a,m)$, and so f is a bijection. Let B_1 be $A_1 \times \mathbb{Z}$ endowed with the only group operation $\odot$ making f a group isomorphism (i.e. $b_1 \odot b_2 = f(f^{-1}(b_1) + f^{-1}(b_2))$ for all $b_1, b_2 \in B_1$). Let γ be the following injective homomorphism

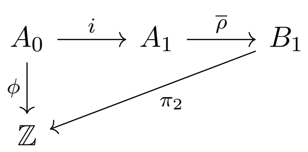
$$\gamma : B_0 \longrightarrow \widehat{B}_1 \quad \text{such that} \quad \gamma(a,n) = (i(a), n + \phi(a)).$$

The map $f \circ \gamma$ gives an injective homomorphism from B_0 to B_1, hence B_1 is an extension of B_0.
Consider the following exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow B_1 \xrightarrow{\overline{\pi}} A_1 \longrightarrow 0,$$

where the first map is the inclusion $n \mapsto (0,n)$ and $\overline{\pi}(a,m) = a$ is the projection on the first coordinate. Obviously, $\overline{\pi}$ is an extension of π, since both are projections on the first coordinate. Moreover we have that $(a,n) \in B_1 \setminus B_0$ if and only if $\overline{\pi}(a,n) = a \in A_1 \setminus A_0$.

It remains to argue that $\overline{\pi}$ does not have a section which extends ρ. Suppose by way of contradiction that there exists a $\overline{\rho}$ which is a section of $\overline{\pi}$ and extends ρ. This means that $\overline{\rho}|_{A_0} = \overline{\rho} \circ i = \gamma \circ \rho$. Consider the following diagram



where π_2 is the projection on the second coordinate. Given $a \in A_0$

$$\pi_2 \circ \overline{\rho} \circ i(a) = \pi_2 \circ \gamma \circ \rho(a) = \pi_2 \circ \gamma(a, 0) = \pi_2(i(a), \phi(a)) = \phi(a),$$

and so the diagram commutes. This is a contradiction, since ϕ does not factorize through the inclusion i. □

The following theorem is the key to the solution of Whitehead's Problem assuming Diamond.

Theorem 3.2.2. *Assume* $\diamondsuit_{\aleph_1}(E)$*. Let* $\{A_\alpha\}_{\alpha \in \aleph_1}$ *be a free strictly increasing* $\aleph_1$*-filtration of a group* A *of cardinality* $\aleph_1$ *such that* $E = \{\alpha \in \aleph_1 : A_{\alpha+1}/A_\alpha$ *is not free*$\}$ *is stationary in* $\aleph_1$*. Then* A *is not a W-group.*

Proof. Suppose that A is actually a W-group. By Theorem 2.4.1 the subgroups of a W-group are themselves W-groups: then $Ext^1(A_\alpha, \mathbb{Z}) = 0$ for all $\alpha \in \aleph_1$.
By the second part of Theorem 1.2.20, let B_α be $A_\alpha \oplus \mathbb{Z}$ and B equal to $A \oplus \mathbb{Z}$, we get the family $\{g_\alpha\}_{\alpha \in E}$, where $g_\alpha : A_\alpha \longrightarrow A_\alpha \oplus \mathbb{Z}$ and is such that for every function $f : A \longrightarrow A \oplus \mathbb{Z}$, the set $\{\alpha \in E : f|_{A_\alpha} = g_\alpha\}$ is stationary in $\aleph_1$. We now define by transfinite induction a proper exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow B_\alpha \stackrel{\psi_\alpha}{\longrightarrow} A_\alpha \longrightarrow 0$$

for all countable ordinals α so that for all $\beta < \alpha$:

- $\psi_\alpha \restriction B_\beta = \psi_\beta$,
- $\psi_\alpha[B_\alpha \setminus B_\beta] = A_\alpha \setminus A_\beta$.

For $\alpha = 0$, consider the exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow B_0 \longrightarrow A_0 \longrightarrow 0$ with the obvious inclusion and projection. Now suppose that the sequences have been defined for all $\beta \in \alpha$. Consider $\alpha = \delta + 1$, a successor ordinal.

$\delta \in E$: we first check whether $g_\delta : A_\delta \longrightarrow A_\delta \oplus \mathbb{Z}$ is a section for ψ_δ, and in this case we let $\psi_{\delta+1}$ be one of the extension of ψ_δ assured by Lemma 3.2.1 (observe that in this way g_δ cannot be extended to a section for $\psi_{\delta+1}$). Otherwise we define $\psi_{\delta+1}$ as an extension of ψ_δ whose existence is again ensured by Lemma 3.2.1.

$\delta \notin E$: then $A_{\delta+1}/A_\delta$ is free, and so $A_{\delta+1} \cong A_\delta \oplus A_{\delta+1}/A_\delta$. Choosing $B_{\delta+1} \cong B_\delta \oplus A_{\delta+1}/A_\delta$, we get $\psi_{\delta+1} : (b, [a]) \mapsto (\psi_\delta(b), [a])$ is a coherent extension of ψ_δ such that

$$0 \longrightarrow \mathbb{Z} \longrightarrow B_{\delta+1} \stackrel{\psi_{\delta+1}}{\longrightarrow} A_{\delta+1} \longrightarrow 0$$

is exact and $\psi_{\delta+1}[B_{\delta+1} \setminus B_\delta] = A_{\delta+1} \setminus A_\delta$.

α limit: since $B_\alpha = \bigcup_{\beta \in \alpha} B_\beta$, we choose ψ_α as the coherent union of all the ψ_β.

Notice that our inductive construction ensures that $\psi_\beta[B_\beta \setminus B_\alpha] = A_\beta \setminus A_\alpha$ for all $\beta > \alpha$.

Let $\psi : B \longrightarrow A$ be the union of the chain $\{\psi_\alpha\}_{\alpha \in \aleph_1}$. $Ext^1(A, \mathbb{Z}) = 0$, so there is a section $\rho : A \longrightarrow A \oplus \mathbb{Z}$. Then (since $\psi_\beta[B_\beta \setminus B_\alpha] = A_\beta \setminus A_\alpha$ for all $\beta > \alpha$), $\rho[A_\alpha] \subseteq B_\alpha$ for all countable ordinals α, hence $\rho \restriction A_\alpha$ is a section of $\psi_\alpha = \psi \restriction B_\alpha$ for all countable ordinals α. Therefore the set $\{\alpha \in E : \rho|_{A_\alpha} = g_\alpha\}$ is stationary and thus it is not empty. For each α in this latter set $\rho_{A_{\alpha+1}}$ is a section of $\psi_{\alpha+1}$ which extends g_α, that is absurd for the construction of $\psi_{\alpha+1}$.

□

Lemma 3.2.3. *Assume $\diamondsuit_{\omega_1}(E)$ for some stationary set E. Assume A is a W-group of cardinality $\aleph_1$; then it satisfies the Chase($\aleph_1$)-condition.*

Proof. Suppose by absurd that the thesis is not true. Since a W-group is $\aleph_1$-free, then if A does not satisfy Chase($\aleph_1$)-condition there exists a countable subgroup B of A such that for every countable subgroup A' which contains it there exists A'' such that A''/A' is not free. For each $\alpha \in \aleph_1$ define A_α by transfinite induction as follows.

For $\alpha = 0$ we set $A_0 = B$. Suppose that A_β has been already defined for every $\beta \in \alpha$.

If $\alpha = \delta + 1$, let $A_{\delta+1}$ be a countable subgroup such that $A_{\delta+1}/A_\delta$ is not free.

If α is limit, then set $A_\alpha = \bigcup_{\beta \in \alpha} A_\beta$.

$\{A_\alpha\}_{\alpha \in \aleph_1}$ is a strictly increasing $\aleph_1$-free filtration such that $\{\alpha \in \aleph_1 : A_{\alpha+1}/A_\alpha$ is not free $\} = \aleph_1 \supseteq E$. This is impossible by previous theorem. □

Corollary 3.2.4. *$\diamondsuit_{\aleph_1}(E)$ implies that if A is a W-group of cardinality $\aleph_1$, then it is free.*

Proof. By Lemma 3.2.3 A satisfies the Chase($\aleph_1$)-condition. Therefore (by Lemma 3.1.9) $\Gamma(A) = [E_{\mathcal{F}}]$ is well-defined, where $\mathcal{F} = \{A_\alpha\}_{\alpha \in \omega_1}$ is some (any) $\aleph_1$-free filtration such that $A_{\alpha+1}$ is $\aleph_1$-pure for each $\alpha \in \aleph_1$. Clearly

$$E \subseteq E_{\mathcal{F}} = \{\alpha \in \aleph_1 : A_\alpha \text{ is not } \aleph_1\text{-pure}\}.$$

(if A/A_α is $\aleph_1$-free, $A_{\alpha+1}/A_\alpha$ is free, being a countable subgroup of it). Moreover $A_\beta/A_{\alpha+1} \cong (A_\beta/A_\alpha)/(A_{\alpha+1}/A_\alpha)$ for each $\beta > \alpha$, and $A_\beta/A_{\alpha+1}$ is free. If $A_{\alpha+1}/A_\alpha$ is free, then $A_\beta/A_\alpha \cong A_\beta/A_{\alpha+1} \oplus A_{\alpha+1}/A_\alpha$: thus A_β/A_α is free too. By Lemma 3.1.3 A/A_α is $\aleph_1$-free, and so $E = E_{\mathcal{F}}$ and $\Gamma(A) = [\varnothing]$. By Theorem 3.2.2 E is non-stationary, hence $[E] = [E_{\mathcal{F}}] = [\emptyset]$. Therefore $\Gamma(A) = [E_{\mathcal{F}}] = [\emptyset]$. By Theorem 3.1.11 we conclude that A is free.

□

3.3 Martin's Axiom implies Whitehead's conjecture is false

Now we deal with the latter part of Theorem 3.0.2, by proving that Martin's axiom entails that there exist W-groups which are not free.

REMARK 12. Recall that a subgroup $B \subseteq A$ is *pure* if A/B is torsion-free. Moreover, given any subset $X \subseteq A$, the smallest pure subgroup containing X is $Y = \{a : na \in \langle X \rangle \text{ for some } n \in \mathbb{Z}^*\}$.

Proof. Let P be a pure subgroup containing X (equivalently $\langle X \rangle$). If $a \in Y$ then $na \in \langle X \rangle$, which is contained in P. Hence the equivalence class $n \cdot [a] = [na]$ is the trivial one in A/P, and so $[a] = [0]$ by torsion-freeness of A/P. Therefore $Y \subseteq P$. On the other hand, A/Y is torsion-free: if there is $m \in \mathbb{Z}^*$ such that $m \cdot [a] = [ma] = [0]$ then there exists $n \in \mathbb{Z}^*$ such that $mn \cdot a \in \langle X \rangle$. Thus $a \in Y$ and Y is pure. By minimality of P it follows that $Y = P$. □

We start proving the following theorem:

Theorem 3.3.1. *Assume MA+$2^{\aleph_0} > \aleph_1$. Let A be a group of cardinality $\aleph_1$ which satisfies Chase($\aleph_1$)-condition. Then A is a W-group.*

Proof. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow B \xrightarrow{\psi} A \longrightarrow 0.$$

We must prove that there exists a section $\rho : A \longrightarrow B$ of ψ. Consider the set

$$P = \{\rho_S : S \longrightarrow B \ : \rho_S \text{ is a homomorphism from a finitely-generated pure subgrop of A such that } \psi \circ \rho_S = id_S\}$$

partially ordered by $\rho_S \leq \rho_{S'}$ if $\rho_S \subseteq \rho_{S'}$. We want to show that P satisifies the three assumptions of Lemma 1.2.30. First of all remark that any $\rho_S \in P$ is injective, being a partial section of ψ, hence any $\rho_S \in P$ (having a free group as domain) is uniquely determined by its values on a set of generators of S.

- By definition P satisfies the first hypothesis of Lemma 1.2.30.
- Consider $\rho_S \in P$ and $s \in A$ and define

  $$S' = \{a \in A : \text{there exists } n \in \mathbb{Z}^* \text{ such that } na \in \langle S \cup \{s\} \rangle \}.$$

  By Remark 12, S' is a pure subgroup containing $S \cup \{s\}$, or analogously $\{a_0, \dots, a_m, s\}$, where $B_0 = \{a_0, \dots, a_m\}$, is a minimal set of generators of S. Moreover S' is a countable subgroup of A, and (since A is $\aleph_1$-free by the Chase($\aleph_1$)-condition) we have that S' is free. However either B_0 is a maximal independent family in S' or $\{a_0, \dots, a_m, s\}$ is. In either cases $r_0(S') \leq m + 1$, hence S' is finitely generated. Therefore S' is a finitely-generated pure subgroup of A.

  Now assume $s \notin S$. Since $S \subseteq S'$ and S is pure, we have that S'/S is a finitely generated torsion-free group. By Proposition 1.1.8 S'/S is free, and $S' = S \oplus S'/S$ (since the exact sequence

  $$0 \longrightarrow S' \longrightarrow S \longrightarrow S'/S \longrightarrow 0$$

splits). Therefore there exists a basis of S', $X \cup Y$, such that X is a basis of S. In order to define an injective homomorphism $\rho_{S'} : S' \longrightarrow B$, it is enough to define it on a basis (since S' is free). For all $x \in X$ set $\rho_{S'}(x) = \rho_S(x)$, and for all $y \in Y$ define $\rho_{S'}(y)$ to be some element in $\psi^{-1}(\{y\})$. We have just built a homomorphism such that $\psi \circ \rho_{S'}$ is the identity on S'. Hence $\rho_{S'}$ belongs to P, and the second hypothesis of Lemma 1.2.30 is true for P.

We are left with the proof that P respects the third assumption of Lemma 1.2.30. To proceed further we need the following:

Claim. *For any uncountable subset P' of P there is an uncountable free pure subgroup A' of A and an uncountable subset P'' of P' such that* $\mathrm{dom}(\rho_S) \subseteq A'$ *for every $\rho_S \in P''$.*

Proof. Enumerate the elements of P', $\{\rho_\alpha : S_\alpha \longrightarrow B\}_{\alpha \in \aleph_1}$. Define P'_n as the subset of P' such that for all $\rho_\alpha \in P'_n$ S_α has a basis of cardinality n. The collection $\{P'_n\}_{n \in \omega}$ gives a partition of P', and thus there is one P'_n whose cardinality is $\aleph_1$. Define T as a pure subgroup of A maximal with respect to the property of being contained in uncountably many $S_\alpha \in P'_n$. Fixing such a T we can further refine P'_n to an uncountable subset P'' given by the S_α in which T is contained. Fix for each $\alpha < \omega_1$ X_α, Y_α such that X_α is a base for T and $X_\alpha \cup Y_\alpha$ is a base for S_α. Since T is countable we can find an uncountabel set I and a fixed X such that $X_\alpha = X$ for all $\alpha \in I$. Hence letting $P^* = \{\rho_\alpha : \alpha \in I\}$. We reenumerate P^* so that:

- $P^* = \{\rho_\alpha : S_\alpha \to B : \alpha < \omega_1\}$ with each S_α a pure subgroup of rank n,
- $T \subseteq S_\alpha$ for all $\alpha < \omega_1$,
- T' is not contained in some S_α for any $T' \supset T$ pure subgroup of A,
- X is a base of T and $X \cup Y_\alpha$ is a base for S_α for all $\alpha < \omega_1$.

Define a chain of countable pure subsets $\{A_\alpha\}_{\alpha \in \aleph_1}$ by transfinite induction as follows: $A_0 = T$, which is free and pure. Next suppose we have defined $\{A_\beta\}_{\beta \in \alpha}$ so that there exists a strictly increasing sequence of ordinals $\{\gamma_\beta\}_{\beta \in \alpha}$ such that for all $0 \in \beta \in \alpha$, Y_{γ_β} is contained in A_β.

If $\alpha = \delta + 1$, let B_δ be an $\aleph_1$-pure countable subgroup containing A_δ, whose existences is ensured by the Chase($\aleph_1$)-condition for A. Suppose that $B_\delta \cap (S_\alpha \setminus T) \neq \emptyset$ for uncountably many $\alpha \in \aleph_1$. Since B_δ is countable, there exists $b \in B_\delta \setminus \{0\}$ such that $b \in B_\delta \cap (S_\alpha \setminus T)$ for uncountably many α. This means that $\langle T \cup \{b\} \rangle$ is contained in uncountably many S_α, with $b \notin T$; as previously done, define $T' = \{a \in A :$ there is $n \in \mathbb{N}$ such that $na \in \langle T \cup \{b\} \rangle\}$. We can again argue that T' is a pure subgroup of A containing $T \cup \{b\}$ with $r_0(T') = r_0(T) + 1$. Since S_α is pure for all $\alpha \in \aleph_1$, it follows that $T' \subseteq S_\alpha$ for uncountably many α (i.e for all α such that $b \in S_\alpha$). This contradicts the maximality property of T (remind that T was chosen to be maximal with the property of being contained in uncountably many S_α). Therefore (since $\langle Y_\alpha \rangle \setminus \{0\} \subseteq S_\alpha \setminus T$) $B_\delta \cap \langle Y_\alpha \rangle = \{0\}$ for eventually all α.

Let $\gamma_{\delta+1}$ be the first ordinal for which it happens. Define $A_{\delta+1}$ as the smallest pure subgroup which contains $A_\delta \cup Y_{\gamma_{\delta+1}}$. Clearly $A_\delta \subseteq A_{\delta+1} \cap B_\delta$. Conversely, if $a \in A_{\delta+1} \cap B_\delta$ then $a \in A_{\delta+1}$, and so there is $n \in \mathbb{N}$ such that $na = a_\delta + y$ with $a_\delta \in A_\delta$ and $y \in Y_{\gamma_{\delta+1}}$. Since $a \in B_\delta$ too, $na = a_\delta + y \in B_\delta$, or rather $y \in B_\delta$. It follows that $y = 0$ and $a \in A_\delta$, hence $A_{\delta+1} \cap B_\delta = A_\delta$. Since $B_\delta + A_{\delta+1}$ is countable and A/B_δ is $\aleph_1$-free, we get that $(B_\delta + A_{\delta+1})/B_\delta$ is also free. Now

$$A_{\delta+1}/A_\delta \cong A_{\delta+1}/(B_\delta \cap A_{\delta+1}) \cong (B_\delta + A_{\delta+1})/B_\delta,$$

hence $A_{\delta+1}/A_\delta$ is free.

For α a limit ordinal, set $A_\alpha = \bigcup_{\beta\in\alpha} A_\beta$. A_α is countable and therefore free. Moreover it is a union of pure subgroups and so it is pure too.

Let $A' = \bigcup\{A_\alpha : \alpha < \omega_1\}$. Then A' is pure, since union of pure subgroups, and free by Lemma 3.1.10 applied to the filtration $\{A_\alpha : \alpha < \omega_1\}$. If we consider $P'' = \{\rho_{\gamma_{\alpha+1}}\}_{\alpha\in\aleph_1}$ we have that $\mathrm{dom}(\rho_{\gamma_{\alpha+1}}) = S_{\gamma_{\alpha+1}} \subseteq A'$.

□

We continue with the proof of the third condition for P. Without loss of generality it is enough to prove it for a P' that satisfies the hypotheses of the previous Claim and consists of elements of the same size.

Following the notation of Claim 3.3.1, let $Z = \{z_\alpha\}_{\alpha\in\aleph_1}$ be a basis of A' and consider $\mathrm{dom}(\rho_{\gamma_\beta})$. Each element of the basis $X \cup Y_{\gamma_\beta}$ can be written as a finite linear combination of $\{z_\alpha\}_{\alpha\in\aleph_1}$, thus $\mathrm{dom}(\rho_{\gamma_\beta})$ is contained in $\bigoplus_{i\in n} \langle z_{\alpha_i}\rangle$ for some fixed set $\{\alpha_1, \ldots, \alpha_n\}$. Since $\mathrm{dom}(\rho_{\gamma_\beta})$ is pure, we have that $\bigoplus_{i\in n} \langle z_{\alpha_i}\rangle \big/ \mathrm{dom}(\rho_{\gamma_\beta})$ is free, hence

$$\mathrm{dom}(\rho_{\gamma_\beta}) \oplus \left(\bigoplus_{i\in n} \langle z_{\alpha_i}\rangle \Big/ \mathrm{dom}(\rho_{\gamma_\beta})\right) \cong \bigoplus_{i\in n} \langle z_{\alpha_i}\rangle.$$

ρ_{γ_β} can be extended to a function ρ^*_β in P whose domain is a finite subset of Z such that $\psi \circ \rho^*_\beta$ is the identity, as follows: fix $\{[z_{\alpha_{i_1}}], \ldots, [z_{\alpha_{i_k}}]\}$ base for $\left(\bigoplus_{i\in n} \langle z_{\alpha_i}\rangle \Big/ \mathrm{dom}(\rho_{\gamma_\beta})\right)$. For each z_{α_i} find b_i in its preimage under ψ and let

$$\rho^*_\beta(a + \sum_{j=1}^{k} n_j[z_{\alpha_j}]) = \rho_\beta(a) + \sum_{j=1}^{k} n_j b_j.$$

By taking an uncountable subset of $\{\rho^*_\beta : \rho_{\gamma_\beta} \in P'\}$, one can assume that the cardinality of $r_0(\mathrm{dom}(\rho^*_\beta))$ is some fixed n for all functions in P'. Let

$$P^{**} = \{\rho^*_\beta : T_\beta \to B : \beta \in I\}$$

be this latter set. Then each of the elements of P^{**} is a condition in P which extends a unique condition in P' and whose domain has a basis of size n contained in the uncountable and pure free subgroup A' of A.

Let $Z_\beta \subseteq Z$ be a basis of T_β for each $\beta \in I$. As in the proof of Lemma 3.3.1, define Z^* as a subset of Z maximal with respect to the property of being contained in uncountably many Z_α. We can refine I to an uncountable J such that $Z^* \subseteq Z_\alpha$ for all $\alpha \in J$. Notice that $\ker(\psi)$ is countable since it is isomorphic to $\mathbb{Z}$. Therefore $\psi^{-1}(z_i)$ is a countable set for all $z_i \in Z$. This means that each $\rho^*_\beta \restriction Z^*$ has range contained in a countable set. In particular we can refine J to an uncountable set K such that $\rho^*_\beta \restriction Z^* = \rho$ for all $\beta \in K$ and for some fixed injective homomorphism $\rho : \langle Z^* \rangle \to B$.

Now for all $z \in Z_0 \setminus Z^*$ there exists only countably many α such that $z \in Z_\alpha$, by the maximality property of Z^*. Hence, there is $\alpha \neq 0$ such that $Z_\alpha \cap Z_0 = Z^*$. Since $\rho^*_0 \restriction Z^* = \rho^*_\alpha \restriction Z^*$ we can define a common extension of ρ^*_0 and ρ^*_α to a section ρ' of ψ defined on $S' = \langle Z^{**} \rangle$ (where $Z^{**} = Z_0 \cup Z_\alpha$) letting $\rho'(a+b+c) = \rho^*_0(a+b) + \rho^*_\alpha(c)$ for $a \in \langle Z^* \rangle, b \in \langle Z_0 \setminus Z^* \rangle, c \in \langle Z_\alpha \setminus Z^* \rangle$ since $S' = \langle Z^* \rangle \oplus \langle Z_0 \setminus Z^* \rangle \oplus \langle Z_\alpha \setminus Z^* \rangle$.

Notice that $A'/S' \cong \langle Z \setminus Z^{**} \rangle$ is free; moreover $(A/S')/(A'/S') \cong A/A'$ is torsion free; we can conclude that also A/S' is torsion-free by Fact 1.1.17.
Hence S' is a finitely-generated pure subgroup of A. We conclude that $\rho' \in P$ refines ρ_{γ_0} and ρ_{γ_α} in P'. Therefore P satisfies also the third property.

By Lemma 1.2.30, there exists $\rho : A \longrightarrow B$ such that for each finite $F \subseteq A$ there is $\rho_S \in P$ such that $F \subseteq \mathrm{dom}(\rho_S)$ and $\rho \restriction_F = \rho_S \restriction_F$. It follows that ρ is a homomorphism and in particular a section of ψ defined on A.

□

We can conclude our discussion proving right away in ZFC that there is a group which is not free but satisfies Chase($\aleph_1$)-condition. Assuming $MA + 2^{\aleph_0} > \aleph_1$ such a group is a W-group of cardinality $\aleph_1$. It follows that under this assumption there exists a non-free W-group of cardinality $\aleph_1$.

Theorem 3.3.2. *There exists a non-free group A of cardinality $\aleph_1$ which satisfies the Chase($\aleph_1$)-condition.*

Proof. We define by transfinite induction a family $\mathcal{F} = \{A_\alpha\}_{\alpha \in \aleph_1}$ satisfying the following properties:

1. A_α is countable and free for every $\alpha \in \aleph_1$;
2. $A_\beta < A_\alpha$ for all $\beta \in \alpha \in \aleph_1$;
3. $A_\alpha = \bigcup_{\beta \in \alpha} A_\beta$ for all limit ordinals $\alpha \in \aleph_1$;
4. $A_\alpha / A_{\beta+1}$ is free for all $\beta \in \alpha \in \aleph_1$;
5. $A_{\alpha+1}/A_\alpha$ is not free for all limit ordinals $\alpha \in \aleph_1$.

Suppose we succeed. Then we let $A = \bigcup_{\alpha < \omega_1} A_\alpha$. By Lemma 3.1.6 $\mathcal{F}$ is an $\aleph_1$-filtration witnesses the Chase(($\aleph_1$))-condition for A: for all α A_α is countable and free, and

$$A/A_{\alpha+1} = \bigcup_{\beta > \alpha+1} A_\beta / A_{\alpha+1}$$

is also $\aleph_1$-free, being an increasing union of countable free groups. We get that $\Gamma(A) = [E_{\mathcal{F}}] = \{\beta < \omega_1 : \beta \text{ is limit}\} = [\omega_1] \neq [\emptyset]$. By Lemma 3.1.11, we conclude that A is not free.

We define the filtration as follows:

- $A_0 = 0$
- Suppose that A_β has been already defined for every $\beta \in \alpha$ so to satisfy the above clauses. We define A_α according to the following cases:

  α is a limit ordinal: $A_\alpha = \bigcup_{\beta\in\alpha} A_\beta$. Then A_α is equal to $\bigcup_{n\in\omega} A_{\beta_n}$, where $\{\beta_n\}_{n\in\omega}$ is a filtration of α such that every β_n is a successor ordinal. Then A_α is a countable torsion-free group: any of its subgroups of finite rank is in some A_{β_n} and thus each of them is free. By Pontryagin's Criterion it follows that also A_α is free. Again by Pontryagin's Criterion we conclude that $A_\alpha/A_{\delta+1}$ is free for all $\delta \in \alpha$ since so are all $A_{\beta_n}/A_{\delta+1}$ by inductive assumption.

  $\alpha = \delta+1$ and δ is not a limit ordinal: $A_\alpha = A_\delta \oplus \mathbb{Z}$. It is immediate to check that $\{A_\gamma : \gamma \leq \alpha\}$ still satisfies all the required clauses.

  $\alpha = \delta + 1$ and δ is a limit ordinal: We fix an increasing sequence β_n converging to δ with $\beta_0 = 0$, and β_n a successor ordinal for all $n > 0$. Then $A_\delta = \bigcup_{n\in\omega} A_{\beta_n}$. Let $\{B_n\}_{n\in\omega}$ be a an increasing family of subsets of A_δ with B_n a basis for A_{β_n} and $B_n \subsetneq B_{n+1}$. For each $n \geq 1$, choose $b_n \in B_n \setminus B_{n-1}$ and define $B'_0 = B_0 = 0$ and $B'_n = B_n \setminus \{b_1, \dots, b_n\}$. Let B' be the subgroup of A_δ generated by $\langle \bigcup_{n\in\omega} B'_n \rangle$. Observe that $\bigcup_{n\in\omega} B'_n = \left(\bigcup_{n\in\omega} B_n\right) \setminus \{b_n : n \in \omega\}$. Since $\bigcup_{n\in\omega} B_n$ is a basis (for A_δ), then $B' \cap \prod_{n\geq 1} \langle b_n \rangle = \{0_{A_\delta}\}$. Let $P = \prod_{n\geq 1} \langle b_n \rangle$.

  It will be convenient in what follows to denote an element $\langle n_i b_i : i \in \mathbb{N}\rangle$ of P by $\sum_{i\in\mathbb{N}} n_i b_i$ and an element $(c, \langle n_i b_i : i \in \mathbb{N}\rangle)$ of $B' \oplus P$ by $c + \sum_{i\in\mathbb{N}} n_i b_i$, using in both cases an additive notation.

  Using this convention, it is immediate to identify A_δ with the subgroup of $B' \oplus P$ given by generalized sums of elements of $B' \cup \{b_n : n \in \mathbb{N}\}$ with a finite number of non-zero coefficients.
  We define $A_{\delta+1}$ as the subgroup of $B' \oplus P$ generated by B' and $\{a_m\}_{1\leq m\in\omega}$, where

  $$a_m = \sum_{n\geq m} \frac{n!}{m!} b_n.$$

  First we observe that $b_n = a_n - (n+1)a_{n+1}$ for each $n \in \mathbb{N}^*$, therefore $B' \cup \{a_n\}_{n\in\mathbb{N}^*}$ generates $A_{\delta+1}$ and $A_\delta \subseteq A_{\delta+1}$. We also claim that $\bigcup_{n\in\omega} B'_n \cup \{a_n\}_{n\in\mathbb{N}^*}$ forms a basis for $A_{\delta+1}$: it is a generating set for $A_{\delta+1}$, and moreover no linear combination of the a_n can belong to $B' \setminus \{0\}$. Therefore it suffices to show that no non-trivial linear combination of the a_n can be 0_P. Suppose to the contrary that there exist $n_1 < ... < n_k$ indices in $\mathbb{N}^*$ and $z_1, ..., z_m$ non-zero integers such that

$$\sum_{k=1}^{m} z_k a_{n_k} = 0_P.$$

Then

$$0_P = \sum_{k=1}^{m} z_k a_{n_k} = \sum_{k=1}^{m} z_k \sum_{n \geq n_k} \frac{n!}{n_k!} b_n = \sum_{k=1}^{m} \sum_{n \geq n_k} z_k \frac{n!}{n_k!} b_n = 0_P.$$

Now the n_1-th coordinate of

$$\sum_{k=1}^{m} \sum_{n \geq n_k} z_k \frac{n!}{n_k!} b_n$$

is exactly $z_1 b_1$, giving that $z_1 = 0$, which is impossible. Thus, $\{a_n\}_{n \in \mathbb{N}^*}$ forms an independent subset of P, or rather $\underset{n \in \omega}{\cup} B'_n \cup \{a_n\}_{n \in \omega}$ is a basis for $A_{\delta+1}$, which is consequently free and countable.

Claim. *$A_\alpha / A_{\beta+1}$ is free for all $\beta \in \delta$.*

Proof. Being $\{\beta_n\}_{n \in \omega}$ a filtration of δ there exists $\beta_n > \beta + 1$ for all $\beta \in \delta$. Hence we get an exact sequence

$$0 \longrightarrow A_{\beta_n}/A_{\beta+1} \longrightarrow A_\alpha/A_{\beta+1} \longrightarrow (A_\alpha/A_{\beta+1})/(A_{\beta_n}/A_{\beta+1}) \longrightarrow 0.$$

$A_{\beta_n}/A_{\beta+1}$ is free by inductive assumptions on $\beta + 1 < \beta_n$ for all $n \in \omega$. By Lemma 1.1.21 to prove the Claim it suffices to prove that $A_\alpha/A_{\beta_n} \cong (A_\alpha/A_{\beta+1})/(A_{\beta_n}/A_{\beta+1})$ is free, and therefore projective for all $n \in \omega$. Let us prove it: since $A_{\beta_n} = \langle B_n \rangle = \langle a_1, \ldots, a_n \rangle$, A_α/A_{β_n} is generated by

$$\bigcup_{m \in \omega} \{B'_m \setminus B_n\} \cup \{[a_m]\}_{n \in m \in \omega}.$$

Now observe that

$$C = \langle \bigcup_{m \in \omega} (B'_m \setminus B_n) \cup \{a_m\}_{n \in m \in \omega} \rangle$$

is a subgroup of $A_{\delta+1}$ in direct sum with B_n, hence the map $a \mapsto [a]$ is an isomorphism of C with A_α/A_{β_n}. Since C is free so is A_α/A_{β_n}.
The proof of the Claim is completed. □

Finally, $A_{\delta+1}/A_\delta$ is not free because $\overline{a_1}$ is an element which is divisible for all $n \in \omega$:

$$a_1 - n!\, a_n = \sum_{i=1}^{n} i!\, b_i \in A_\delta$$

for all $n \in \mathbb{N}$, hence $\overline{a_1} = n!\, \overline{a_n}$ in $A_{\delta+1}/A_\delta$ for all $n \in \mathbb{N}$.

□

The second part of Theorem 3.0.2 may be generalized as follows.

Proposition 3.3.3. *Martin's Axiom plus* $2^{\aleph_0} > \aleph_1$ *imply that for every uncountable cardinal* κ *there exists a non-free W-group of cardinality* κ.

Proof. By Theorem 2.4.1, if A is a non-free W-group of cardinality $\aleph_1$ then the direct sum of κ copies of A is a W-group since

$$Ext^1\left(\bigoplus_{\alpha\in\kappa} A, \mathbb{Z}\right) \cong \prod_{\alpha\in\kappa} Ext^1(A, \mathbb{Z}) = 0.$$

□

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