Generic absoluteness and full absoluteness for the theory of $H_{\aleph_2}$.

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This talk will focus just on the generic absoluteness part of its title. I will also start with a series of naif considerations, these might be enough irritating for some of us so to help to keep awake the attention of the audience.
Essentially I will outline several reasons why I believe forcing axioms are viable candidates for being “correct axioms of set theory”.
I’m sure that most of my arguments will not and cannot stand to all possible objections, but I think I will be able to make at least one point:

> *Generic absoluteness results for the theory of strong forcing axioms are a natural outcome of the investigation of the metatheory of standard forcing axioms and explain why these axioms are so effective to describe the theory of $H_{\aleph_2}$.*

In the same way Woodin’s generic absoluteness result for $L(\mathbb{R})$ explain why large cardinals are so effective in settling the theory of projective sets.

Why forcing axioms should be reasonable axioms?

- **Argument 1: The axiom of choice is a global forcing axiom.**
  By means of topology we can present forcing axioms as natural strenghtenings of the axiom of choice.
- **Argument 2: Forcing axioms are succesful.**
  By means of basic features of the category language we can formulate most forcing axioms as density properties of certain class partial orders.
  Using this formulation of forcing axioms we can obtain the generic invariance of the theory of $H_{\aleph_2}$ with respect to many of these forcing axioms.
  This gives an a posteriori explanation of the success forcing axioms have met in solving problems.

The axiom of choice is a global forcing axiom

This observation has been handled to me by Stevo Todorčević. I hope he will not blame me for presenting it here.
This observation needs an almost self evident argument to be spelled out and I'm surprised that I've not been aware of this long before.

The axiom of choice is a global forcing axiom

Definition

Let λ be an infinite cardinal. DC_λ holds if for all sets X and all functions $F : X^{<\lambda} \to P(X)$ there exists $g : \lambda \to X$ such that $g(\alpha) \in F(g \restriction \alpha)$ for all $\alpha < \lambda$.

Fact

The axiom of choice is equivalent over the theory ZF *to the assertion that* DC_λ *holds for all* λ.

This is a local statement, i.e. there is a level by level correspondance between the amount of choice and of dependent choice available in the universe.

The axiom of choice is a global forcing axiom

Definition

Let P be a partial order. $\mathrm{FA}_\lambda(P)$ holds if for all family $\{D_\alpha : \alpha < \lambda\}$ of dense subsets of P, there exists a filter $G \subset P$ which has non-empty intersection with all the D_α.

Let Γ be a class of partial orders. Then $\mathrm{FA}_\lambda(\Gamma)$ holds if $\mathrm{FA}_\lambda(P)$ holds for all $P \in \Gamma$.

Fact

$\mathrm{DC}_{\aleph_0}$ *is equivalent over the theory* ZF *to the assertion that* $\mathrm{FA}_{\aleph_0}(P)$ *holds for all partial orders* P.

The axiom of choice is a global forcing axiom.

Sketch of proof. I show just the direction I want to bring forward:
Assume $F : X^{<\omega} \to P(X)$ is a function. Let T be the subtree of $X^{<\omega}$ given by finite sequences $s \in X^{<\omega}$ such that $s(i) \in F(s \upharpoonright i)$ for all $i < |s|$.
Consider the family given by the dense sets

$$D_n = \{s \in T : |s| > n\}.$$

If G is a filter on T meeting the dense sets of this family, $\cup G$ works.

The axiom of choice is a global forcing axiom.

More generally:

Definition

A partial order P is $< \lambda$-closed if all chains in P of length less than λ have a lower bound.

Let $\mathsf{AC} \restriction \lambda$ abbreviate DC_γ *holds for all* $\gamma < \lambda$ and Γ_λ be the class of $< \lambda$-closed posets.

Fact

DC_λ *is equivalent to* $\mathsf{FA}_\lambda(\Gamma_\lambda)$ *over the theory* $\mathsf{ZF} + \mathsf{AC} \restriction \lambda$.

The axiom of choice is a global forcing axiom.

Conclusion:

Fact

The axiom of choice is equivalent over the theory ZF *to the assertion that* $\mathsf{FA}_\lambda(\Gamma_\lambda)$ *holds for all* λ*.*

Forcing axioms are natural strenghtenings of the axiom of choice. They aim to isolate a maximal strengthening of $\mathsf{AC} \upharpoonright \omega_2$ enlarging the family Γ for which $\mathsf{FA}_{\aleph_1}(\Gamma)$ holds.

Forcing axioms are density properties of class posets.

Definition

Let Γ be a class of complete boolean algebras and Θ be a class of complete homomorphisms between elements of Γ and closed under composition and identity maps.

- $\mathbb{B} \leq_{\Theta} \mathbb{Q}$ if there is a complete homomorphism $i : \mathbb{B} \to \mathbb{Q}$ in Θ.
- $\mathbb{B} \leq^{*}_{\Theta} \mathbb{Q}$ if there is a complete and *injective* homomorphism $i : \mathbb{B} \to \mathbb{Q}$ in Θ.

With these definitions $(\Gamma, \leq_{\Theta})$ and $(\Gamma, \leq^{*}_{\Theta})$ are class partial orders.

Forcing axioms are density properties of class posets.

In particular we can now look at these class partial orders as forcing notions, and check whether they are interesting forcing notions.

The order $\leq^*_\Theta$ is the one we use to study iterated forcing and captures the notion of complete embedding for partial orders.

$\leq_\Theta$ has been neglected so far but is sufficient to grant that whenever $i : \mathbb{B} \to \mathbb{Q}$ witnesses $\mathbb{Q} \leq_\Theta \mathbb{B}$ and G is V-generic for $\mathbb{Q}$, then $i^{-1}[G]$ is V-generic for $\mathbb{B}$.

Forcing axioms are density properties of class posets.

Theorem

The following holds:

- ***Woodin:*** *Assume there are class many Woodin cardinals. Then the family of presaturated towers is dense in* $(\Omega, \leq_\Omega)$ *where* Ω *stands both for the class of all complete boolean algebras and for the class of all complete homomorphisms between complete boolean algebras.*
- ***Woodin:*** *Assume there are class many Woodin cardinals. Then Martin's maximum is equivalent to the assertion that the family of presaturated towers is dense in* $(\mathsf{SSP}, \leq_\Omega)$*.*
- ***V.:*** *Assume there are class many Woodin cardinals Then* MM^{++} *(a strong form of* MM*) is equivalent to the assertion that the family of presaturated towers is dense in* $(\mathsf{SSP}, \leq_{\mathsf{SSP}})$*, where* $\mathbb{B} \geq_{\mathsf{SSP}} \mathbb{Q}$ *iff there is* $i : \mathbb{B} \to \mathbb{Q}$ *complete homomorphism such that*

$$[\![\mathbb{Q}/i[\dot{G}_\mathbb{B}] \in \mathsf{SSP}]\!]_\mathbb{B} = 1_\mathbb{B}.$$

A strongest forcing axiom?

Definition (V.)

MM^{+++} holds if the class of *strongly* presaturated towers is dense in $(\mathrm{SSP}, \leq_{\mathrm{SSP}})$.

Fact

$\mathrm{MM}^{+++} \Rightarrow \mathrm{MM}^{++} \Rightarrow \mathrm{MM}$.

Theorem (V.)

MM^{+++} *is consistent relative to the existence of a huge cardinal.*

I will skip the definition of strongly presaturated tower for this talk......

Remark

MM^{+++} *will be forced by any of the standard forcings which yield* MM^{++} *provided that* δ *is superhuge (i.e. the supercompactness of* δ*) is witnessed by huge embeddings).*

Intermezzo – Why I like the category forcing $(\mathsf{SSP}, \leq_{\mathsf{SSP}})$:

This category has many surprising and nice features:

Theorem (V.)

Assume that δ is supercompact. Then $(\mathsf{SSP} \cap V_\delta, \leq_{\mathsf{SSP}} \restriction V_\delta)$ is an SSP partial order $\mathbb{U}_\delta$.
Moreover:

- $\mathbb{B} \geq_{\mathsf{SSP}} \mathbb{U}_\delta \restriction \mathbb{B}$ *for all* $\mathbb{B} \in \mathsf{SSP} \cap V_\delta$.
- $\mathbb{U}_\delta$ *forces* MM^{++}.

Theorem (V.)

Assume δ is a reflecting cardinal and MM^{+++} holds (i.e. there are densely many strongly presaturated towers in SSP).
Then $\mathbb{U}_\delta$ is itself a strongly presaturated tower.

In general the following holds for suitable properties $\phi(x)$:

$\phi(\mathbb{U}_\delta)$ holds if and only if the following set

$$\{\mathbb{B} \in \mathbb{U}_\delta : \phi(\mathbb{B}) \textit{ holds}\}$$

is dense in $\mathbb{U}_\delta$.

Generic absoluteness: standard formulation

Assume $T \subseteq \text{ZFC}$, T_0 is a family of sentences in the language of T, Γ is a class of forcing notions.
T_0 is Γ-generically invariant for T if for all $S \supseteq T$ and any $\phi \in T_0$ any of the following three equivalent conditions holds:

1. $S \vdash \phi$
2. $S \vdash$ there is $\mathbb{B} \in \Gamma$ such that $[\![\phi]\!] = 1_{\mathbb{B}}$ and $[\![T]\!] = 1_{\mathbb{B}}$.
3. $S \vdash [\![\phi]\!] = 1_{\mathbb{B}}$ for all $\mathbb{B} \in \Gamma$ such that $[\![T]\!] = 1_{\mathbb{B}}$.

Theorem (Woodin)

Let T_0 be the theory with real parameters of $L[\mathrm{Ord}^\omega]$ and $T = \mathrm{ZFC}$+large cardinals.
Then T_0 is Γ-generically invariant for T where Γ is the class of all posets.

Fact (Shelah)

Assume T_0 is the first order theory (with parameters) of $H_{\aleph_2}$ and $T \supseteq \mathrm{ZFC}$.
Then T_0 cannot be Γ-generically invariant for T unless Γ is contained in SSP*, the class of stationary set preserving posets.*

Strongest generic absoluteness result for third order number theory.

Theorem (V.)

Let T_0 *be the theory with parameters in* $P(\omega_1)$ *of* $L[\mathrm{Ord}^{\omega_1}]$ *and*

$$T = \mathrm{ZFC} + \textit{ large cardinals } + \mathrm{MM}^{+++}.$$

Then T_0 *is* SSP*-generically invariant for* T*.*

I shall from now on focus on a weaker family of forcing axioms leading to similar (but weaker) generic absoluteness results.

Iterated resurrection axioms and generic absoluteness

These results are inspired by Hamkins and Johnstone's resurrection axioms, and by Tsaprounis elaborations on their work.
THEY ARE JOINT WORK WITH GIORGIO AUDRITO, A PH.D. STUDENT IN TORINO.

Iterated resurrection axioms and generic absoluteness

Let Γ be a definable class of forcing closed under two step iterations. Let us denote by Γ also the family of complete homomoprhisms $i : \mathbb{B} \to \mathbb{Q}$ such that

$$[\![\mathbb{Q}/i[\dot{G}_{\mathbb{B}}] \in \Gamma]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

By recursion on α we define the iterated resurrection axiom $\mathsf{RA}_\alpha(\Gamma)$ as follows:

Definition (Audrito, V.)

$\mathsf{RA}_\alpha(\Gamma)$ holds if for all $\beta < \alpha$

$$\{\mathbb{B} \in \Gamma : H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V^{\mathbb{B}}} \text{ and } [\![\mathsf{RA}_\beta(\Gamma)]\!]_{\mathbb{B}} = 1_{\mathbb{B}}\}$$

is dense in $(\Gamma, \leq_\Gamma)$.

Remark

$\mathsf{RA}_1(\Gamma)$ *is the resurrection axiom for the class* Γ *introduced by Hamkins and Johnstone.*

$\mathsf{RA}_\omega(\Gamma)$ yields generic absoluteness.

Theorem (Audrito, V.)

$\mathsf{ZFC} + \mathsf{RA}_\omega(\Gamma)$ *has generic absoluteness for the formulae relativized to* $H_\mathfrak{c}$ *and all forcings in* Γ.

With respect to the generic absoluteness about $\mathsf{ZFC} + \mathsf{MM}^{+++} + \mathsf{LC}$:

- it covers a smaller theory, since $H_\mathfrak{c} \subset L([\mathrm{Ord}]^{\aleph_1})$,
- it is more general since it holds for any Γ (not only SSP),
- it has (for most classes Γ) much lower consistency strength.

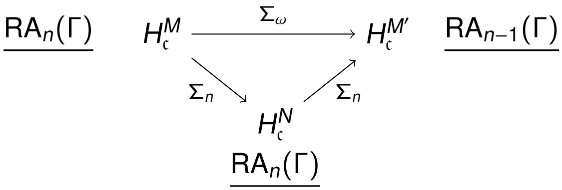
It suffices to show:

Lemma

$\mathsf{ZFC} + \mathsf{RA}_n(\Gamma)$ *has generic absoluteness for the* Σ_n *formulae relativized to* $H_\mathfrak{c}$ *with respect to forcings in* Γ.

Proof.

By induction on n, consider a Σ_n formula $\phi = \exists x \psi(x)$ and draw the following:



- $M \models \psi^{H_{\mathfrak{c}}}(a) \Rightarrow N \models \psi^{H_{\mathfrak{c}}}(a)$ so $M \models \exists x \psi^{H_{\mathfrak{c}}}(x) \Rightarrow N \models \exists x \psi^{H_{\mathfrak{c}}}(x)$,
- $N \models \exists x \psi^{H_{\mathfrak{c}}}(x) \Rightarrow M' \models \exists x \psi^{H_{\mathfrak{c}}}(x)$ (same argument) $\Rightarrow M \models \exists x \psi^{H_{\mathfrak{c}}}(x)$ (elementarity).

□

Generic absoluteness from iterated resurrection axioms

Definition

Let Γ be a class of partial orders closed under two steps iterations. Γ is *iterable* if:

- Γ is closed under lottery sums,
- The order $\leq^*_\Gamma$ is closed under set sized descending sequences.

Remark

CCC, Axiom A, proper, semiproper, $< \lambda$-closed posets all define iterable classes. Notable exceptions are SSP *and the class of posets P for which* $\mathsf{FA}_{\aleph_1}(P)$ *holds which are just closed under two step iterations and lottery sums.*

Generic absoluteness from iterated resurrection axioms

Theorem (Audrito, V.)

The following holds:

- $\mathsf{RA}_{\mathsf{Ord}}(\Gamma)$ *for iterable* Γ *is consistent relative to a Mahlo cardinal,*
- $\mathsf{RA}_{\mathsf{Ord}}(\mathsf{SSP})$ *is consistent relative to a stationary limit of supercompact cardinals,*
- $\mathsf{MM}^{+++} \Rightarrow \mathsf{RA}_{\mathsf{Ord}}(\mathsf{SSP})$.

Generic absoluteness from iterated resurrection axioms

Sketchy proof: To prove consistency of $\mathsf{RA}_\alpha(\Gamma)$ with Γ iterable (as for $\mathsf{FA}_{\omega_1}(\Gamma)$ and variations), we use lottery iteration forcing with respect to suitable fast-growing (Menas) function $f : \kappa \to \kappa$ for a large enough cardinal κ.

$$\mathbb{B}_0 = \{0, 1\}$$

$$\mathbb{B}_{\alpha+1} = \mathbb{B}_\alpha * \dot{\mathbb{Q}}_\alpha \text{ where } \dot{\mathbb{Q}}_\alpha = \prod\{\Gamma \cap H_{f(\alpha)}\}$$

$\mathbb{B}_\alpha$ for α limit is a lower bound in Γ for the chain $(\mathbb{B}_\beta : \beta < \alpha)$ of $\leq^*_\Gamma$.

For $\Gamma = \mathsf{SSP}$ we use the category forcing $\mathbb{U}^{\mathsf{SSP}}_\kappa$ for a large enough cardinal κ.

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