An introduction to OCA

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Introduction

These notes are extracted from the lectures on forcing axioms and applications held by professor Matteo Viale at the University of Turin in the academic year 2011-2012. Our purpose is to give a brief account of the axiom OCA, introduced by Todorčević in [8], which can be seen as a sort of two-dimensional perfect set property. It is a basic result of descriptive set theory that every analytic set is either countable or it contains a perfect subset. It might be surprising but a similar dichotomy can be stated in a two dimensional version.

Let X be a separable metric space and by $[X]^2$ denote the family of all unordered pairs of elements of X,

$$[X]^2 = \{\{x, y\} : x \neq y \text{ and } x, y \in X\}.$$

Subsets of $[X]^2$ can be seen as the symmetric subsets of X^2 minus the diagonal. A subset K of $[X]^2$ is open if for every $\{x, y\}$ in K there are disjoint neighborhoods U of x and V of y such that $\{\{x', y'\} : x' \in U, y' \in V\}$ is contained in K. We call **(open) coloring** of X every (open) subset of $[X]^2$.

Definition 1. *Let X be a separable metric space, $K \subseteq [X]^2$ and $Y \subseteq X$. Y is said to be K-**homogeneous** if $[Y]^2$ is contained in K. Instead, we say that Y is K-**countable** if $Y = \bigcup\{Y_n : n \in \omega\}$ where each Y_n is K^c- homogeneous.*

Example 1. *Let us examine two typical examples of open colorings of $\mathbb{R}$:*

1. *For all n, $K_n = \{(x, y) \in \mathbb{R}^2 : y < x - 1/n\}$ is an open coloring of $\mathbb{R}$ and $\mathbb{R}$ is K_n-countable.*
2. *If $K = \{(x, y) \in \mathbb{R}^2 : y < x \ \& \ \forall n(y \neq x - 1/n)\}$, K is an open coloring of $\mathbb{R}$, but $\mathbb{R}$ is not K-countable. In fact, a K^c-homogeneous set is at most countable. (For more details see Remark **??** of Section 2).*

Drawing pictures of these sets might be of great help.

Remark 1. *Let X be a separable metric space and let K be a coloring of X. The following properties are trivial to check:*

(a) Let $Y \subseteq X$. If X is K-countable, then also Y is K-countable. (Thus, if Y is not K-countable, then so is X).

(b) If $\{X_n : n \in \omega\}$ is a family of $K-$countable subsets of X, then also $\bigcup_{n\in\omega} X_n$ is K-countable.

Let us introduce the Open Coloring Axiom (OCA). From now on, if not specified, X will be a separable metric space.

Axiom 1. $\mathbf{OCA_P(X)}$. *For any K open colouring of X exactly one of the following holds:*

- *X is $K-$countable,*
- *There exists a perfect subset (i.e. a nonempty compact subset without isolated points) P of X that is homogeneous for K.*

As we wil see in Section 1, it is a ZFC theorem that the conclusion of $OCA_P(X)$ holds for every open graph on an analytic set X of a Polish space.

Axiom 2. $\mathbf{OCA_P}$ *$OCA_P(X)$ holds for all X separable metric spaces.*

After examinig some properties of colorings in Section 2, we will show that the stated above is a natural consequence of AD (Section 3).

Is it possible to push such a dichotomy even further, in order to cover classes of sets for which AD fails. This is possible if we slightly weaken the dichotomy.

Axiom 3. $\mathbf{OCA(X)}$ *For any K open colouring of X exactly one of the following holds:*

- *X is $K-$countable,*
- *There exists an uncountable subset Z of X that is homogeneous for K.*

Axiom 4. OCA *$OCA(X)$ holds for all X separable metric spaces.*

This last dichotomy is strong enough to decide many questions on the continuum. In Section 4, indeed, we will prove that under OCA in $(P(\omega), \subseteq^*)$ there are only Haussdorff gaps or (κ, ω)-gaps where $\kappa \geq b$ and that $b = \omega_2$.

1 Principle of Open Coloring for analytic sets

One reason why OCA_P, and then OCA, can be considered a natural axiom is the following Principle of Open Coloring for analytic sets, which we will prove in this section.

Theorem 1. *Let X be an analytic set and K an open colouring of X. Then exactly one of the following holds:*

- *X is K-countable, or*

- *X contains a perfect $K-$homogeneous set.*

The proof of the next lemma is straightforward, since we have assumed by definition that a perfect subset of a topological space is compact.

Lemma 1. *Let X and Y be topological spaces. If P is a perfect subset of X and $f : X \to Y$ is a continuous function injective on P, then $f[P]$ is perfect in Y.*

Proposition 1. *Let X and Y be metric spaces. If $OCA_P(X)$ ($OCA(X)$) holds and Y is a continuous image of X, then $OCA_P(Y)$ ($OCA(Y)$) holds.*

Proof. Let K be an open colouring of Y. Observe that

$$H = \{\{x, y\} : f(x) \neq f(y) \text{ and } \{f(x), f(y)\} \in K\}$$

is open in $[X]^2$. Notice that trivially the image by f of a $H-$homogeneous (H^c-homogeneous) subset of X is a $K-$homogeneous (K^c-homogeneous) subset of Y.

By $OCA_P(X)$, either $X = \bigcup\{X_n : n \in \omega\}$, where each X_n is homogeneous for H^c, or there exists a perfect subset P of X homogeneous for H. In the first case, $Y = f[X] = \bigcup\{f[X_n] : n \in \omega\}$ and each $f[X_n]$ is homogeneous for K^c. Otherwise, notice that f in injective on P, since P is homogeneous for H. Thus, by Lemma 1, $f[P]$ is perfect in Y. □

Proposition 1 plays a crucial role in the study of OCA. In fact, we know that every T_2 second countable space, and thus even every separable metric space, is a 1-1 continuous image of a set of reals (i.e. a subset of 2^ω). Therefore OCA for sets of reals, which can be proved to be consistent with ZFC, implies OCA for separable metric spaces.

Moreover, by Proposition 1, in order to prove Theorem 1 we just need to prove it for $X = \omega^\omega$, since every analityc set is a continuous image of ω^ω.

Notation 1. *Let $s, t \in \omega^n$ for some $n \in \omega$. Let*

$$|x| = n,$$

$$[s] = \{f \in \omega^\omega : f \restriction n = s\},$$

$$[s] \otimes [t] = \{(f, g) \in \omega^\omega \times \omega^\omega : (f \in [s] \ \& \ g \in [t]) \ or \ (f \in [t] \ \& \ g \in [s])\}.$$

Theorem 2. *(ZFC) $OCA_P(\omega^\omega)$.*

Proof. Let K be an open colouring of ω^ω such that ω^ω is not K-countable. We have to find in ω^ω a perfect set P homogeneous for K.

We will construct a perfect subtree of ω^ω whose body is homogeneous for K. Proving the following claims, we will tacitly use Remark 1.

Claim 1. *If $s \in \omega^{<\omega}$ is such that $[s]$ is not K-countable, then there exists t extension of s such that $[t]$ is not K-countable.*

Proof. Suppose the claim is false for some s. In particular, we would have that $[s\hat{}n]$ is $K-$countable for every $n \in \omega$, but this contradicts our hypothesis on s. $\square$

Claim 2. *If $s \in \omega^{<\omega}$ is such that $[s]$ is not K-countable, then there exist t, u incompatible extensions of s such that $[t]$ and $[u]$ are not K-countable.*

Proof. By Claim 1, it is possible to define by induction a succession t_n with $t_0 = s$, t_{n+1} proper extension of t_n and such that each $[t_n]$ is not $K-$countable. Let f be the limit of the t_n. If the claim is false, f is the common limit of all the succession of this type and $[s] \setminus \{f\}$ is a countable union of K-countable sets. So $[s] \setminus \{f\}$ is $K-$countable set. Since K^c is closed and contains $[s] \setminus \{f\}$, then even f is in K^c. Therefore all $[s]$ is $K-$countable, a contradiction. $\square$

Notice that Claim 2 says that

$$T_K = \{s \in \omega^{<\omega} : [s] \text{ is not } K\text{-countable}\}$$

is a perfect subtree of $\omega^{<\omega}$.

Claim 3. *For every $s \in T_K$ there are $t, u \in T_K$ incompatible extensions of s such that $[t] \otimes [u] \subseteq K$.*

Proof. Observe that K is an open colouring even of $[T_K]$. Moreover, notice that for each $s \in T_K$,

$$[s] \cap [T_K] \text{ is not } K\text{-countable in } [T_K].$$

In fact, we have that $[s] \cap (\omega^\omega \setminus [T_K])$ is $K-$countable, since it can be expressed as the union of countably many sets as $[t]$ with t extension of s and $[t]$ be $K-$countable. Thus, if $[s] \cap [T_K]$ were $K-$countable so would $[s]$ be.

Now suppose that $s \in T_k$ contradicts the claim, i.e. for all $t, u \in T_K$ extending s we have that $[t] \otimes [u] \nsubseteq K$. This implies that $([t] \otimes [u]) \cap ([T_K] \otimes [T_K]) \subseteq K^c$, since K is open in $[T_K]^2$. But this is a contradiction, because it would mean that $[s] \cap [T_K]$ is homogeneous for K^c, while $[s] \cap [T_K]$ is not even K-countable. $\square$

Now we build a tree $T_P \subseteq T_K$ by induction as follows. Let $s_\emptyset = \emptyset$ be in T_P and, given $s_\sigma \in T_P$, by Claim 3 we can choose $p_0, p_1 \in T_K$ which are incompatible, extend s_σ and such that $[p_0] \otimes [p_1] \subseteq K$. Let $s_{\sigma^\frown i} = p_i$ be in T_P. It is clear that $P = [T_P]$ is perfect subset of ω^ω homogeneous for K. This conclude the proof of Theorem 2. $\square$

Now Theorem 1 is completely proved.

Corollary 1. *For every analytic set X either X is countable or there is $P \subseteq X$ perfect.*

Proof. Apply Theorem 1 to $K = [X]^2$. $\square$

2 Basic properties of open colourings of a separable metric space

In this section we will show some basic properties of open colourings of a polish space (or more generally of a separable metric space).

Remark 2. *Let $X \subset \mathbb{R}$ and let K be an open colouring of $\mathbb{R}$. If X is homogeneous for $K^c \cap [Y]^2$ then $\overline{[X]^2} = [\overline{X}]^2$ is homogeneous for K^c, since K is open. Thus it is easy to see, using Remark 1, that X is $K-$countable iff there exists $Z \in F_\sigma(\mathbb{R})$ which contains X and is $K-$countable. Moreover by refining the topology on $\mathbb{R}$ to a new polish topology τ in which such a Z is closed, we get that X is K-countable iff the topological closure of X with respect to τ is K-countable. All in all we have that for Polsh space....any separable metric space X with topology τ and any K open coloring of X, there is a possibly finer topology τ', and a metric ρ on X compatible with τ'?, such that X is K-countable with respect to τ iff the completion of X with respect to τ' is K-countable.*

Remark 3. *Let $S \subseteq 2^\omega$ and $K \subseteq [S]^2$. Then the following statements are equivalent:*

(i) There is a separable metric topology on S such that K is open in $[S]^2$,

(ii) there are two families of Borel sets of 2^ω $\{A_n : n < \omega\}$, $\{B_n : n < \omega\}$ such that $K = \bigcup\{(A_n \cap S) \otimes (B_n \cap S) : n < \omega\}$.

Proof. *(i) $\to$ (ii)* by definition. We proceed with the proof of the converse implication; we can suppose $S \subseteq 4^\omega$; let τ be the topology on S be defined by the base $\{N_\sigma : \sigma \in 4^{<\omega}\}$ as follows:

- $N_\emptyset = S$,
- if σ has even length $2n$,

$$N_{\sigma^\frown 0} = N_\sigma \cap A_n,$$
$$N_{\sigma^\frown 1} = N_\sigma \cap (S \setminus A_n),$$
$$N_{\sigma^\frown 2} = N_\sigma \cap B_n,$$
$$N_{\sigma^\frown 3} = N_\sigma \cap (S \setminus B_n)$$

- if σ has odd length $2n+1$,

$$N_{\sigma^\frown i} = N_\sigma \cap [\langle \sigma(2j) : j \leq n/2 \rangle^\frown i].$$

We have that τ is a separable metric topology on S and A_n, B_n are clopen in τ for all n (for example, $A_n = \bigcup\{N_{\sigma^\frown 0} : |\sigma| = 2n\}$). □

Definition 2. *Given K open colouring of a separable metric space X, $x \in X$ is a K-**accumulation point** if for every open neighborhood U of x, $K(x) \cap U$ is not K-countable. If Y is a subset of X, we say that x is a K-**accumulation point for** Y if if for every open neighborhood U of x, $K(x) \cap U \cap Y$ is not K-countable.*

The following holds:

Property 1. *(K-density property) Given K open colouring of $\mathbb{R}$ and $X \subseteq \mathbb{R}$, the following are equivalent:*

(i) X is $K-$countable,

(ii) $A_X = \{y \in X : y$ is a K-accumulation point for $X\}$ is K-countable.

Proof. If X is K-countable, then A_X is clearly K-countable, being A_X a subset of X. Conversely, suppose X is not K-countable while A_X is K-countable. Then also $B_X = X \setminus A_X$ is not K-countable.

Notice that if $x \in B_X$, then there is an integer n_x such that $[x - 1/n_x, x + 1/n_x] \cap K(x) \cap X$ is K-countable. Let $A_n = \{x \in B_X : n_x = n\}$ for every $n \in \omega$.

Claim 4. *For every $n \in \omega$, for every (a, b) open interval such that $|b-a| < 1/2n$, the set $(a, b) \cap A_n$ is K-countable.*

Proof. Notice that:

1. The set $(a, b) \cap K(x) \cap X$ is K-countable for all $x \in A_n \cap (a, b)$, since $x - 1/n < a < x < b < x + 1/n$.

2. Let $\{x_n : n < \omega\}$ be dense in $(a, b) \cap A_n$. If $y \in K(x)$ for some $x \in A_n \cap (a, b)$, then $y \in K(x_n)$ for some n. In fact, since K is open, there exists $U \subseteq (a, b)$ open neighbourhood of x such that $\{y\} \otimes U \subseteq K$. Then there is n such that $x_n \in U$, so $y \in K(x_n)$. Therefore, for all $x \in A_n \cap (a, b)$,
$$K(x) \subseteq \bigcup_{n\in\omega} K(x_n).$$

3. For all n the set $Z_n = ((a, b) \cap A_n) \setminus \bigcup\{K(y) : y \in (a, b) \cap A_n\}$ is homogeneous for K^c, by definition.

By 3, to prove the claim we only need to show that the set
$$A = \bigcup\{K(y) : y \in (a, b) \cap A_n\} \cap (a, b) \cap A_n.$$
is $K-$countable. Using 2, we have that $A \subseteq \bigcup\{K(x_n) \cap (a, b) \cap A_n : n \in \omega\}$ and thus, applying 1, A is cointained in a countable union of $K-$countable sets. □

By the claim, each A_n is K-countable, since it can be expressed as the union of the K-countable sets of the type of $I \cap A_n$, where I is a rational interval of diameter less then $1/2n$.

Finally, also B_X is K-countable since it is the union of all the A_n. This is a contradiction. □

3 $AD \to OCA_P$

In this section we will show that OCA_P is a straigthforward consequence of AD using a simple refinement of the perfect set game.

Theorem 3. *(ZFC) Assume AD. Then $OCA_P(X)$ holds for all $X \subseteq 2^\omega$.*

Let $X \subseteq 2^\omega$ and K be an open colouring of 2^ω. We define the game $G(K, X)$ as follows. Each of player's I moves is a pair s_0^n, s_1^n from $2^{<\omega}$ such that $[s_0^n] \otimes [s_1^n] \subseteq K$ and each of player's II moves is some $i_n \in \{0, 1\}$. We insist on the following rules:

- for all $n < \omega$, s_0^{n+1} and s_1^{n+1} are incompatible extensions of $s_{i_n}^n$,
- I wins iff $x = \bigcup_{n<\omega} s_{i_n}^n$ belongs to X.

The theorem follows from the following claim:

Claim 5. *In $G(K, X)$ the following holds:*

(i) I has a winning strategy iff X contains a perfect subset homogeneous for K,

(ii) If II has a winning strategy, then X is K-countable.

Proof. For (ii), notice that a winning strategy for I is essentially a perfect tree T_P and that the rules of the game force $[T_P]$ to be an homogeneous set for K.

For what concerns (ii), let σ_{II} be a winning strategy for II. Given $x \in X$, we say that a position $P = \langle (s_0^0, s_1^0), i_0, \cdots, (s_0^n, s_1^n), i_n \rangle \in \sigma_{II}$ is **good for** x if $x \in [s_{i_n}^n]$. Since σ_{II} is a winning strategy for II, for every $x \in X$ the tree T_x of good positions for x is well-founded. Given P good position for x, let

$$\begin{aligned} A_P \quad = \quad & \{y \in [s_{i_n}^n] : \text{ for all } (s_0^{n+1}, s_1^{n+1}) \text{ legal moves of I after } i_n, \text{ if } i \text{ is what } \sigma_{II} \\ & \text{requires II to play next, then } y \notin s_i^{n+1}\}. \end{aligned}$$

Notice that if P is an end leaf of T_x, then $x \in A_P$. Moreover, $[A_P]^2 \cap K = \emptyset$. Otherwise, if there were $z, y \in A_P$ such that $\{z, y\} \in K$, then $[z \upharpoonright n] \otimes [y \upharpoonright n] \subseteq K$ for some n, since K is open. Therefore $(z_{\upharpoonright n}, y_{\upharpoonright n})$ would be a legal move of I answering to P but either z or y would be in $s_{i_{n+1}}^{n+1}$, contrary to the definition of A_P. Finally, notice that $X \subseteq \bigcup_{P \in G(K,X)} A_P$, so X is a countable union of K^c-homogeneous sets. □

Remark 4. *It is not straigthforward to strengthen (ii) in the above claim to an equivalence as for (i). Probably we need to refine the rules of the game or eventually even the Axiom OCA_P.*

The dichotomy of OCA_P that holds for open colourings of subsets of $\mathbb{R}$ does not hold for generic closed colourings as shown by the following:

Remark 5. *If we assume AD, there is a closed colouring K of $\mathbb{R}$ such that neither $\mathbb{R}$ is K-countable nor admits a perfect $K-$homogeneous subset. On the other hand, assuming AC we have that $\mathbb{R}$ is K-countable with respect to the same colouring K.*

Proof. Consider on $[\mathbb{R}]^2 = \{(x,y) : x > y\}$ the lines $l_n = \{(x,y) : y = x - 1/n\}$ and let $K = \bigcup_{n<\omega} l_n$; then K is closed in $[\mathbb{R}]^2$. For all $x \in \mathbb{R}$, let us consider the K-fiber of x, i.e. the set $K(x) = \{y : \{x,y\} \in K\}$. Notice that $K(x)$ is the set $\{x - 1/n : n < \omega\}$, so it is countable. If $Y \subseteq \mathbb{R}$ is homogeneous for K and $x \in Y$, then $Y \subseteq K(x)$, so Y can be at most countable. This shows that $\mathbb{R}$ cannot have a perfect $K-$ homogeneous subset.

Let us assume AC and let A be a Vtali set. We have that the countable family $\{A + r : r \in \mathbb{Q}\}$ give us a cover of $\mathbb{R}$ consisting of K^c-homogeneous sets.

To complete the proof, we wil show that every countable family which is a cover of $\mathbb{R}$ made by K^c-homogeneous sets contains a set that has not the Baire property. Suppose $\mathbb{R} = \bigcup_{n\in\omega} X_n$, with X_n K^c-homogeneous for all n. Since $\mathbb{R}$ is Baire, there exists n such that $X = X_n$ is not meager.

Let us look at $\mathbb{R}$ as the topological group $(\mathbb{R}, +)$. If X had the Baire property, by the proof of the Pettis theorem for topological group (see [9]), we would have that there exists an open neighborhood V of 0 such that $X \cap (X + h) \neq \emptyset$ for all h in V. Thus, in particular, there exists an $m \in \omega$ such that $X \cap (X + 1/m) \neq \emptyset$, a contadiction to tha fact that X is K^c-homogeneous.

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If X had the Baire property, there would exist a nonempty open set $U = (a,b)$ on wich $X \cap U$ is comeager. Let $r = 1/m < b - a$ for some $m \in \omega$. Obviously, we have that $(X \cap U) + r$ is comeager in $U + r = (a + r, b + r)$, and thus $(X \cap U) + r$ is comeager in $(a + r, b)$, which implies that $(X \cap U) + r$ is not meager in U. But this is a contradiction to tha fact that $U \setminus X$ is meager, since $((X \cap U) + r) \cap U \subseteq U \setminus X$, being X K^c-homogeneous.

□

4 OCA and gaps in ω^ω

In this section we will show some powerful applications of OCA to problems concerning properties of the continuum.

On ω^ω set $f <^* g$ if the set $\{n \in \omega : f(n) \geq g(n)\}$ is finite. A set $A \subseteq \omega^\omega$ is called **bounded** if there is a $g \in \omega^\omega$ such that $f <^* g$ for all $f \in A$. Let us define the cardinal b as the minimal size of an unbounded set of ω^ω.

Definition 3. *Let $A = \{f_\alpha : \alpha < \kappa\}$ and $B = \{g_\beta : \beta < \lambda\}$ be subsets of ω^ω. (A,B) is said to be a (κ, λ^*)-**pregap** in ω^ω if*

- *$f_\alpha <^* f_\gamma$ for all $\alpha < \gamma < \kappa$,*
- *$g_\rho <^* g_\beta$ for all $\beta < \rho < \lambda$,*
- *$f_\alpha <^* g_\beta$for all $\alpha < \kappa$ and for all $\beta < \lambda$.*

*A pregap (A, B) is **filled** if there exists $h \in \omega^\omega$ such that $f <^* h <^* g$, for all $f \in A$ and for all $g \in B$. Otherwise, we say that (A, B) is **unfilled**. Finally, a pregap (A, B) is called **gap** if it is an unfilled pregap.*

These are basic facts and folklore results about gaps:

Theorem 4. *There are no (ω, ω^*)-gaps.*

Theorem 5. *There exist a (ω_1, ω_1^*)-gaps and a (b, ω^*)-gap on ω^ω. These gaps are called **Haussdorff gaps**.*

In ZFC this is the best possible existence result, while under OCA the Haussdorff gaps are essentially the only kind of gaps that exist:

Theorem 6. *Assume OCA. Then the only type of gaps in ω^ω are either (ω_1, ω_1^*) or (κ, ω^*) where κ is a cardinal of size at least b.*

Proof. Suppose not, i.e. there exists a gap $(\{f_\alpha : \alpha < \kappa\}, \{g_\beta : \beta < \lambda\})$ in $(\omega^\omega, <^*)$, with κ, λ regular and uncountable cardinals, and such that $\kappa > \omega_1$.

We can modify the gap as follows. Notice that for every $\alpha \in \kappa$ there is an $m_\alpha \in \omega$ such that $|\{\beta \in \lambda : f_\alpha(n) < g_\beta(n) \text{ for all } n \geq m_\alpha\}| = \lambda$ and for κ-many α the integer m_α will be the same. So we take from A an unique element for every m_α and by rescaling the f_α's and the g_β's in order to have $m_\alpha = 0$ for all $\alpha < \kappa$, we obtain the following subset of ω^ω:

$$X = \{(f_\alpha, g_\beta) : f_\alpha(n) < g_\beta(n) \text{ for all } n \in \omega, \alpha \in \kappa, \beta \in \lambda\}.$$

Let us consider the colouring of X

$$K = \{\{(f_\alpha, g_\beta), (f_\xi, g_\eta)\} : \exists n\, f_\alpha(n) \geq g_\eta(n) \text{ or } \exists n\, f_\xi(n) \geq g_\beta(n)\}.$$

Notice that K is an open coloring of X, since if $\{(f_\alpha, g_\beta), (f_\xi, g_\eta)\} \in K$ and n witnesses this fact, we just need to fix the first $n + 1$ coordinates of each function to obtain an open neighboorhood of $\{(f_\alpha, g_\beta), (f_\xi, g_\eta)\}$ contained in K.

Then, by OCA, to prove the theorem it is sufficient to show that X is neither $K-$countable, nor admits an uncountable subset homogeneous for K.

Claim 6. *X is not K-countable.*

Proof. If $X = \bigcup_{n \in \omega} X_n$ with each X_n homogeneous for K^c. For each n, set $A_n = \{\alpha : \exists \beta\, (f_\alpha, g_\beta) \in X_n\}$ and $B_n = \{\beta : \exists \alpha\, (f_\alpha, g_\beta) \in X_n\}$. Suppose there is an n such that $|A_n| = \kappa$ and $|B_n| = \lambda$ and set $g(m) = min\{g_\beta(m) : \beta \in B_n\}$. Notice that g fills the gap, a contradiction. The other possibility is that for all n either $|A_n| < \kappa$ or $|B_n| < \lambda$. Let $\alpha_0 = \bigcup\{A_n : |A_n| < \kappa\} < \kappa$ and $\beta_0 = \bigcup\{B_n : |B_n| < \lambda\} < \lambda$. Let n_0 such that $(f_{\alpha_0}, g_\beta) \in X_{n_0}$ for some $\beta \geq \beta_0$. If $|A_{n_0}| < \kappa$ then $\alpha_0 \notin A_{n_0}$, if $|B_{n_0}| < \lambda$ then $\beta \notin B_{n_0}$, a contradiction. ☐

Claim 7. *X has no uncountable K-homogeneous subset.*

Proof. If not, let $Y \subseteq X$ be uncountable and homogeneous for K. Notice that for every $(f_\alpha, g_\beta), (f_\xi, g_\eta) \in Y$, we have that $\alpha \neq \xi$ and $\beta \neq \eta$. Otherwise, if for example $\alpha = \xi$, since Y is $K-$homogeneous, then there is an n such that $f_\alpha(n) \geq g_\eta(n)$, i.e. $f_\xi(n) \geq g_\eta(n)$, a contradiction. Recall that by the Dushnik-Miller Theorem (see [5]), we have that for every $F : [Y]^2 \to \{0, 1\}$ either there exists an $H_1 \subset Y$ of order type $\omega_1$such that $F = 0$ on $[H_1]^2$ or there exists an $H_2 \subset Y$ of order type ω such that $F = 1$ on $[H_2]^2$. Let us consider the application on $[Y]^2$ defined by setting

$$F(\{(f_\alpha, g_\beta), (f_\xi, g_\eta)\}) = \begin{cases} 0, & \text{if } \alpha < \xi \leftrightarrow \beta < \eta \\ 1, & \text{otherwise.} \end{cases}$$

If H_2 were a countable subset of $[Y]^2$ on which F is costantly equal to 1, then we would have an infinite discending of elements of ω^ω with respect to $<^*$. Therefore, by the Dushnik-Miller Theorem, there exists an $H_1 \subset Y$ of order type $\omega_1$such that $F = 0$ on $[H_1]^2$. Let $\{(f_{\alpha_\nu}, g_{\beta_\nu}) : \nu < \omega_1\}$ such that if $\rho < \gamma$, $f_{\alpha_\rho} <^* f_{\alpha_\gamma} <^* g_{\beta_\gamma} <^* g_{\beta_\rho}$.

Since $\kappa > \omega_1$ there is an η such that $f_{\alpha_\nu} <^* f_\eta$ for all ν. Choose n_0 such that $A = \{\nu : \forall n \geq n_0\, f_{\alpha_\nu}(n) < f_\eta(n)\}$ is uncountable and find $n_1 \geq n_0$ such that $B = \{\nu \in A : \forall n \geq n_1\, g_{\alpha_\nu}(n) > f_\eta(n)\}$ is uncountable. Find $u_0 \in \omega^{n_1}$ such that $C = \{\nu \in B : f_{\alpha_\nu} \in [u_0]\}$ is uncountable and finally find $u_1 \in \omega^{n_1}$ such that $D = \{\nu \in C : g_{\beta_\nu} \in [u_1]\}$ is uncountable; then for all $\rho, \gamma \in D$ if $k < n_1$ $f_{\alpha_\rho}(k) = f_{\alpha_\gamma}(k) < g_{\beta_\gamma}(k)$, while if $k \geq n_1$ then $f_{\alpha_\rho}(k) < f_\eta(k) < g_{\beta_\gamma}(k)$; so for all $\rho, \gamma \in D$, for all $k \in \omega$, $f_{\alpha_\rho}(k) < g_{\beta_\gamma}(k)$.

This means that $\{(f_{\alpha_\rho}, g_{\beta_\rho}), (f_{\alpha_\gamma}, g_{\beta_\gamma})\} \notin K$.

□

The theorem is completely proved. □

Theorem 7. *Assume OCA. Then* $b = \omega_2$.

Lemma 2. *OCA implies that* $b > \omega_1$.

Proof. Let $A = \{f_\alpha : \alpha < b\}$ be an unbounded family of strictly increasing functions in ω^ω, let $\{f_\alpha, f_\beta\} \in K$ if there are n, m such that either: $f_\alpha(m) < f_\beta(m)\,\&\, f_\alpha(n) > f_\beta(n)$ or $f_\alpha(m) > f_\beta(m)\,\&\, f_\alpha(n) < f_\beta(n)$. K is open in $[A]^2$, since if $\{f_\alpha, f_\beta\} \in K$ and $k > n, m$, $[f_\alpha \restriction k] \otimes [f_\beta \restriction k] \subseteq K$.

A is not K-countable, else there must be an A' homogeneous for K^c and uncountable; then $(A', <_{lex})$ would be an uncountable well order inside $(\omega^\omega, <_{lex})$ which is not possible.

So, by OCA, A has an uncountable K-homogeneous subset Y. We show that Y is bounded, so that $|A| > |Y| \geq \omega_1$, and the lemma holds.

Suppose Y is unbounded and for each $t \in \omega^{<\omega}$ such that $[t] \cap Y \neq \emptyset$ choose α_t such that $f_{\alpha_t} \in Y$; let $\gamma > sup\{\alpha_t : t \in \omega^{<\omega}\}$ such that $f_\gamma \in Y$. Choose k_0 in order that $Z = \{f \in Y : \forall k \geq k_0\, f(k) > f_\gamma(k)\}$ is still unbounded and $k_1 \geq k_0$ such that $\{f(k_1) : f \in Z\}$ is infinite. Now choose $u \in \omega^{k_1}$ such that $Z \cap [u]$ is unbounded.

Let $k_2 \geq k_1$ such that for all $k \geq k_2$, $f_{\alpha_u}(k) < f_\gamma(k)$. and $f \in Z \cap [u]$ such that $f(k_1) > f_\gamma(k_2)$.

Then for $k < k_1$, $f_{\alpha_u}(k) = f(k)$, for $k_1 \leq k \leq k_2$, $f_{\alpha_u}(k) \leq f_{\alpha_u}(k_2) < f_\gamma(k_2) < f(k_1) \leq f(k)$; for $k > k_2$ $f_{\alpha_u}(k) < f_\gamma(k) < f(k)$. But this is a contradiction, since $f, f_{\alpha_u} \in Y$ but $\{f, f_{\alpha_u}\} \notin K$. □

Therefore, by Theorem 6, to conclude the proof of Theorem 7 we just need the following lemma:

Lemma 3. *If $b > \omega_2$ then there is an (ω_2, λ) gap for some λ uncountable.*

Proof. Let $A = \{f_\alpha : \alpha < \omega_2\}$ be a family of strictly increasing functions in ω^ω, and consider $\mathcal{F} = \{g : \forall\alpha\, f_\alpha <^* g\}$.

Let $B = \{g_\alpha : \alpha < \lambda\} \subseteq \mathcal{F}$ a maximal chain under $>^*$.

Claim 8. $cof(\lambda) > \omega$

If the claim holds (A, B) is a gap as required by the lemma.

Proof. We just have to show that if $\{g_n : n \in \omega\}$ is a decreasing chain under $<^*$ in $\mathcal{F}$ then there is a $g <^* g_n$ in $\mathcal{F}$.

For each $\alpha < \omega_2$ let $m_\alpha \in \omega^\omega$ be a strictly increasing function such that for all $k \geq m_\alpha(n)$, $f_\alpha(k) < min\{g_i(k) : i \leq n\}$; let $m^* >^* m_\alpha$ for all α; this is possible since $b > \omega_2$. Set for all $k \in [m^*(n); m^*(n+1))$, $g(k) = min\{g_i(k) : i \leq n\}$.

Now given f_α, let n be large enough in order that $m^*(k) > m_\alpha(k)$ for all $k \geq n$ then for all $j > n$ if $j \in [m^*(l), m^*(l+1))$, then $j \in [m_\alpha(l'), m_\alpha(l'+1))$ for some $l' \geq l$ so $f_\alpha(j) < min\{g_i(k) : i \leq l'\} \leq min\{g_i(k) : i \leq l\} = g(k)$. So $g \in \mathcal{F}$ and $g <^* g_n$ for all n.

□

This completes the proof of the lemma. □

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