NOTES ON MODEL COMPLETENESS AND MODEL COMPANIONSHIP

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Abstract. We give a self-contained and detailed account of the main results on model completeness and model companionship.

1. Existentially closed structures, model completeness, model companionship

We present this topic expanding on [4, Sections 3.1-3.2]. We decided to include detailed proofs since the presentation of [4] is (in some occasions) rather sketchy, and the focus is not exactly ours. The unique section which contains (to our knowledge) original results is 1.5.

The first objective is to isolate necessary and sufficient conditions granting that some τ-structure $\mathcal{M}$ embeds into some model of some τ-theory T.

We introduce the following terminology:

Notation 1.1.

- $\sqsubseteq$ denotes the substructure relation between structures.
- $\mathcal{M} \prec_n \mathcal{N}$ indicates that $\mathcal{M}$ is a Σ_n-elementary substructure of $\mathcal{N}$, we omit the n to denote full-elementarity.
- Given a first order signature τ, $\tau_\forall$ denotes the universal τ-sentences; likewise we interpret $\tau_\exists, \tau_{\forall\exists}, \ldots$. $\tau_{\forall\vee\exists}$ denotes the boolean combinations of universal τ-sentences; likewise we interpret $\tau_{\forall\exists\vee\exists\forall}, \ldots$.
- Given a first order theory T, $T_\forall$ denotes the sentences in $\tau_\forall$ which are consequences of T, likewise we interpret $T_\exists, T_{\forall\exists}, T_{\forall\vee\exists}, \ldots$.
- We often denote a τ-structure $(M, R^M : R \in \tau)$ by (M, τ^M).
- We often identify a τ-structure $\mathcal{M} = (M, \tau^M)$ with its domain M and an ordered tuple $\vec{a} \in \mathcal{M}^{<\omega}$ with its set of elements.
- We often write $\mathcal{M} \models \phi(\vec{a})$ rather than $\mathcal{M} \models \phi(\vec{x})[\vec{x}/\vec{a}]$ when $\mathcal{M}$ is τ-structure $\vec{a} \in \mathcal{M}^{<\omega}$, ϕ is a τ-formula.
- We let the atomic diagram $\Delta_0(\mathcal{M})$ of a τ-model $\mathcal{M} = (M, \tau^M)$ be the family of quantifier free sentences $\phi(\vec{a})$ in signature $\tau \cup M$ such that $\mathcal{M} \models \phi(\vec{a})$.

The following is a first basic fact about byembeddability of models of theories S, T:

Lemma 1.2. *Let τ be a signature and T, S be τ-theories. TFAE:*

(1) $T_\forall \supseteq S_\forall$.
(2) For any $\mathcal{M}$ model of T there is $\mathcal{N}$ model of S superstructure of $\mathcal{M}$.

Proof.

1 implies 2: Assume $\mathcal{M}$ models T and is such that no $\mathcal{N}$ model of S is a superstructure of $\mathcal{M}$. Then $S \cup \Delta_0(\mathcal{M})$ is not consistent (where $\Delta_0(\mathcal{M})$ is the atomic diagram of $\mathcal{M}$). By compactness find $\psi(\vec{a}) \in \Delta_0(\mathcal{M})$ quantifier-free sentence such that $S + \psi(\vec{a})$ is inconsistent. This gives that

$$S \models \forall\vec{x}\, \neg\psi(\vec{x})$$

since $\vec{a}$ is a string of constant symbols all outside of τ. Therefore $\forall \vec{x}\, \neg\psi(\vec{x}) \in S_\forall \subseteq T_\forall$. Hence

$$\mathcal{M} \models \forall \vec{x}\, \neg\psi(\vec{x}) \wedge \psi(\vec{a}),$$

a contradiction.

2 implies 1: Left to the reader.

□

We can get more refined versions of the above Lemma as follows:

Definition 1.3. Given τ-theories T, S, a τ-sentence ψ separates T from S if $T \vdash \psi$ and $S \vdash \neg\psi$.

T is Π_n-separated from S if some Π_n-sentence for τ separates T from S.

Lemma 1.4. *Assume S, T are τ-theories. TFAE:*

(1) T is not Π_1-separated from S (i.e. no universal sentence ψ is such that $T \vdash \psi$ and $S \vdash \neg\psi$).

(2) There is some *τ-model $\mathcal{M}$ of S which can be embedded in some τ-model $\mathcal{N}$ of T.*

See also [4, Lemma 3.1.1, Lemma 3.1.2, Thm. 3.1.3]

Proof. We assume T, S are closed under logical consequences.

(2) implies (1): By contraposition we prove $\neg(1) \to \neg(2)$.

Assume some universal sentence ψ separates T from S. Then for any model of T, all its substructures model ψ, therefore they cannot be models of S.

(1) implies (2): By contraposition we prove $\neg(2) \to \neg(1)$.

Assume that for any model $\mathcal{M}$ of S and $\mathcal{N}$ of T $\mathcal{M} \not\sqsubseteq \mathcal{N}$. We must show that T is Π_1-separated from S.

Given a τ-structure $\mathcal{M} = (M, \tau^M)$ which models S, let $\Delta_0(\mathcal{M})$ be the atomic diagram[1] of $\mathcal{M}$ in the signature $\tau \cup M$.

The theory $T \cup \Delta_0(\mathcal{M})$ is inconsistent, otherwise $\mathcal{M}$ embeds into some model of T: let $\bar{\mathcal{Q}}$ be a $\tau \cup \mathcal{M}$-model of $\Delta_0(\mathcal{M}) \cup T$ and $\mathcal{Q}$ be the τ-structure obtained from $\bar{\mathcal{Q}}$ omitting the interpretation of the constants not in τ. Clearly $\mathcal{Q}$ models T. The interpretation of the constants in $\tau \cup \mathcal{M}$ inside $\bar{\mathcal{Q}}$ defines a τ-substructure of $\mathcal{Q}$ isomorphic to $\mathcal{M}$.

By compactness (since $\Delta_0(\mathcal{M})$ is closed under finite conjunctions) there is a quantifier free τ-formula $\psi_{\mathcal{M}}(\vec{x})$ and $\vec{a} \in \mathcal{M}^{<\omega}$ such that $T + \psi_{\mathcal{M}}(\vec{a})$ is inconsistent. This gives that $T \vdash \neg\psi_{\mathcal{M}}(\vec{a})$. Since $\vec{a}$ is a family of constants never occurring in T, we get that $T \vdash \forall\vec{x}\neg\psi_{\mathcal{M}}(\vec{x})$ and $\mathcal{M} \models \exists\vec{x}\psi_{\mathcal{M}}(\vec{x})$.

The theory

$$S \cup \{\neg\exists\vec{x}\psi_{\mathcal{M}}(\vec{x}) : \mathcal{M} \models S\}$$

is inconsistent, since $\neg\exists\vec{x}\psi_{\mathcal{M}}(\vec{x})$ fails in any model $\mathcal{M}$ of S.

By compactness there is a finite set of formulae $\psi_{\mathcal{M}_1} \dots \psi_{\mathcal{M}_k}$ such that

$$S + \bigwedge \{\neg\exists\vec{x}_i\psi_{\mathcal{M}_i}(\vec{x}_i) : i = 1, \dots, k\}$$

is inconsistent. This gives that

$$S \vdash \bigvee_{i=1}^{k} \exists\vec{x}_i\psi_{\mathcal{M}_i}(\vec{x}_i).$$

The τ-sentence $\psi := \bigvee_{i=1}^{k} \exists\vec{x}_i\psi_{\mathcal{M}_i}(\vec{x}_i)$ holds in all models of S and its negation

$$\bigwedge \{\neg\exists\vec{x}_i\psi_{\mathcal{M}_i}(\vec{x}_i) : i = 1, \dots, k\}$$

[1] We let the atomic diagram of a τ-model $\mathcal{M} = (M, \tau^M)$ be the family of quantifier free formulae in signature $\tau \cup M$ which holds in the natural expansion of $\mathcal{M}$ to $\tau \cup M$.

is a conjunction of universal sentences (hence —modulo logical equivalence— universal) derivable from T. Hence $\neg\psi$ separates T from S.

$\square$

The following Lemma shows that models of $T_\forall$ can always be extended to superstructures which model T.

Lemma 1.5. *Let T be a τ-theory and $\mathcal{M}$ be a τ-structure. TFAE:*

(1) $\mathcal{M}$ is a τ-model of $T_\forall$.
(2) There exists $\mathcal{N} \sqsupseteq \mathcal{M}$ which models T.

Proof. (2) implies (1) is trivial.
Conversely:

Claim 1. *T is not Π_1-separated from $\Delta_0(\mathcal{M})$ (in the signature $\tau \cup \mathcal{M}$).*

Proof. If not there are $\vec{a} \in \mathcal{M}^{<\omega}$, and a quantifier free τ-formula $\phi(\vec{x}, \vec{z})$ such that

$$T \vdash \forall\vec{z}\phi(\vec{a}, \vec{z}),$$

while

$$\Delta_0(\mathcal{M}) \vdash \neg\forall\vec{z}\phi(\vec{a}, \vec{z}).$$

The latter yields that

$$\Delta_0(\mathcal{M}) \vdash \exists\vec{x}\exists\vec{z}\neg\phi(\vec{x}, \vec{z}),$$

and therefore also that

$$\mathcal{M} \models \exists\vec{x}\exists\vec{z}\neg\phi(\vec{x}, \vec{z}).$$

On the other hand, since the constants $\vec{a}$ do not appear in any of the sentences in T, we also get that

$$T \vdash \forall\vec{x}\forall\vec{z}\phi(\vec{x}, \vec{z}).$$

This is a contradiction since $\mathcal{M}$ models $T_\forall$. $\square$

By the Claim and Lemma 1.4 some $\tau\cup\mathcal{M}$-model $\bar{\mathcal{P}}$ of $\Delta_0(\mathcal{M})$ embeds into some $\tau\cup\mathcal{M}$-model $\bar{\mathcal{Q}}$ of T. Let $\mathcal{Q}$ be the τ-structure obtained from $\bar{\mathcal{Q}}$ omitting the interpretation of the constants not in τ. Then $\mathcal{Q}$ models T and contains a substructure isomorphic to $\mathcal{M}$. $\square$

Corollary 1.6 (Resurrection Lemma)**.** *Assume $\mathcal{M} \prec_1 \mathcal{N}$ are τ-structures. Then there is $\mathcal{Q} \sqsupseteq \mathcal{N}$ which is an elementary extension of $\mathcal{M}$.*

Proof. Let T be the elementary diagram $\Delta_\omega(\mathcal{M})$ of $\mathcal{M}$ in the signature $\tau \cup \mathcal{M}$. It is easy to check that any model of T when restricted to the signature τ is an elementay extension of $\mathcal{M}$. Since $\mathcal{M} \prec_1 \mathcal{N}$, the natural extension of $\mathcal{N}$ to a $\tau \cup \mathcal{M}$-structure realizes the Π_1-fragment of T in the signature $\tau \cup \mathcal{M}$. Now apply the previous Lemma. $\square$

The Resurrection Lemma motivates the resurrection axioms introduced by Hamkins and Johnstone in [3], and their iterated versions introduced by the author and Audrito in [2].

1.1. **Existentially closed structures.** The objective is now to isolate the "generic" models of some universal theory T (i.e. all axioms of T are universal sentences). These are described by the T-existentially closed models.

Definition 1.7. Given a first order signature τ, let T be any consistent τ-theory. A τ-structure $\mathcal{M}$ is T-existentially closed (T-ec) if

(1) $\mathcal{M}$ can be embedded in a model of T.
(2) $\mathcal{M} \prec_{\Sigma_1} \mathcal{N}$ for all $\mathcal{N} \sqsupseteq \mathcal{M}$ which are models of T.

In general T-ec models need not be models[2] of T, but only of their universal fragment. A standard diagonalization argument shows that for any theory T there are T-ec models, see Lemma 1.10 below or [4, Lemma 3.2.11].

A trivial observation which will come handy in the sequel is the following:

Fact 1.8. *Assume $\mathcal{M}$ is a T-ec model and $S \supseteq T$ is such that some $\mathcal{N} \sqsupseteq \mathcal{M}$ models S. Then $\mathcal{M}$ is S-ec.*

Proposition 1.9. *Assume a τ-structure $\mathcal{M}$ is T-ec. Then:*

(1) $\mathcal{M} \models T_\forall$.
(2) $\mathcal{M}$ is also $T_\forall$-ec.
(3) If $\mathcal{N} \prec_{\Sigma_1} \mathcal{M}$, then $\mathcal{N}$ is also T-ec.
(4) Let $\forall \vec{x} \exists \vec{y} \psi(\vec{x}, \vec{y}, \vec{a})$ be a Π_2-sentence with $\psi(\vec{x}, \vec{y}, \vec{z})$ quantifier free τ-formula and parameters $\vec{a}$ in $\mathcal{M}^{<\omega}$. Assume it holds in some $\mathcal{N} \sqsupseteq \mathcal{M}$ which models $T_\forall$, then it holds in $\mathcal{M}$.
(5) Let S be the τ-theory of $\mathcal{M}$. For any Π_2-sentence ψ in the signature τ TFAE:
  - *ψ holds in some model of $S_\forall$.*
  - *ψ holds in $\mathcal{M}$.*

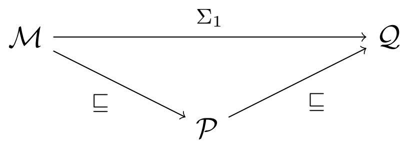
Proof.

(1): There is at least one super-structure of $\mathcal{M}$ which models T, and any $\psi \in T_\forall$ holds in this superstructure, hence in $\mathcal{M}$.

(2): Assume $\mathcal{M} \sqsubseteq \mathcal{P}$ for some model $\mathcal{P}$ of $T_\forall$. We must argue that $\mathcal{M} \prec_1 \mathcal{P}$.

By Lemma 1.5, there is $\mathcal{Q} \sqsupseteq \mathcal{P}$ which models T.

Since $\mathcal{M}$ and $\mathcal{Q}$ are both models of T and $\mathcal{M}$ is T-ec, we get the following diagram:



Then any Σ_1-formula $\psi(\vec{a})$ with $\vec{a} \in \mathcal{M}^{<\omega}$ realized in $\mathcal{P}$ holds in $\mathcal{Q}$, and is therefore reflected to $\mathcal{M}$. We are done by Tarski-Vaught's criterion.

(3): Assume $\mathcal{N} \sqsubseteq \mathcal{P}$ for some model of $T_\forall$ $\mathcal{P}$. Let $\Delta_0(\mathcal{P})$ be the atomic diagram of $\mathcal{P}$ in the signature $\tau \cup \mathcal{P} \cup \mathcal{M}$ and $\Delta_0(\mathcal{M})$ be the atomic diagram of $\mathcal{M}$ in the same signature[3].

Claim 2. *$T_\forall \cup \Delta_0(\mathcal{P}) \cup \Delta_0(\mathcal{M})$ is a consistent $\tau \cup \mathcal{M} \cup \mathcal{P}$-theory.*

Proof. Assume not. Find $\vec{a} \in (\mathcal{P} \setminus \mathcal{N})^{<\omega}$, $\vec{b} \in (\mathcal{M} \setminus \mathcal{N})^{<\omega}$, $\vec{c} \in \mathcal{N}^{<\omega}$ and τ-formulae $\psi_0(\vec{x}, \vec{z})$, $\psi_1(\vec{y}, \vec{z})$ such that:
- $\psi_0(\vec{a}, \vec{c}) \in \Delta_0(\mathcal{P})$,
- $\psi_1(\vec{b}, \vec{c}) \in \Delta_0(\mathcal{M})$,
- $T \cup \left\{\psi_0(\vec{a}, \vec{c}), \psi_1(\vec{b}, \vec{c})\right\}$ is inconsistent.

Then
$$T \vdash \neg\psi_0(\vec{a}, \vec{c}) \vee \neg\psi_1(\vec{b}, \vec{c}).$$

[2]For example let T be the theory of commutative rings with no zero divisors which are not fields in the signature $(+, \cdot, 0, 1)$. Then the T-ec structures are exactly all the algebraically closed fields, and no T-ec model is a model of T.

[3]We are considering $\mathcal{P} \cup \mathcal{M}$ as the union of the domains of the structure $\mathcal{P}, \mathcal{M}$ amalgamated over $\mathcal{N}$; in particular we add a new constant for each element of $\mathcal{P} \setminus \mathcal{N}$, a new constant for each element of $\mathcal{M} \setminus \mathcal{N}$, a new constant for each element of $\mathcal{N}$.

Since the constants appearing in $\vec{a}, \vec{b}, \vec{c}$ are never appearing in sentences of T, we get that

$$T \vdash \forall \vec{z}\, (\forall \vec{x} \neg \psi_0(\vec{x}, \vec{z})) \vee (\forall \vec{y} \neg \psi_1(\vec{y}, \vec{z})).$$

Since $\mathcal{P}$ models $T_\forall$, and

$$\mathcal{P} \models \psi_0(\vec{x}, \vec{z})[\vec{x}/\vec{a}, \vec{z}/\vec{c}],$$

we get that

$$\mathcal{P} \models \forall \vec{y} \neg \psi_1(\vec{y}, \vec{c}).$$

Therefore

$$\mathcal{N} \models \forall \vec{y} \neg \psi_1(\vec{y}, \vec{c})$$

being a substructure of $\mathcal{P}$, and so does $\mathcal{M}$ since $\mathcal{N} \prec_1 \mathcal{M}$. This contradicts $\psi_1(\vec{b}, \vec{c}) \in \Delta_0(\mathcal{M})$. □

If $\bar{\mathcal{Q}}$ is a model realizing $T_\forall \cup \Delta_0(\mathcal{P}) \cup \Delta_0(\mathcal{M})$, and $\mathcal{Q}$ is the τ-structure obtained forgetting the constant symbols not in τ, we get that:

- $\mathcal{P}$ and $\mathcal{M}$ are both substructures of $\mathcal{Q}$ containing $\mathcal{N}$ as a common substructure;
- $\mathcal{N} \prec_1 \mathcal{M} \prec_1 \mathcal{Q}$, since $\mathcal{Q}$ realizes $T_\forall$ and $\mathcal{M}$ is $T_\forall$-ec.

We can now conclude that if a Σ_1-formula $\psi(\vec{c})$ for $\tau \cup \mathcal{N}$ with parameters in $\mathcal{N}$ holds in $\mathcal{P}$, it holds in $\mathcal{Q}$ as well (since $\mathcal{Q} \sqsupseteq \mathcal{P}$), and therefore also in $\mathcal{N}$ (since $\mathcal{N} \prec_1 \mathcal{Q}$).

(4): Observe that for all $\vec{b} \in \mathcal{M}^{<\omega}$, $\exists \vec{y}\, \psi(\vec{b}, \vec{y}, \vec{a})$ holds in $\mathcal{N}$, and therefore in $\mathcal{M}$, since $\mathcal{M}$ is T-ec; hence $\mathcal{M} \models \forall \vec{x} \exists \vec{y} \psi(\vec{x}, \vec{y}, \vec{a})$.

(5): First of all note that $\mathcal{M}$ is S-ec since $S \supseteq T$ (by Fact 1.8). By Lemma 1.5 (applied to $S_\forall + \psi$ and $\mathcal{M}$) any Π_2-sentence ψ for τ which holds in some model of $S_\forall$ holds in some model of $S_\forall$ which is a superstructure of $\mathcal{M}$. Now apply 4.

□

In particular a structure is T-ec if and only if it is $T_\forall$-ec, and a T-ec structure realizes all Π_2-sentences which are consistent with its Π_1-theory.

We now show that any structure $\mathcal{M}$ can always be extended to a T-ec structure for any T which is not separated from the Π_1-theory of $\mathcal{M}$.

Lemma 1.10. *[4, Lemma 3.2.11] Given a first order τ-theory T, any model of $T_\forall$ can be extended to a τ-superstructure which is T-ec.*

Proof. Given a model $\mathcal{M}$ of T, we construct an ascending chain of $T_\forall$-models as follows. Enumerate all quantifier free τ-formulae as $\{\phi_\alpha(y, \vec{x}_\alpha) : \alpha < |\tau|\}$. Let $\mathcal{M}_0 = \mathcal{M}$ have size $\kappa \geq |\tau| + \aleph_0$. Fix also some enumeration

$$\begin{aligned} \pi :&\kappa \to |\tau| \times \kappa^2 \\ &\alpha \mapsto (\pi_0(\alpha), \pi_1(\alpha), \pi_2(\alpha)) \end{aligned}$$

such that $\pi_2(\alpha) \leq \alpha$ for all $\alpha < \kappa$ and for each $\xi < |\tau|$, and $\eta, \beta < \kappa$ there are unboundedly many $\alpha < \kappa$ such that $\pi(\alpha) = (\xi, \eta, \beta)$.

Let now $\mathcal{M}_\eta$ with enumeration $\left\{\vec{m}^\xi_\eta : \xi < \kappa\right\}$ of $\mathcal{M}_\eta^{<\omega}$ be given for all $\eta \leq \beta$. If $\mathcal{M}_\beta$ is T-ec, stop the construction. Else check whether $T_\forall \cup \Delta_0(\mathcal{M}_\beta) \cup \left\{\exists y \phi_{\pi_0(\alpha)}(y, \vec{m}^{\pi_1(\alpha)}_{\pi_2(\alpha)})\right\}$ is a consistent $\tau \cup \mathcal{M}_\beta$-theory; if so let $\mathcal{M}_{\beta+1}$ have size κ and realize this theory. At limit stages γ, let $\mathcal{M}_\gamma$ be the direct limit of the chain of τ-structures $\{\mathcal{M}_\beta : \beta < \gamma\}$. Then all $\mathcal{M}_\xi$ are models of $T_\forall$, and at some stage $\beta \leq \kappa$ $\mathcal{M}_\beta$ is $T_\forall$-ec (hence also T-ec), since all existential τ-formulae with parameters in some $\mathcal{M}_\eta$ will be considered along the construction, and realized along the way if this is possible, and all $\mathcal{M}_\eta$ are always models of $T_\forall$ (at limit stages the ascending chain of $T_\forall$-models remains a $T_\forall$-model). □

Compare the above construction with the standard consistency proofs of bounded forcing axioms as given for example in [1, Section 2]. In the latter case to preserve $T_\forall$ at limit stages we use iteration theorems[4].

1.2. **The Kaiser hull of a first order theory.** The Kaiser Hull of a theory T describes the smallest elementary class containing all the "generic" structures for T. For most theories T the models of the respective Kaiser hulls realize exactly all Π_2-sentences which are consistent with the universal fragment of any extension of T.

Definition 1.11. [4, Lemma 3.2.12, Lemma 3.2.13] Given a theory T in a signature τ, its Kaiser hull $\mathrm{KH}(T)$ is given by the Π_2-sentences of τ which holds in all T-ec structures.

Definition 1.12. A τ-theory T is Π_n-complete, if it is consistent and for any Π_n-sentence either $\phi \in T$ or $\neg\phi \in T$.

By Proposition 1.9.5 we get:

Fact 1.13. *Given a Π_1-complete first order τ-theory T, its Kaiser Hull is a Π_2-complete τ-theory defined by the request that for any Π_2-sentence ψ*

$$\psi \in \mathrm{KH}(T) \quad \textit{if and only if} \quad \{\psi\} \cup T_\forall \textit{ is consistent.}$$

In particular any model of the Kaiser hull of a Π_1-complete T realizes simultaneously all Π_2-sentences which are individually consistent with $T_\forall$.

For theories T of interests to us their Kaiser hull can be described in the same terms, but the proof is much more delicate. We start with the following weaker property which holds for arbitrary theories:

Fact 1.14. *Given a τ-theory T, its Kaiser hull $\mathrm{KH}(T)$ contains the set of Π_2-sentences ψ for τ such that for all complete $S \supseteq T$, $S_\forall \cup \{\psi\}$ is consistent.*

Proof. Assume ψ is a Π_2-sentence such that for all complete $S \supseteq T$, $S_\forall \cup \{\psi\}$ is consistent. We must show that ψ holds in all T-ec models.

Fix $\mathcal{M}$ an existentially closed model for T (it exists by Lemma 1.10); we must show that $\mathcal{M} \models \psi$. Let $\mathcal{N} \sqsupseteq \mathcal{M}$ be a model of T and S be the τ-theory of $\mathcal{N}$. Then S is a complete theory and $\mathcal{M} \models S_\forall$ since $\mathcal{M} \prec_1 \mathcal{N}$ (being T-ec). Since $S \supseteq T$, $\mathcal{M}$ is also S-ec (by Fact 1.8). Since $S_\forall \cup \{\psi\}$ is consistent, and $S_\forall$ is Π_1-complete, we obtain that $\mathcal{M}$ models ψ, being an $S_\forall$-ec model, and using Fact 1.13. □

We will show in Lemma 1.26 that the set of Π_2-sentences described in the Fact provides an equivalent characterization of the Kaiser hull for many theories admitting a model companion.

1.3. **Model completeness.** It is possible (depending on the choice of the theory T) that there are models of the Kaiser hull of T which are not T-ec. Robinson has come up with two model theoretic properties (model completeness and model companionship) which describe the case in which the models of the Kaiser hull of T are exactly the class of T-ec models (even in case T is not a complete theory).

Definition 1.15. A τ-theory T is *model complete* if for all τ-models $\mathcal{M}$ and $\mathcal{N}$ of T we have that $\mathcal{M} \sqsubseteq \mathcal{N}$ implies $\mathcal{M} \prec \mathcal{N}$.

Remark that theories admitting quantifier elimination are automatically model complete. On the other hand model complete theories need not be complete[5]. However for

[4] Assume G is V-generic for a forcing which is a limit of an iteration of length ω of forcings $\{P_n : n < \omega\}$. In general $H_{\omega_2}^{V[G]}$ is not given by the union of $H_{\omega_2}^{V[G\cap P_n]}$, hence a subtler argument is needed to maintain that $H_{\omega_2}^{V[G]}$ preserves $T_\forall$.

[5] For example the theory of algebraically closed fields is model complete, but algebraically closed fields of different characteristics are elementarily inequivalent.

theories T which are Π_1-complete, model completeness entails completeness: any two models of a Π_1-complete, model complete T share the same Π_1-theory, therefore if $T_1 \supseteq T$ and $T_2 \supseteq T$ with $\mathcal{M}_i$ a model of T_i, we can suppose (by Lemma 1.4) that $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$. Since they are both models of T, model completeness entails that $\mathcal{M}_1 \prec \mathcal{M}_2$.

Lemma 1.16. *[4, Lemma 3.2.7] (Robinson's test) Let T be a τ-theory. The following are equivalent:*

(a) T is model complete.
(b) Any model of T is T-ec.
(c) Each existential *τ-formula $\phi(\vec{x})$ in free variables $\vec{x}$ is T-equivalent to a universal τ-formula $\psi(\vec{x})$ in the same free variables.*
(d) Each τ-formula $\phi(\vec{x})$ in free variables $\vec{x}$ is T-equivalent to a universal τ-formula $\psi(\vec{x})$ in the same free variables.

Remark that (d) (or (c)) shows that being a model complete τ-theory T is expressible by a $\Delta_0(\tau, T)$-property in any model of ZFC, hence it is absolute with respect to forcing.

Proof.

(a) implies (b): Immediate.

(b) implies (c): Fix an existential formula $\phi(\vec{x})$ in free variables $x_1, \ldots, x_n$. If $\phi(\vec{x})$ is not consistent with T it is T-equivalent to the trivial formula $\forall y(y \neq y)$ in free variables $\vec{x}$. Hence we may assume that $T \cup \phi(\vec{x})$ is a consistent theory. Let $\vec{c} = (c_1, \ldots, c_n)$ be a finite set of new constant symbols. Then $T \cup \phi(\vec{c})$ is a consistent $\tau \cup \{c_1, \ldots, c_n\}$-theory.

Let Γ be the set of universal τ-formulae $\theta(\vec{x})$ such that

$$T \vdash \forall \vec{x}\, (\phi(\vec{x}) \rightarrow \theta(\vec{x})).$$

Note that Γ is closed under finite conjunctions and disjunctions. Let $\Gamma(\vec{c}) = \{\theta(\vec{c}) : \theta(\vec{x}) \in \Gamma\}$. Note that $T \cup \Gamma(\vec{c})$ is a consistent $\tau \cup \{c_1, \ldots, c_n\}$-theory, since it holds in any $\tau \cup \{c_1, \ldots, c_n\}$-model of $T \cup \phi(\vec{c})$.

It suffices to prove

$$T \cup \Gamma(\vec{c}) \models \phi(\vec{c}); \tag{1}$$

if this is the case, by compactness, a finite subset $\Gamma_0(\vec{c})$ of $\Gamma(\vec{c})$ is such that

$$T \cup \Gamma_0(\vec{c}) \models \phi(\vec{c});$$

letting $\bar{\theta}(\vec{x}) := \bigwedge \{\psi(\vec{x}) : \psi(\vec{c}) \in \Gamma_0(\vec{c})\}$, the latter gives that

$$T \models \forall \vec{x}\, (\bar{\theta}(\vec{x}) \rightarrow \phi(\vec{x}))$$

(since the constants $\vec{c}$ do not appear in T).

$\bar{\theta}(\vec{x}) \in \Gamma$ is a universal formula witnessing (c) for $\phi(\vec{x})$.

So we prove (1):

Proof. Let $\mathcal{M}$ be a $\tau \cup \{c_1, \ldots, c_n\}$-model of $T \cup \Gamma(\vec{c})$. We must show that $\mathcal{M}$ models $\phi(\vec{c})$.

The key step is to prove the following:

Claim 3. *$T \cup \Delta_0(\mathcal{M}) \cup \{\phi(\vec{c})\}$ is consistent (where $\Delta_0(\mathcal{M})$ is the $\tau \cup \{c_1, \ldots, c_n\}$-atomic diagram of $\mathcal{M}$ in signature $\tau \cup \{c_1, \ldots, c_n\} \cup \mathcal{M}$).*

Assume the Claim holds and let $\mathcal{N}$ realize the above theory. Then

$$\mathcal{M} \sqsubseteq \mathcal{N} \restriction (\tau \cup \{c_1, \ldots, c_n\}).$$

Hence

$$\mathcal{M} \restriction \tau \sqsubseteq \mathcal{N} \restriction \tau.$$

By (b)

$$\mathcal{M} \restriction \tau \prec_1 \mathcal{N} \restriction \tau.$$

Now let $b_1, \dots, b_n \in \mathcal{M}$ be the interpretations of $c_1, \dots, c_n$ in the $\tau \cup \{c_1, \dots, c_n\}$-structure $\mathcal{M}$. Then

$$\mathcal{N} \restriction \tau \models \phi(x_1, \dots, x_n)[b_1, \dots, b_n].$$

Since $\phi(\vec{x})$ is Σ_1 for τ and $b_1, \dots, b_n \in \mathcal{M}$, we get that

$$\mathcal{M} \restriction \tau \models \phi(x_1, \dots, x_n)[b_1, \dots, b_n],$$

hence

$$\mathcal{M} \models \phi(c_1, \dots, c_n),$$

and we are done.

So we are left with the proof of the Claim.

Proof. Let $\psi(\vec{x}, \vec{y})$ be a quantifier free τ-formula such that $\psi(\vec{c}, \vec{a}) \in \Delta_0(\mathcal{M})$ for some $\vec{a} \in \mathcal{M}$.

Clearly $\mathcal{M}$ models $\exists \vec{y} \psi(\vec{c}, \vec{y})$.

Then the universal formula $\neg \exists \vec{y} \psi(\vec{c}, \vec{y}) \notin \Gamma(\vec{c})$, since $\mathcal{M}$ models its negation and $\Gamma(\vec{c})$ at the same time.

This gives that

$$T \not\vdash \forall \vec{x} \, (\phi(\vec{x}) \to \neg \exists \vec{y} \psi(\vec{x}, \vec{y})),$$

i.e.

$$T \cup \{\exists \vec{x} \, [\phi(\vec{x}) \wedge \exists \vec{y} \psi(\vec{x}, \vec{y})]\}$$

is consistent.

We conclude that

$$T \cup \{\phi(\vec{c}) \wedge \psi(\vec{c}, \vec{a})\}$$

is consistent for any tuple $a_1, \dots, a_k \in \mathcal{M}$ and formula ψ such that $\mathcal{M}$ models $\psi(\vec{c}, \vec{a})$ (since $\vec{c}, \vec{a}$ are constants never appearing in the formulae of T).

This shows that $T \cup \Delta_0(\mathcal{M}) \cup \{\phi(\vec{c})\}$ is consistent. □

(1) is proved. □

(c) implies (d): We prove by induction on n that Π_n-formulae and Σ_n-formulae are T-equivalent to a Π_1-formula.

(c) gives the base case $n = 1$ of the induction for Σ_1-formulae and (trivially) for Π_1-formulae.

Assuming we have proved the implication for all Σ_n formulae for some fixed $n > 0$, we obtain it for Π_{n+1}-formulae $\forall \vec{x} \psi(\vec{x}, \vec{y})$ (with $\psi(\vec{x}, \vec{y})$ Σ_n) applying the inductive assumptions to $\psi(\vec{x}, \vec{y})$; next we observe that a Σ_{n+1}-formula is equivalent to the negation of a Π_{n+1}-formula, which is in turn equivalent to the negation of a universal formula (by what we already argued), which is equivalent to an existential formula, and thus equivalent to a universal formula (by (c)).

(d) implies (a): By (d) every formula is T-equivalent both to a universal formula and to an existential formula (since its negation is T-equivalent to a universal formula).

This gives that $\mathcal{M} \prec \mathcal{N}$ whenever $\mathcal{M} \sqsubseteq \mathcal{N}$ are models of T, since truth of universal formulae is inherited by substructures, while truth of existential formulae pass to superstructures.

□

We will also need the following:

Fact 1.17. *Let τ be a signature and T a model complete τ-theory. Let $\sigma \supseteq \tau$ be a signature and $T^* \supseteq T$ a σ-theory such that every σ-formula is T^*-equivalent to a τ-formula. Then T^* is model complete.*

Proof. By the model completeness of T and the assumptions on T^* we get that every σ-formula is equivalent to a Π_1-formula for $\tau \subseteq \sigma$. We conclude by Robinson's test. □

Later on we will show that in most cases model complete theories maximize the family of Π_2-sentences compatible with any Π_1-completion of their universal fragment. This will be part of a broad family of properties for first order theories which require a new concept in order to be properly formulated, that of model companionship.

1.4. **Model companionship.** Model completeness comes in pairs with another fundamental concept which generalizes to arbitrary first order theories the relation existing between algebraically closed fields and commutative rings without zero-divisors. As a matter of fact, the case described below occurs when T^* is the theory of algebraically closed fields and T is the theory of commutative rings with no zero divisors.

Definition 1.18. Given two theories T and T^* in the same language τ, T^* is the *model companion* of T if the following conditions holds:

(1) Each model of T can be extended to a model of T^*.
(2) Each model of T^* can be extended to a model of T.
(3) T^* is model complete.

Different theories can have the same model companion, for example the theory of fields and the theory of commutative rings with no zero-divisors which are not fields both have the theory of algebraically closed fields as their model companion.

Theorem 1.19. *[4, Thm 3.2.14] Let T be a first order theory. If its model companion T^* exists, then*

(1) $T_\forall = T^*_\forall$.
(2) T^* is the theory of the existentially closed models of $T_\forall$.

Proof.

(1) By Lemma 1.5.
(2) By Robinson's test 1.16 T^* is the theory realized exactly by the T^*-ec models; by Proposition 1.9(2) $\mathcal{M}$ is T^*-ec if and only if it is $T^*_\forall$-ec; by (1) $T^*_\forall = T_\forall$.

□

An immediate by-product of the above Theorem is that the model companion of a theory does not necessarily exist, but, if it does, it is unique and is its Kaiser hull.

Theorem 1.20. *[4, Thm. 3.2.9] Assume T has a model companion T^*. Then T^* is axiomatized by its Π_2-consequences and is the Kaiser hull of $T_\forall$.*

Moreover T^* is the unique model companion of T and is characterized by the property of being the unique model complete theory S such that $S_\forall = T_\forall$.

Proof. For quantifier free formulae $\psi(\vec{x},\vec{y})$ and $\phi(\vec{x},\vec{z})$ the assertion

$$\forall\vec{x}\,[\exists\vec{y}\psi(\vec{x},\vec{y}) \leftrightarrow \forall\vec{z}\phi(\vec{x},\vec{z})]$$

is a Π_2-sentence.

Let T^{**} be the theory given by the Π_2-consequences of T^*.

Since T^* is model complete, by Robinson's test 1.16(c), for any Σ_1-formula $\exists\vec{y}\psi(\vec{x},\vec{y})$ there is a universal formula $\forall\vec{z}\phi(\vec{x},\vec{z})$ such that

$$\forall\vec{x}\,[\exists\vec{y}\psi(\vec{x},\vec{y}) \leftrightarrow \forall\vec{z}\phi(\vec{x},\vec{z})]$$

is in T^{**}.

Again by Robinson's test 1.16(c) T^{**} is model complete.

Now assume S is a model complete theory such that $S_\forall = T_\forall$. Clearly $T^*_\forall = T_\forall = S_\forall$. By Robinson's test 1.16(b) and Proposition 1.9(2), $S_\forall$ holds exactly in the $T_\forall$-ec models, but these are exactly the models of T^*. Hence $T^* = S$.

This shows that any model complete theory is axiomatized by its Π_2-consequences, that the model companion T^* of T is unique, that T^* is also the Kaiser hull of T (being axiomatized by the Π_2-sentences which hold in all T-ec-models), and is characterized by the property of being the unique model complete theory S such that $T_\forall = S_\forall$. □

Thm. 1.20 provides an equivalent characterization of model companion theories (which is expressible by a Δ_0-property in parameters T and T^*, hence absolute for transitive models of ZFC).

Note also that Robinson's test 1.16(d) gives an explicit axiomatization of a model complete theory T:

Fact 1.21. *Assume T is a model complete τ-theory. Let $\psi \mapsto \theta_\psi^T$ be a function assigning to each Σ_1-formula $\psi(\vec{x})$ for τ a Π_1-formula $\theta_\psi^T(\vec{x})$ which is T-equivalent to $\psi(\vec{x})$.*

Then T is axiomatized by $T_\forall$ and the Π_2-sentences

$$\mathsf{AX}_\psi^T \equiv \forall \vec{x}(\psi(\vec{x}) \leftrightarrow \theta_\psi^T(\vec{x}))$$

as $\psi(\vec{x})$ ranges over the Σ_1-formulae for τ.

Proof. First of all

$$T^* = \left\{\mathsf{AX}_\psi^T : \psi \text{ a } \tau\text{-formula}\right\}$$

is a model complete theory, since T^* satisfies Robinson's test 1.16(d). Let $S = T^* + T_\forall$. Note that S is also model complete (by Robinson's test 1.16(d)). Moreover $S \subseteq T$ (since $\mathsf{AX}_\psi^T \in T$ for all Σ_1-formulae ψ), and $S_\forall \supseteq T_\forall$ (since $T_\forall$ is certainly among the universal consequences of S). We conclude that $S_\forall = T_\forall$. Therefore S is the model companion of T. $S = T$ by uniqueness of the model companion. □

1.5. **Absolutely model companionable theories.** There is a further reinforcement of the notion of model companionship which we need to analyze the model companions of partial Morleyizations of set theory. Our aim is to isolate those (possibly non-complete) theories T whose model companion (or Kaiser Hull) is axiomatized by the Π_2-sentences which are consistent with the universal fragment of any model of T. We also show that this property is strictly stronger than model companionship.

Notation 1.22. Let τ be a signature and T be a τ-theory. $T_{\forall\vee\exists}$ consists of all logical consequences ψ of T which are boolean combinations of universal τ-sentences.

Note that any sentence in $T_{\forall\vee\exists}$ is either logically equivalent to $\theta \vee \psi$ or equivalent to $\theta \wedge \psi$ with θ universal and ψ existential.

Note also that $T_{\forall\vee\exists}$ may contain more information than $T_\forall + T_\exists$ as there could be a universal $\theta \notin T_\forall$ and an existential $\psi \notin T_\exists$ with $\theta \vee \psi \in T_{\forall\vee\exists}$.

Lemma 1.23. *Let τ be a signature and T, S be τ-theories. TFAE:*

(1) $T_{\forall\vee\exists} \supseteq S_{\forall\vee\exists}$.
(2) *For any $\mathcal{M}$ model of T there is $\mathcal{N}$ model of S superstructure of $\mathcal{M}$ realizing exactly the same universal sentences.*
(3) *For every boolean combination of universal sentences θ, $T + \theta$ is consistent only if so is $S + \theta$.*

Proof.

1 implies 2: Assume $\mathcal{M}$ models T and is such that no $\mathcal{N}$ model of S which is a superstructure of $\mathcal{M}$ realizes exactly the same universal sentences.

For any such $\mathcal{N}$ with $\mathcal{M} \sqsubseteq \mathcal{N} \models S$ we get that some universal sentence $\theta_\mathcal{N}$ true in $\mathcal{M}$ fails in $\mathcal{N}$. We claim that the $\tau \cup \mathcal{M}$-theory

$$S^* = \Delta_0(\mathcal{M}) \cup S \cup \{\theta_\mathcal{N} : \mathcal{M} \sqsubseteq \mathcal{N}, \mathcal{N} \models S\}$$

is inconsistent. If not let $\mathcal{P}^*$ be a model of S^*. Then $\mathcal{P} = (\mathcal{P}^* \restriction \tau) \sqsupseteq \mathcal{M}$ is a model of S, hence it models $\theta_{\mathcal{P}}$ and $\neg\theta_{\mathcal{P}}$ at the same time.

By compactness we can find a universal sentence $\phi_{\mathcal{M}}$ given by the conjunction of a finite set

$$\{\theta_{\mathcal{P}_i} : i = 1, \dots, n, \mathcal{P}_i \sqsupseteq \mathcal{M}\}$$

and a quantifier free sentence $\psi_{\mathcal{M}}(\vec{a})$ of $\Delta_0(\mathcal{M})$ such that

$$S + \psi_{\mathcal{M}}(\vec{a}) + \phi_{\mathcal{M}}$$

is inconsistent. Hence

$$S \models \neg\phi_{\mathcal{M}} \vee \neg\exists\vec{x}\,\psi_{\mathcal{M}}(\vec{x}).$$

Now observe that:
- $\neg\phi_{\mathcal{M}} \vee \neg\exists\vec{x}\,\psi_{\mathcal{M}}(\vec{x})$ is a boolean combination of universal sentences,
- $\mathcal{M} \models T + \exists\vec{x}\,\psi_{\mathcal{M}}(\vec{x}) \wedge \phi_{\mathcal{M}}$.

Therefore we get that $\neg\phi_{\mathcal{M}} \vee \neg\exists\vec{x}\,\psi_{\mathcal{M}}(\vec{x})$ is in $S_{\forall\vee\exists} \setminus T_{\forall\vee\exists}$.

2 implies 3: Left to the reader.

3 implies 1: If $T_{\forall\vee\exists} \not\supseteq S_{\forall\vee\exists}$ there is $\theta \in S_{\forall\vee\exists} \setminus T_{\forall\vee\exists}$. Then $\neg\theta$ is inconsistent with S and consistent with T.

□

Definition 1.24. Let τ be a signature and T, S be τ-theories.
- T and S are *cotheories* if $T_\forall = S_\forall$.
- T and S are *absolute cotheories* if $T_{\forall\vee\exists} = S_{\forall\vee\exists}$.

Definition 1.25. T is the absolute model companion of S, if it is its model companion and T and S are absolute cotheories.

This is the characterization of absolute model companionship which brought our attention to this notion.

Fact 1.26. *T^* is the absolute model companion of T if and only if it is model complete and is axiomatized by the Π_2-sentences ψ for τ such that $\psi + S_\forall$ is consistent for any τ-theory $S \supseteq T$.*

Proof. Assume $\psi \in T^*$ and take $\mathcal{M}$ a model of some completion R of T. Since T, T^* are absolute cotheories, there exists $\mathcal{N}$ model of T^* satisfying exactly the same universal theory of $\mathcal{M}$, hence $\psi + R_\forall$ holds in $\mathcal{N}$.

For the converse assume $S_\forall + \psi$ is consistent whenever S is an extension of T. We must show that $\psi \in T^*$: pick $\mathcal{M}$ model of T^* and let S be its complete theory. Since T^* is the model companion of T there is a superstructure $\mathcal{N}$ of $\mathcal{M}$ which models T. Then $\mathcal{N}$ models $S_\forall + T$, since $\mathcal{M} \prec_1 \mathcal{N}$. By assumption $S_\forall + \psi$ is consistent, since $S_\forall + T \supseteq T$ and $S_\forall$ is the universal fragment of $S_\forall + T$. Since $S_\forall \supseteq T_\forall$ is Π_1-complete and $\mathcal{M}$ is $T_\forall$-ec, $\mathcal{M}$ is also $S_\forall$-ec, hence $\mathcal{M} \models \psi$ by Proposition 1.9. □

Absolute model companionship is strictly stronger than model companionship. Note that if T is model complete, T is the model companion of $T_\forall$ and the absolute model companion of $T_{\forall\vee\exists}$. The two notions do not coincide whenever $T_\forall \supsetneq T_{\forall\vee\exists}$.

Remark 1.27. If T^* is the model companion of T, $T^*_{\forall\vee\exists} \supseteq T_{\forall\vee\exists}$: assume $\mathcal{M} \models T^*$, then there is a superstructure $\mathcal{N}$ of $\mathcal{M}$ which models T (since T^* is the model companion of T). Now $\mathcal{M} \prec_1 \mathcal{N}$, since $\mathcal{M}$ is T-ec. Hence $\mathcal{N}$ has the same Π_1-theory of $\mathcal{M}$.

The inclusion can be strict: consider the theories S of rings with no zero-divisors and T of algebraically closed fields in signature $\{+, \cdot, 0, 1\}$. Then T is the model companion (even the model completion) of S, but $T_{\forall\vee\exists} \supsetneq S_{\forall\vee\exists}$: $S + \forall x\,\neg(x \cdot x + 1 = 0)$ holds in $\mathbb{Q}$ and fails in any algebraically closed field. We conclude that T is not the absolute model companion of S (and also that the Kaiser hull of S is not axiomatized by the Π_2-sentences which are consistent with any extension of S).

1.6. **Is model companionship a tameness notion?** As we already outlined in the introduction model completeness and model companionship are "tameness" notion for first order theories which must be handled with care. We spell out the details in this small section.

Proposition 1.28. *Given a signature τ consider the signature τ^* which adds an n-ary predicate symbol R_ϕ for any τ-formula $\phi(x_1, \dots, x_n)$ with displayed free variables.*

Let T_τ be the following τ^*-theory:

- $\forall \vec{x}\, (\phi(\vec{x}) \leftrightarrow R_\phi(\vec{x}))$ *for all quantifier free τ-formulae $\phi(\vec{x})$,*
- $\forall \vec{x}\, [R_{\phi\wedge\psi}(\vec{x}) \leftrightarrow (R_\psi(\vec{x}) \wedge R_\phi(\vec{x}))]$ *for all τ-formulae $\phi(\vec{x}), \psi(\vec{x})$,*
- $\forall \vec{x}\, [R_{\neg\phi}(\vec{x}) \leftrightarrow \neg R_\phi(\vec{x})]$ *for all τ-formulae $\phi(\vec{x})$,*
- $\forall \vec{x}\, [\exists y R_\phi(y, \vec{x}) \leftrightarrow R_{\exists y\phi}(\vec{x})]$ *for all τ-formulae $\phi(y, \vec{x})$.*

Then any τ-structure $\mathcal{N}$ admits a unique extension to a τ^*-structure $\mathcal{N}^*$ which models T_τ. Moreover every τ^*-formula is T_τ-equivalent to an atomic τ^*-formula. In particular for any τ-model $\mathcal{N}$, the algebras of its τ-definable subsets and of the τ^*-definable subsets of $\mathcal{N}^*$ are the same.

Therefore for any consistent τ-theory T, $T \cup T_\tau$ is consistent and admits quantifier elemination, hence is model complete.

Proof. By an easy induction one can prove that any τ-formula $\phi(\vec{x})$ is T_τ-equivalent to the atomic τ^*-formula $R_\phi(\vec{x})$.

Another simple inductive argument brings that any τ^*-formula $\phi(\vec{x})$ is T_τ-equivalent to the τ-formula obtained by replacing all symbols $R_\psi(\vec{x})$ occurring in ϕ by the τ-formula $\psi(\vec{x})$. Combining these observations together we get that any τ^*-formula is equivalent to an atomic τ^*-formula.

T_τ forces the $\mathcal{M}^*$-interpretation of any relation symbol $R_\phi(\vec{x})$ in $\tau^* \setminus \tau$ to be the $\mathcal{M}$-interpretation of the τ-formula $\phi(\vec{x})$ to which it is T_τ-equivalent. □

Observe that the expansion of the language from τ to τ^* behaves well with respect to several model theoretic notions of tameness distinct from model completeness: for example T is a *stable* τ-theory if and only if so is the τ^*-theory $T \cup T_\tau$, the same holds for NIP-theories, or for o-minimal theories, or for κ-categorical theories.

The passage from τ-structures to τ^*-structures which model T_τ can have effects on the embeddability relation; for example assume $\mathcal{M} \sqsubseteq \mathcal{N}$ is a non-elementary embedding of τ-structures; then $\mathcal{M}^* \not\sqsubseteq \mathcal{N}^*$: if the non-atomic τ-formula $\phi(\vec{a})$ in parameter $\vec{a} \in \mathcal{M}^{<\omega}$ holds in $\mathcal{M}$ and does not hold in $\mathcal{N}$, the atomic τ^*-formula $R_\phi(\vec{a})$ holds in $\mathcal{M}^*$ and does not hold in $\mathcal{N}^*$.

However if T is a model complete τ-theory, then for $\mathcal{M} \sqsubseteq \mathcal{N}$ τ-models of T, we get that $\mathcal{M} \prec \mathcal{N}$; this entails that $\mathcal{M}^* \sqsubseteq \mathcal{N}^*$, which (by the quantifier elimination of $T \cup T_\tau$) gives that $\mathcal{M}^* \prec \mathcal{N}^*$. In particular for a model complete τ-theory T and $\mathcal{M}, \mathcal{N}$ τ-models of T, $\mathcal{M} \sqsubseteq \mathcal{N}$ if and only if $\mathcal{M}^* \sqsubseteq \mathcal{N}^*$.

Let us now investigate the case of model companionship. If T is the model companion of S with $S \neq T$ in the signature τ, $T \cup T_\tau$ and $S \cup T_\tau$ are both model complete theories in the signature τ^*. But $T \cup T_\tau$ cannot be the model companion of $S \cup T_\tau$, by uniqueness of the model companion, since each of these theories is the model companion of itself and they are distinct. Moreover if T and S are also complete, no τ^*-model of $S \cup T_\tau$ can embed into a τ^*-model of $T \cup T_\tau$: since T is the model companion of S and $S \neq T$, $T_\forall = S_\forall$ and there is some Π_2-sentence ψ $\forall x \exists y \phi(x, y)$ with ϕ-quantifer free in $T \setminus S$. Therefore $\forall x\, R_{\exists y\phi}(x) \in (T \cup T_\tau)_\forall \setminus (S \cup T_\tau)_\forall$; we conclude by Lemma 1.4, since $T \cup T_\tau$ and $S \cup T_\tau$ are complete, hence the above sentence separates $(T \cup T_\tau)_\forall$ from $(S \cup T_\tau)_\forall$.

1.7. **Summing up.** The results of this section gives that for any τ-theory T:

- The universal fragment of T describes the family of substructures of models of T, and (in most cases, e.g. if T is Π_1-complete) the T-ec models realize all Π_2-sentences which are "absolutely" consistent with $T_\forall$ (i.e. consistent with the universal fragment of any extension of T).
- Model companionship and model completeness describe (almost all) the cases in which the family of Π_2-sentences which are "absolutely" consistent with T (as defined in the previous item) describes the elementary class given by the T-ec structures.
- One can always extend τ to a signature τ^* so that T has a conservative extension to a τ^*-theory T^* which is model complete, but this process may be completely uninformative since it may completely destroy the substructure relation existing between τ-models of T (unless T is already model complete).
- On the other hand for certain theories T, one can unfold their "tameness" by carefully extending τ to a signature τ^* in which only certain τ-formulae are made equivalent to atomic τ^*-formulae. In the new signature T can be extended to a conservative extension T^* which has a model companion $\bar{T}$, while this process has mild consequences on the τ^*-substructure relation for models of $T^*_\forall$ (i.e. for the pairs of interest of τ-models $\mathcal{M}_0 \sqsubseteq \mathcal{M}_1$ of a suitable fragment of T, their unique extensions to τ^*-models $\mathcal{M}^*_i$ are still models of $T^*_\forall$ and maintain that $\mathcal{M}^*_0 \sqsubseteq \mathcal{M}^*_1$ also for τ^*). This gives useful structural information on the web of relations existing between τ^*-models of $T^*_\forall$.
- Our conclusion is that model completeness and model companionship are tameness properties of elementary classes $\mathcal{E}$ defined by a theory T rather than of the theory T itself: these model-theoretic notions outline certain regularity patterns for the substructure relation on models of $\mathcal{E}$, patterns which may be unfolded only when passing to a signature distinct from the one in which $\mathcal{E}$ is first axiomatized (much the same way as it occurs for Birkhoff's characterization of algebraic varieties in terms of universal theories).

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