Notes on forcing

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Contents

Chapter 1

Introduction

1.1 Content

These notes are meant for students in mathematics who wish to follow the set theory course in the master program of Torino university. The course program covers large portions of Kunen's book [7] (or its new edition [8]). We focus in particular on:

- Axiomatic set theory, including basics of cardinal and ordinal arithmetic, transfinite recursion and the Mostowski collapsing theorem, the reflection theorems, the absoluteness properties of transitive structures, the development of basic model theory (including the Löwenheim-Skolem theorem for set structures) inside the standard model V for ZFC. This material is extensively covered in [7, Chapters I,III,IV,V] or [8, Chapter I, and Sections II.1–II.5].

- Forcing and combinatorics of partial orders ([7, Chapters II,VII] or [8, Chapters III, IV]). Since the course presents this material in a way which differs substantially from the approaches taken in [7] or [8], we decided to write up these notes to cover this part of the program.

These notes are divided in six chapters and two appendixes. Apart from the introduction (first chapter) the second, third and fourth chapters recall standard material on partial orders and boolean algebras which can be found in several textbooks. The fifth introduces boolean valued semantics as a natural generalization of Tarski semantics for first order logic. The sixth chapter is the heart of these notes. It gives a detailed presentation of the forcing method via boolean valued models. It features the most celebrated application of this method, namely the undecidability of the continuum hypothesis CH. The first appendix gives some more details on the parts of our course for which the reference texts are [7, Chapters III,IV,V] or [8, Chapter I, and Sections II.1–II.5] specifically on the basic properties of transitive structures and on absolute properties. The second appendix cover some basic facts about partial orders and topological spaces and includes a proof of the Stone-Cech compactification theorem. The reader already familiar with the material covered in chapters 2-3-4 of these notes may just skim through this part or refer back to it if needed while reading chapters 5 and 6.

- Chapters 2 and 3 introduce basic notions regarding boolean algebras. Some basic properties of the dualities which link these apparently distinct categories of mathematical objects are recalled. We prove in particular that every partial order admits, up to isomorphism, a unique complete boolean algebra in which it embeds as a dense subset, and we link the basic theory of boolean algebras to that of compact Haussdorff spaces by means of the Stone representation theorem. We bring to the attention of the reader that these chapters present material on boolean algebras which she/he may already have encountered in other courses. We decided to include it here in order to present these results with a focus aimed towards their use within set theory and their role in the development of the forcing method.

- Chapter 4 introduces some basic combinatorial properties of partial orders which will be needed to prove the independence of the continuum hypothesis by means of the forcing method. In particular we focus on CCC partial orders, we prove the Δ-system Lemma, and we use it to prove that the notion of forcing which can be used to produce a model of ZFC where CH fails is CCC. Most (if not all) of these results can be found in [8, III.1-III.2-III.3] or [7, II] , however these sections of both books contain a large amount of material which is not strictly necessary to present the two basic applications of the forcing method we aim for in these notes. The role of this chapter is to collect the minimal amount of information needed to run our applications of the forcing method. We also aim to outline the connections between the combinatorial properties of certain partial orders and well known topological properties of the Stone spaces associated to them (among other things we prove the Baire category theorem for compact Hausdorff spaces).

- Chapter 5 introduces the boolean valued semantics for first order theories and shows that certain function spaces which naturally occur in functional analysis produce natural examples of boolean valued models. The boolean valued semantic selects a given complete boolean algebra B and assigns to every statement ϕ a boolean value in B. The boolean operations reflect the behavior of the propositional connectives; it requires more attention to give a meaning to atomic formulae and to quantifiers, and we need a certain amount of completeness for B in order to be able to interpret quantifiers in the boolean semantic.

- Chapter 6 develops the theory of forcing by means of boolean valued models giving detailed proofs of the basic properties of boolean valued models for set theory, of Cohen's forcing theorem, and the two basic applications of the method which suffice to prove the independence of CH from the standard axioms of set theory. Our presentation of forcing departs completely from the approach taken by Kunen and is more keen to that taken by Bell [2], Jech [6]. We decide to follow this different approach for two reasons:

  1. In our eyes, the boolean valued models approach makes the metamathematical arguments needed to understand the forcing method easier to grasp and greatly simplifies some proofs.

2. The boolean valued model approach makes more transparent what is the role played by *generic filters* in the development of the forcing method and where the hypothesis that a filter is suitably generic is essential. Moreover it enlightens the link existing between the notion of generic filter arising in forcing with the corresponding topological notion of generic point of a topological space which is at the heart of the Baire category arguments.

Typographical conventions All over these notes in some occasions we introduce some arguments and material which are not central for the development of the core results. We adopt the typographical convention to put these parts of our notes in a smaller font.

1.1.1 Prerequisites.

We assume that the reader of these notes has familiarity with the content of [7, Chapter I, Sections II.1–II.5] or [8, Chapter III,IV,V]. This familiarity is *not* of vital importance for the comprehension of the first four chapters of these notes, where just a basic knowledge of the axioms of set theory is required and no familiarity with the formal first order development of ZFC is needed. On the other hand to understand the fifth chapter on forcing the reader *must know*:

- the basic facts about cardinal arithmetic and well orders (what is done for example in [7, Sections I.7, I.10] is more than sufficient),
- the first order axiomatization of set theory ZFC (which for us is the first order axiom system introduced in [7, Introduction, Section 7]),
- what is an absolute property between transitive sets $M \subseteq N \subseteq V$, where V is meant to be the "standard" model of ZFC and M, N are definable classes or sets (see Section 7.1 of the appendix for more details),
- that all properties which are provably Δ_1 in a theory T with respect to parameters $a_1, \dots a_n \in M$ are absolute for transitive (class or set) models $M \subseteq N \subseteq V$ of T,
- which standard set theoretic objects (eventually defined by transfinite recursion) are defined by properties which are absolute for transitive structures which models large fragments of ZFC, i.e.: the notion of relation, function, ordinal, algebraic structure, etc... or the functions defined by transfinite recursion using absolutely defined functions to be generated (for example the rank function, the Mostowski collapsing function, the transitive closure operation, etc....)
- which standard set theoretic properties are *not* absolute for transitive structures which models large fragments of ZFC, for example the notion of cardinality, of power set, etc...
- the standard facts about Mostowski collapsing theorem.

All these facts (and much more) are covered in [8, Chapter I, and Sections II.1–II.5] or [7, Chapters I, III, IV, V]. Some of these facts are also covered and expanded in the appendix of these notes (see Section 7.1 of the appendix).

1.2 Some remarks on the onthologyy of mathematics

The first chapters (up and including chapter 5) do not require on our side any special commitment on the onthology of mathematical entities and can be considered as a standard textbook on a mathematical theory which is developed much in the same way as one develops the theory of other fields of mathematics, i.e. we are in the situation common to most of mathematics where onthological considerations on the nature of mathematical entities do not play a significant role in our reasonings. On the other hand in the sixth chapter on forcing we have to give a neat explanation of the type of onthological assumptions we are making, in order to make transparent many of our arguments. Below we give a concise account of the point of view on the nature of mathematics we pursue in these notes. We do this in the following form: we list a series of basic questions on the onthology of mathematics, and we explain in few words what are the possible stances and the one that we choose to adopt.

1. **What is a mathematical reasoning?** For us a mathematical reasoning is a process expressed in a *natural language* (i.e. italian, english, french, chinese, whatever is most suited) which from given premises (hypotheses) produces a certain conclusion (thesis) which is *mathematically rigorous*.

2. **What does it mean *mathematically rigorous*?** For us it means that there is a first order language such that the premises and the conclusion can be formalized by first order formulae in that language ($\phi_1, \dots \phi_n$ for the premises and ψ for the conclusion) and such that there is a sound and complete first order calculus which allow to prove ψ by premises $\phi_1, \dots \phi_n$ on the basis of the calculus rules and axioms.

3. **What is the *meaning* of premises and conclusion?** There are various possible stances in this regard which range from:

    - *Extreme formalism:* there is no clear meaning in the premises and the conclusion of our reasoning as expressed in the natural language, since there cannot be a precise semantic interpretation of natural languages. What we know for sure is that with respect to the formalized counterpart $\phi_1, \dots \phi_n$ of the premises and ψ of the conclusion, for what we know so far, from premises $\phi_1, \dots \phi_n$ on the one hand, using a first order sound and complete calculus, we have not been able to derive a contradiction, on the other hand, we have been able to derive ψ.

    - *Extreme platonism:* There is a *hyperuranium* of mathematical entities, the premises and the conclusion define clear mathematical properties which can be predicated of objects in this hyperuranium. Our reasoning show that if the premises assert true properties of the hyperuranium, so does its conclusion. The fact that our reasoning (which we express in a natural language) can be formalized in a first order calculus gives a proof check of the correctness of our reasoning process establishing truths of the hyperuranium.

In this course we adopt a stance of *extreme platonism* when dealing with mathematical reasoning.

4. **What does it mean $\mathsf{CON}(T)$ for a first order theory T in a language $\mathcal{L} = (R_i, i \in I, f_j : j \in J, c_k : k \in K)$ to which Gödel's incompleteness theorem applies?**

   - For the extreme formalist it means just that so far nobody has been able to derive a contradiction using a sound a correct first order calculus and starting from the axioms of T as premises.
   - For a platonist it means that there is a *set* M in V and relations $R_i^M : i \in I$ on M^{n_i}, for $i \in I$, functions $f_j^M : M^{n_j} \to M$ for $j \in J$, and elements of M c_k^M for $k \in K$ also all in V such that $(M, R_i^M : i \in I, f_j^M : j \in J, c_k^M : k \in K)$ is a Tarski model for T as well as an element of V.

5. **What is the status of the first order theory ZFC?**

   - For an extreme formalist it is not different from the status of any other first order theory T to which Gödel's incompleteness theorem applies: the only sure thing we know so far is that a deduction of the false has not been found using a sound and correct first order calculus in which the premises are axioms of ZFC.
   - For an extreme platonist, there is among the elements of the hyperuranium a well defined mathematical entity V consisting of all those mathematical entities which are *sets*. V is not all of the hyperuranium, for example Russell's class $R = \{x \in V : x \notin x\}$ is a well defined mathematical entity belonging to the hyperuranium but is not an element of V (i.e. R is not a *set*!). Nonetheless V is very large and contains as elements most (if not all) mathematical entities we commonly use to do mathematics such as the natural numbers, the complex and real numbers, most topological spaces, the spaces of functions used in functional analysis, etc..... Moreover V is closed under many set-theoretic operations, i.e. if $(a_i : i \in I) \in V$ then also its product $\prod_{i \in I} a_i \in V$, if $a \in V$, $\cup a$ and $\mathcal{P}(a) \in V$ as well, if $B \in V$ and $\Phi(x)$ is a property which makes sense to be asked whether it is true of mathematical entities $\{a \in B : \Phi(a) \textit{ holds}\}$ is also in V, if $F : V \to V$ is a function and $A \in V$ is a set then $F[A]$ is also a set in V, etc...... In particular the first order structure $(V, \in)$ models ZFC. So for an extreme platonist, ZFC is not only a consistent theory (since it holds in the Tarski model $(V, \in)$, though this model is not a set), but it formalizes in a first order language a true state of affairs of the large portion of the hyperuranium given by V.

6. **Are there independence results over V?**

   - For an extreme platonist there are no independence results over V, given that V is a well defined coherent mathematical entity and thus the first order theory of the Tarski structure $(V, \in)$ is complete and coherent. In particular the continuum hypothesis is either true or false in V, even though currently our imperfect knowledge of V makes it impossible to ascertain which is the case. On the other hand ZFC is just a recursive list of first order properties which reflects true properties of V, but which we know that they cannot give a complete first order axiomatization of the theory of V in the first order language $\{\in\}$ due to Gödel's incompleteness theorem. It is well possible (and it is actually the case) that there can be models (M_i, E_i) which are sets in V for $i = 0, 1$ and are first order $\in$-models of the first order $\in$-theory ZFC with the following property: there is a $\in$-formula ϕ in the language of ZFC such that $(M_0, E_0) \models \phi$ and $(M_1, E_1) \models \neg\phi$. Actually the aim of these notes is to show that this is the case for ϕ being the first order formalization of the continuum hypothesis CH in signature $\in$.
   - For an extreme formalist the above question is void of content given that V is a meaningless concept.

7. **How do we proceed to prove that CH is independent from the axioms of ZFC?** We really commit ourselves to the extreme platonist stance. First of all ZFC is consistent, since $(V, \in, =)$ is a model of ZFC. Moreover $(V, \in)$ is a model of $\in$-sentence formalizing the completeness theorem we saw in model theory. We consider just the the foolowing form of the completeness theorem in V: For every first order language $\mathcal{L} = \{R_1, \dots, R_n, f_1, \dots, f_l, c_1, \dots, c_k\}$ given by a finite set of relations, functions, and constants, there is a recursive set of natural numbers $\mathsf{Form}_{\mathcal{L}} \subseteq \omega$ in V which is a code for the formulae in $\mathcal{L}$. There is also a recursive subset $\mathsf{Sent}_{\mathcal{L}}$ of $\mathsf{Form}_{\mathcal{L}}$ consisting of the formulae without free variables (i.e. its sentences). There are also:

   - A recursive predicate $\mathsf{DER}_{\mathcal{L}} \subseteq \mathsf{Form}_{\mathcal{L}}^{<\omega}$ which says that $(\phi_1, \dots \phi_n, \psi) \in \mathsf{DER}_{\mathcal{L}}$ iff there is a derivation in first order calculus of ψ from premises $\phi_1, \dots \phi_n$;
   - A definable satisfaction predicate (i.e. a class definable in V)

     $$\mathrm{Sat} : \mathsf{Form}_{\mathcal{L}} \times \mathcal{L}\text{-structures} \times V^{<\omega} \to 2 \cup \{*\}$$

     such that for all $\mathcal{L}$-structure $\mathcal{M} = (M, R_1^M, \dots, R_n^M, f_1^M, \dots, f_l^M, c_1^M, \dots, c_k^M) \in V$ and $\vec{s} \in M^{<\omega}$,

     $$(V, \in) \models \mathrm{Sat}(\phi, M, \vec{s}) = 1$$

     if and only if

     $$\mathcal{M} \models \phi(\vec{s})$$

     is true (in the latter case according to the rules of Tarski semantics for the $\mathcal{L}$-structure $\mathcal{M}$, and in the former case according to the rules of Tarski semantics for the structure $(V, \in)$ to interpret the definable class function Sat).

Now the correctness and completeness theorem in V says that $(V, \in)$ models the following formula in parameter $\mathsf{DER}_\mathcal{L}$, $\mathsf{Form}_\mathcal{L}$, for any set of sentences $T \subseteq \mathsf{Sent}_\mathcal{L}$:

> There is no $(\phi_1, \ldots \phi_n, \psi \wedge \neg\psi)$ in $\mathsf{DER}_\mathcal{L}$ with $\phi_1, \ldots, \phi_n \in T$ if and only if there is an $\mathcal{L}$-structure $(M, R_1^M, \ldots, R_n^M, f_1^M, \ldots, f_l^M, c_1^M, \ldots, c_k^M)$ such that $\mathrm{Sat}(\phi, M, \emptyset) = 1$ for all $\phi \in T$.

It can be checked that the above expression can be formulated as a $\in$-formula (more on this will be said in Chapter 7).

This means that in V, there is (M, E) which is a set and is model of ZFC, since we know that ZFC is consistent given that we assume that $(V, \in, =)$ is a Tarski model of ZFC.

Nonetheless in these notes we want more than this. We want that in V there is a transitive countable model M such that $(M, \in)$ is a model of ZFC. This can be achieved if for example we assume that in V there is a strongly inaccessible cardinal (more on this will be said in Chapter 7). The existence of an inaccessible cardinal is an axiom which an extreme platonist consider true. So we will from now on work in the first order theory ZFC^+ extending ZFC with the statement *There is a countable transitive set $M \in V$ such that $(M, \in)$ is model of ZFC*.

We will use the forcing method to build from the transitive and countable ZFC-model M new countable transitive models $N_0, N_1 \in V$ of ZFC such that $(N_0, \in) \models \mathsf{CH}$ and $(N_1, \in) \models \neg\mathsf{CH}$.

8. **What will an extreme formalist think of this proof of the independence of CH from the axioms of set theory?** Our proof does not make any sense for a formalist! Nonetheless, even if we will not spell out the details, by means of standard logical arguments, our proof can be converted in a proof that $\mathsf{CON}(\mathsf{ZFC}^+)$ implies also $\mathsf{CON}(\mathsf{ZFC} + \mathsf{CH})$ and $\mathsf{CON}(\mathsf{ZFC} + \neg\mathsf{CH})$. This is meaningful for a firmalist in the following sense: if we know that no contradiction can be derived in a sound and complete first order calculus from the axioms of ZFC^+, then such a contradiction can be derived in the same calculus neither from the axioms $\mathsf{ZFC} + \neg\mathsf{CH}$, nor from the axioms $\mathsf{ZFC} + \mathsf{CH}$. Moreover by means of arguments which are more sophisticated (and are rooted in the reflection theorem for V), one can rework our proof of the independence of CH from the axioms of set theory and obtain a proof (also for a formalist) that $\mathsf{CON}(\mathsf{ZFC})$ implies also $\mathsf{CON}(\mathsf{ZFC} + \mathsf{CH})$ and $\mathsf{CON}(\mathsf{ZFC} + \neg\mathsf{CH})$.

9. **Why the existence of a ZFC-model which is a set in V is not enough for our purposes, and we want to work with countable transitive models of ZFC?** We want to avoid to work with ill-founded models of ZFC. The pathologies we do not want to run into are well explained by the non-standard models of the first order theory of Peano's arithmetic. We know

that the structure of natural numbers $\mathbb{N}$ is a set in V, for example $(\mathbb{N}, <)$ can be presented as the model $(\omega, \in)$ and the sum, product, exponentiation of natural numbers can also be presented as suitable operations on elements of ω which can be defined in V using formulae in the parameter ω. So let the operations $+, \cdot$ be such that $(\omega, \in, +, \cdot)$ is a representative of the isomorphism type of the structure $(\mathbb{N}, <, +, \cdot)$. Let S be the first order complete theory of the structure $(\omega, \in, +, \cdot)$ in the language $\mathcal{L} = \{+, \cdot, <\}$. It is well known that there are ill-founded models of S, i.e. structures $(M, <_M, +_M, \cdot_M) \in V$ such that the order type of $(M, <_M)$ is not isomorphic to $(\omega, \in)$ and is ill founded: it can be shown that the order type of $(M, <_M)$ is isomorphic to an order of type $\mathbb{N} + I \times \mathbb{Z}$, where I is a dense linear order without end-points and all $a \in \mathbb{N}$ precede any $(b, c) \in I \times \mathbb{Z}$, and the order between elements in $I \times \mathbb{Z}$ is the lexicographic order. Notice that $\mathbb{N} + I \times \mathbb{Z}$ contains many non empty sets without minimum, for example $I \times \mathbb{Z}$. So this is the case also for $(M, <_M)$. On the other hand $(M, <_M, +_M, \cdot_M)$ is a model of Peano's arithmetic, in particular it models the principle of induction, which in this case amounts to say that for all formulae $\phi(x, y_1, \ldots, y_n)$ in $\mathcal{L}$ and $a_1, \ldots, a_n \in M$ if

$$(M, <_M, +_M, \cdot_M) \models \exists x \phi(x, a_1, \ldots, a_n),$$

then

$$(M, <_M, +_M, \cdot_M) \models \exists x(\phi(x, a_1, \ldots, a_n) \wedge \forall y[y <_M x \to \neg\phi(x, a_1, \ldots, a_n)]).$$

This means that there are ill-founded subsets of M with respect to the order $<_M$, but that the model $(M, <_M, +_M, \cdot_M)$ is not able to define any such ill-founded subset. Similar arguments can occur for models of ZFC. In particular there can be models $(N, E_N) \in V$ of ZFC which are ill-founded: i.e. there is $X = \{a_n : n \in \omega\} \in V$ subset of N such that $a_{n+1}\ E_N\ a_n$ for all $n \in \omega$. However, since (N, E_N) models the axiom of foundation, such a set $X \in V$ cannot be a definable subset of N, otherwise this X would contradict that N models the foundation axiom. On the other hand if we assume that $(N, \in, =)$ is a transitive model of ZFC, we have that $(N, \in, =)$ is really a well-founded model of ZFC in V or even in the hyperuranium. This adherence between the first order theory of $(N, \in)$ (where the axiom of foundation states that all ordered non empty sets in N have a minimal $\in$-element) and its true properties from the point of view of V (or of the even larger hyperuranium) is important because it will enormously simplifies many of our considerations and calculations on such type of ZFC-models $(N, \in)$.

10. **Why do we work with ZFC as a first order counterpart of the theory of V, rather than with Morse Kelley MK+*choice*?** It is just a matter of habits, since there is a well developed study of the first order theory of ZFC, in particular for what concerns the analysis of forcing. On the other hand, this is not the case for the Morse-Kelley axiomatization of set theory.

Chapter 2

Boolean algebras

The core of of this chapter develops the basic properties of boolean algebras: first we prove the Stone duality theorem linking boolean algebras to the category of compact 0-dimensional Hausdorff spaces. Next we give several different presentations of these algebras in terms of their logical properties (Def. 2.1.1), of their ring structure (Theorem 2.9.3), and as partial orders (Lemma 2.10.2). Finally we address the theory of complete boolean algebras: we show that any complete boolean algebra (*cba* in the sequel) can be represented as the family of regular open sets of a compact topological space (Theorem 3.1.5 and Proposition 3.1.16), and we prove that every partial order can be completed to a cba, which is unique up to isomorphism (Theorem 3.2.5). Towards this aim we first recall some definitions about orders, topological spaces while setting up the notation. Throughout the five sections of this chapter the reference text for unexplained details on most of these matters is [1, Section 24-Boolean algebras]; an introductoy and exhaustive text on boolean algebras is [4].

2.1 Basic definitions

We give the following *equational characterization* of a boolean algebra:

Definition 2.1.1. Let $(\mathsf{B}, \wedge, \vee, \neg, 0, 1)$ be a sextuple consisting of a set B, two total binary operations $\wedge$ and $\vee$ on B, a total unary operation $\neg$ on B and two elements 0 and 1 of B.

$(\mathsf{B}, \wedge, \vee, \neg, 0, 1)$ is a boolean algebra if it satisfies the following equations:

$$a \vee (b \vee c) = (a \vee b) \vee c \qquad \text{associativity}$$
$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad \text{distributivity}$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \vee b = b \vee a \qquad \text{commutativity}$$
$$a \wedge b = b \wedge a$$

$$a \vee 0 = a \qquad \text{identity}$$
$$a \wedge 1 = a$$

$$a \vee \neg a = 1 \qquad \text{complements}$$
$$a \wedge \neg a = 0$$

A bounded distributive lattice is a structure $(\mathsf{B}, \wedge, \vee, 0, 1)$ such that its two operations $\wedge, \vee$ satisfy the identity laws, the commutativity and associativity laws, and the distributivity laws.

Example 2.1.2. Given a (non-empty) set X and a topology τ on X:

- Let 0, 1, $\vee$, $\wedge$, $\neg$ and $\leq$ be respectively $\emptyset, X, \cup, \cap, \neg$ and $\subseteq$, then the power set $\mathcal{P}(X)$ of X is a (complete) boolean algebra and $A \subseteq B$ if and only if $A \cap B = A$ if and only if $A \cup B = B$.

- The family τ and the family of closed sets τ^c are bounded distributive sublattices of $\mathcal{P}(X)$ (with the same operations we have on $\mathcal{P}(X)$).

- The family $\mathsf{CLOP}(X, \tau)$ of clopen set of τ (with the same operations we have on $\mathcal{P}(X)$) is a boolean subalgebra of $\mathcal{P}(X)$ (though in general it is not complete).

Notation 2.1.3. It is often convenient to introduce further operations on a boolean algebra. For example given a boolean algebra B and $a, b \in \mathsf{B}$ $a \setminus b = a \wedge \neg b$, and $a \Delta b = (a \setminus b) \vee (b \setminus a) = (a \vee b) \setminus (a \wedge b)$.

Notice that if B is $\mathcal{P}(X)$, the above operations turn out to be the natural set theoretic operations on subsets of X.

The main results of this section are the following:

- Boolean algebras admits maximal ideals with corresponding dual ultrafilters (the Prime ideal Theorem 2.7.4). This is crucial for many classical results on first order logic whose proof depends on the axiom of choice (for example in the proof of the compactness and/or completeness theorems for first order logic).

- Every boolean algebra is isomorphic to the algebra of clopen sets of some compact 0-dimensional space (The Stone duality Theorem 2.8.2). This is a cornerstone result linking the logical and algebraic point of view on boolean algebras to the topological analysis of compactness.

- Boolean algebras can also be described as the class of commutative rings with idempotent multiplication (see Def. 2.9.1 and Theorem 2.9.3). This characterization allows to infuse the study of boolean algebra with methods coming from algebra and ring theory; for example we will see that it greatly simplifies certain computations, among which those regarding the properties of boolean ideals and of boolean quotients.

In the remainder of this section we assume the reader is familiar with the basic properties of orders and topological spaces. We refer the reader to section 8 for the missing details.

2.2 The order on boolean algebras

We start with the following:

Proposition 2.2.1. *Let* $(\mathsf{B}, \wedge, \vee, \neg, 0, 1)$ *be a boolean algebra. Define* $a \leq b$ *by* $a \wedge b = a$ *for* $a, b \in \mathsf{B}$. *Then:*

(i) $\leq$ *is an order relation on* B,

(ii) $a \wedge b$ *defines the infimum of* $\{a, b\}$,

(iii) $a \vee b$ *defines the supremum of* $\{a, b\}$,

(iv) $a \leq b$ *if and only if* $a \vee b = b$.

Proof.

(i) $\leq$ **is reflexive:**

$$a = a \wedge 1 = a \wedge (a \vee \neg a) = (a \wedge a) \vee (a \wedge \neg a) = (a \wedge a) \vee 0 = a \wedge a,$$

hence $a \leq a$.

$\leq$ **is transitive:** Assume $a \leq b$ (i.e. $a \wedge b = a$) and $b \leq c$ (i.e. $b \wedge c = b$). Then

$$a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a.$$

$\leq$ **is antisymmetric:** Assume $a \leq b \leq a$, then $a = a \wedge b = b \wedge a = b$.

(ii) First of all $a \wedge b \wedge a = a \wedge b$ and $a \wedge b \wedge b = a \wedge b$, hence $a \wedge b$ is a lower bound for a, b.

Assume $c \leq a, b$. Then $c \wedge a = c$ and $c \wedge b = c$, hence

$$c \wedge (a \wedge b) = (c \wedge a) \wedge b = c \wedge b = c,$$

therefore our thesis.

(iii) First of all we show that $a \vee b$ is an upper bound for b, a:

$$b = b \vee 0_{\mathsf{B}} = b \vee (a \wedge \neg a) = (b \vee a) \wedge (b \vee \neg a) \leq b \vee a, \tag{2.1}$$

where in the latter inequality we used the fact that $c \wedge d \leq c$ (being the infimum of $\{c, d\}$ by the previous item) for all $c, d \in \mathsf{B}$; similarly we can prove $a \leq a \vee b$.

The second observation is the following:

$$\text{Assume } c, d \leq e, \text{ then } c \vee d \leq e. \tag{2.2}$$

This holds since:

$$(c \vee d) \wedge e = (c \wedge e) \vee (d \wedge e) = c \vee d.$$

By 2.1, 2.2 we get that $a \vee b$ is the supremum of $\{a, b\}$ (2.1 grants that it is an upper bound, and 2.2 that is the smallest such).

(iv) $a \leq b$ if and only if $b = \max\{a, b\} = \sup\{a, b\} = a \vee b$.

□

2.3 Boolean identities

Proposition 2.3.1. *The following holds on a boolean algebra:*

(i) $a = a \vee a = a \wedge a$ *(Idempotence laws).*

(ii) $\neg a$ *is the unique* $b \in \mathsf{B}$ *such that* $b \wedge a = 0_{\mathsf{B}}$ *and* $b \vee a = 1_{\mathsf{B}}$
(Law of uniqueness for complements).

(iii) $\neg\neg a = a$ *(Double negation law).*

(iv) $\neg(a \wedge b) = \neg a \vee \neg b$ *(First De Morgan law).*

(v) $\neg(a \vee b) = \neg a \wedge \neg b$ *(Second De Morgan law).*

Proof.

(i) Immediate since $a \wedge a = \min\{a, a\} = a = \max\{a, a\} = a \vee a$.

(ii) Assume $b \wedge a = 0_{\mathsf{B}}$ and $b \vee a = 1_{\mathsf{B}}$, we show that $b = \neg a$.

$$b = b \wedge 1_{\mathsf{B}} = b \wedge (a \vee \neg a) = (b \wedge a) \vee (b \wedge \neg a) = 0_{\mathsf{B}} \vee (b \wedge \neg a) = (b \wedge \neg a).$$

Therefore $b \leq \neg a$. On the other hand:

$$b = b \vee 0_{\mathsf{B}} = b \vee (a \wedge \neg a) = (b \vee a) \wedge (b \vee \neg a) = 1_{\mathsf{B}} \wedge (b \vee \neg a) = (b \vee \neg a).$$

Therefore $b \geq \neg a$.

(iii) By (ii) $a = \neg\neg a$, since both satisfy the equations defining the complement of $\neg a$.

(iv) Remark that

$$(a \wedge b) \vee (\neg a \vee \neg b) = (a \vee \neg a \vee \neg b) \wedge (b \vee \neg a \vee \neg b) \geq 1.$$

Similarly one can prove that

$$(a \wedge b) \wedge (\neg a \vee \neg b) \leq 0.$$

By (ii) we get that $\neg(a \wedge b) = (\neg a \vee \neg b)$.

(v) Left to the reader (along the lines of the proof of the previous item).

□

2.4 Ideals and morphisms of boolean algebras

Ideals on boolean algebras are the kernel of boolean algebra morphisms. The usual algebraic properties of morphisms of rings and groups works equally well for boolean algebras, the reason being that boolean algebras are axiomatized by equational theories, as rings and groups are.

Definition 2.4.1. Let B, C be boolean algebras. A map $k : \mathsf{B} \to \mathsf{C}$ is a *homomorphism* of boolean algebras if it preserves the boolean operations, an *isomorphism* if it is a bijective homomorphism.

A subalgebra of a boolean algebra $(\mathsf{B}, \wedge, \vee, \neg, 0, 1, \leq)$ is a subset A of B such that the inclusion map of A into B defines an injective homomorphism.

Fact 2.4.2. *A map $\phi : \mathsf{B} \to \mathsf{C}$ is an homomorphism of boolean algebras if it preserves $\vee, \neg$ or if it preserves $\wedge, \neg$.*

Proof. Left to the reader. (**Hint:** Use De Morgan's laws). □

Exercise 2.4.3. Prove that a boolean morphism $\phi : \mathsf{B} \to \mathsf{C}$ preserves the operation of symmetric difference Δ.

Remark 2.4.4. Since boolean algebras are axiomatized by an equational theory, the class of boolean algebras is closed under homomorphic images, products and substructures (by the easy direction of Birkhoff's theorem, see [**?**, Theorem XXX]).

Definition 2.4.5. Let B be a boolean algebra. $I \subseteq \mathsf{B}$ is an ideal if it is closed under the $\vee$ operation and is downward closed (i.e. $b \in I$ and $a \leq b$ gives that $a \in I$ as well).

Exercise 2.4.6. I is an ideal if and only if the following two conditions are simultaneously met:

Da svolgere per verificare se diffici Una direzione OK ma l'altra? – *M*

- $a \Delta b \in I$ for all $a, b \in I$,
- $a \wedge b \in I$ for all $b \in I$ and $a \in \mathsf{B}$.

– *M*

2.5 Atomic and finite boolean algebras

Notation 2.5.1. Given a boolean algebra B, $\mathsf{B}^+ = \mathsf{B} \setminus \{\emptyset\}$. We will often look at B as the partial order $(\mathsf{B}^+, \leq)$.

A subset X of B^+ is *dense* if for all $b \in \mathsf{B}^+$ there is $a \in X$ such that $a \leq b$.

Definition 2.5.2. Let B be a boolean algebra. The atoms of B are the minimal elements[1] of $(\mathsf{B}^+, \leq)$ (if they exists).

- B is atomic if its atoms form a dense subsets of B^+.
- B is atomless if it has no atoms.

Remark 2.5.3. The following holds:

- Let B be a boolean algebra. The following are equivalent:
  - $a \in \mathsf{B}$ is an atom,
  - for all $b \in \mathsf{B}$ it is not the case that $0 < b < a$,
  - $a \wedge b = a$ or $a \wedge b = 0$ for all $b \in \mathsf{B}$.

  To see this oberve that if $b \leq a$, then either $b = 0$ or $b = a$, hence $a \wedge b = a$ or $a \wedge b = 0$; if $b \geq a$, $a \wedge b = a$; if $b \not\geq a$, $a \wedge b \neq a$; since $a \wedge b \leq a$, we must have that $a \wedge b = 0$.

- Let X be a non-empty set, then $\mathcal{P}(X)$ is atomic:

  The order relation on $\mathcal{P}(X)$ given by $X \leq Y$ if $X \cap Y = X$ is the inclusion relation; all singletons $\{x\}$ for $x \in X$ are atoms of $\mathcal{P}(X)$; $\mathcal{P}(X)^+ = \mathcal{P}(X) \setminus \{\emptyset\}$, and any non-empty set $Y \subseteq X$ has some $y \in Y$ with $\{y\} \subseteq Y$.

- All finite boolean algebras B are atomic:

  Assume $b_0 \in \mathsf{B}$ is not refined by any atom. Inductively define a chain

  $$\{b_n : n \in \mathbb{N}\}$$

  such that $b_{n+1} < b_n$ is not refined by any atom. The procedure cannot terminate, otherwise some b_{n+1} is refined by some atom a, hence so is b_0, contradicting our assumptions on b_0. Hence $\{b_n : n \in \mathbb{N}\}$ is an infinite subset of B, a contradiction.

  We may formalize properly this proof as follows: fix b_0 such that no atom refines b_0. Consider the family

  $$\begin{aligned}\{f : \mathrm{dom}(f) \to \mathsf{B} \ : \quad & \mathrm{dom}(f) \in \mathbb{N} \text{ or } \mathrm{dom}(f) = \mathbb{N} \text{ and } f \text{ is such that} \\ & f(0) = b_0 \text{ and } f(i) > f(i+1) \text{ for all } i+1 \in \mathrm{dom}(f)\}\end{aligned}$$

  ordered by inclusion. This partial order has upper bounds for all its subchains. By Zorn's Lemma the family has a maximal element h. It is easy to check that $\mathrm{dom}(h) = \mathbb{N}$ and $h(i) > h(i+1)$ for all $i \in \mathbb{N}$.

[1] Recall that a is a minimal element of an order $(P, \leq)$ if $b < a$ for no $b \in P$.

The structure of the class of finite boolean algebras is described by the following:

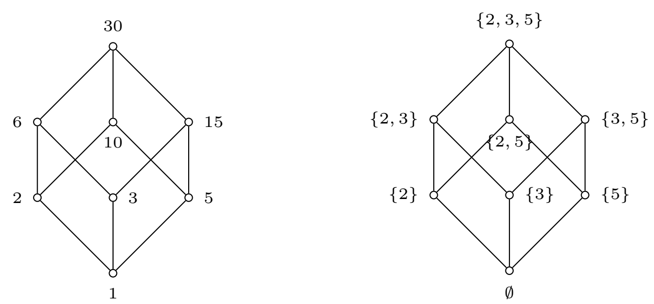
Proposition 2.5.4. *Assume* B *is a finite boolean algebra. Then* $\mathsf{B} \cong \mathcal{P}(A_{\mathsf{B}})$*, where* A_{B} *is the set of atoms of* B.

The following exercise provide a concrete example of how this isomorphism can be defined for a finite boolean algebra.

Exercise 2.5.5. Consider the set $\mathrm{Div}(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ with operations $\wedge, \vee, \neg$ given by $n \wedge m = \mathrm{MCD}(n, m)$, $n \vee m = \mathrm{mcm}(n, m)$, $\neg(n) = \frac{30}{n}$.

1. Check that:

    - $\mathsf{B} = \langle \mathrm{Div}(30), \wedge, \vee, \neg, 1, 30\rangle$ is a boolean algebra (look at the picture below to understand what is the order structure of B, where the vertexes are ordered according to whether one is below another and there is a line connecting the two).
    - $n \leq m$ if and only if n divides m.
    - The atoms of $\mathrm{Div}(30)$ are $2, 3, 5$.
    - The map $F : \mathrm{Div}(30) \to \mathcal{P}(\{2, 3, 5\})$ of the proposition is given by $n \mapsto \{p : p \text{ is a prime number and divides } n\}$ and implements an isomorphism of B with $\langle \mathcal{P}(\{2, 3, 5\}), \cap, \cup, A \mapsto \{2, 3, 5\} \setminus A, \emptyset, \{2, 3, 5\}\rangle$.



2. For which n $\mathrm{Div}(n)$ with the above operations is a boolean algebra? (**Hint:** Show that $\mathrm{Div}(18)$ is not a boolean algebra. Notice that 18 has a prime factor which divides it in power 2).

Now we prove the Proposition.

Proof. By the previous remark B is atomic. Define

$$\begin{aligned} F : \mathsf{B} &\to \mathcal{P}(A_{\mathsf{B}}) \\ b &\mapsto \{a \in A_{\mathsf{B}} : a \leq b\} \end{aligned}$$

F is a morphism:

- F preserves $\wedge$:

$$\begin{aligned} F(b \wedge c) = \{a \in A_{\mathsf{B}} : a \leq b \wedge c\} &= \{a \in A_{\mathsf{B}} : a \leq b \text{ and } a \leq c\} = \\ &\{a \in A_{\mathsf{B}} : a \leq b\} \cap \{a \in A_{\mathsf{B}} : a \leq c\} = F(b) \cap F(c), \end{aligned}$$

where the second equality follows since $b \wedge c = \inf\{b, c\}$.

- F maps 1_{B} to A_{B} and 0_{B} to $\emptyset$ (useful exercise for the reader).
- F preserves $\neg$:

$$F(\neg b) = \{a \in A_{\mathsf{B}} : a \leq \neg b\} = \{a \in A_{\mathsf{B}} : a \not\leq b\} = A_{\mathsf{B}} \setminus F(b),$$

where the second equality follows by the following argument: For $a \in A_{\mathsf{B}}$ we have that:

$$a \not\leq b$$

if and only if (since a is an atom)

$$a \wedge b = 0$$

if and only if

$$a = a \wedge 1 = a \wedge (b \vee \neg b) = (a \wedge b) \vee (a \wedge \neg b) = 0 \vee (a \wedge \neg b) = a \wedge \neg b$$

if and only if

$$a \leq \neg b.$$

F **is an injection:** If $b \neq c$, assume $b \wedge \neg c > 0_{\mathsf{B}}$, and let $a \in A_{\mathsf{B}}$ refine $b \wedge \neg c$. Then $a \in F(b) \setminus F(c)$ since $a \leq b$ while $a \not\leq c$.

F **is surjective:** Given $X \subseteq A_{\mathsf{B}}$, let $b_X = \bigvee X$. The following holds:

For any $a \in A_{\mathsf{B}}$ $a \leq b_X$ if and only if $a \in X$.

If $a \in X$ clearly $a \leq b_X$. On the other hand if $a \notin X$, then $a \wedge u = 0_{\mathsf{B}}$ for all $u \in X$ (since distinct atoms are pairwise incompatible), hence (by applying $|X|$-many times the distributive law)

$$a \wedge \bigvee X = \bigvee \{a \wedge u : u \in X\} = 0_{\mathsf{B}}.$$

Therefore $a \wedge b_X = 0_{\mathsf{B}} \neq a$, i.e. $a \not\leq b_X$.

We get that $F(b_X) = X$.

□

2.6 Examples of boolean algebras

The first two examples comes from propositional logic, the third and fourth from first order logic, the fifth from Lebesgue measure, the sixth from general topology. We assume the reader is familiar with the background material needed to analyze each of these examples.

Example 2.6.1 (Lindenbaum algebras on finitely many propositional variables). The reader should be familiar with the basic concepts of propositional calculus to follow this and the next example; a possible reference is [**?**, Section1]. Below, as a guiding example for the discussion to follow, we give the truth table of the propositional formula ϕ

$$((\mathrm{B} \to \mathrm{A}) \wedge ((\mathrm{B} \vee \mathrm{C}) \to \mathrm{A}))$$

in propositional variables $\mathrm{A}, \mathrm{B}, \mathrm{C}$ (as well of all its propositional subformulae):

A	B	C	(B → A)	(B ∨ C)	((B ∨ C) → A)	ϕ
1	1	1	1	1	1	1
1	1	0	1	1	1	1
1	0	1	1	1	1	1
1	0	0	1	0	1	1
0	1	1	0	1	0	0
0	1	0	0	1	0	0
0	0	1	1	1	0	0
0	0	0	1	0	1	1

Recall that:

- for $\phi_1, \ldots, \phi_n, \psi$ propositional formulae in propositional variables $A_1, \ldots, A_n$, $\phi_1, \ldots, \phi_n \models \phi$ if the truth tables of $\phi_1, \ldots, \phi_n$ and ψ in the variables $A_1, \ldots, A_n$ are such that every time a row assigns value 1 to each of the formulae $\phi_1, \ldots, \phi_n$, that row assigns 1 also to ψ.

- ϕ is logically equivalent to ψ ($\phi \equiv \psi$) if and only if $\phi \models \psi$ and $\psi \models \phi$ i.e. the two formulae have the same truth table in variables $A_1, \ldots, A_n$.

Let $\mathcal{A}_n = \{A_1, \ldots, A_n\}$ be a set of n-many propositional variables; the Lindenbaum algebra B_n on $\mathcal{A}_n$ is defined as follows:

- Its domain is $\{[\phi] : \phi \text{ an } \mathcal{A}_n\text{-formula}\}$, where $[\phi]$ is the equivalence class given by all formulae ψ logically equivalent to ϕ);

- $1_{\mathsf{B}_n} = [A_1 \vee \neg A_1]$, $0_{\mathsf{B}_n} = [A_1 \wedge \neg A_1]$;

- $[\phi] \wedge_{\mathsf{B}_n} [\psi] = [\phi \wedge \psi]$, $[\phi] \vee_{\mathsf{B}_n} [\psi] = [\phi \vee \psi]$, $\neg_{\mathsf{B}_n}[\psi] = [\neg\psi]$.

Remark that:

$[\phi] \leq [\psi]$ if and only if $[\phi \wedge \psi] = [\phi]$ if and only if

$$\phi \wedge \psi \models \phi \text{ and } \phi \models \phi \wedge \psi.$$

Now $\phi \models \phi \wedge \psi$ if and only if $\phi \models \psi$. Hence $[\phi] \leq [\psi]$ if and only if $\phi \models \psi$. We get the following very nice fact:

B_n *is the quotient of the preorder given by* $\models$ *on the propositional* $\mathcal{A}_n$*-formulae.*

A second nice observation is the following:

> *The domain of B_n is in bijective correspondence with the set of truth tables for $\mathcal{A}_n$-propositional formulae. Moreover the atoms of B_n are in bijective correspondance with the truth tables containing exactly one row in which a 1 appears. Hence B_n has 2^n-many atoms and $2^{(2^n)}$-many elements.*

We can see it by the following argument: Clearly any $\mathcal{A}_n$-formula has a truth table and two formulae are logically equivalent if and only if they have the same truth table.

Given n-many propositional variables there are 2^n-possible assignments of truth values to the propositional formulae. Hence each truth table on an $\mathcal{A}$-propositional formula has 2^n-rows.

There are $2^{(2^n)}$ possible truth tables in 2^n-rows, and each such truth table identifies a unique equivalence class. Therefore:

> *The Lindenbaum algebra B_n has $2^{(2^n)}$-elements.*

B_n, being finite, is isomorphic to $\mathcal{P}(A_{\mathsf{B}_n})$. We already computed the size of B_n has being $2^{2^n} = |\mathcal{P}(2^n)|$, hence B_n has exactly 2^n atoms.

Let us identify which truth tables define atoms:

> *The atoms of B_n are identified by the truth tables with exactly one* 1.

Assume $[\phi] \leq [\psi]$ for some ψ whose truth table has exactly one 1 in the relevant column. Then the truth table of ϕ can have 1 only in the places where these occurs in the truth table of ψ, i.e. in at most one place, therefore either ϕ is not satisfiable (i.e. its truth table consists just of 0 in the relevant column) or ϕ is logically equivalent to ψ. This means that $[\psi]$ is an atom.

Since there are 2^n such atoms, one for each of the possible truth tables where 1 appears in exactly one row, we get that the truth tables with exactly one 1 appearing in them define all the possible atoms of $\mathcal{A}$.

Example 2.6.2 (Lindenbaum algebras on infinitely many propositional variables). Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be an infinite set of propositional variables. Recall that for $\mathcal{A}$-propositional formulae:

- $\phi_1, \ldots, \phi_n \models \psi$ if and only if whenever n is large enough so that all the propositional variables occurring in $\phi \cup \psi$ are among $A_1, \ldots, A_n$, the truth tables of $\phi_1, \ldots, \phi_n$ and ψ as computed with respect to $A_1, \ldots, A_n$ are such that every time a row assigns value 1 to each of the formulae $\phi_1, \ldots, \phi_n$, that row assigns 1 also to ψ.

- ϕ is logically equivalent to ψ ($\phi \equiv \psi$) if and only if $\phi \models \psi$ and $\psi \models \phi$ i.e. the two formulae have the same truth table in variables $A_1, \ldots, A_n$ for any large enough n.

The Lindenbaum algebra B_∞ on $\mathcal{A}$ is defined as follows:

- Its domain is $\{[\phi] : \phi$ an $\mathcal{A}$-formula$\}$, where $[\phi]$ is the equivalence class given by all formulae ψ logically equivalent to ϕ.

- $1_{\mathsf{B}_\infty} = [A_1 \vee A_1]$, $0_{\mathsf{B}_\infty} = [A_1 \wedge \neg A_1]$;

- $[\phi] \wedge_{\mathsf{B}_\infty} [\psi] = [\phi \wedge \psi]$, $[\phi] \vee_{\mathsf{B}_\infty} [\psi] = [\phi \vee \psi]$, $\neg_{\mathsf{B}_\infty}[\psi] = [\neg\psi]$.

This is an infinite atomless boolean algebra.

To prove that it is atomless, proceed as follows: Given a satisfiable formula ϕ, assume that $\{A_1, \ldots, A_m\}$ contains all the propositional variables occuring in ϕ. Then the truth table of ϕ in variables $A_1, \ldots, A_m$ has at least a 1 in the relevant column. We can show that $\phi \not\models \phi \wedge A_{m+1}$: Since ϕ is satisfiable, find a row l in the truth table of ϕ over $A_1, \ldots, A_m$ such that ϕ gets value 1 in row l. Consider now the truth table of $\phi \wedge A_{m+1}, \phi$ as computed in variables $A_1, \ldots, A_{m+1}$. Let k be the row which assigns 0 to A_{m+1} and to each A_j for $j = 1, \ldots, m$ exactly the same value (0 or 1) it gets in row l of the truth table of ϕ over $A_1, \ldots, A_m$. Then in this row k, $\phi \wedge A_{m+1}$ gets value 0, while ϕ gets value 1. Hence $\phi \not\models \phi \wedge A_{m+1}$. Clearly $\phi \wedge A_{m+1} \models \phi$. Therefore $[\phi \wedge A_{m+1}] < [\phi]$.

This shows that $[\phi]$ is not an atom.

The inclusion map of the Lindenbaum algebrae B_n into $\mathsf{B}_{\mathcal{A}}$ is an injective homomorphism for all n and $\mathsf{B}_{\mathcal{A}}$ is the union (and direct limit) of the algebrae B_n for $n \in \mathbb{N}$.

Example 2.6.3 (Lindenbaum algebras of $\mathcal{L}$-theories). The reader should be familiar with the basic concepts of first order logic to follow this and the next example; a possible reference is [**?**, Sections 2 and 3]. Let $\mathcal{L}$ be a first order signature and T be a satisfiable $\mathcal{L}$-theory consisting of $\mathcal{L}$-sentences. Recall that $\phi_1, \ldots, \phi_n \models_T \psi$ if $\phi_1, \ldots, \phi_n, T \models \psi$, and $\phi \equiv_T \psi$ if $\phi \models_T \psi$ and $\psi \models_T \phi$.

The Lindenbaum algebra B_T is defined as follows:

- Its domain is $\{[\phi]_T : \phi$ an $\mathcal{L}$-sentence$\}$ where $[\phi]_T$ is the equivalence class of ϕ with respect to the equivalence relation $\equiv_T$.

- $1_{\mathsf{B}_T} = [\phi \vee \neg\phi]_T$, $0_{\mathsf{B}_T} = [\phi \wedge \neg\phi]_T$.

- $[\phi] \wedge_{\mathsf{B}_T} [\psi] = [\phi \wedge \psi]_T$, $[\phi] \vee_{\mathsf{B}_T} [\psi] = [\phi \vee \psi]_T$, $\neg_{\mathsf{B}_T}[\psi] = [\neg\psi]_T$.

Remark that $[\phi]_T = 1_{\mathsf{B}_T}$ if and only if $T \models \phi$.

Exercise 2.6.4. T is complete if and only if B_T has only 2-elements.

Exercise 2.6.5. Let $S \subseteq T$ be satisfiable $\mathcal{L}$-theories (consisting only of sentences) for a given language $\mathcal{L}$.

- Show that $I_T = \{[\phi]_S : T \models \neg\phi\}$ is an ideal on the Lindenbaum algebra B_S.

- Show also that $\mathsf{B}_T \cong \mathsf{B}_S/_{I_T}$ via the map $[[\phi]_S]_{I_T} \mapsto [\phi]_T$.

Example 2.6.6 (**The algebras of definable subset of an $\mathcal{L}$-structure.**). Let $\mathcal{M}$ be an $\mathcal{L}$-structure with domain M.

The algebra $\mathsf{B}^n_{\mathcal{M}}$ of n-dimensional $\mathcal{L}$-definable subsets of M^n has domain

$$\left\{T^{\mathcal{M}}_{\phi(x_1,\ldots,x_n,y_1,\ldots y_k),\langle b_1,\ldots,b_k\rangle} : \phi(x_1,\ldots,x_n,y_1,\ldots y_k) \text{ an } \mathcal{L}\text{-formula}, \ \{b_1,\ldots,b_k\} \subseteq M\right\}$$

with operations inherited as a subalgebra of $\mathcal{P}(M^n)$.

Recall that $T^{\mathcal{M}}_{\phi(x_1,\ldots,x_n,y_1,\ldots y_k),\langle b_1,\ldots,b_k\rangle}$ is the set

$$\{\langle a_1,\ldots,a_n\rangle \in M^n : \mathcal{M} \models \phi(x_1,\ldots,x_n,y_1,\ldots,y_k)[x_i/a_i, y_j/b_j]\}.$$

Example 2.6.7 (**The boolen algebras of characteristic functions**). Let (X,τ) be a topological space and consider the ring B_X of characteristic 2 given by the $f : X \to \mathbb{Z}_2$ which are continuous with respect to τ (where $\mathbb{Z}_2$ is endowed with the discrete topology). The boolean operations on B_X are defined as follows: $f \vee g = \max(f,g)$, $\neg f = \chi_X - f$, $f \wedge g = f \cdot g$. The top elements is χ_X and the bottom element is $\chi_\emptyset$.

When X is endowed with the discrete topology, B_X is $\mathcal{P}(X)$. When X is endowed with a connected topology $\mathsf{B}_X = \{\chi_X, \chi_\emptyset\}$.

See Section 2.9 for more details.

Exercise 2.6.8. Assume (X,τ) is a topological space and consider the boolean algebra B_X defined in the above example 2.6.7. $I \subseteq \mathsf{B}_X$ is an ideal with respect to the boolean algebra structure on B_X if and only if it is an ideal with respect to the ring structure on B_X (**Hint:** what is $f\Delta g$ in the boolean structure of B_X? Once you find out, use exercise 2.4.6).

Example 2.6.9 (**The algebra of Lebesgue measurable subsets of** $[0,1]$**, and its quotient algebra modulo the ideal of null sets**). The boolean algebra $\mathcal{M}([0;1])$ given by the Lebesgue measurable subsets of $[0;1]$ is an example of an atomic boolean algebra properly contained in $\mathcal{P}([0;1])$ which is not isomorphic to $\mathcal{P}([0,1])$: $\mathcal{M}([0;1])$ is not complete, while $\mathcal{P}([0,1])$ is complete (see Section 3 for a definition of completeness.). A counterexample to the completeness of $\mathcal{M}([0;1])$ is given by any non Lebesgue-measurable set $V \notin \mathcal{M}([0;1]$; V is a supremum in $\mathcal{P}([0,1])$ of the family $\{\{r\} : r \in V\}$ of atoms of $\mathcal{M}([0;1])$.

$\mathsf{MALG} = \mathcal{M}([0;1])/_{\mathrm{Null}}$ (where Null is the ideal of measure 0-subsets of $[0;1]$) is an example of an atomless boolean algebra which is also complete.

See Section 3.3.2, Prop. 3.3.5, Cor. 3.3.6 for more details on MALG.

Example 2.6.10 (**The clopen sets of the Cantor set**). The clopen sets on $2^{\mathbb{N}}$ with product topology form an atomless countable boolean algebra.

The family of sets $N_s = \left\{f \in 2^{\mathbb{N}} : s \subseteq f\right\}$ as s ranges in $2^{<\mathbb{N}}$ describe a basis consisting of clopen sets of $2^{\mathbb{N}}$.

Any clopen set can be uniquely described as a finite union of sets in this basis.

It can be shown that this boolean algebra is isomorphic to the Lindenbaum algebra on infinitely many propositional variables.

2.7 The Prime Ideal Theorem

Definition 2.7.1. Let B be a boolean algebra[2].

- $G \subset \mathsf{B}$ is a *prefilter* in B if and only if for every $a_1, \ldots, a_n \in G$, $a_1 \wedge \cdots \wedge a_n > 0_{\mathsf{B}}$.
- $G \subset \mathsf{B}$ is *ultra* if for all $b \in \mathsf{B}$ either $b \in G$ or $\neg b \in G$.
- $G \subset \mathsf{B}$ is a *filter* if it contains all its finite meets and is upward closed (i.e. $a \wedge b = a \in G$ entails that $b \in G$ as well).
- $G \subset \mathsf{B}$ is an *ultrafilter* if it is a filter and is ultra.
- A filter G is *principal* if $G = \{c : c \geq b\}$ for some $b \in \mathsf{B}$.
- Given $A \subseteq \mathsf{B}$, $\breve{A} = \{\neg a : a \in A\}$.

Exercise 2.7.2. Assume G is a principal filter on a boolean algebra B with $a \in G$ an atom of B. Show that $G = G_a = \{b \in \mathsf{B}^+ : a \leq b\}$, and that G_a is a principal ultrafilter.

Exercise 2.7.3. Let B be a boolean algebra. Show that

- $I \subseteq \mathsf{B}$ is an ideal if and only if $\breve{I}$ is a filter.
- I is a prime or maximal ideal if and only if one among $a, \neg a \in I$ (maximality) if and only if one among $a, b \in I$ whenever $a \wedge b \in I$ (primality).

Theorem 2.7.4 (Prime ideal theorem)**.** *Assume F is a prefilter on a boolean algebra B. Then F can be extended to an ultrafilter G on B.*

Proof. Let $\mathcal{A}$ be the family of prefilters on B containing F. We show that:

- any chain under inclusion contained in the partial order $(\mathcal{A}, \subseteq)$ admits an upper bound in $\mathcal{A}$,
- a maximal element of $\mathcal{A}$ is an ultrafilter on B containing F.

By Zorn's Lemma, $\mathcal{A}$ has a maximal elements, hence the thesis.

A chain under inclusion of $(\mathcal{A}, \subseteq)$ admits an upper bound:

Assume $\{F_i : i \in I\} \subseteq \mathcal{A}$ is a chain (i.e. for $i, j \in I$ either $F_i \subseteq F_j$ or $F_j \subseteq F_i$). Let $H = \bigcup_{i \in I} F_i$. We show that H is a prefilter. Assume $b_1, \ldots, b_n \in H$. Then each $b_i \in F_{j_i}$ for some $j_i \in I$. Since $\{F_i : i \in I\}$ is a chain, there is some $k \leq n$ such that $F_{j_k} \supseteq F_{j_i}$ for all $i = 1, \ldots, n$. Hence each $b_i \in F_{j_k}$ for all $i = 1, \ldots, n$. Since F_{j_k} is a prefilter, $b_1 \wedge \cdots \wedge b_n > 0_{\mathsf{B}}$. Hence H is a prefilter.

[2]This definition of filter generalizes the usual definition of a filter on a set X. In that case, $\mathsf{B} = \mathcal{P}(X)$.

Any maximal element of $(\mathcal{A}, \subseteq)$ is an ultrafilter:

Assume G is a maximal element of $(\mathcal{A}, \subseteq)$, we must show that G is upward closed, closed under meets, and contains either b or $\neg b$ for all $b \in \mathsf{B}$. First of all we prove that if a prefilter G is ultra (i.e. such that for all $b \in \mathsf{B}$ either $b \in G$ or $\neg b \in G$, then it is a ultrafilter:

- Assume $b \in G$ and $a \geq b$ (i.e. $a \wedge b = b$). Then $\neg a \notin G$, since
$$\neg a \wedge b = \neg a \wedge (a \wedge b) = 0_{\mathsf{B}}$$
and G is a prefilter. Hence $a \in G$, since G is ultra.
- Assume $a, b \in G$. Then $\neg(a \wedge b) \notin G$, otherwise $\neg(a \wedge b) \wedge a \wedge b = 0_{\mathsf{B}}$. Since G is ultra we get that $(a \wedge b) \in G$.

Now assume G is a maximal prefilter of $\mathcal{A}$. We show that G is ultra: assume not as witnessed by b. Then $G \cup \{b\}$ and $G \cup \{\neg b\}$ are not prefilters. Hence there are $a_1, \ldots, a_n \in G$ and $b_1, \ldots, b_k \in G$ such that
$$a_1 \wedge \cdots \wedge a_n \wedge b = 0_{\mathsf{B}}$$
and
$$b_1 \wedge \cdots \wedge b_k \wedge \neg b = 0_{\mathsf{B}}.$$
Hence
$$\begin{aligned} 0_{\mathsf{B}} < a_1 \wedge \cdots \wedge a_n \wedge b_1 \wedge \cdots \wedge b_k = \\ a_1 \wedge \cdots \wedge a_n \wedge b_1 \wedge \cdots \wedge b_k \wedge (b \vee \neg b) \leq \\ \leq (a_1 \wedge \cdots \wedge a_n \wedge b) \vee (b_1 \wedge \cdots \wedge b_k \wedge \neg b) = 0_{\mathsf{B}}, \end{aligned}$$
where the first inequality holds because G is a prefilter and $a_1, \ldots, a_n, b_1, \ldots, b_k \in G$. We reached a contradiction. Hence G is a prefilter which is ultra and thus an ultrafilter.

The theorem is proved. □

2.8 Stone spaces of boolean algebras

There is a natural functor that attaches to a boolean algebra the Stone space of its ultrafilters. We prove here that these spaces are exactly the family of compact, Hausdorff, 0-dimensional topological spaces. Finally we prove the Stone duality theorem, which represents any boolean algebra as the family of clopen sets of its Stone space.

Let B be a boolean algebra. We define
$$\mathrm{St}(\mathsf{B}) = \{G \subseteq \mathsf{B} : G \text{ is an ultrafilter}\},$$
and
$$\tau_{\mathsf{B}} \text{ to be the topology on } \mathrm{St}(\mathsf{B}) \text{ generated by}^{3} \{N_b = \{G \in \mathrm{St}(\mathsf{B}) : b \in G\} : b \in \mathsf{B}\}.$$
The topological space $(\mathrm{St}(\mathsf{B}), \tau_{\mathsf{B}})$ is the *Stone space* of B. We have:

[3] I.e., the smallest topology that contains $\{N_b : b \in \mathsf{B}\}$.

1. for all $b \in \mathsf{B}$, $N_b \cap N_{\neg b} = \emptyset$;
2. for all $b \in \mathsf{B}$, $N_b \cup N_{\neg b} = \mathrm{St}(\mathsf{B})$;
3. for all $b_1, \ldots, b_n \in \mathsf{B}$, $N_{b_1} \cap \ldots \cap N_{b_n} = N_{b_1 \wedge \ldots \wedge b_n}$;
4. for all $b_1, \ldots, b_n \in \mathsf{B}$, $N_{b_1} \cup \ldots \cup N_{b_n} = N_{b_1 \vee \ldots \vee b_n}$.

Exercise 2.8.1. Prove the above facts for $\mathrm{St}(\mathsf{B})$.

Now we outline the key properties about the Stone space of a boolean algebra B.

Theorem 2.8.2 (Stone duality for boolean algebras)**.** *Given a boolean algebra* B*, we have that:*

1. $(\mathrm{St}(\mathsf{B}), \tau_{\mathsf{B}})$ *is a Hausdorff* 0*-dimensional, compact topological space.*
2. *The map*
$$\begin{aligned} \phi : &\mathsf{B} \to \mathsf{CLOP}(\mathrm{St}(\mathsf{B})) \\ &b \mapsto N_b \end{aligned}$$
is an isomorphism, hence the clopen sets of τ_{B} *are the sets* N_b *for* $b \in \mathsf{B}$*, and form a basis for* τ_{B}*.*
3. *There is a natural correspondence between open (closed) subsets of* $\mathrm{St}(\mathsf{B})$ *and ideals (filters) on* B*:*
    - *For* $X \subseteq \mathrm{St}(\mathsf{B})$
    $$I_X = \{c \in \mathsf{B} : N_c \subseteq X\} \text{ is an ideal on } \mathsf{B},$$
    and
    $$F_X = \{c \in \mathsf{B} : N_c \supseteq U\} \text{ is a filter on } \mathsf{B}.$$
    - $U \subseteq \mathrm{St}(\mathsf{B})$ *is open if and only if*
    $$\bigcup \{N_c \in \mathsf{B} : c \in I_U\} = U.$$
    - $C \subseteq \mathrm{St}(\mathsf{B})$ *is closed if and only if*
    $$\bigcap \{N_c \in \mathsf{B} : c \in F_C\} = C.$$
4. G *is an isolated point of* $\mathrm{St}(\mathsf{B})$ *if and only if* $G = G_a = \{b \in \mathsf{B} : a \leq b\}$ *is a principal ultrafilter generated by some atom* $a \in \mathsf{B}$*.*

Proof. We prove all items as follows:

1. *Topological properties of* $\mathrm{St}(\mathsf{B})$*:*

    0-dimensional: We have already observed that $N_b \cup N_{\neg b} = \mathrm{St}(\mathsf{B})$ and $N_b \cap N_{\neg b} = \emptyset$; thus these sets are clopen; they form a semibasis by definition of τ_{B}; since $N_{b_1} \cap \ldots \cap N_{b_n} = N_{b_1 \wedge \ldots \wedge b_n}$, this semibasis is closed under finite intersections, hence it is a basis.

Hausdorff: If G, H are two different points of $\mathrm{St}(\mathsf{B})$, then there is $b \in G \Delta H$; assume $b \in G \setminus H$, then $G \in N_b$ and $H \in N_{\neg b}$, since H is ultra and $b \notin H$.

Compact: Fix $\mathcal{F}$ a family of closed sets with the finite intersection property. Let

$$\mathcal{G} = \{N_b : \exists C \in \mathcal{F}\,(N_b \supseteq C)\}.$$

- We first claim that $\bigcap \mathcal{F} = \bigcap \mathcal{G}$ holds.
  Since every set in $\mathcal{G}$ contains some set in $\mathcal{F}$, we get the inclusion $\subseteq$. Conversely, if $C \in \mathcal{F}$, since C is a closed set and $\{N_b : b \in \mathsf{B}\}$ is a basis of clopen, then we can write $C = \bigcap\{N_b : b \in A\}$ for some[4] $A \subseteq \mathsf{B}$. Thus $C \supseteq \bigcap \mathcal{G}$ (since $C = \bigcap\{N_b : b \in A\}$, every N_b with $b \in A$ contains C, hence belongs to $\mathcal{G}$ by definition of $\mathcal{G}$). Since this holds for all $C \in \mathcal{F}$, we get the other inclusion.
- Second claim: $\mathcal{G}$ has the finite intersection property. In fact, let $N_{b_1}, \dots, N_{b_k}$ be in $\mathcal{G}$, and let $C_1, \dots, C_k \in \mathcal{F}$ such that $C_i \subseteq N_{b_i}$. Then
  $$\emptyset \neq \bigcap_{i=1,\dots,k} C_i \subseteq \bigcap_{i=1,\dots,k} N_{b_i}.$$

Now we can conclude the proof: let $H = \{b \in \mathsf{B} : N_b \in \mathcal{G}\}$, since $\mathcal{G}$ has the finite intersection property,

$$N_{b_1 \wedge \dots \wedge b_k} = N_{b_1} \cap \dots \cap N_{b_k} \neq \emptyset \text{ for every } b_1, \dots, b_k \in H,$$

so

$$\forall b_1, \dots, b_k \in H (b_1 \wedge \dots \wedge b_k > 0_{\mathsf{B}}).$$

Thus H is a prefilter, so - by the prime ideal theorem - there exists an ultrafilter G on B such that $G \supseteq H$. $G \in \bigcap \mathcal{G}$ because $\forall b \in H (b \in G)$, so $G \in N_b$ for all $b \in H$, hence $G \in N_b$ for all $N_b \in \mathcal{G}$. We conclude that

$$\emptyset \neq \bigcap \mathcal{G} = \bigcap \mathcal{F}.$$

□

A clopen set is of the form N_b for some $b \in \mathsf{B}$: Let U be clopen in $\mathrm{St}(\mathsf{B})$. Then $U = \bigcup\{N_b : N_b \subseteq U\}$, since it is open. Also U is a closed subset of a compact space, hence U is compact and any of its open covering admits a finite subcovering. Therefore there are $b_1, \dots, b_k$ such that $U = N_{b_1} \cup \dots \cup N_{b_k} = N_{b_1 \vee \dots \vee b_k}$.

2. B *is isomorphic to the clopen subset of* $\mathrm{St}(\mathsf{B})$*:*

   ϕ is an homomorphism: Observe that $N_{b \wedge c} = N_b \cap N_c$, $N_{b \vee c} = N_b \cup N_c$, $N_{\neg b} = \mathrm{St}(\mathsf{B}) \setminus N_b$.

[4] If C is closed, $C = \mathrm{St}(\mathsf{B}) \setminus U$ for some U open. By definition $U = \bigcup\{N_{b_j} : j \in J\}$ for some family J, since $\{N_b : b \in \mathsf{B}\}$ is a basis for τ_{B}; hence

$$C = \mathrm{St}(\mathsf{B}) \setminus \bigcup\{N_{b_j} : j \in J\} = \bigcap\{\mathrm{St}(\mathsf{B}) \setminus N_{b_j} : j \in J\} = \bigcap\{N_{\neg b_j} : j \in J\}.$$

ϕ is injective: Assume $b \neq c$. Then either $b \wedge \neg c > 0_{\mathsf{B}}$ or $c \wedge \neg b > 0_{\mathsf{B}}$, assuming the first option, an ultrafilter G extending $\{b \wedge \neg c\}$ is in $N_b \setminus N_c$, assuming the second option holds we can find $G \in N_c \setminus N_b$.

ϕ is surjective: Immediate since any clopen set is of the form $N_b = \phi(b)$ for some $b \in \mathsf{B}$.

3. *Correspondence between closed (open) subsets of* $\mathrm{St}(\mathsf{B})$ *with filters (ideals) on* B*:* Useful exercise for the reader, in essence it has already been proved when we established the compactness of $\mathrm{St}(\mathsf{B})$.

4. $G \in \mathrm{St}(\mathsf{B})$ *is an isolated point (i.e. such that* $\{G\}$ *is open) of* $\mathrm{St}(\mathsf{B})$ *if and only if* G *is a principal ultrafilter:*

   G is isolated if and only if $\{G\}$ is clopen, hence $\{G\} = N_a$ for some $a \in \mathsf{B}$.

   a must be an atom: Otherwise there is $0_{\mathsf{B}} < b < a$. Let $c = \neg b \wedge a$; then $b \wedge c = 0_{\mathsf{B}}$ and $0_{\mathsf{B}} < c < a$. Find G_0 and G_1 in $\mathrm{St}(\mathsf{B})$ with $c \in G_0$ and $b \in G_1$. Then $a \in G_0, G_1$, but $G_0 \neq G_1$, since $b \in G_1$, $\neg b \in G_0$; we reached a contradiction with $\{G\} = N_a$.

We can also go the other way round, i.e. we take a topological space and attach to it a boolean algebra.

Proposition 2.8.3. *Let* (X, τ) *be a* 0*-dimensional compact topological space. Then* (X, τ) *is homeomorphic to the Stone space of* $\mathsf{CLOP}(X, \tau)$ *via the map*

$$\begin{aligned} \pi : X &\longrightarrow \mathrm{St}(\mathsf{CLOP}(X, \tau)) \\ x &\longmapsto G_x = \{U \in \mathsf{CLOP}(X, \tau) : x \in U\} \in \mathrm{St}(\mathsf{CLOP}(X, \tau)). \end{aligned}$$

Proof. We show that π is a well defined continuous bijection which is also open (i.e. maps open sets in open sets), this suffices to prove the Proposition.

Well defined and injective: the fact that G_x is an ultrafilter is an easy exercise. When x and y are distinct, since τ is Hausdorff and 0-dimensional, there is a clopen set U containing x and not y. Then $U \in G_x$ and $U \notin G_y$.

Surjective: Let $G \in \mathrm{St}(\mathsf{CLOP}(X, \tau))$, set $C = \bigcap G$. We claim that C is a singleton. C is non-empty since X is compact and G is a family of closed sets with the finite intersection property, thus it must have a non-empty intersection. Now assume $x \neq y \in C$. Find as in previous item U clopen such that $x \in U$ and $y \notin U$. Then $U \in G$ iff $(X \setminus U) \notin G$, which gives that $x \in C$ iff $y \notin C$, a contradiction. Let x be the unique element of C. Then it is easily checked that $G = G_x$.

Continuous and open: Notice that for any clopen set U and $x \in X$ $x \in U$ iff $U \in G_x$, thus $\pi[U] = N_U$, from this we easily infer that π is continuous and open. ☐

In particular we have shown that the map $\mathsf{B} \mapsto \mathrm{St}(\mathsf{B})$ defines a natural bijection between the class of boolean algebras and the class of compact 0-dimensional, Hausdorff spaces. It can be shown that this map is a contravariant functor between these two categories which identifies homomorphisms $i : \mathsf{B} \to \mathsf{C}$ with continuos maps $f : \mathrm{St}(\mathsf{C}) \to \mathrm{St}(\mathsf{B})$. But we won't pursue this direction further here.

We give two other different presentations of the notion of boolean algebra, one axiomatizable in the first order language for rings and another in the first order language for partial orders.

2.9 Boolean rings

Throughout this section we assume the reader is familiar with the notion of commutative ring, of an ideal on it, and of their basic properties.

We want to show that boolean algebras can also be described as commutative rings with idempotent multiplication.

Definition 2.9.1. Let $\mathsf{R} = \langle R, +, \cdot, 0, 1\rangle$ be a commutative ring. R is boolean if it has idempotent multiplication (i.e. $a^2 = a$ for all $a \in \mathsf{R}$).

Remark 2.9.2. A commutative ring R with idempotent multiplication has automatically characteristic 2: $(a + a)^2 = a + a$ for all $a \in \mathsf{R}$, hence

$$0 = (a + a)^2 - (a + a) = a^2 + a^2 + 2a - 2a = a^2 + a^2 = a + a$$

for all $a \in \mathsf{R}$. Hence in boolean rings $a = -a$ and the sum operation and the difference operation coincide.

Theorem 2.9.3. *Let*

$$(\mathsf{B}, \wedge, \vee, \neg, 0, 1)$$

be a boolean algebra. Then

$$(\mathsf{B}, \Delta, \wedge, 0, 1)$$

is a boolean ring.

Given a boolean ring

$$\mathsf{R} = \langle \mathsf{R}, +, \cdot, 0, 1\rangle ,$$

define

- $a \vee b = a + b + a \cdot b$,
- $a \wedge b = a \cdot b$,
- $\neg a = 1 + a$

for all $a, b \in \mathsf{R}$. Then

$$\mathsf{R} = \langle \mathsf{R}, \vee, \wedge, \neg, 0, 1\rangle$$

is a boolean algebra.

We split each of the two statement of the theorem in separate Lemmas. We first show that interpreting in a boolean algebra the boolean operation of symmetric difference Δ as a sum and that of meet $\wedge$ as a multiplication, any boolean algebra is naturally identified with the boolean ring of characteristic functions of clopen sets of $\mathrm{St}(\mathsf{B})$.

Exercise 2.9.4. Let (X, τ) be a topological space.

$$C(X, 2) = \{f : X \to \mathbb{Z}_2 \mid f \text{ is continuous}\},$$

where $\mathbb{Z}_2 = \{0, 1\}$ is the two elements ring endowed with discrete topology.

Show that $C(X, 2)$ is a boolean ring when endowed with operations defined pointwise (i.e. $f * g(H) = f(H) * g(H)$, for $*$ among $+, \cdot$), and the constant functions $c_0 : H \mapsto 0$, $c_1 : H \mapsto 1$ as 0 and 1 of the ring.

Lemma 2.9.5. *Let* $(\mathsf{B}, \wedge, \vee, \neg, 0, 1)$ *be a boolean algebra. Then* $(\mathsf{B}, \Delta, \wedge, 0, 1)$ *is a boolean ring isomorphic to the ring* $C(\mathrm{St}(\mathsf{B}), 2)$*.*

Proof. We leave to the reader to check that $C(\mathrm{St}(\mathsf{B}), 2)$ is a boolean ring. Let $\theta : \mathsf{B} \to C(\mathrm{St}(\mathsf{B}), 2)$ be defined by $b \mapsto f_b$, with $f_b(G) = 1$ iff $b \in G$.

f_b is continuous for any b since $f_b^{-1}(\{1\}) = N_b$ and $f_b^{-1}(\{0\}) = N_{\neg b}$. It is also easy to check that θ is injective. Let us now check that θ is surjective. Let $g : \mathrm{St}(\mathsf{B}) \to \{0, 1\}$ be continuous. Let $A_i = g^{-1}(\{i\})$, $i = 0, 1$. Each A_i is clopen, by continuity of g. Clearly $A_0 \cup A_1 = \mathrm{St}(\mathsf{B})$ and $A_0 \cap A_1 = \emptyset$. Hence $A_0 = N_b$ and $N_1 = N_{\neg b}$ for some $b \in \mathsf{B}$. Then $\theta(b) = g$.

We now check that:

$$f_a + f_b = f_{a\Delta b}; \quad f_a \cdot f_b = f_{a\wedge b},$$

so that the proof is completed: given an ultrafilter $G \subseteq \mathsf{B}$, we have:

$$\begin{aligned} f_a(G) + f_b(G) = 1 &\iff (f_a(G) = 0 \wedge f_b(G) = 1) \vee (f_a(G) = 1 \wedge f_b(G) = 0) \iff \\ &\iff (a \in G \ \wedge \ b \notin G) \vee (a \notin G \ \wedge \ b \in G) \iff \\ &\iff a\Delta b \in G, \end{aligned}$$

and

$$f_a(G) \cdot f_b(G) = 1 \iff (f_a(G) = 1 \wedge f_b(G) = 1) \iff (a \in G \wedge b \in G) \iff a \wedge b \in G.$$

□

Remark 2.9.6. The above Lemma greatly simplifies the proofs of certain properties of boolean operations and of boolean morphisms: for example try to prove the associativity of the symmetric difference operation Δ on a boolean algebra B using the equational presentation of boolean algebras given in Def 2.1.1, and compare your attempts, with the argument that Δ is associative being (modulo the above isomorphism) the sum operation of a commutative ring.

Proposition 2.9.7. *For a boolean ring* B *the following holds:*

1. *I is an ideal on the boolean algebra $\langle \mathsf{B}, \wedge, \vee, \neg, 0_{\mathsf{B}}, 1_{\mathsf{B}} \rangle$ if and only if I is an ideal on the ring $\langle \mathsf{B}, +, \cdot, 0, 1 \rangle$.*

2. *B does not have 0-divisors if and only if it is isomorphic to $\mathbb{Z}_2$.*

3. *The dual of a subset A of B is unambiguosly defined as $\breve{A} = \{\neg a : a \in A\} = \{1 - a : a \in A\}$. Moreover I is a (prime) ideal on the ring B if and only if $\breve{I}$ is an ultrafilter on the boolean algebra B.*

Proof.

1. Assume I is an ideal on the ring $\langle \mathsf{B}, +, \cdot, 0, 1 \rangle$, we show that I is an ideal on the boolean algebra $\langle \mathsf{B}, \wedge, \vee, \neg, 0_{\mathsf{B}}, 1_{\mathsf{B}} \rangle$: assume $a, b \in I$, then $a + b, a \cdot b \in I$ as well, hence $a \vee b = a + b + a \cdot b \in I$; moreover $a \in I$ and $b \leq a$ entails that $b = b \cdot a \in I$.

   Conversely assume I is an ideal on the boolean algebra $\langle \mathsf{B}, \wedge, \vee, \neg, 0_{\mathsf{B}}, 1_{\mathsf{B}} \rangle$, we show that I is an ideal on the ring B: if $a, b \in I$, $a+b = a \vee b \wedge \neg(a \wedge b) \leq a \vee b \in I$, moreover if $a \in I$ and $b \in \mathsf{B}$ $a \cdot b = a \wedge b \leq a \in I$, hence $a \cdot b \in I$ as well.

2. $x^2 = x$ entails that the equation $x(x-1) = 0$ holds for all x in a boolean ring. If x belongs to a boolean ring with no zero-divisors, either $x = 0$ or $x - 1 = 0$. Hence the boolean ring is $\{0, 1\} = \mathbb{Z}_2$.

3. The first observation is trivial, given that sum and subtraction are the same operation on a boolean ring and $\neg a = 1 - a$ by definition.

   The second observation follows by the fact that the unique boolean ring without zero divisors is $\mathbb{Z}_2$. Now recall that I is a prime ideal on a ring B if and only if $\mathsf{B}/_I$ has no zero-divisors, and I is a maximal ideal on a ring B if and only if $\mathsf{B}/_I$ is a field. By the previous item, the quotient of the boolean ring B by a prime ideal is $\mathbb{Z}_2$ which is a field. This entails that all prime ideals of B are maximal, i.e. their dual is an ultrafilter.

$\square$

The next proposition shows that ideals on boolean algebras and kernel of boolean morphisms are the same, its proof takes advantage of the characterization of boolean algebras as boolean rings.

Proposition 2.9.8. *Let B, C be boolean algebras, $\phi : \mathsf{B} \to \mathsf{C}$ be a boolean morphism, $I \subseteq \mathsf{B}$ an ideal. Then:*

- $\ker \phi = \{b \in \mathsf{B} : \phi(b) = 0_{\mathsf{C}}\}$ *is an ideal on B,*

- *the map $\pi_I : b \mapsto [b]_I = \{c \in \mathsf{B} : c \Delta b \in I\}$ defines a surjective morphism of B onto the quotient boolean algebra $\mathsf{B}/_I$ (with operations in $\mathsf{B}/_I$ defined by $[b]_I \wedge [c]_I = [b \wedge c]_I$, $[b]_I \vee [c]_I = [b \vee c]_I$, $\neg [b]_I = [\neg b]_I$;*

- *the map $\phi/_{\ker(\phi)} : [b]_{\ker \phi} \mapsto \phi(b)$ is a well defined injective morphism of $\mathsf{B}/_{\ker(\phi)}$ into C;*

- $\phi = (\phi/_{\ker(\phi)}) \circ \pi_{\ker(\phi)}$.

Proof. By Lemma 2.9.5 and exercises 2.4.3, 2.4.6 it suffices to prove the Proposition for the usual notions of ideal and morphism on rings, since:

- Lemma 2.9.5 show that boolean algebras are boolean rings.
- Exercise 2.4.6 gives that I is an ideal on a boolean algebra $(\mathsf{B}, \vee_{\mathsf{B}}, \wedge_{\mathsf{B}}, \neg_{\mathsf{B}}, 0_{\mathsf{B}}, 1_{\mathsf{B}})$ if and only if it is an ideal on the boolean ring $(\mathsf{B}, \Delta_{\mathsf{B}}, \wedge_{\mathsf{B}}, 0, 1)$.
- Exercise 2.4.3 shows that a boolean morphism is also a ring morphism.

The proposition for ring morphisms and ring ideals is a standard result in ring theory. □

We now prove the converse:

Lemma 2.9.9. *Assume* $\mathsf{R} = \langle \mathsf{R}, +, \cdot, 0, 1\rangle$ *is a boolean ring. Let* $a \vee b = a + b + a \cdot b$, $a \wedge b = a \cdot b$, $\neg a = 1 + a$ *for all* $a, b \in \mathsf{R}$. *Then* $\mathsf{R} = \langle \mathsf{R}, \vee, \wedge, \neg, 0, 1\rangle$ *is a boolean algebra.*

Proof. Let us go through the equations of 2.1.1 with $\cdot$ in the place of $\wedge$. The associativity and commutativity law for $\cdot$, and the identity laws for $0, 1$ are ring axioms. The commutativity law for $\vee$ is trivially checked. We are left to check the associativity law for $\vee$, and the laws of complementation and distributivity:

The associativity law for $\vee$ holds since

$$a \vee (b \vee c) = a \vee (b + c + b \cdot c) = a + b + c + b \cdot c + a \cdot b + a \cdot c + a \cdot b \cdot c$$

while

$$(a \vee b) \vee c = c \vee (b + a + b \cdot a) = c + b + a + b \cdot a + c \cdot b + c \cdot a + c \cdot b \cdot a.$$

The complementation laws are also immediate to check:

$$a \cdot (1 - a) = a - a^2 = a - a = 0$$

while

$$a \vee (1 - a) = a + 1 - a + a \cdot (1 - a) = 1 + 0 = 1.$$

Now

$$\begin{aligned} (a \vee b) \cdot c = (a + b + a \cdot b) \cdot c = \\ a \cdot c + b \cdot c + a \cdot b \cdot c = a \cdot c + b \cdot c + a \cdot b \cdot c^2 = \\ a \cdot c + b \cdot c + (a \cdot c) \cdot (b \cdot c) = (a \cdot c) \vee (b \cdot c) \end{aligned}$$

for all a, b, c, hence the first distributivity law $(a \vee b) \cdot c = (a \cdot c) \vee (b \cdot c)$ holds.

Also the second distributivity law $(a \cdot b) \vee c = (a \vee c) \cdot (b \vee c)$ holds:

$$(a \cdot b) \vee c = a \cdot b + c + a \cdot b \cdot c,$$

while

$$\begin{aligned}(a \vee c) \cdot (b \vee c) &= \\ &= (a + c + a \cdot c) \cdot (b + c + b \cdot c) = \\ &= a \cdot b + a \cdot c + a \cdot b \cdot c + b \cdot c + c^2 + b \cdot c^2 + a \cdot b \cdot c + a \cdot c^2 + a \cdot b \cdot c^2 = \\ &= a \cdot b + a \cdot c + a \cdot b \cdot c + b \cdot c + c + b \cdot c + a \cdot b \cdot c + a \cdot c + a \cdot b \cdot c = \\ &= a \cdot b + a \cdot b \cdot c + c\end{aligned}$$

for all a, b, c. □

2.10 Boolean algebras as complemented distributive lattices

We also give another characterization of boolean algebras in term of their order relation. These axioms for boolean algebras can be expressed in a first order theory with a binary relation symbol for the order relation. Nonetheless we expand the language adding symbols for the operations $\wedge, \vee, \neg$ and constants $0, 1$ definable in this axiom system for boolean algebras and leave to the reader to check that this is not necessary.

A *join-semilattice* $(P, \leq)$ is a partial order such that every pair of elements (x, y) of P admits an unique least upper bound denoted by $x \vee y$, the *join* of x and y.

Dually, a partial order $(P, \leq)$ is a *meet-semilattice* when any two elements x and y in P have an unique greatest lower bound denoted by $x \wedge y$, the *meet* of x and y.

A partial order $(P, \leq)$ is a *lattice* if it is both a join-semilattice and a meet-semilattice.

A lattice $(P, \leq)$ is *bounded* if it has a greatest element 1_P and a least element 0_P which satisfy $0 \leq x \leq 1$ for every x in P.

A lattice $(P, \leq)$ is *distributive* if for all x, y and z in P we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Let $(P, \leq)$ be a bounded lattice. A *complement* of an element $a \in P$ is an element $b \in P$ such that $a \vee b = 1$ and $a \wedge b = 0$.

Remark 2.10.1. In a distributive lattice, if a has a complement it is unique. In this case we denote by $\neg a$ the complement of a.

A lattice is *complemented* if it is bounded and every element has a complement.

A lattice $(P, \leq)$ is *complete* if every subset $X = \{x_i : i \in I\}$ of P has a meet (or infimum) $\bigwedge_{i \in I} x_i$ and a join (or supremum) $\bigvee_{i \in I} x_i$.

Notice that if $X = \emptyset$, then $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$, so a complete lattice is always bounded.

Lemma 2.10.2. $(\mathsf{B}, \wedge, \vee, \neg, 0, 1)$ *is a boolean algebra if and only if* $(\mathsf{B}, \leq)$ *is a complemented distributive lattice.*

Proof. One direction is clear: say that a boolean algebra B is a field of sets if it is a subalgebra of

$$\langle \mathcal{P}(X), \cap, \cup, A \mapsto X \setminus A, \emptyset, X \rangle$$

for some set X. It is an easy exercise to check that boolean algebras which are fields of sets are complemented distributive lattices. By Stone's duality theorem any boolean algebra is isomorphic to a field of sets. The converse direction is left to the reader. – *M* □

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2.11 Suprema and infima of subsets of a boolean algebra

Notation 2.11.1. Given a boolean algebra B, we denote by $\bigvee A$ the supremum (least upper bound) under $\leq$ of a subset A of B (i.e. the least element $a \in \mathsf{B}$ such that $a \geq b$ for all $b \in A$ - if this least element exists), and by $\bigwedge A$ its infimum (i.e. the largest element $a \in \mathsf{B}$ such that $a \leq b$ for all $b \in A$ - if this largest element exists). Similarly $\bigwedge A$ denotes the infimum of some $A \subseteq \mathsf{B}$.

The following proposition gives a simple topological method to compute the supremum of a subset of a boolean algebra:

Proposition 2.11.2. *Let B be a boolean algebra and $X \subseteq \mathsf{B}$. Then $a = \bigvee X$ if and only if $\bigcup \{N_b : b \in X\}$ is a dense open subset of N_a in the relative topology of N_a as a subset of $(\mathrm{St}(\mathsf{B}), \tau_{\mathsf{B}})$.*

Proof. Assume $A = \bigcup \{N_b : b \in X\}$ is a dense open subset of N_a but $a \neq \bigvee X$, we will reach a contradiction.

The first assumption on A gives that $N_b \subseteq N_a$ for all $b \in X$, which occurs if and only if $a \geq b$ for all $b \in X$, i.e. a is an upper bound of X. Since $a \neq \bigvee X$, there must be some e which is still an upper bound for X with $e \not\geq a$. Now if e is an upper bound for X, then so is $c = e \wedge a$. Since $e \not\geq a$ and $e \wedge a \leq a$, we conclude that $c = e \wedge a < a$ is an upper bound for X. Hence if a is not the least upper bound for X, there must some $0 < c < a$ which is still an upper bound for X. This gives that $a > d = a \wedge \neg c > 0$, and also that for all $b \in X$ $N_d \cap N_b \subseteq N_d \cap N_c = \emptyset$. We get that $A \cap N_d = \emptyset$. But N_d is an open non-empty subset of N_a, hence A is not an open dense subset of N_a, the desired contradiction.

Conversely assume $A = \bigcup \{N_b : b \in X\}$ is not a dense open subset of N_a, we must argue that $a \neq \bigvee X$. If $a \not\geq b$ for some $b \in X$, certainly $a \neq \bigvee X$, therefore we can assume $N_a \supseteq N_b$ for all $b \in X$. Since A is not a dense open subset of N_a, we can find $0_{\mathsf{B}} < d \leq a$ such that $N_d \cap A = \emptyset$. We conclude that $c = a \wedge \neg d$ is such that $N_a \supset N_c \supseteq N_b$ for all $b \in X$, i.e. c witnesses that a is not the least upper bound of X. □

Corollary 2.11.3. *Let B be a boolean algebra, the following holds for any X subset of B:*

1. $r \wedge \bigvee X = \bigvee \{r \wedge b : b \in X\}$ *if* $\bigvee X$ *is well defined.*

2. $\neg \bigvee X = \bigwedge \{\neg b : b \in X\}$ *if any among* $\bigvee X$ *or* $\bigwedge \{\neg b : b \in X\}$ *is well defined.*

Proof.

1. Let $a = \bigvee X$. By the previous proposition we get that $A = \bigcup \{N_b : b \in X\}$ is a dense open subset of N_a. Therefore $N_r \cap A$ is a dense open subset of $N_r \cap N_a = N_{r \wedge a}$. Now
$$N_r \cap A = N_r \cap \bigcup \{N_b : b \in X\} = \bigcup \{N_b \cap N_r : b \in X\} = \bigcup \{N_{b \wedge r} : b \in X\}.$$
By the previous proposition we conclude that $a \wedge r = \bigvee \{r \wedge b : b \in X\}$.

2. Remark that $a \leq b$ if and only if $\neg a \geq \neg b$. Now $a = \bigvee X$ if and only if $\bigcup \{N_b : b \in X\}$ is a dense open subset of N_a. Therefore $N_{\neg b} \supseteq N_{\neg a}$ for all $b \in X$. Now assume there is some $c > \neg a$ such that $\neg b \geq c$ for all $b \in X$. Then $\neg c < a$ and $b \leq \neg c$ for all $b \in X$, contradicting $a = \bigvee X$. We leave to the reader to handle the case in which we assume $\bigwedge \{\neg b : b \in X\}$ is well defined.

□

Remark 2.11.4. Take the boolean algebra B of clopen subsets of the Cantor space C identified as the subset of $[0; 1]$ given by
$$C = \left\{ a_f = \sum_{i=0}^{\infty} \frac{2 \cdot f(i)}{3^{i+1}} \, : \, f \in 2^{\mathbb{N}} \right\},$$
endowed with the euclidean topology. Let $a_n = \frac{3^{n+1}-3}{3^{n+1}}$, $b_n = a_n + \frac{1}{3^{n+1}}$, $U_n = [a_n; b_n] \cap C$. We get that
$$X = \{U_n \, : \, n \in \mathbb{N}\} \subseteq \mathsf{B}$$
is such that $C = \bigvee_{\mathsf{B}} X$, but $\cup X = C \setminus \{1\}$ is a proper dense open subset of C in the euclidean topology.

Moreover if we let $X_0 = \{U_{2n} : n \in \mathbb{N}\}$ we get that $\bigcup X_0$ is open, but also that no clopen subset of C contains X_0 as a dense subset: Let $A \supseteq X_0$ be a closed subset of C. Then it must contain 1, which is an accumulation point of X_0. If $A \subseteq C$ is also open, then $A \supseteq C \cap [\frac{3^n-1}{3^n}; 1]$ for some large enough n, since $\left\{C \cap [\frac{3^n-1}{3^n}; 1] : n \in \mathbb{N}\right\}$ is a base of clopen neighborhood of 1 in C. This gives that $U_{2k+1} \subseteq A$ for some large enough k. But U_{2k+1} is disjoint from X_0, hence X_0 is not a dense open subset of any clopen A containing it.

In particular X_0 has no supremum in B.

Notation 2.11.5. Given a boolean algebra B, we often consider B^+ when referring to B as an order, otherwise some definitions could indeed become trivial.

For $X \subseteq B$, $\downarrow X = \{b : \exists c \in X \, b \leq_{\mathsf{B}} c\}$, and $\uparrow X = \{b : \exists c \in X \, b \geq_{\mathsf{B}} c\}$.

For $b \in \mathsf{B}^+$, the boolean algebra $\mathsf{B} \upharpoonright b$ is given by $\{a \in \mathsf{B} : a \leq_{\mathsf{B}} b\} = \downarrow \{b\}$, with the operations inherited from B. The top element of $\mathsf{B} \upharpoonright b$ is b.

$X \subseteq \mathsf{B}$ is predense if $\downarrow X$ is dense in B^+ with respect to $\leq_{\mathsf{B}}$.

$X \subseteq \mathsf{B}$ is predense below $b \in \mathsf{B}$ if $\downarrow X$ is dense in $(\mathsf{B} \upharpoonright b)^+$.

We will need the following property of the Stone spaces of boolean algebras:

Fact 2.11.6. *Assume* B *is a boolean algebra and* $X \subseteq \mathsf{B}$*. The following holds:*

1. $\bigvee X = \bigvee \downarrow X$ *whenever one of the two members is well defined.*

2. *For all* $r \in \mathsf{B}$ $r \wedge \bigvee X > 0_{\mathsf{B}}$ *if and only if* $r \wedge b > 0_{\mathsf{B}}$ *for some* $b \in X$ *whenever* $\bigvee X$ *is well defined.*

3. $\bigvee X = 1_{\mathsf{B}}$ *iff* $X \cap \mathsf{B}^+$ *is a predense subset of* B^+ *in the sense of the order.*

4. *More generally for any dense set* $D \subseteq \mathsf{B}^+$ *and any* $a \in \mathsf{B}^+$ $a = \bigvee \{q \in D : q \leq_{\mathsf{B}} a\}$.

Proof.

1. $\bigvee X \leq_{\mathsf{B}} \bigvee \downarrow X$ since $X \subseteq \downarrow X$. For the converse inequality, if $d \geq b$ for all $b \in X$ we also have that $d \geq c$ for all $c \in \downarrow X$, hence $\bigvee X$ is an upper bound for $\downarrow X$, hence $\bigvee \downarrow X \leq_{\mathsf{B}} \bigvee X$.

2. Left to the reader: use Corollary 2.11.3.

3. Left to the reader: use Corollary 2.11.3.

4. Left to the reader: use Corollary 2.11.3.

□

Chapter 3

Complete boolean algebras

Definition 3.0.1. A boolean algebra $(\mathsf{B}, 0, 1, \vee, \wedge, \neg, \leq)$ is complete (or *cba* for short) if it admits suprema and infima with respect to all of its subsets for the order relation $\leq$.

Complete boolean algebras can be split in two pieces:

Lemma 3.0.2. *A complete boolean algebra B can be split in the disjoint sum of an atomic boolean algebra and of an atomless boolean algebra. I.e. there is $c \in \mathsf{B}$ such that $\mathsf{B} \restriction \neg c$ is atomless, and $B \restriction c$ is atomic.*

Proof. Let $A = \{a \in \mathsf{B} : a \text{ is an atom of } \mathsf{B}\}$ and $c = \bigvee A$. Then $b \wedge c = 0_\mathsf{B}$ entails that b is not an atom of $\mathsf{B} \restriction \neg c$ (otherwise b would also be an atom of B and thus be a refinement of c), while $b \leq c$ entails that for some atom $a \in A$, $a \wedge b > 0_\mathsf{B}$ which occurs only if $a \leq b$, since a is an atom. This gives that $\mathsf{B} \restriction c$ is atomic and $\mathsf{B} \restriction \neg c$ is atomless. □

Definition 3.0.3. Let B, C be boolean algebras A map $k : \mathsf{B} \to \mathsf{C}$ is a *complete homomorphism* if it maps predense subsets of B^+ to predense subsets of C^+, or equivalently if it preserves suprema and infima.

Exercise 3.0.4. Any complete homomorphism is also a homomorphism in the usual sense.

Fact 3.0.5. *An isomorphism of boolean algebras preserves suprema and infima. Hence isomorphic images of complete boolean algebras are complete boolean algebras.*

Proof. Left to the reader. □

The following is less trivial and useful:

Lemma 3.0.6. *Let B, C be complete boolean algebras and $i : \mathsf{B} \to \mathsf{C}$ be an order preserving morphism which respects suprema. Then i is a complete homomorphism also for the boolean algebraic structure.*

Proof. Note that $i(a \vee b) = i(a) \vee i(b)$ by assumption. Now observe that $\neg a = \bigvee \{b \in \mathsf{B} : b \wedge a = 0_\mathsf{B}\}$. □

3.1 Complete boolean algebras of regular open sets

We prove that every complete boolean algebra can be represented as the family of regular open sets of some given topological space, and we characterize complete boolean algebras as those whose Stone spaces have the property that their regular open sets are clopen. The first step in this direction is to show that the regular open sets of a given topological space have a natural structure of complete boolean algebra.

Notation 3.1.1. Given a topological space (X, τ) and an arbitrary subset A of X, we denote by $\mathsf{Cl}(A)$ (the *closure* of A) the smallest closed set containing A. We denote by $\mathsf{Int}(A)$ (the *interior* of A) the biggest open set that is contained in A. An open set A is *regular open* if $A = \mathsf{Int}(\mathsf{Cl}(A))$. For any $A \subseteq X$ $\mathsf{Reg}(A) = \mathsf{Int}(\mathsf{Cl}(A))$ denotes the *regularization* of the set A. We denote by $\mathsf{RO}(X, \tau)$ the collection of regular open sets in X with respect to τ. If no confusion can arise we write $\mathsf{RO}(X)$ instead of $\mathsf{RO}(X, \tau)$.

Remark 3.1.2. Any clopen subset of a topological space is regular. Any open interval of $\mathbb{R}$ with the usual topology is regular, a standard example of an open non regular set in the euclidean topology on $\mathbb{R}$ is $(1;2) \cup (2;3)$: its closure is $[1;3]$ and the interior of its closure is $(1;3)$. We will see that if U and V are open regular then so is $U \cap V$. Moreover any isolated point $x \in X$ of a topological space X is such that $\{x\}$ is clopen and thus regular.

Example 3.1.3. Let τ be the euclidean topology on $\mathbb{R}$; then any interval is a regular open set.

If $a < b < c$, we have that $(a;b)$, $(b;c)$ are regular open while $(a;b) \cup (b;c)$ is not with its regularization being $(a;c)$.

In general regular open sets are those open sets which can be written in the form $\bigcup_{j \in J}(a_j; b_j)$ with the family $\{(a_j; b_j) : j \in J\}$ consisting of pairwise disjoint open intervals such that $a_i \neq b_j$ for any $i, j \in J$.

Definition 3.1.4. Given a topological space (X, τ), we equip $\mathsf{RO}(X)$ with the following operations:

$$\begin{aligned}
U \vee V =& \mathsf{Reg}(U \cup V), \\
U \wedge V =& U \cap V, \\
\bigvee_{i \in I} U_i =& \mathsf{Reg}(\bigcup_{i \in I} U_i), \\
\bigwedge_{i \in I} U_i =& \mathsf{Reg}(\bigcap_{i \in I} U_i), \\
\neg U =& X \setminus \mathsf{Cl}(U).
\end{aligned}$$

We prove the following:

Theorem 3.1.5. *Assume* (X, τ) *be a topological space. Then* $(\mathsf{RO}(X), \vee, \wedge, \neg, \emptyset, X)$ *is a complete boolean algebra.*

We will need several facts on regular open sets, the first of which is the following characterization:

Lemma 3.1.6. *Let (X, τ) be a topological space. For any open $A \in \tau$ we have:*

$$\mathsf{Reg}(A) = \{x \in X : \exists U \in \tau \text{ open set containing } x \text{ such that } A \cap U \text{ is dense in } U\}.$$

Proof. For one inclusion, take $x \in \mathsf{Reg}(A)$. The set $U = \mathsf{Reg}(A)$ is an open set contaning x, and $A \cap U$ is dense in U because A is dense in $\mathsf{Cl}(A)$ and $U \subseteq \mathsf{Cl}(A)$ is open.

For the converse inclusion, take $x \in X$ and U an open set containing x such that $A \cap U$ is dense in U, then $A \cap U$ is dense also in $\mathsf{Cl}(U)$, thus $\mathsf{Cl}(A \cap U) = \mathsf{Cl}(U)$ holds. So U is an open subset of $\mathsf{Cl}(A)$, and we obtain $x \in \mathsf{Reg}(A)$. □

Exercise 3.1.7. The above Lemma explains why (for the euclidean topology on $\mathbb{R}$) 2 belongs to $\mathsf{Reg}((1;2) \cup (2;3))$ while 1 and 3 do not. Work out the details of why 2 satisfies the above characterization for points of $\mathsf{Reg}((1;2) \cup (2;3))$, while 1 and 3 do not.

Remark 3.1.8. If U is an open neighborhood of x witnessing that $x \in \mathsf{Reg}(A)$ any $V \subseteq U$ open neighborhood of x is equally well a witness of $x \in \mathsf{Reg}(A)$, since $A \cap W = A \cap U \cap W$ is a dense open subset of W for any open $W \subseteq U$.

Exercise 3.1.9. Prove the above observation.

We also need the following crucial property:

Fact 3.1.10. *Given a topological space (X, τ), assume U, V are open sets in τ. Then $U \cap V$ is a dense open subset of V if and only if $\mathsf{Reg}(V) \subseteq \mathsf{Reg}(U)$. In particular $\mathsf{Reg}(V) = \mathsf{Reg}(U)$ iff $U \cap V \supseteq W$ for some W open dense subset of U and open dense subset of V.*

Proof. Assume $U \cap V$ is a dense open subset of V. Let $x \in \mathsf{Reg}(V)$. Let W be an open neighborhood of x such that $V \cap W$ is a dense subset of W. Then $U \cap V \cap W$ is also a dense open subset of W (if $P \subseteq W$ is open non-empty, $V \cap P$ is a non-empty open subset of V, since $V \cap W$ is a dense open subset of W; thus $U \cap V \cap P$ is also a non-empty open set, given that $U \cap V$ is dense in V), and so a fortiori also $U \cap W$ is a dense open subset of W. In particular W witnesses that $x \in \mathsf{Reg}(U)$.

Conversely assume $\mathsf{Reg}(V) \subseteq \mathsf{Reg}(U)$. Since U is a dense open subset of $\mathsf{Reg}(U)$ and V is a dense open subset of $\mathsf{Reg}(V)$, $U \cap V$ is a dense open subset of $\mathsf{Reg}(V)$, hence also of V (since the intersection of two open dense subsets of some topological space is still open dense). □

Fact 3.1.11. *Given a topological space (X, τ), assume U, V are open sets in τ. Then $U \cap V$ is non-empty if and only if so is $U \cap \mathsf{Reg}(V)$.*

Proof. One implication is trivial. For the right to left deirection assume $x \in U \cap \mathsf{Reg}(V)$. Find $W \in \tau$ with $x \in W$ and $W \cap V$ dense in W. Since $x \in U \cap W$, we get that $U \cap W$ is non-empty hence so is $U \cap V \cap W$ by density of $W \cap V$ in W and we are done. □

Notation 3.1.12. Given a topological space (X, τ) and $V \subseteq X$,

$$V^{\perp} = X \setminus \mathsf{Cl}(V).$$

For us priority is on the left, hence for example $U^{\perp\perp\perp}$ is a shorthand for $((U^{\perp})^{\perp})^{\perp}$.

Fact 3.1.13. *Given a topological space (X, τ), the following holds for any $U, V \subseteq X$:*

1. *for all $U, V \subseteq X$, $U \subseteq V$ implies $V^{\perp} \subseteq U^{\perp}$,*
2. *$(U \cup V)^{\perp} = U^{\perp} \cap V^{\perp}$,*
3. *$\mathsf{Reg}(V) = V^{\perp\perp}$,*
4. *$U^{\perp\perp\perp} = U^{\perp}$,*
5. *If U, V are open $(U \cap V)^{\perp\perp} = U^{\perp\perp} \cap V^{\perp\perp}$.*

In particular we also have:

(A) $\mathsf{Reg} : \mathcal{P}(X) \to \mathcal{P}(X)$ *is an idempotent operator, i.e.*

$$\mathsf{Reg}(\mathsf{Reg}(V)) = \mathsf{Reg}(V)$$

for any $V \subseteq X$.

(B) The intersection of any two regular open sets of (X, τ) is regular open.

Proof.

1: easy exercise.

2: We need the following basic topological fact:

For any topological space (X, τ) and $A, B \subseteq X$

$$\mathsf{Cl}(A \cup B) = \mathsf{Cl}(A) \cup \mathsf{Cl}(B).$$

Proof. Clearly $\mathsf{Cl}(A \cup B) \supseteq \mathsf{Cl}(A) \cup \mathsf{Cl}(B)$. For the converse inclusion observe that $A \cup B \subseteq \mathsf{Cl}(A) \cup \mathsf{Cl}(B)$ and $A \cup B$ is a dense subset of $\mathsf{Cl}(A \cup B)$. Hence

$$\mathsf{Cl}(A \cup B) = \mathsf{Cl}(\mathsf{Cl}(A) \cup \mathsf{Cl}(B)) = \mathsf{Cl}(A) \cup \mathsf{Cl}(B),$$

where:

- the first equality holds because

$$A \cup B \subseteq \mathsf{Cl}(A) \cup \mathsf{Cl}(B) \subseteq \mathsf{Cl}(A \cup B)$$

with $A \cup B$ a dense subset of $\mathsf{Cl}(A \cup B)$;
- the second equality holds because $\mathsf{Cl}(\mathsf{Cl}(Y)) = \mathsf{Cl}(Y)$ for all $Y \subseteq X$.

□

Hence

$$(U \cup V)^{\perp} = X \setminus \mathsf{Cl}(U \cup V) = X \setminus (\mathsf{Cl}(U) \cup \mathsf{Cl}(V)) = (X \setminus \mathsf{Cl}(U)) \cap (X \setminus \mathsf{Cl}(V)) = U^{\perp} \cap V^{\perp}.$$

3:

$$\begin{aligned} x \in \mathsf{Reg}(V) &\Leftrightarrow \text{there is an open neighborhood } N \text{ of } x \text{ fully contained in } \mathsf{Cl}(V) \\ &\Leftrightarrow \text{there is an open neighborhood } N \text{ of } x \text{ disjoint from } X \setminus \mathsf{Cl}(V) \\ &\Leftrightarrow x \notin \mathsf{Cl}(X \setminus \mathsf{Cl}(V)) \\ &\Leftrightarrow x \in V^{\perp\perp}. \end{aligned}$$

4: Assume U is open, then we have $U \subseteq \mathsf{Reg}(U)$. So, as $\mathsf{Reg}(U) = U^{\perp\perp}$ holds, we have

$$U \subseteq U^{\perp\perp} \tag{3.1}$$

Now, if U is open, applying the first point to (3.1) we get $U^{\perp\perp\perp} \subseteq U^{\perp}$. Conversely, applying (3.1) to $U^{\perp}$ we get $U^{\perp} \subseteq U^{\perp\perp\perp}$, which concludes the proof.

5: We use Lemma 3.1.6.

Assume first $x \in U^{\perp\perp} \cap V^{\perp\perp} = \mathsf{Reg}(U) \cap \mathsf{Reg}(V)$. Then there are N_0, N_1 open neighborhoods of x such that $U \cap N_0$ and $V \cap N_1$ are open dense subsets respectively of N_0 and N_1. Since $x \in N_0 \cap N_1$, we get that U, V have both a dense open intersection with $N_0 \cap N_1$. Hence $N_0 \cap N_1$ witnesses that $x \in \mathsf{Reg}(U \cap V) = U^{\perp\perp}$ as $U \cap V$ has a dense open intersection with it.

For the converse inclusion let $x \in (U \cap V)^{\perp\perp} = \mathsf{Reg}(U \cap V)$. Then there is N open neighborhood of x such that $U \cap V \cap N$ is dense in N, thus $U \cap N$ and $V \cap N$ are both dense subsets of N; this gives that $x \in U^{\perp\perp} \cap V^{\perp\perp}$, as was to be shown.

For the last two assertions:

(A). By 3 and 4

$$\mathsf{Reg}(\mathsf{Reg}(V)) = V^{\perp\perp\perp\perp} = V^{\perp\perp} = \mathsf{Reg}(V)$$

for any $V \subseteq X$.

(B). Combining (A) with 5

$$(U^{\perp\perp} \cap V^{\perp\perp})^{\perp\perp} = (U \cap V)^{\perp\perp\perp\perp} = (U \cap V)^{\perp\perp} = U^{\perp\perp} \cap V^{\perp\perp}.$$

□

We prove first that $\mathsf{RO}(X)$ is a boolean algebra using Definition 2.1.1, and then we prove that it is complete.

Proposition 3.1.14. *The family* $\mathsf{RO}(X)$*, with the operations defined above, is a boolean algebra.*

Proof. We take $U, V, W \in \mathsf{RO}(X)$ and we go through the equations of Definition 2.1.1.

- Associativity of $\vee$. By Fact 3.1.10 it is enough to check that $U \cup V \cup W$ is a dense open subset of $\mathsf{Reg}(U \cup \mathsf{Reg}(V \cup W))$ and of $\mathsf{Reg}(W \cup \mathsf{Reg}(V \cup U))$. This is immediate from the definitions. Alternatively we can use the following algebraic identities:

$$\begin{aligned} U \vee (V \vee W) &= (U \cup (V \cup W)^{\perp\perp})^{\perp\perp} \\ &= (U^{\perp} \cap (V \cup W)^{\perp\perp\perp})^{\perp} \\ &= (U^{\perp} \cap (V \cup W)^{\perp})^{\perp} \\ &= (U^{\perp} \cap (V^{\perp} \cap W^{\perp}))^{\perp} \\ &= ((U^{\perp} \cap V^{\perp}) \cap W^{\perp})^{\perp} \\ &= ((U \cup V)^{\perp} \cap W^{\perp})^{\perp} \\ &= ((U \cup V)^{\perp\perp\perp} \cap W^{\perp})^{\perp} \\ &= ((U \cup V)^{\perp\perp} \cup W)^{\perp\perp} \\ &= (U \vee V) \vee W. \end{aligned}$$

- The associativity of $\wedge$ is just the associativity of $\cap$.

- Distributivity. We only show that

$$(U \wedge V) \vee (U \wedge W) = U \wedge (V \vee W)$$

holds, the other equation is similar. To this aim observe that

$$\begin{aligned} &V \cup W \text{ is dense in } \mathsf{Reg}(V \cup W) \\ \Rightarrow\ &U \cap (V \cup W) \text{ is dense in } U \cap \mathsf{Reg}(V \cup W) \\ \Rightarrow\ &\mathsf{Reg}(U \cap (V \cup W)) = \mathsf{Reg}(U \cap \mathsf{Reg}(V \cup W)) = U \cap \mathsf{Reg}(V \cup W), \end{aligned}$$

where the last equality holds because the intersection of open regular sets is open regular. Hence $U \cap (V \cup W) = (U \cap V) \cup (U \cap W)$ is a dense open subset both of $U \cap \mathsf{Reg}(V \cup W) = U \wedge (V \vee W)$, as well as of $\mathsf{Reg}((U \cap V) \cup (U \cap W)) = (U \wedge V) \vee (U \wedge W)$. By Fact 3.1.10 we conclude.

- Commutativity. $U \wedge V = U \cap V = V \cap U = V \wedge U$ and $U \vee V = \mathsf{Reg}((U \cup V)) = \mathsf{Reg}((V \cup U)) = V \vee U$.

- Identity. $U \vee 0 = \mathsf{Reg}(U \cup \emptyset) = \mathsf{Reg}(U) = U$ and $U \wedge 1 = U \cap X = U$.

- Complements. $U \vee \neg U = \mathsf{Reg}(U \cup (X \setminus \mathsf{Cl}(U))) = X$ (since $U \cup (X \setminus \mathsf{Cl}(U))$ is a dense subset of X: for A open, $A \cap U$ is empty iff $A \cap \mathsf{Cl}(U)$ is empty, so either $A \cap U$ is non-empty or $A \subseteq U^{\perp}$); while $U \wedge \neg U = U \cap (X \setminus \mathsf{Cl}(U)) = \emptyset$. □

It now remains to prove that $\mathsf{RO}(X)$ is complete.

Proposition 3.1.15. *The algebra* $\mathsf{RO}(X)$ *is complete.*

Proof. Given a family $K = \{U_i : i \in I\}$ in $\mathsf{RO}(X)$ define $V = (\bigcup_{i\in I} U_i)^{\perp\perp}$. For any $i \in I$ we have $U_i \subseteq \bigcup_{j\in I} U_j$, so that

$$U_i = U_i^{\perp\perp} \subseteq (\bigcup_{j\in I} U_j)^{\perp\perp} = V$$

holds. This shows that V is an upper bound for the elements of K. If W is another such upper bound, then $U_i \subseteq W$, so that $\bigcup_{i\in I} U_i \subseteq W$, whence

$$V = (\bigcup_{i\in I} U_i)^{\perp\perp} \subseteq W^{\perp\perp} = W.$$

The proof for $\wedge$ is similar. $\square$

We have shown that for a given topology τ on X there are two natural boolean algebras we can attach to it: $\mathsf{CLOP}(X,\tau)$ and $\mathsf{RO}(X,\tau)$. Observe that $\mathsf{CLOP}(X,\tau)^+$ is always contained in $\mathsf{RO}(X,\tau)^+$ and that if τ is 0-dimensional, any open set contains a clopen set, thus $\mathsf{CLOP}(X,\tau)^+$ is a dense subset of $\mathsf{RO}(X,\tau)^+$.

The next lemma gives a necessary and sufficient condition so that $\mathsf{CLOP}(X,\tau)$ and $\mathsf{RO}(X,\tau)$ coincide.

Proposition 3.1.16. *Assume* B *is a boolean algebra.* B *is complete if and only if the regular open sets of* $\mathrm{St}(\mathsf{B})$ *overlap with the clopen subsets of* $\mathrm{St}(\mathsf{B})$*.*

Proof. Assume B is complete. Let A be an arbitrary open set, then:

$$A = \bigcup_{i\in I} N_{b_i}$$

for a given family $\{b_i : i \in I\} \subseteq \mathsf{B}$. Since B is complete, let:

$$b = \bigvee_{i\in I} b_i.$$

Then N_b is clopen, and thus regular open. We show that $N_b = \mathsf{Cl}(A)$.

First we observe that

$$A = \bigcup_{i\in I} N_{b_i} \subseteq N_b.$$

In particular since N_b is closed $\mathsf{Cl}(A) \subseteq N_b$.

To prove the converse inclusion we proceed as follows: first we observe that for all $c \in \mathsf{B}$

$$c \wedge b = 0 \text{ iff } c \wedge b_i = 0 \text{ for all } i \in I.$$

This gives that

$$N_c \cap N_b = \emptyset \text{ iff } N_c \cap A = \emptyset \text{ iff } N_c \cap \mathsf{Cl}(A) = \emptyset.$$

Thus

$$X \setminus \mathsf{Cl}(A) = \bigcup\{N_c : N_c \cap A = \emptyset\}$$

is disjoint from N_b. We can conclude that

$$N_b \subseteq \mathsf{Cl}(A).$$

The converse follows immediately, since $\mathsf{B} \cong \mathsf{CLOP}(\mathrm{St}(\mathsf{B})) = \mathsf{RO}(\mathrm{St}(\mathsf{B}))$, which is complete. $\square$

We say that a topological space (X, τ) is *extremally disconnected* (or *extremely disconnected*) if $\mathsf{CLOP}(X, \tau) = \mathsf{RO}(X, \tau)$.

3.2 Boolean completions

In this section we prove that every pre-order can be completed to a complete boolean algebra.

Notation 3.2.1. Let $(Q, \leq_Q)$ be a pre-order (i.e. $\leq_Q$ is a transitive and reflexive relation on Q).

For $X \subseteq Q$

$$\downarrow X = \{p \in Q : \exists a \in X (p \leq_Q a)\}.$$

is the downard closure of X ($\downarrow p$ stands for $\downarrow \{p\}$).

Exercise 3.2.2. Let $(Q, \leq_Q)$ be a pre-order. Show that:

- The family τ_Q of downward closed subsets of Q form a family of sets closed under arbitrary unions and arbitrary interesections.
- The family $\{\downarrow q : q \in Q\}$ is a base for the topological space (Q, τ_Q) with the property that for each $q \in Q$, $\downarrow q$ is its smallest open neighborhood in τ_Q.
- (Q, τ_Q) is T_0 if and only if $\leq_Q$ is an order (i.e. $\leq_Q$ is antysimmetric).

Recall that (X, τ) is T_0 if given points $x \neq y$ in X there is an open set which contains one but not the other.

Definition 3.2.3. Let $(Q, \leq_Q)$ be a pre-order. The order topology on Q is τ_Q.

Notation 3.2.4. Given a pre-order $(Q, \leq_Q)$, we denote by $\mathsf{RO}(Q)$ (or $\mathsf{RO}(Q, \tau_Q)$ in case confusion can arise) the algebra of regular open sets of the order topology on $(Q, \leq_Q)$.

Theorem 3.2.5. *Let $(Q, \leq_Q)$ be a pre-order. There exists an unique (up to isomorphism) cba B_Q and a map $j : Q \to \mathsf{B}_Q$ such that:*

1. *j preserves order and incompatibility (i.e. both $a \leq_Q b \Rightarrow j(a) \leq_{\mathsf{B}_Q} j(b)$ and $a \perp b \Leftrightarrow j(a) \wedge j(b) = 0_{\mathsf{B}_Q}$ hold).*
2. *$j[Q]$ is a dense subset of the partial order $(\mathsf{B}_Q^+, \leq)$.*

Note that while B_Q is unique, there can be many $j : Q \to \mathsf{B}_Q$ which satisfy the above requirements.

We split the proof in two lemmas, one for the existence part and the other for the uniqueness part.

Lemma 3.2.6. *Let $(Q, \leq_Q)$ be a pre-order. The map*

$$\begin{aligned} j_Q :& Q \to \mathsf{RO}(Q, \tau_Q) \\ & q \mapsto \mathsf{Reg}(\downarrow q) \end{aligned}$$

is such that:

1. *j_Q preserves order and incompatibility.*

2. *$j_Q[Q]$ is a dense subset of the partial order* $(\mathsf{RO}(Q, \tau_Q)^+, \subseteq)$.

Proof. By Lemma 3.1.6, we have that for all open sets $A \in \tau_Q$

$$\mathsf{Reg}(A) = \{p \in Q : \downarrow p \cap A \text{ is a dense subset of } \downarrow p\} \tag{3.2}$$

since $\downarrow p$ is the smallest open neighborhood of p (if the property given in Lemma 3.1.6 holds for some open neighborhood of p it holds as well for $\downarrow p$).

We will repeatedly use the above characterization of regular open sets.

j_Q **is order preserving:** if $p \leq_Q q$, then $\downarrow p = \downarrow q \cap \downarrow p$ and clearly $\downarrow p$ is dense in $\mathsf{Reg}(\downarrow p)$, so $j_Q(p) \leq j_Q(q)$ by Fact 3.1.10.

j_Q **is incompatibility preserving:** Note that p, q are compatible if and only if $\downarrow p, \downarrow q$ (which are open sets of τ_P) have non-empty intersection, if and only if

$$\mathsf{Reg}(\downarrow p) \cap \mathsf{Reg}(\downarrow q)$$

is non-empty (which is the case by Fact 3.1.11). Hence the thesis.

j_Q **has a dense image:** let $X \subseteq Q$ be non-empty. Let $p \in X$; clearly $\downarrow X \supseteq \downarrow p$, hence $j_Q(p) \leq \mathsf{Reg}(\downarrow X)$.

□

We are left to show the uniqueness of this boolean completion. It suffices to prove the following:

Lemma 3.2.7. *Assume* B *is a cba and* $k : Q \to \mathsf{B}$ *preserves order and incompatibility and is such that* $k[Q]$ *is dense in* B^+. *Then the map:*

$$\begin{aligned} \pi : \mathsf{RO}(Q) &\longrightarrow \mathsf{B} \\ A &\longmapsto \bigvee \{k(p) : p \in A\} \end{aligned}$$

is an isomorphism.

Assume the Lemma holds and $j_i : Q \to \mathsf{B}_i$ for $i < 2$ preserve order and incompatibility and are such that $j_i[Q]$ is dense in B_i^+ (with both B_i cbas), we can compose the isomorphisms given by the Lemma to get an isomorphism of B_1 onto B_2.

We prove the Lemma.

Proof. We prove the Lemma in several steps as follows:

π **is order preserving:** by definition.

$\pi \circ j_Q = k$**:** Let $q \in Q$. Then

$$\pi \circ j_Q(q) = \bigvee \{k(r) : r \in \mathsf{Reg}(\downarrow q)\} \geq k(q)$$

since $q \in \mathsf{Reg}(\downarrow q)$. Assume the inequality is strict. Then

$$\pi \circ j_Q(q) \wedge \neg k(q) > 0_{\mathsf{B}}.$$

Now $k[Q]$ is dense in B^+, thus we can find $r \in Q$ such that $k(r) \wedge k(q) = 0_{\mathsf{B}}$ and $k(r) \leq \pi \circ j_Q(q)$.

Since $k(r) \wedge k(q) = 0_{\mathsf{B}}$, r and q are orthogonal in Q.

On the other hand we also have that $k(r) \wedge k(s) > 0_{\mathsf{B}}$ for some $s \in \mathsf{Reg}(\downarrow q)$, since

$$k(r) \leq \pi \circ j_Q(q) = \bigvee \{k(s) : s \in \mathsf{Reg}(\downarrow q)\}.$$

This occurs only if $\downarrow r \cap \downarrow s \neq \emptyset$. Since $s \in \mathsf{Reg}(\downarrow q)$ we have that $\downarrow s \cap \downarrow q$ is a dense subset of $\downarrow s$. In particular $\downarrow s \cap \downarrow q \cap \downarrow r$ is non-empty. Thus there is $t \leq r, q$. This contradicts the orthogonality of q, r in Q.

π **is surjective:** let $b \in \mathsf{B}_2$, by the density of $k[Q]$ we have that (see Proposition 2.11.2 and note that $k[Q]$ is dense in B^+)

$$b = \bigvee \{k(p) \in Q : k(p) \leq b\}.$$

It is enough to show that $A = \{p \in Q : k(p) \leq b\}$ is regular open to get that $\pi(A) = b$. Clearly A is downward closed and thus open. Now assume $r \in \mathsf{Reg}(A) \setminus A$. Then $k(r) \not\leq b$. This gives that $k(r) \wedge \neg b > 0$ and thus that some s is such that $k(s) \leq k(r) \wedge \neg b$. Since $k(s)$ and $k(r)$ are compatible in B^+ we have that some $t \in Q$ refines r and s. In particular $t \leq r$ and $0_{\mathsf{B}} < k(t) \leq k(r) \wedge \neg b$. Since $r \in \mathsf{Reg}(A) \setminus A$, $A \cap \downarrow r$ is dense in $\downarrow r$; since $t \leq r$, we can find $t^* \in A$ such that $t^* \leq t$. In conclusion $t^* \in A$ is incompatible with all elements of A since $k(t^*) \leq k(t)$ is incompatible with b, a contradiction.

π **is injective:** if $A \neq B$ are regular open, we may assume w.l.o.g. that there is $q \in Q$ such that $j_Q(q) \subseteq A$ and $j_Q(q)$ is orthogonal to B, which occurs if and only if $\downarrow q \cap B = \emptyset$. The latter gives that q is orthogonal to all elements in B. Then $\pi(A) \geq \pi(j_Q(q))$ and $\pi(j_Q(q)) = k(q)$ is orthogonal to $\bigvee k[B] = \pi(B)$, since k is order and incompatibility preserving. We get that $\pi(A) \neq \pi(B)$.

By Lemma 3.0.6 the proof of is completed. □

The proof of Theorem 3.2.5 is completed.

Corollary 3.2.8. *$(Q, \leq)$ is a separative partial order if and only if the map $j : Q \to \mathsf{RO}(Q)$ of Theorem 3.2.5 is an injection.*

Proof. It is enough to show that Q is separative iff $\downarrow p = \mathsf{Reg}(\downarrow p)$ for all $p \in Q$.

($\Rightarrow$): Assume Q is separative, and towards a contradiction let $r \in \mathsf{Reg}(\downarrow p) \setminus \downarrow p$. Then we can refine r to an $s \perp p$ still in $\mathsf{Reg}(\downarrow p)$ since Q is separative and $r \not\leq p$. This is the desired contradiction.

($\Leftarrow$): Assume $\downarrow p = \mathsf{Reg}(\downarrow p)$. And let $p \not\leq q$. Assume towards a contradiction that for all $r \leq p$, r and q are compatible in Q, i.e. $\downarrow q \cap \downarrow r$ is non-empty. Then $\downarrow q \cap \downarrow p$ is dense in $\downarrow p$. Which (by Fact **??**) gives that

$$\downarrow p = \mathsf{Reg}(\downarrow p) \subseteq \mathsf{Reg}(\downarrow q) = \downarrow q,$$

i.e. $p \in \downarrow q$, contradicting our assumption that $p \not\leq q$.

$\square$

Corollary 3.2.9. *Using the terminology of 3.2.5, $(Q, \leq)$ is an atomless pre-order iff B_Q is atomless.*

Proof. It follows since the map j of the theorem preserves the order and incompatibility relation and has a dense image. $\square$

We conclude this section with the following observation:

Remark 3.2.10. There is a nice theorem (see [1, Theorem 22.14] for a proof) asserting that up to isomorphism there is a unique atomless complete boolean algebra B such that B^+ contains a countable dense subset. Here is a list of partial orders $(P, \leq)$ and topological spaces (X, τ) such that B is isomorphic to the regular open sets in the relevant topology:

1. The atomless partial order $(\tau \setminus \{\emptyset\}, \subseteq)$, where τ is the standard euclidean topology on $\mathbb{R}$.

2. The partial order $(D, \subseteq)$ given by open intervals with rational endpoints: D is countable and is a dense subset of $\tau \setminus \{\emptyset\}$ under inclusion. This gives that $\mathsf{RO}(D)$ and $\mathsf{RO}(\tau \setminus \{\emptyset\})$ are isomorphic atomless complete boolean algebras admitting a countable dense subset.

3. The boolean completion $\mathsf{RO}(2^{<\omega})$ of the partial order $(2^{<\omega}, \supseteq)$ (the latter is a separative countable atomless partial order and is contained in its boolean completion as a dense subset).

4. The regular open subsets of $\mathbb{R}$ in the euclidean topology τ (the map $A \mapsto \mathsf{Reg}(A)$ surjects the partial order $(\tau \setminus \{\emptyset\}, \subseteq)$ onto $\mathsf{RO}(\mathbb{R}, \tau)^+$ preserving order and incompatibility).

5. The regular open sets of the product topology τ^* on 2^ω (the map $s \mapsto N_s = \{f \in 2^\omega : s \subseteq f\}$ is order and incompatibility preserving and maps $2^{<\omega}$ in a dense subset of $\mathsf{RO}(2^\omega, \tau^*)^+$).

Notice that in each case the regular open sets considered refer to different topological spaces: the first three algebra of regular open sets are induced by the order topology respectively on $(D, \subseteq)$, on $(\tau \setminus \{\emptyset\}, \subseteq)$, on $(2^{<\omega}, \supseteq)$; while in the fourth and the fifth case these algebras are given by the regular open sets of the topological space $(\mathbb{R}, \tau)$ or of the space $(2^\omega, \tau^*)$. Remark also that the map $j : \tau \setminus \{\emptyset\} \to \mathsf{RO}(\tau \setminus \{\emptyset\})$ given by Theorem 3.2.5 identifies two open sets iff they have dense intersections. In particular in this case the relevant j is not injective but it is still order and incompatibility preserving.

In conclusion we get that the same atomless complete boolean algebra can be obtained as the algebra of regular open sets of five distinct topologies on five distinct topological spaces whose topologies are not always isomorphic when seen as partial orders, but whose algebras of regular open sets on the other hand are all isomorphic.

This reflects a common state of affairs for all complete atomless boolean algebras.

3.2.1 Some remarks on partial orders and their boolean completions

Summing up, in these first sections, we have proved among other things, the following results:

Let $(Q, \leq_Q)$ a pre-order. There exists an unique (up to isomorphism) cba B_Q such that exists a map $j : Q \to \mathsf{B}_Q$ such that:

1. j preserves order and incompatibility.
2. $j[Q]$ is dense in B_Q^+.
3. Q is a separative partial order iff j is an injection.
4. Q is atomless iff B_Q is atomless.
5. B_Q is the cba given by the regular open sets of many topological spaces.
6. $\mathrm{St}(\mathsf{B}_Q) = \{G \subseteq \mathsf{B}_Q : G \text{ is an ultrafilter}\}$ with the topology τ_{B_Q} generated by the sets $\{N_b : b \in \mathsf{B}_Q\}$ is such that
$$\mathsf{RO}(\mathrm{St}(\mathsf{B}_Q), \tau_{\mathsf{B}_q}) = \mathsf{CLOP}(\mathrm{St}(\mathsf{B}_Q), \tau_{\mathsf{B}_q}) \cong \mathsf{RO}(Q, \tau_Q)$$
and $(\mathrm{St}(\mathsf{B}_Q)$ is an extremally disconnected compact Hausdorff topological space.
7. $\mathsf{B}_Q \cong C(\mathrm{St}(\mathsf{B}_Q), 2)$ where the latter is
$$\{f : \mathrm{St}(\mathsf{B}_Q) \to \mathbb{Z}_2,\ f \text{ is continous}\}$$
(with $\mathbb{Z}_2 = \{0, 1\}$ endowed of the discrete topology).

3.3 Completeness and the measure algebra

3.3.1 κ-completeness and κ-CC imply completeness

Lemma 3.3.1. *Let B a boolean algebra. If B is $<\kappa$-cc and B is $<\kappa$-complete then B is complete.*

Proof. Let λ be the the least cardinal for which there exists $\{b_\alpha : \alpha < \lambda\}$ a sequence of elements of B such that $\bigvee_{\alpha<\lambda} b_\alpha$ does not exists aiming for a contradiction. For each $\beta < \lambda$, let $c_\beta = \bigvee_{\alpha<\beta} b_\alpha$, it exists since B is $<\lambda$-complete. Without loss of generality, by refining the sequence if necessary, we can assume that the sequence $\{c_\alpha : \alpha < \lambda\}$ is not eventually constant and therefore that $c_{\alpha+1} \setminus c_\alpha \neq 0_{\mathsf{B}}$, for every $\alpha < \lambda$. Now, define another sequence $\{a_\alpha : \alpha < \lambda\}$ as follows:

$$\begin{aligned} &a_0 = c_0, \\ &a_\alpha = c_{\alpha+1} \setminus c_\alpha \text{ if } \alpha > 0. \end{aligned}$$

The set $\{a_\alpha : \alpha < \lambda\}$ turns out to be an antichain in B, which is a contradiction because $\lambda \geq \kappa$ and B has the $<\kappa$-cc.

3.3.2 The algebra of Lebesgue measurable sets modulo null sets.

Recall that a subset of $[0, 1]$ is *Borel* if it can be obtained in countably many steps starting from the basic open intervals applying the operations of countable unions and taking the complement. We say that a $A \subseteq [0, 1]$ is *null* (or *measure-zero*) if for every $\epsilon > 0$ there exists a family $\{I_i : i < \omega\}$ of open intervals such that $A \subseteq \bigcup\{I_i : i < \omega\}$ and $\sum_{i<\omega} I_i < \epsilon$. For every $A \subseteq [0, 1]$, we say A is *Lebesgue measurable* if and only if $A \mathbin{\Delta} X$ is null, for some Borel set $X \subseteq [0, 1]$. For every Lebesgue measurable $A \subseteq [0, 1]$, we denote the Lebesgue measure of A with $\mu(A)$ and we define it as the infimum of $\sum_{n\in\mathbb{N}} I_n$, where $\{I_i : i < \omega\}$ is a covering of A consisting of basic open intervals.

Exercise 3.3.2. Let $\mathcal{M}([0, 1])$ be the boolean algebra of Lebesgue measurable subsets of $[0, 1]$ with usual boolean operations of union, intersection and taking the complement. The set Null of all null subsets of $[0, 1]$, is an ideal of $\mathcal{M}([0, 1])$.

We can consider $\mathcal{M}([0, 1])/\mathrm{Null}$, the boolean algebra of the Lebesgue measurable subsets of $[0, 1]$ modulo the ideal of null sets. The elements of $\mathcal{M}([0, 1])/\mathrm{Null}$ are equivalence classes of Lebesgue measurable subsets of the unit interval

$$[X]_{\mathrm{Null}} = \{Y \subseteq [0, 1] : X \mathbin{\Delta} Y \text{ is null}\}.$$

This is also known as the *measure algebra* and sometimes it is denoted with MALG.

Proposition 3.3.3. *The measure algebra* MALG *is ccc, i.e.,* MALG *has no uncountable antichains.*

Proof. Let $\mathcal{A}$ be an antichain of MALG. This means that $[A]_{\mathrm{Null}} \cap [B]_{\mathrm{Null}} \in \mathrm{Null}$ i.e. that $\mu(A \cap B) = 0$ for all $[A]_{\mathrm{Null}}, [B]_{\mathrm{Null}} \in \mathcal{A}$. For every $n \in \omega$, let

$$\mathcal{A}_n = \{[X]_{\mathrm{Null}} \in A : \mu(X) \geq 1/n\}.$$

We claim that $|\mathcal{A}_n| \leq n$. For, if $|\mathcal{A}_n| > n$ and $[X_1]_{\text{Null}}, \ldots, [X_{n+1}]_{\text{Null}} \in \mathcal{A}_n$, then

$$\mu(\bigcup_{j=1}^{n+1} X_j) > 1,$$

though $\bigcup_{j=1,\ldots,n+1} X_j \subseteq [0,1]$ which has measure 1, a contradiction. So, $\mathcal{A} = \bigcup_{n<\omega} \mathcal{A}_n$ is a countable union of finite sets which implies that $\mathcal{A}$ is countable. □

Proposition 3.3.4. *The measure algebra* MALG *is countably complete, i.e. if* $\{A_n : n \in \omega\} \subseteq$ MALG, $\bigvee_{n\in\omega} A_n$ *exists in* MALG.

Proof. Let for each n, $A_n = [B_n]_{\text{Null}}$ for some measurable set $B_n \subseteq [0,1]$. Check that $[\bigcup_{n\in\omega} B_n]_{\text{Null}}$ is in MALG and is an exact upper bound of $\{A_n : n \in \omega\}$. □

Proposition 3.3.5. *The measure algebra* MALG *is atomless.*

Proof. It suffices to show that if $\mu(A) > 0$, then A can be split in two pieces of positive measure. Assume not and build by induction sets A_n and intervals $I_n = [i_n/2^n, i_n + 1/2^n]$ such that $\mu(A_n) = \mu(A)$ and $A_n = A \cap [i_n/2^n, i_n + 1/2^n]$ as follows:

- $A_0 = A$, $i_0 = 0$, hence $I_0 = [0,1]$.
- Given A_n and $I_n = [i_n/2^n, i_n + 1/2^n]$, let $j = 2i_n + 1$. Then $A_n = (A \cap [i_n/2^n, j/2^{n+1})]) \cup (A \cap [j/2^{n+1}, i_n + 1/2^n])$ with

  $$\mu(A \cap [i_n/2^n, j/2^{n+1})] \cap (A \cap [j/2^{n+1}, i_n + 1/2^n]) = \mu(\{j/2^{n+2}\}) = 0.$$

  Hence by assumption on A either $\mu(A) = \mu(A \cap [i_n/2^n, j/2^{n+1})])$ or $\mu(A) = \mu(A \cap [j/2^{n+1}, i_n + 1/2^n])$. We let $i_{n+1} = i_n$ and $I_{n+1} = [i_n/2^n, j/2^{n+1})]$ if the first case occurs, and $i_{n+1} = j$ and $I_{n+1} = [j/2^{n+1}, i_n + 1/2^n)]$ if the second case occurs. We let $A_{n+1} = A_n \cap I_{n+1}$.

We obtain that $\mu(A_n) = \mu(A)$ for all n, and that $\bigcap_{n\in\mathbb{N}} A_n \subseteq \{x\}$ for a unique point x given by the intersection of all intervals I_n. By the countable completeness of μ we get that

$$0 = \mu(\{x\}) \geq \mu(\bigcap_{n\in\mathbb{N}} A_n) = \inf_{n\in\mathbb{N}}(\mu(A_n)) = \mu(A) > 0,$$

a contradiction. Hence A can be split in two pieces of positive measure, concluding the proof. □

□

Corollary 3.3.6. *The measure algebra is complete, atomless, and CCC.*

Chapter 4

Partial orders

In this chapter we analyze certain combinatorial properties of partial-orders; in particular we focus on the one hand on the relations existing between a partial-order and its boolean completion, and on the other hand on the quasi-order introduced by Cohen to obtain the consistency of the failure of CH by means of forcing, and we outline the key combinatorial features used to prove this result. The material of this chapter overlaps with some parts of [8, Chapter III] or [7, Chapter II].

4.1 Basic definitions

A *quasi-order*, also called *pre-order* or qo, is a set P equipped with a reflexive and transitive binary relation denoted by $\leq_P$. An antisymmetric qo is a *partial-order*, or even just *po*. Every qo has an associated *strict* relation denoted by $<_P$ and defined by $x <_P y$ if and only if $x \leq_P y$ and $y \not\leq_P x$.

Driving examples of the kind of partial-orders we will focus on are given by $(\tau \setminus \{\emptyset\}, \subseteq)$, where τ is a topology on some space X with no isolated points.

Exercise 4.1.1. Let (X, τ) be a topological space. Show that $(\tau \setminus \{\emptyset\}, \subseteq)$ is a partial order.

Exercise 4.1.2. Let τ be the euclidean topology on $\mathbb{R}$. Let for $A, B \in \tau$ $A \subseteq^* B$ if $A \cap B$ is a dense subset of A. Show that $(\tau \setminus \{\emptyset\}, \subseteq^*)$ is a qo but not a po. (HINT: the transitive and reflexive property of $\subseteq^*$ are basic topological facts about density. To see that $\subseteq^*$ is not anti-symmetric consider an open interval I, and the same interval I without a point).

Remark that if P is a partial order then the strict relation $<_P$ is just $\leq_P \setminus \Delta_P$, where Δ_P stands for the diagonal in P^2. Remark also that this is far from being true in any qo, since for instance the total relation P^2 on P is a qo.

In a clear context we write $\leq$ instead of $\leq_P$.

When $x \leq y$ holds we say that x is *below* y. When x is either below or above y, we say that x and y are *comparable*. An order $(P, \leq)$ is *total* or *linear* when any two elements are comparable.

Let $(P, \leq)$ be a qo.

We say that two elements x, y in P are *compatible* and we write $x||y$ if there is $z \in P$ such that both $z \leq x$ and $z \leq y$ hold. Otherwise x and y are *incompatible*, which is denoted by $x \perp y$.

Exercise 4.1.3. Following the notation of Exercise 4.1.2, show that $A, B \in \tau \setminus \{\emptyset\}$ are compatible for $\subseteq^*$ if and only if $A \cap B$ is non-empty.

A *chain* of a quasi-order $(P, \leq)$ is a subset of P which is linearly ordered by $\leq$. An *antichain* of $(P, \leq)$ is a subset of P consisting of incompatible elements.

A subset D of P is *dense* in P if for all x in P there is some y in D below x, it is predense if its downward closure

$$\downarrow D = \{q : \exists x \in D, q \leq x\}$$

is dense, it is a maximal antichain if it is a predense antichain.

Exercise 4.1.4. Let τ be the euclidean topology on $\mathbb{R}$. Following the notation of Exercise 4.1.2, show that:

- The intervals with rational end-points form a dense subset both for $(\tau \setminus \{\emptyset\}, \subseteq)$ and for $(\tau \setminus \{\emptyset\}, \subseteq^*)$.
- The set $\{(q; q + 1/n) : n \in [1; 100] \cap \mathbb{N}, q \in \mathbb{Q}\}$ is predense but not dense both for $(\tau \setminus \{\emptyset\}, \subseteq)$ and for $(\tau \setminus \{\emptyset\}, \subseteq^*)$.
- The set $\{(n; n + 1) : n \in \mathbb{Z}\}$ is a maximal antichain both in $(\tau \setminus \{\emptyset\}, \subseteq)$ and for $(\tau \setminus \{\emptyset\}, \subseteq^*)$.
- Show that any antichain of $(\tau \setminus \{\emptyset\}, \subseteq)$ or of $(\tau \setminus \{\emptyset\}, \subseteq^*)$ must be countable (HINT: an antichain $\mathcal{A}$ for both orders consists of pairwise disjoint non-empty sets; by the first item any element of $\mathcal{A}$ must contain an interval with rational end-points; if $A \neq B \in \mathcal{A}$ can they contain the same interval with rational end-points? how many such intervals there are?).

Exercise 4.1.5. Let (X, τ) be a topological space. Show that any base for τ is a dense subset of $(\tau \setminus \{\emptyset\}, \sqsubseteq)$.

The following remark will play a crucial role in many of the arguments of these notes:

Fact 4.1.6. *Let (X, τ) be a topological space. Then:*

- *$D \subset X$ is dense and open for τ if and only if $\sigma_D = \{O \in \tau : O \subseteq D\}$ is a dense and open subset of the quasi-order $(\tau \setminus \{\emptyset\}, \subseteq)$.*
- *$\sigma \subseteq \tau \setminus \{\emptyset\}$ is predense in the quasi-order $(\tau \setminus \{\emptyset\}, \subseteq)$ if and only if $\cup\sigma = D_\sigma$ is an open dense subset of X with respect to the topology τ.*

We leave the proof as an exercise for the reader.

We say that $(P, \leq)$ is *separative* if for all x and y in P, if x is not below y then there is some z below x that is incompatible with y. Formally,

$$\forall x \in P \, \forall y \in P \, (\, x \not\leq y \rightarrow \exists z \leq x (z \perp y)) \, .$$

Exercise 4.1.7. Following the notation of Exercise 4.1.2, show that neither $(\tau \setminus \{\emptyset\}, \subseteq)$ nor $(\tau \setminus \{\emptyset\}, \subseteq^*)$ are separative.

$(P, \leq)$ is *atomless* if it does not have minimal elements in the following strong sense: given any p in P there are elements $q \perp r$ of P strictly below p.

An atom of a quasi order $(P, \leq)$ is an element $p \in P$ such that any two q, r refining p are compatible.

Exercise 4.1.8. Following the notation of Exercise 4.1.2, show that $(\tau \setminus \{\emptyset\}, \subseteq)$ and $(\tau \setminus \{\emptyset\}, \subseteq^*)$ are atomless.

Exercise 4.1.9. Following the notation of Exercise 4.1.2, let σ be the family of open sets of τ which have non-empty intersection with $(0;1) \cup \{2\}$. Show that the interval $(1;3)$ is an atom of $(\sigma \setminus \{\emptyset\}, \subseteq)$ and $(\sigma \setminus \{\emptyset\}, \subseteq^*)$.

Exercise 4.1.10. Let (X, τ) be a Hausdorff topological space. Show that $a \in X$ is an isolated point if and only if $\{a\}$ is an atom of $(\tau \setminus \{\emptyset\}, \subseteq)$.

Exercise 4.1.11. Let $2^{<\omega}$ be the set of finite sequences of 0s and 1s, more precisely:

$$2^{<\omega} = \bigcup_{n \in \omega} 2^n$$

where 2^n is the set of functions with domain n and range 2. Let $s \leq t$ if $t \subseteq s$, that is if t is an initial segment of s. Then $(2^{<\omega}, \leq)$ is a separative and atomless quasi-order. (HINT: First prove that $s \perp t$ iff $s \cup t$ is not a function and $s || t$ iff $s \cup t = s$ or $s \cup t = t$).

It can be seen that the quasi-orders given in examples 4.1.11, 4.1.2 are quite similar: they give rise to isomorphic boolean completions, (see Theorem 3.2.5 and Remark 3.2.10).

Fact 4.1.12. *Assume a be a minimal element of a quasi-order $(P, \leq)$, and $D \subseteq P$ be dense. Then $a \in P$.*

Proof. Since D is dense, $D \cap (\downarrow \{a\}) \neq \emptyset$. But a is a minimal element of P, hence $\{a\} = \downarrow \{a\} \subseteq D$. □

Fact 4.1.13. *Assume $D \subseteq E \subseteq F$ with $(F, \leq_F)$ a quasi-order. Assume D is a dense subset of the quasi-order $(E, \leq_F)$, and E is a dense subset of the quasi-order $(F, \leq_F)$. Then D is a dense subset of the quasi-order $(F, \leq_F)$. I.e the property of being dense is transitive.*

Proof. Exercise for the reader. □

Zorn's Lemma

Assume $(P, \leq)$ is a pre order and $\mathcal{A}$ is a subset of P. $p \in P$ is an *upper bound* for $\mathcal{A}$ if $p \geq a$ for all $a \in \mathcal{A}$. p is an *exact upper bound* for $\mathcal{A}$ or a *supremum* of $\mathcal{A}$ it is an upper bound for $\mathcal{A}$ and $q \geq p$ for all upper bounds q for $\mathcal{A}$. Exchanging $\leq$ with $\geq$ one obtains the notions of *lower bound* and *exact lower bound* or *infimum*.

$p \in \mathcal{A}$ is a *maximal element* for $\mathcal{A}$ if it is an upper bound for $\mathcal{A}$. Dually p is a *minimal element* for $\mathcal{A}$ if it is a lower bound for $\mathcal{A}$.

We recall the following equivalent of the axiom of choice: Da spostare-eliminare M

Definition 4.1.14 (Zorn's lemma)**.** Let X be a non-empty set. Assume $\mathcal{A} \subseteq \mathcal{P}(X)$ is non-empty and such that all chains in the quasi-order $(\mathcal{A}, \subseteq)$ have an upper bound. Then $\mathcal{A}$ admits a maximal element.

– *M*

4.1.1 The order topology

A quasi-order is equipped with a canonical topological structure. Let $(P, \leq)$ be a quasi-order. For each $p \in P$ we let:

$$\downarrow p := \downarrow \{p\} = \{q \in P : q \leq p\}.$$

The sets $\downarrow p$ form a semi-basis for a topology τ_P on P, which we call the *order topology*. We remark the following:

- The open sets of P in this topology are the downward closed subsets of P with respect to the order $\leq$ (dually it is easily checked that the closed sets in τ_P^c are exactly the upward closed subsets of P).

- For any $p \in P$, $\downarrow p$ is the smallest open set to which p belongs.

- A subset D of P is dense in the sense of the order iff it is dense in P with respect to the order topology.

- The family of open sets of this order topology is closed under arbitrary intersections, since the family of downward closed subsets of P has this property. In particular the order topologies are always complete and distributive sublattices of $\mathcal{P}(P)$ (see Section 2.10 for a definition of complete and distributive lattice).

Remark 4.1.15. This topology is not to be confused with the one commonly associated to a linear order. For example the family of open sets for the order topology induced by the linear order $(\mathbb{R}, <)$ is given by the intervals of the form $(-\infty, a)$ or $(-\infty, a]$ as a ranges in $\mathbb{R} \cup \{+\infty, -\infty\}$, this topology is clearly not the euclidean topology on $\mathbb{R}$, which is the one usually associated to the canonical linear order of $\mathbb{R}$. The order topology we introduced corresponds to the Alexandrov topology on a quasi order, when reversing the order on $(P, \leq)$ (i.e. we consider as open sets what are the closed sets for the Alexandrov topology). In these notes we are interested in order topologies for orders which are *not* linear. For any quasi-order $(P, \leq)$ containing $p \neq q$ with $p \leq q$ the induced order topology is not Hausdorff: $p \in U$ for any open neighborhood of q, since $p \in N_q$.

4.2 Filters, antichains, and predense sets on quasi-orders

Notation 4.2.1. Let $(P, \leq)$ be a quasi-order and $X \subseteq P$.

- X is dense if it is dense in the order topology on P, i.e. if and only if for all $p \in P$ there exists $q \in X$ $q \leq p$.
- X is open if it is open in the order topology on P, i.e. if it is downward closed.
- X is *predense* if $\downarrow X$ is a dense open set of P in the order topology.
- X is a *maximal antichain* if it is a pre-dense antichain.
- X is *dense below* $p \in P$ if $X \cap P \restriction p$ is a dense subset of the quasi-order $P \restriction p$.
- X is *predense below* $p \in P$ if for all $q \leq p$ there is $r \in X$ compatible with q, i.e. if $\downarrow X$ is dense below p.

Exercise 4.2.2. Assume B is a complete boolean algebra. Then:

- $X \subseteq \mathsf{B}^+$ is predense in $(\mathsf{B}^+, \leq_{\mathsf{B}})$ if and only if $\bigvee X = 1_{\mathsf{B}}$. (HINT: If not $a = \neg \bigvee X > 0_{\mathsf{B}}$ and $b \wedge a = 0_{\mathsf{B}}$ for all $b \in \downarrow X$, i.e. $\downarrow X$ is not dense in B^+).
- X is predense below $b \in \mathsf{B}^+$ if and only if $\downarrow X \cap \downarrow b$ is a dense subset of $\downarrow b$ if and only if $b = \bigvee_{\mathsf{B}} \{\downarrow\} X \cap \downarrow b$.
- If $D \subseteq \mathsf{B}^+$ is dense, there exists $A \subseteq D$ maximal antichain of B^+. (HINT: Apply Zorn's Lemma to the antichains contained in D ordered by inclusion, a maximal element of this quasi-order is a maximal antichain $A \subseteq D$).

Definition 4.2.3. Let $(P, \leq_P)$, $(Q, \leq_Q)$ be quasi orders. A map $i : P \to Q$ between quasi-orders is:

- a *morphism* if it preserves the order relation,
- an *embedding* if it preserves the order and the incompatibility relations,
- a *complete embedding* if it maps predense subsets of P in predense subsets of Q.

Remark that an embedding need not be injective, examples of non-injective complete embeddings will be given later on (cfr. for example Remark 3.2.10).

Exercise 4.2.4. Consider the space 2^ω endowed with the product topology τ. Let for $s \in 2^{<\omega}$ $N_s = \{f \in 2^\omega : s \subseteq f\}$. Show that the map $s \mapsto N_s$ is an embedding of $(2^{<\omega}, \leq)$ into $(\tau \setminus \{\emptyset\}, \subseteq)$ with a dense image (HINT: the map $s \mapsto N_s$ is order reversing and preserve incompatibility, hence it is an embedding of partial orders. Prove that $\{N_s : s \in 2^{<\omega}\}$ is a base for τ).

Definition 4.2.5. Let $(P, \leq)$ be a quasi-order.

- I is an ideal on P if it is a downward closed subset of P such that $a, b \in I$ entail that for some $c \in I$, $a, b \leq c$. Dually a *filter* G on P is an upward closed subset of P such that any two elements of G are compatible, otherwise said:
  1. for all $p, q \in G$, there is $r \in G(r \leq p, q)$.

2. for all $p \in G$ and $q \geq p$, $q \in G$.

- A *prefilter* on P is a subset H of P such that

$$\uparrow H = \{q : \exists p \in H\, p \leq q\}$$

is a filter, equivalently a prefilter H is a subset of P such that any of its finite subset has a lower bound in[1] H.

Exercise 4.2.6. Recall that for $s \in 2^{<\omega}$ $N_s = \{f \in 2^\omega : s \supseteq f\}$.

- Show that if G is an ultrafilter on $\mathsf{RO}(2^\omega)$, then $\{s : N_s \in G\}$ is a filter on $(2^{<\omega}, \supseteq)$.
- Conversely for any $f \in 2^\omega$, show that $G_f = \{N_s : s \subseteq f\}$ is a prefilter on the boolean algebra $\mathsf{CLOP}(2^\omega)$ whose upward closure in $\mathsf{CLOP}(2^\omega)$ is a ultrafilter in $\mathrm{St}(\mathsf{CLOP}(2^\omega))$;
- *S*how also that for any $f \in 2^\omega$ the upward closure in $\mathsf{RO}(2^\omega)$ of G_f is just a filter on the boolean algebra $\mathsf{RO}(2^\omega)$ (HINT: to show that G_f does not generate a ultrafilter on $\mathsf{RO}(2^\omega)$ look at Fact 4.3.3 to argue that *Even* and *Odd* are regular open set not in $\uparrow G_{c_0}$, where c_0 is the constant sequence of 0).

Proposition 4.2.7. *Let P be a quasi-order. Let G be a filter on P and $X \subseteq P$. Then*

$$G \cap X \neq \emptyset \Leftrightarrow G \cap \downarrow X \neq \emptyset.$$

Proof. if $r \in G \cap \downarrow X$, then $\exists q \geq r$ such that $q \in X$. So, since G is a filter, $q \in G \cap X$. ☐

Definition 4.2.8. Let $(P, \leq)$ be a quasi-order. Let $\mathcal{F} = \{D_i : i \in I\}$ be a family of subsets of P. Let G be a filter. G is *$\mathcal{F}$-generic* if $G \cap D_i \neq \emptyset$, for all $i \in I$.

The following is a useful equivalent of Baire's category theorem:

Lemma 4.2.9 (Generic filter Lemma)**.** *Let $(P, \leq)$ be a quasi-order and $\mathcal{F} = \{D_i : i \in \omega\}$ be a family of predense subsets of P. Then for every $p \in P$ there exists a filter G on P $\mathcal{F}$-generic with $p \in G$.*

Proof. Using AC and recursion on ω, choose $p_n \in P$ for $n \in \omega$ so that $p_0 = p$, $p_{n+1} \leq p_n$ and $p_{n+1} \in \downarrow D_n$. Let

$$G = \uparrow \{p_n : n \in \omega\}.$$

G is upward closed by definition. We check now it is a filter. Let $r_0, r_1 \in G$ and let m_i such that $r_i \geq p_{m_i}$, for $i = 0, 1$. Then $r_i \geq p_n$ for all $n \geq m_0, m_1$ and $i = 0, 1$. ☐

[1] The notions of filter, ideal, prefilter generalize to quasi-orders the corresponding notions introduced just for boolean algebras.

Actually the notion of prefilter we introduced for quasi orders is slightly stronger than the notion of prefilter on a boolean algebra: if B is a boolean algebra and H is a prefilter for the boolean algebra B, it might not be a prefilter for the quasi-order $(\mathsf{B}^+, \leq)$, since it might not contain lower bounds for its finite subsets, but just have the property that its finite subsets have positive meet.

Corollary 4.2.10 (Baire's category Theorem)**.** *Assume (X, τ) is a compact Hausdorff space. Then the intersection of any countable family of dense open subsets is dense.*

Proof. Let $\{D_n : n \in \mathbb{N}\}$ be a countable family of dense open subsets of X. Let A be an open non-empty subset of X, we must find a point $x \in A \cap \bigcap_{n\in\mathbb{N}} D_n$.

We use the following property of compact Hausdorff spaces (normality): any non empty open set O admits an open subset B such that $\mathsf{Cl}(B) \subseteq O$.

So fix B non-empty and open such that $\mathsf{Cl}(B) \subseteq A$. Let σ be the restriction of τ to $\mathsf{Cl}(B)$ so that $(\mathsf{Cl}(B), \sigma)$ is also a compact Hausdorff space. Notice that $E_n = D_n \cap B$ is a dense open subset of $\mathsf{Cl}(B)$ for all $n \in \mathbb{N}$. Consider now the quasi-order $(\sigma \setminus \{\emptyset\}, \subseteq)$ and the sets

$$F_n = \{O \in \sigma \setminus \{\emptyset\} : \mathsf{Cl}(O) \subseteq E_n\}$$

Claim 4.2.10.1. *F_n is open dense in $(\sigma \setminus \{\emptyset\}, \subseteq)$.*

Proof. Clearly F_n is open. Let $C \in \sigma$ be open non-empty. Hence $E_n \cap C$ is an open non-empty subset of C. Since $(\mathsf{Cl}(B), \sigma)$ is compact Hausdorff, there is $U \in \sigma \setminus \{\emptyset\}$ such that $\mathsf{Cl}(U) \subseteq E_n \cap C$. Then $U \in F_n$ refines C. Hence F_n is dense since C was chosen arbitrarily in $\sigma \setminus \{\emptyset\}$. □

Now let G be a filter on $(\sigma \setminus \{\emptyset\}, \subseteq)$ such that $G \cap F_n \neq \emptyset$ for all $n \in \mathbb{N}$, which exists by Lemma 4.2.9. Notice that each $B_n \in G \cap F_n$ is such that $\mathsf{Cl}(B_n) \subseteq \mathsf{Cl}(B) \cap E_n \subseteq A \cap D_n$. Notice also that the family $\{\mathsf{Cl}(B_n) : n \in \mathbb{N}\}$ has the finite intersection property, since any finite subset of this family $\mathsf{Cl}(B_{i_1}) \ldots \mathsf{Cl}(B_{i_k})$ is such that

$$\mathsf{Cl}(B_{i_1}) \cap \cdots \cap \mathsf{Cl}(B_{i_k}) \supseteq B_{i_1} \cap \cdots \cap B_{i_k} \supseteq U \neq \emptyset$$

for some $U \in G$, since G is a filter and $B_{i_1}, \ldots B_{i_k} \in G$. Since $\mathsf{Cl}(B)$ is compact, $\bigcap \{\mathsf{Cl}(B_n) : n \in \mathbb{N}\}$ is non-empty. Any point in this intersection belongs to $A \cap \bigcap_{n\in\mathbb{N}} D_n$. □

The two exercises below show that the generic filter Lemma is non-trivial only if we are considering atomless quasi-orders. We will see in Chapter 6 that the forcing method invented by Cohen stems from a careful analysis of the notion of generic filter.

The following exercises show that atoms of preorders give rise to trivial generic filters.

Exercise 4.2.11. Let P be a preorder and a an atom of P. Then $G_a = \uparrow \{a\}$ is a $\mathcal{D}$-generic filter, where $\mathcal{D}$ is the collection of dense subsets of P. (HINT: An atom of P belongs to all dense subsets of P).

The following exercise outlines in more details the relations existing between atoms of a boolean algebra and the notion of genericity.

Exercise 4.2.12. Assume C is a boolean algebra. Then $\mathrm{St}(\mathsf{C}) \setminus \{G\}$ is open dense for any $G \in \mathrm{St}(\mathsf{C})$ which is a non-principal ultrafilter (i.e. such that $a \notin G$ for any a atom of $\mathrm{St}(\mathsf{C})$). (HINT: recall (or prove) that G is a non-principal ultrafilter if and only if G is not an isolated point of $\mathrm{St}(\mathsf{C})$, moreover any non-isolated point of a Hausdorff topological space has a complement which is open dense in $\mathrm{St}(\mathsf{C})$).

Show the following:

1. Assume C is atomless, then the intersection of *all* dense open subsets of $\mathrm{St}(\mathsf{C})$ is empty (HINT: already the intersection of

$$\{St(\mathsf{C}) \setminus \{G\} : G \in \mathrm{St}(\mathsf{C})\}$$

is empty).

2. If $a \in \mathsf{C}$ is an atom, then $G_a = \{b \in \mathsf{B} : a \leq b\}$ is a ultrafilter in $\mathrm{St}(\mathsf{C})$ meeting all the dense open subsets of $\mathrm{St}(\mathsf{C})$.

We now come to a basic application of the generic filter Lemma which is at the heart of Cohen's forcing method.

Exercise 4.2.13. Show that the following sets are dense open in $2^{<\omega}$.

- For $f \in 2^\omega$, $D_f = \{s \in 2^{<\omega} : s \perp f\}$,
- $E_n = \{s \in 2^{<\omega} : n \in \mathrm{dom}(s)\}$.

Prove that there is no filter G on $2^{<\omega}$ which is $\{D_f : f \in 2^\omega\} \cup \{E_n : n \in \omega\}$.

(HINT: Assume towards a contradiction that there exists a filter G such that $G \cap D_f \neq \emptyset$ for every $f \in 2^\omega$ and $G \cap E_n \neq \emptyset$ for every $n \in \omega$. Let $\bigcup\{s : s \in G\} = g \in 2^\omega$. Then $G \cap D_g \neq \emptyset$ and so there should be $t \in G$ such that $t \perp g$, i.e. $\exists n(t(n) \neq g(n))$. But

$$G \ni t \subseteq g = \bigcup G,$$

a contradiction.)

We can even show that certain quasi-orders have a family of $\aleph_1$-many dense sets which cannot be met in a filter:

Fact 4.2.14. *Consider the partial order $((\omega_1)^{<\omega}; \supseteq)$ ordered by reverse inclusion. There exists a family $\{D_\alpha : \alpha < \omega_1\}$ of dense sets such that for every filter $G \subseteq \omega_1^{<\omega}$ there exists α such that $G \cap D_\alpha = \emptyset$.*

Proof. Set

$$\begin{aligned} B_\alpha =&\{s \in (\omega_1)^{<\omega} : \exists n\; s(n) = \alpha\} \\ E_n =&\{s \in (\omega_1)^{<\omega} : |s| \geq n\}. \end{aligned}$$

For all α and n, B_α and E_n are open by definition, let us see that they are dense. Take $s \in (\omega_1)^{<\omega}$. If there exists $n < |s|$ such that $s(n) = \alpha$ then $s \in B_\alpha$, otherwise $s^\frown\alpha = s \cup \{(|s|, \alpha)\} \in B_\alpha$. Hence B_α is dense. We leave to the reader the proof that E_n is dense for every n.

Assume now that there exists a filter G such that $G \cap B_\alpha \neq \emptyset$ for every $\alpha \in \omega_1$ and $G \cap E_n \neq \emptyset$ for every $n \in \omega$, then $\bigcup G : \omega \to \omega_1$ is a surjection, a contradiction. So the family $\{B_\alpha : \alpha < \omega_1\} \cup \{E_n : n \in \omega\}$ is the one we were looking for. □

At this point we can already bring forward something that we will formalize in the last chapter of these notes. Let M be a transitive countable model of ZFC and assume that $P \in M$ is atomless and separative. It can be seen that the family of dense sets of P is uncountable. On the other hand there are only countably many dense sets of P which can belong to M. The generic filter Lemma guarantees that there exists a filter G that intersects all the dense sets of P which are in M.

Now observe the following:

Fact 4.2.15. *Assume M is a countable transitive model of ZFC, $P \in M$ is atomless and separative and G is an M-generic filter, i.e. G meets all the dense subsets of P which belong to M. Then $G \notin M$.*

Proof. It is always the case that $P \setminus G$ is an open dense subset of P whenever G is a filter on P and P is atomless and separative (Given any $p \in P$ find $r, q \leq p$ and incompatible, then at least one between r and q is not in G). Thus $G \in M$ implies $P \setminus G \in M$. However $G \cap (P \setminus G) = \emptyset$, thus G cannot be M-generic. $\square$

Hence whenever M is a countable transitive model of ZFC, $P \in M$ is atomless and separative, and G is an M-generic filter, we can define

$$M[G] = \bigcap\{N \supseteq M : N \text{ is transitive } \wedge\ N \models \mathsf{ZFC}\ \wedge\ G \in N\}.$$

Our arguments show already that $M[G]$ strictly contains M (since $G \in M[G] \setminus M$), provided that there is some transitive set $N \supseteq M \cup \{G\}$ which is a model of ZFC. We will further show that $M[G]$ is itself a model of ZFC and that (depending on the choice of the $P \in M$ for which G is M-generic) we can define $M[G]$ so that it satisfies CH or its negation by carefully choosing P.

4.3 The quasi-orders $Fn(X, Y)$

Definition 4.3.1. Given sets X, Y and a cardinal κ, let $Fn(X, Y, \kappa)$ be the quasi-order of functions with domain a subset of X of size less than κ and ranging in Y. The order on $Fn(X, Y, \kappa)$ is given by the reverse inclusion.

We write simply $Fn(X, Y)$ instead of $Fn(X, Y, \omega)$ and for any $p \in Fn(X, Y)$, we put

$$\downarrow p = \{f \in Y^X : p \subset f\}.$$

So $2^{<\omega}$ is the set of functions in $Fn(\omega, 2)$ whose domain is a natural number.

Remark 4.3.2.

1. The order $(2^{<\omega}, \subseteq)$ is a dense suborder of $(Fn(\omega, 2), \subseteq)$, in particular they have the same boolean completion, which can be represented as $\mathsf{RO}(2^\omega)$ the family of regular open sets in 2^ω with the product topology. The map $s \mapsto N_s = \{f \in 2^\omega : s \subseteq f\}$ implements an order and incompatibility preserving embedding of $Fn(\omega, 2)$ into $\mathsf{RO}(2^\omega)$ with a dense image, since the family

$$\{N_s : s \in 2^{<\omega}\}$$

   forms a basis of clopen sets (and thus regular open) for the product topology on 2^ω. We leave to the reader to check that this map is order and incompatibility preserving.

2. If Y is finite, the space Y^X endowed with the product topology is a compact 0-dimensional Hausdorff space with no isolated points, in particular any clopen set in 2^X is a finite union of sets of the the form $\downarrow p$ for some $p \in Fn(X, 2)$. For any $p \in Fn(X, 2)$, we can write $\downarrow p$ as a closed set:

$$N_p = \bigcup\{2^X \setminus N_t \mid t \neq p, \mathrm{dom}(t) = \mathrm{dom}(p)\},$$

   since there are only finitely many t ranging in 2 with the same domain as p.

   The compact Hausdorff space 2^ω endowed with the product topology is also known in the literature as the *Cantor space*.

3. The family of clopen sets in the product topology on 2^X is a boolean algebra with the standard set-theoretic operations and the sets $\downarrow p$ as p ranges in $Fn(X,2)$ form a dense subset of the positive elements of this boolean algebra. Its boolean completion is the space of regular open sets of 2^X with the product topology.

Fact 4.3.3. *Some regular open sets of 2^ω are not closed.*

Proof. A counterexample is given by Odd and $Even$, where Odd (resp.$Even$) is the set of sequences in 2^ω that differ from 0^ω and start with an odd (resp. even) number of zeros.

These two sets are open, disjoint and their closures intersect only in 0^ω.

In particular, 0^ω is the unique point in the closure of Odd and $Even$ such that no open set containing it has a dense intersection with Odd or a dense intersection with $Even$. While any element of Odd (resp. $Even$) has a clopen neighboorhood fully contained in Odd (resp. $Even$). This means that Odd and $Even$ are regular and open, but they are not closed. □

Exercise 4.3.4.

- The map

$$\begin{aligned} i : 2^{<\omega} &\longrightarrow 2^\omega \\ s &\longmapsto s^\frown 1^\frown 0^\omega \end{aligned}$$

  is continuous, injective and has a dense image in the Cantor space.

- The map

$$\begin{aligned} i^* : \mathsf{RO}(2^{<\omega}) &\longrightarrow \mathsf{RO}(2^\omega) \\ A &\longmapsto \bigcup\{N_s \mid s \in A\}. \end{aligned}$$

  is an isomorphism of complete boolean algebras.

In particular, $\mathsf{RO}(2^\omega)$ is another possible representation of the boolean completion of the quasi-orders $2^{<\omega}$, $Fn(\omega,2)$ and as a boolean algebra $\mathsf{RO}(2^\omega)$ is a proper superalgebra of the boolean algebra given by the clopen subsets of 2^ω. The positive elements of the latter however form a dense suborder of $\mathsf{RO}(2^\omega)^+$.

The latter observation outlines a distinction between 2^ω and the Stone space of the Boolean completion of the quasi-order $2^{<\omega}$, a distinction which is common to the Stone spaces of a Boolean algebra and the Stone space of its boolean completion. We spell out the details in the following observation:

Remark 4.3.5. Let $\mathsf{B} = \mathsf{RO}(2^\omega)$ be the boolean completion of $2^{<\omega}$ and $\mathrm{St}(\mathsf{B})$ its associated Stone space. Then $\mathrm{St}(\mathsf{B})$ is a 0-dimensional compact Haussdorff space, and there is a natural projection

$$\begin{aligned} \pi : \mathrm{St}(\mathsf{B}) &\to 2^\omega \\ G &\mapsto f_G = \bigcup\{s \in 2^{<\omega} : N_s \in G\} \end{aligned}$$

This projection is:

- Continuous closed and open, since $N_s \in G$ iff $s \subset f_G$ for all $s \in 2^{<\omega}$.

- Surjective: given $f \in 2^\omega$, consider an ultrafilter G that contains N_s for every $s \subset f$; then $\pi(G) = f$.
- However π is not injective: for example there are G and H ultrafilters in $\mathrm{St}(\mathsf{B})$ such that $\pi(G) = \pi(H) = 0^\omega$, but $Odd \in G$, $Even \in H$.

This occurs since B can be identified with the family of regular open sets of 2^ω and $\mathrm{St}(\mathsf{B})$ is a Stone space whose clopen sets overlap with its regular open sets, while we already remarked that the clopen subsets of 2^ω form a strictly proper subalgebra of the regular open subsets of 2^ω.

4.3.1 The quasi-order $Fn(\omega_2 \times \omega, 2)$

$$Fn(\omega_2 \times \omega, 2) = \{s : s : \omega_2 \times \omega \to 2 \ \wedge \ \mathrm{dom}(s) \text{ is finite}\}.$$

We can naturally identify

$$(2^\omega)^{\omega_2} = \{f : \mathrm{dom}(f) = \omega_2 \ \wedge \ \forall i \in \mathrm{dom}(f)(f(i) \in 2^\omega)\}$$

with the space $2^{\omega_2 \times \omega}$. With this identification its product topology is generated by the family $\{N_s : s \in Fn(\omega_2 \times \omega, 2)\}$, where in this case we use this natural identification to let

$$N_s = \{f \in (2^\omega)^{\omega_2} : \forall(\alpha, n) \in \mathrm{dom}(s) \ f(\alpha)(n) = s(\alpha, n)\}.$$

Moreover the following holds:

Lemma 4.3.6. *The map $s \mapsto N_s$ defines a dense embedding of the quasi-order $Fn(\omega_2 \times \omega, 2)$ into $\mathsf{RO}(2^{\omega_2 \times \omega})$. In particular $\mathsf{RO}(Fn(\omega_2 \times \omega, 2))$ and $\mathsf{RO}(2^{\omega_2 \times \omega})$ are isomorphic complete boolean algeberas.*

Proof. Notice that the family $\{N_s : s \in Fn(\omega_2 \times \omega, 2)\}$ is a base for the product topology on $2^{\omega_2 \times \omega}$ consisting of clopen (and thus also regular open) sets.

In particular this gives that the target of the map is dense. It is an easy exercise to check that the map is also order and incompatibility preserving. $\square$

We define the following subsets of $\mathsf{RO}(2^{\omega_2 \times \omega})$:

- $D_{n,\alpha} = \{N_s : s \in Fn(\omega_2 \times \omega, 2) \ (\alpha, n) \in \mathrm{dom}(s)\}$;
- $E_{\alpha,\beta} = \{N_s : s \in Fn(\omega_2 \times \omega, 2) \ \exists n \ s(\alpha, n) \neq s(\beta, n)\}$.

Let $\mathcal{D}$ be the family

$$\{D_{n,\alpha} : n \in \omega, \alpha \in \omega_2\} \cup \{E_{\alpha,\beta} : \alpha \neq \beta \in \omega_2\}.$$

Assume that we could find a filter G which is $\mathcal{D}$-generic, then, letting $g_\alpha = \bigcup\{\langle n, s(\alpha, n)\rangle : s \in G, n \in \omega\}$, we would have that $\{g_\alpha : \alpha < \omega_2\}$ are different elements of 2^ω, this would entail the failure of CH.

Exercise 4.3.7. Show that $\{N_s : s \in Fn(\omega_2 \times \omega, 2)\}$ is a dense subset of $\mathsf{RO}(2^{\omega_2 \times \omega})$ and that the map $s \mapsto N_s$ is injective and order and incompatibility preserving.

Show also that $E_{\alpha,\beta}$ and $D_{n,\alpha}$ are dense in $\mathsf{RO}(2^{\omega_2 \times \omega})$ for all $\alpha \neq \beta < \omega_2$ and $n < \omega$.

4.4 Quasi-orders with the countable chain condition

Definition 4.4.1. Let $(P, \leq)$ be a quasi-order. P has the ***countable chain condition*** (CCC) if every antichain of P is countable.

Remark 4.4.2. Every countable quasi-order has the CCC.

So, for example, $2^{<\omega}$ has the CCC, while $(\omega_1)^{<\omega}$ does not have it, indeed, the set

$$\{\{(0, \alpha)\} : \alpha < \omega_1\}$$

is an uncountable antichain.

We have only defined the CCC for quasi-orders; actually, this definition can be generalized to topological spaces.

Definition 4.4.3. A topological space has the CCC if the quasi-order $(\tau \setminus \{\emptyset\}, \subseteq)$ has the CCC.

Moreover the following holds:

Lemma 4.4.4. *Assume P is a quasi-order with the CCC. Then $\mathsf{RO}(P)^+$ has the CCC as well.*

Proof. Assume $A \subseteq \mathsf{RO}(P)^+$ is an antichain. For each $a \in A$ find $p_a \in P$ such that $i(p_a) \leq a$ where $i : P \to \mathsf{RO}(P)$ is the canonical immersion of P in its boolean completion. Since i is order and incompatibility preserving $\{p_a : a \in A\}$ is an antichain in P, and thus is countable. Moreover the map $a \mapsto p_a$ is injective since $a \neq b$ entails $a \wedge b = 0$ which gives that p_a and p_b are incompatible in P. We conclude that A is countable as well. □

Definition 4.4.5. If S is a set of finite sets then it is a Δ-***system*** if there is some (possibly empty) r such that for any $a, b \in S$, if $a \neq b$, then $a \cap b = r$. r is the *root* of the system.

Lemma 4.4.6. (Δ-system lemma) *Let κ be an uncountable regular cardinal, and let $\mathcal{A}$ be a family of finite sets with $|\mathcal{A}| = \kappa$. Then there is a $\mathcal{B} \in [\mathcal{A}]^\kappa$ such that $\mathcal{B}$ forms a Δ-system.*

Proof. Since $cf(\kappa) = \kappa > \omega$ and there are only $\aleph_0$ possible $|X|$ for $X \in \mathcal{A}$, we may fix $n \in \omega$ and $\mathcal{D} \in [\mathcal{A}]^\kappa$ such that $|s| = n$ for all $s \in \mathcal{D}$. Now, we prove it by induction on n.

1. $n = 1$: Then $\mathcal{D}$ is already a Δ-system with empty root.

2. $n > 1$: For each $p \in X$, let $\mathcal{D}_p = \{X \in \mathcal{D} : p \in X\}$. There are two cases.

    - Case I: $|\mathcal{D}_p| = \kappa$ for some p. Fix p, and let $E = \{X \setminus \{p\} : X \in \mathcal{D}_p\}$, which is a family of κ sets of size $n - 1$. Applying the lemma inductively, fix $C \in [E]^\kappa$ that forms a Δ-System with some root r. Then $\{Z \cup \{p\} : Z \in C\} \in [\mathcal{D}]^\kappa$ forms a Δ-system with root $r \cup \{p\}$.

- Case II: $|\mathcal{D}_p| < \kappa$ for all p. Then, for any set S with $|S| < \kappa$, $\{X \in \mathcal{D} : X \cap S \neq \emptyset\} = \bigcup_{p \in S} \mathcal{D}_p$ has size less than κ, since κ is regular; thus, there is an $X \in \mathcal{D}$ such that $X \cap S = \emptyset$. Then, by recursion on β, we may choose $X_\beta \in \mathcal{D}$ for $\beta < \kappa$ so that for each β, $X_\beta \cap \bigcup_{\alpha<\beta} X_\alpha = \emptyset$. But then $\{X_\beta : \beta < \kappa\}$ is a Δ-system with empty root.

The proof is completed. □

Corollary 4.4.7. *If $X \subseteq [\omega_1]^{<\omega}$ has cardinality $\aleph_1$, then there exists $Y \subseteq X$, with $|Y| = \omega_1$, such that there exists $r \in [\omega_1]^{<\omega}$ such that $\forall a, b \in Y (a \cap b = r)$.*

We can now prove the following:

Proposition 4.4.8. *For every set X, $Fn(X, 2)$ has the CCC.*

First of all remark the following

Fact 4.4.9. *Assume $f : X \to Y$ is a bijection. Then $\hat{f} : Fn(X, 2) \to Fn(Y, 2)$ is an isomorphism of quasi-orders, where $\hat{f}(s)$ is the sequence with domain $f[\mathrm{dom}(s)]$ such that $\hat{f}(s)(y) = s \circ f^{-1}(y)$ for all y in its domain.*

Proof. A useful exercise for the reader. □

Proof. In view of the above fact it is enough to show that $Fn(\kappa, 2)$ has the CCC for all cardinals κ. If $\kappa \leq \aleph_0$ we are done, since $Fn(\kappa, 2)$ is countable in this case, so we suppose $\kappa > \omega$. Take $\{s_\alpha : \alpha < \omega_1\} \subseteq Fn(\kappa, 2)$ with $s_\alpha \neq s_\beta$ if $\alpha \neq \beta$. We claim that there are at least two compatible elements in $\{s_\alpha : \alpha < \omega_1\}$. First, we find a set $X \subseteq \kappa$ such that $|X| \leq \aleph_1$ and $\mathrm{dom}(s_\alpha) \subseteq X$ for any $\alpha < \omega_1$. Let, for any $\alpha < \omega_1$:

$$\mathrm{dom}(s_\alpha) = \{(\beta_0^\alpha, \dots, \beta_{k_\alpha}^\alpha)\},$$

with k_α less than ω. Let

$$X = \{\beta_j^\alpha : \alpha < \omega_1 \ \wedge \ j \leq \kappa_\alpha\}.$$

Notice that $|X| \leq \aleph_1$ and $\mathrm{dom}(s_\alpha) \subseteq X$ for any $\alpha < \omega_1$. We have to distinguish two cases:

1. $|X| \leq \aleph_0$. We will prove that this case leads to a contradiction. For all $r \in [X]^{<\omega}$, let $Z_r = \{s_\alpha : \mathrm{dom}(s_\alpha) = r\}$. Obviously $Z_r \subseteq 2^r$ and $|2^r| = 2^{|r|} < \omega$. Thus $\forall r \in [X]^{<\omega}(|Z_r| < \omega)$. We have that

$$\{s_\alpha : \alpha < \omega_1\} = \bigcup_{r \in [X]^{<\omega}} Z_r.$$

But $|\bigcup_{r \in [X]^{<\omega}} Z_r| \leq \aleph_0$, since the Z_r's are finite and $[X]^{<\omega}$ is countable. However an uncountable set cannot be equal to a countable one, so we reached a contradiction.

2. $|X| = \aleph_1$. Let n be such that the set

$$Z = \{\alpha : |\operatorname{dom}(s_\alpha)| = n\}$$

has cardinality $\aleph_1$. Such an n must exist due to the regularity of ω_1. Now, for all $\alpha \in Z$, consider

$$s_\alpha = \{(\beta_0^\alpha, i_0^\alpha), \dots, (\beta_{n-1}^\alpha, i_{n-1}^\alpha)\}.$$

Define

$$\mathcal{D} = \{\operatorname{dom}(s_\alpha) : \alpha \in Z\}.$$

We claim that $|\mathcal{D}| = \aleph_1$. To this aim, consider the function $\varphi : Z \to \mathcal{D}$, $\alpha \mapsto \operatorname{dom}(s_\alpha)$. φ is a finite to one function, since if $\operatorname{dom}(s_\alpha) = \operatorname{dom}(s_\gamma)$ then

$$s_\alpha = \{(\beta_0, i_0^\alpha), \dots, (\beta_{n-1}, i_{n-1}^\alpha)\}$$

and

$$s_\gamma = \{(\beta_0, i_0^\gamma), \dots, (\beta_{n-1}, i_{n-1}^\gamma)\}.$$

But $(i_j^\alpha : j < n)$ and $(i_j^\beta : j < n)$ are both sequences in 2^n, so there can be at most 2^n-many of them. Now if $(i_j^\alpha : j < n) = (i_j^\gamma : j < n)$, then $s_\alpha = s_\gamma$, thus $\alpha = \gamma$.
Thus we can apply the Δ-system lemma to $\mathcal{D}$ and we obtain a set $\mathcal{B} \subseteq \mathcal{D}$ of size $\aleph_1$ and a root $r \in [X]^{<n}$ such that (defining $W = \varphi^{-1}[\mathcal{B}]$)

$$\forall \alpha, \gamma \in W \ (\alpha \neq \gamma \Rightarrow \operatorname{dom}(s_\alpha) \cap \operatorname{dom}(s_\gamma) = r).$$

Now let $t_\alpha = s_\alpha \restriction r$ for all $\alpha \in W$. The map $\alpha \mapsto t_\alpha$ has uncountable domain and finite range since $t_\alpha \in 2^r$, so there must exists an uncountable $W' \subset W$ and some $t \in 2^r$ such that $t_\alpha = t$ for all $\alpha \in W'$. In order to complete the proof, it is sufficient to show that for all $\alpha, \gamma \in W'$ $s_\alpha \cup s_\gamma$ is a condition in $Fn(\kappa, 2)$, i.e. that it is a function. Now observe that $\operatorname{dom}(s_\alpha) \cap \operatorname{dom}(s_\gamma) = r$ and that

$$s_\alpha \cup s_\gamma = (s_\alpha \cup s_\gamma \restriction r) \cup (s_\alpha \cup s_\gamma \restriction (\kappa \setminus r)).$$

Notice that for all $\beta \notin r$ at most one among s_α and s_γ is defined on β, thus $(s_\alpha \cup s_\gamma) \restriction (\kappa \setminus r)$ is a function. Notice also that $(s_\alpha \cup s_\gamma) \restriction r = t$ is a function. Thus $s_\alpha \cup s_\gamma$ is also a function since $(s_\alpha \cup s_\gamma) \restriction r$ and $(s_\alpha \cup s_\gamma) \restriction (\kappa \setminus r)$ are functions with a disjoint domain, and thus their union is also a function.

□

Corollary 4.4.10. *The boolean algebra* $\mathsf{RO}(2^{\omega_2 \times \omega})$ *has the CCC.*

Proof. By Lemma 4.4.4, $P = Fn(\omega_2 \times \omega, 2)$ embeds as a dense suborder of $\mathsf{RO}(2^{\omega_2 \times \omega})$ via the map $s \mapsto N_s$. In particular $\mathsf{RO}(P)$ and $\mathsf{RO}(2^{\omega_2 \times \omega})$ are isomorphic boolean algebras, by Theorem 3.2.5. We conclude that $\mathsf{RO}(2^{\omega_2 \times \omega})$ is CCC using Lemma 4.4.4 for $\mathsf{RO}(P)$. □

Chapter 5

Boolean Valued Models

This chapter consists of three sections:

1. In the first section we give the formal definition of boolean semantic for any first order language, and we present the soundness theorem for the semantic for the language of set theory. The boolean valued semantic selects a given complete boolean algebra B and assigns to every statement ϕ a boolean value in B. The boolean operations will reflect the behavior of the propositional connectives; it will require more of attention to give a meaning to atomic formulae and to quantifiers, and we need that B has an high degree of completeness in order to be able to interpret quantifiers in boolean semantics. The standard Tarski semantics will be recovered when we choose the boolean algebra $\{0,1\}$ as B.

2. The second section carves a bit more into the theory of B-valued models M and their Tarski quotient $M/_G$ induced by an ultrafilter $G \in \mathrm{St}(\mathsf{B})$. We supply some guiding examples of such models, among which we analyze the space of analytic functions over the real numbers $C^\omega(\mathbb{R})$. We show that this is a boolean valued model which is not properly behaving, this will lead us to the key property of *fullness*.

3. In the third section we state a necessary and sufficient condition (that of being a *full* model) on a B-valued model M which gives a complete control on how truth in M/G is determined by the topological properties of G as a point of $\mathrm{St}(\mathsf{B})$ via a Łoś theorem for full boolean valued models. We also prove a version of the Forcing theorem relating the boolean value of a formula ϕ in a B-valued model M to the topological density of the family of G such that $M/G \models \phi$. We then provide three interesting distinct examples of full boolean valued models and obtain that Łoś theorem for ultraproducts $\prod_{x\in X} M_x/G$ of Tarski models M_x by an ultrafilter G on $\mathcal{P}(X)$ is a special application of the Łoś theorem for full $\mathcal{P}(X)$-valued models. We also introduce Cohen's forcing relation on a B-valued model M and compare it to the B-valued semantics for M.

5.1 Boolean valued models and boolean valued semantics

In this section we give the formal definition of a boolean valued model for any first order *relational* language (i.e. a language containing non function symbols), and we introduce a sound semantic for these languages. We limit ourselves to analyze relational languages to avoid some technicalities arising in the semantical interpretation of function symbols in boolean valued models.

Definition 5.1.1. Let $\mathcal{L} = \{R_i : i \in I, c_j : j \in J\}$ be a language with no function symbol (a relational language in the sequel) and B a Boolean algebra. A B-*valued model* $\mathfrak{M}$ for $\mathcal{L}$ consists of:

1. A non-empty set M. The elements of M are called *names*.
2. The Boolean value of the equality symbol. That is, a function
$$\begin{aligned} M^2 &\longrightarrow \mathsf{B} \\ \langle \tau, \sigma \rangle &\longmapsto [\![\tau = \sigma]\!]^{\mathfrak{M}}_{\mathsf{B}} \end{aligned}.$$
3. The interpretation of symbols in $\mathcal{L}$. That is:
   - for each n-ary relation symbol $R \in \mathcal{L}$, a function
   $$\begin{aligned} M^n &\longrightarrow \mathsf{B} \\ \langle \tau_1, \dots, \tau_n \rangle &\longmapsto [\![R(\tau_1, \dots, \tau_n)]\!]^{\mathfrak{M}}_{\mathsf{B}} \end{aligned};$$
   - for each constant symbol $c \in \mathcal{L}$, a name $c^{\mathfrak{M}} \in M$.

We require that the following conditions hold:

1. For all $\tau, \sigma, \pi \in M$,
$$[\![\tau = \tau]\!]^{\mathfrak{M}}_{\mathsf{B}} = \mathbb{1}, \tag{5.1}$$
$$[\![\tau = \sigma]\!]^{\mathfrak{M}}_{\mathsf{B}} = [\![\sigma = \tau]\!]^{\mathfrak{M}}_{\mathsf{B}}, \tag{5.2}$$
$$[\![\tau = \sigma]\!]^{\mathfrak{M}}_{\mathsf{B}} \wedge [\![\sigma = \pi]\!]^{\mathfrak{M}}_{\mathsf{B}} \leq [\![\tau = \pi]\!]^{\mathfrak{M}}_{\mathsf{B}}. \tag{5.3}$$
2. If $R \in \mathcal{L}$ is an n-ary relation symbol, for all $\langle \tau_1, \dots, \tau_n \rangle, \langle \sigma_1, \dots, \sigma_n \rangle \in M^n$,
$$\left(\bigwedge_{i=1}^{n} [\![\tau_i = \sigma_i]\!]^{\mathfrak{M}}_{\mathsf{B}} \right) \wedge [\![R(\tau_1, \dots, \tau_n)]\!]^{\mathfrak{M}}_{\mathsf{B}} \leq [\![R(\sigma_1, \dots, \sigma_n)]\!]^{\mathfrak{M}}_{\mathsf{B}}. \tag{5.4}$$

We define now the semantic of a boolean valued model: assume we have fixed an $\mathcal{L}$-structure M, its Tarski semantic can be seen as a function that takes a $\mathcal{L}$-statement φ and assigns 1 or 0 to φ according to the fact that $M \vDash \varphi$ or $M \nvDash \varphi$. We want to generalize this framework letting this evaluation function be defined on arbitrary B-valued models while assigning its values inside B. To deal with the semantics of quantifiers we need to evaluate the formule in $\mathsf{RO}(\mathsf{B})$ rather than B, however only a certain amount of completeness on B and M is needed to assign a correct truth value to all formulae. We adopt the following strategy to define the semantics of a boolean valued structure for $\mathcal{L}$:

- Given $\langle M, =^M, R_i^M : i \in I\rangle$ B-valued model for a relational language $\mathcal{L} = \{R_i : i \in I\}$, we expand $\mathcal{L}$ to $\mathcal{L}_M = \mathcal{L} \cup \{c_a : a \in M\}$ adding constant symbols for all elements of M so that c_a is always assigned to a. In such a way we can interpret in M formulae with constant symbols in the place of free variables.
- $FRV(\mathcal{L})$ denotes the set of free variables for the formulae of the language $\mathcal{L}$, and any map $\nu : FRV(\mathcal{L}) \to M$ is an assignment.
- Given an assignment ν, a free variable x, and $b \in M$, $\nu_{x/b}$ denotes the assignment ν' such that $\nu'(y) = \nu(y)$ for all $y \neq x$ in $FRV(\mathcal{L})$ and such that $\nu'(x) = b$.
- If $\bar{y} = (y_0, \dots, y_{n-1})$ is an n-tuple of free variables $\nu(\bar{y})$ is a short-hand for $(\nu(y_0), \dots, \nu(y_{n-1}))$.
- If $\bar{a} = (a_0, \dots, a_{n-1})$ is an n-tuple of elements of M $c_{\bar{a}}$ is a short-hand for the n-tuple of constant symbols of $\mathcal{L}_M$ $(c_{a_0}, \dots, c_{a_{n-1}})$.

Definition 5.1.2. Let $\mathfrak{M} = \langle M, =_M, R_i^M : i \in I\rangle$ be a B-valued model for the relational language $\mathcal{L} = \{R_i : i \in I\}$.

We identify B as a dense[1] subalgebra of $\mathsf{RO}(\mathsf{B})$ and evaluate all formulae of $\mathcal{L}_M$ without free variables (but possibly with constant symbols) as follows:

- $[\![R(c_{a_1}, \dots, c_{a_n})]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})} = R_i^{\mathfrak{M}}(a_1, \dots, a_n)$.
- $[\![\varphi \wedge \psi]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})} = [\![\varphi]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})} \wedge_{\mathsf{RO}(\mathsf{B})} [\![\psi]\!]^{\mathfrak{M}}_{\mathsf{B}}$.
- $[\![\neg\varphi]\!]^{\mathfrak{M}}_{\mathsf{B}} = \neg_{\mathsf{B}} [\![\varphi]\!]^{\mathfrak{M}}_{\mathsf{B}}$.
- $[\![\varphi \to \psi]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})} = \neg_{\mathsf{B}} [\![\varphi]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})} \vee_{\mathsf{B}} [\![\psi]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})}$.
- $[\![\exists x \varphi(x, c_{\bar{a}})]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})} = \bigvee_{b \in M} [\![\varphi(c_b, c_{\bar{a}})]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})}$.
- $[\![\forall x \varphi(x, c_{\bar{a}})]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})} = \bigwedge_{b \in M} [\![\varphi(c_b, c_{\bar{a}})]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})}$.

If $\phi(x_1, \dots, x_n)$ is a formula of $\mathcal{L}_M$ with free variables $x_1, \dots, x_n$ and ν is an assignment, we let $\nu(\phi(x_1, \dots, x_n)) = \left[\!\left[\phi(c_{\nu(x_1)}, \dots, c_{\nu(x_n)})\right]\!\right]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})}$.

$\mathfrak{M}$ is a *well behaved* B-valued model if $[\![\phi]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})} \in \mathsf{B}$ for all $\mathcal{L}_M$-sentence ϕ.

To simplify notation we shall confuse from now on the constant symbol $c_a \in \mathcal{L}_M$ with its intended interpretation $a \in M$. When working with well behaved B-valued models, we write henceforth $[\![\phi]\!]^{\mathfrak{M}}_{\mathsf{B}}$ rather than $[\![\phi]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})}$. We also feel free to omit subscripts and superscripts if no confusion on the intended meaning can arise.

Remark 5.1.3. Some comments:

- The definition of $[\![\exists x \varphi(x, \bar{a})]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})}$ and $[\![\forall x \varphi(x, \bar{a})]\!]^{\mathfrak{M}}_{\mathsf{RO}(\mathsf{B})}$ requires the evaluation to take values possibly not in B. This motivates the definition of well behaved boolean valued model.

[1] E.g. B^+ seen as a partial order is a dense subset of $\mathsf{RO}(\mathsf{B})^+$, which is the case by Cor. 3.2.8 since $(\mathsf{B}^+, \leq)$ is a separative partial order.

- Clearly the definitions of $[\![\phi \vee \psi]\!]^M_{\mathsf{B}}$ and $[\![\phi \to \psi]\!]^M_{\mathsf{B}}$ is redundant once we have defined $[\![\neg\varphi]\!]^M_{\mathsf{B}}$ and $[\![\varphi \wedge \psi]\!]^M_{\mathsf{B}}$. Also $[\![\forall x\varphi(x, \bar{a})]\!]^M_{\mathsf{B}}$ is redundant once we have defined $[\![\neg\varphi]\!]^M_{\mathsf{B}}$ and $[\![\exists x\varphi(x, \bar{y})]\!]^M_{\mathsf{B}}$.
- If $\mathsf{B} = \{0, 1\}$, the semantic we have just defined is the usual Tarski semantic for first order logic.

We conclude this section showing that the semantic we just defined is a natural generalization of Tarski semantic which is sound with respect to first order calculus.

Definition 5.1.4. A statement φ in the language $\mathcal{L}$ is *valid* in a boolean valued model M for $\mathcal{L}$ and the boolean algebra B if $[\![\varphi]\!] = 1_{\mathsf{B}}$. A theory T is valid in M if every axiom $\varphi \in T$ is valid.

Theorem 5.1.5. (Soundness Theorem) *Let $\mathcal{L}$ be a relational first order language. If a $\mathcal{L}$-formula φ is provable syntatically by a $\mathcal{L}$-theory T, and T is valid in a B-valued model M, then $\nu(\varphi) = 1_{\mathsf{B}}$ for all assignments $\nu : FRV(\mathcal{L}) \to M$.*

To prove the theorem we first need two basic results on boolean algebras:

Exercise 5.1.6. In a boolean algebra B, for any $a, b \in \mathsf{B}$:

$$a \leq b \Leftrightarrow \neg a \geq \neg b.$$

Exercise 5.1.7. Let B be a boolean algebra and define the operation $u \to v = \neg u \vee v$ for $u, v \in \mathsf{B}$. Then

$$u \to v \geq w \Leftrightarrow u \wedge w \leq v.$$

We now prove the soundness theorem:

Proof. First of all we have to fix a deductive system for first order calculus. We choose the following which is taken (with slight modifications) from [11, Section 2.6].

Axioms

1. $x = x$.
2. $\varphi(a) \to \exists x\varphi(x)$.
3. $x = y \to [\varphi(x) \to \varphi(y)]$.

Rules

4. $\varphi \vdash \varphi \vee \psi$.
5. $\varphi \vee \varphi \vdash \varphi$.
6. $(\varphi \vee (\psi \vee \chi)) \vdash ((\varphi \vee \psi) \vee \chi)$.
7. $\varphi \vee \psi, \neg\varphi \vee \chi \vdash \psi \vee \chi$.
8. $\forall x(\varphi(x) \to \psi) \vdash (\exists x\varphi(x)) \to \psi$.

We first prove that for all assignments $\nu : FRV \to M$, and all axioms ϕ in the above list $\nu(\phi) = 1_{\mathsf{B}}$. Regarding the rules, we prove for any rule that

$$\varphi \vdash \psi \Rightarrow [\![\varphi]\!] \leq [\![\psi]\!] .$$

The proof is rather straightforward, we sketch some of its parts:

1. $x = x$. It follows by the definition of boolean valued model that

   $$[\![a = a]\!] = 1_{\mathsf{B}}$$

   for all $a \in M$. We thus get that $\nu(x = x) = 1_{\mathsf{B}}$ for all valuations ν and free variables x.

2. $\varphi(a) \to \exists x \varphi(x)$. We have by definition $[\![\exists x \varphi(x)]\!] = \bigvee_{b \in M} [\![\varphi(b)]\!] \geq [\![\varphi(a)]\!]$, so we conclude using Exercise 5.1.7.

3. $x = y \to [\varphi(x) \to \varphi(y)]$. By Exercise 5.1.7 it is sufficient to show that

   $$[\![a = b]\!] \leq [\![\varphi(a) \leftrightarrow \varphi(b)]\!]$$

   or equivalently

   $$[\![a = b]\!] \wedge [\![\varphi(a)]\!] = [\![a = b]\!] \wedge [\![\varphi(b)]\!]$$

   for all $a, b \in M$. This is proved by induction on the complexity of φ, noticing that for atomic formulae this follows by the definition of boolean valued model. Let in what follows

   $$\nu = (a_1, \ldots, a_{i-1}, a, a_{i+1} \ldots, a_n), \nu' = (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \in M^n.$$

   Negation: If $\phi \equiv \neg\psi$, by induction we have

   $$[\![a = b]\!] \wedge [\![\psi(\nu)]\!] = [\![a = b]\!] \wedge [\![\psi(\nu')]\!]$$

   which clearly holds if and only if

   $$[\![a = b]\!] \wedge [\![\neg\psi(\nu)]\!] = [\![a = b]\!] \wedge [\![\neg\psi(\nu')]\!] .$$

   Conjunction: If $\phi \equiv \psi \wedge \theta$ we have:

   $$[\![a = b]\!] \wedge [\![\phi(\nu)]\!] = ([\![a = b]\!] \wedge [\![\psi(\nu)]\!]) \wedge ([\![a = b]\!] \wedge [\![\theta(\nu)]\!]) =$$

   $$([\![a = b]\!] \wedge [\![\psi(\nu')]\!]) \wedge ([\![a = b]\!] \wedge [\![\theta(\nu')]\!]) = [\![a = b]\!] \wedge [\![\phi(\nu')]\!]$$

   Existential: If $\phi(x_1, \ldots, x_n) \equiv \exists y \psi(y, x_1, \ldots, x_n)$ we have that:

   $$\begin{aligned} [\![a = b]\!] \wedge [\![\phi(\nu)]\!] &= \bigvee_{c \in M} ([\![\psi(y/c, \nu)]\!] \wedge [\![a = b]\!]) = \\ &= \bigvee_{c \in M} ([\![\psi(y/c, \nu')]\!] \wedge [\![a = b]\!]) = \\ &= [\![\phi(\nu')]\!] \wedge [\![a = b]\!] . \end{aligned}$$

4. $\varphi \vdash \varphi \vee \psi$. Immediate since $u \vee v \geq u$ for all $u, v \in \mathsf{B}$.

5. $\varphi \vee \varphi \vdash \varphi$. Immediate since $u \vee u = u$ for all $u \in \mathsf{B}$.

6. $(\varphi \vee (\psi \vee \chi)) \vdash ((\varphi \vee \psi) \vee \chi)$. Immediate since $[\![(\varphi \vee (\psi \vee \chi))]\!] = [\![((\varphi \vee \psi) \vee \chi)]\!]$.

7. $\varphi \vee \psi, \neg\varphi \vee \chi \vdash \psi \vee \chi$.

   This follows easily form the following exercise on boolean algebras:

   Exercise 5.1.8. Show that for all a, b, c in a boolean algebra B

   $$(a \vee b) \wedge (\neg a \vee c) \leq b \vee c.$$

8. $\forall x(\varphi(x) \to \psi) \vdash (\exists x \varphi(x)) \to \psi$.

   $$[\![\forall x(\varphi(x) \to \psi)]\!] = [\![\forall x(\neg\varphi(x) \vee \psi)]\!] = \bigwedge_{b \in M} ([\![\neg\varphi(b)]\!] \vee [\![\psi]\!]) =$$

   (using the fact that x is not free in ψ)

   $$= (\bigwedge_{b \in M} [\![\neg\varphi(b)]\!]) \vee [\![\psi]\!] = (\neg \bigvee_{b \in M} [\![\varphi(b)]\!]) \vee [\![\psi]\!] = [\![(\exists x \varphi(x)) \to \psi]\!] \, .$$

The proof is complete. □

Regarding the completeness theorem for the boolean valued semantics, we have it automatically since (as we already observed) the Tarski models are a subfamily of the boolean valued models. All in all we have:

Theorem 5.1.9. (Soundess and Completeness) *Let $\mathcal{L}$ be a relational first order language. A $\mathcal{L}$-formula φ is provable syntatically by a $\mathcal{L}$-theory T if and only if for all boolean algebras B $\nu(\varphi) \geq \nu(\psi)$ for every assignment $\nu : FRV(\mathcal{L}) \to M$ on a B-valued model M for $\mathcal{L}$ and every $\psi \in T$.*

5.2 Examples of boolean valued models: boolean valued extensions of $\mathbb{R}$

We start to introduce the main ideas behind the forcing method making an excursion in other areas of mathematics and borrowing our language and terminology from analysis and sheaf theory.

First of all we need to introduce the definition of morphism between B-valued models:

Definition 5.2.1. Fix a relational language $\mathcal{L} = \{R_i : i \in I, c_j : j \in J\}$. Let $i : \mathsf{B} \to \mathsf{C}$ be an homomorphism of boolean algebras. Let $\mathfrak{M}$ be a well behaved B-valued model for $\mathcal{L}$ with domain M, and $\mathfrak{N}$ be a well behaved C-valued model for $\mathcal{L}$ with domain N.

- $k : M \to N$ is an *i-morphism* if for all $R \in \mathcal{L}$ of arity n and $a_1, \ldots, a_n \in M$
$$[\![R\ (k(a_1), \ldots k(a_n))]\!]^{\mathfrak{N}}_{\mathsf{C}} \geq i([\![R\ (a_1, \ldots a_n)]\!]^{\mathfrak{M}}_{\mathsf{B}}),$$
and for all $a, b \in M$
$$[\![k(a) = k(b)]\!]^{\mathfrak{N}}_{\mathsf{C}} \geq i([\![a = b]\!]^{\mathfrak{M}}_{\mathsf{B}}),$$
for all $a, b \in M$.
- $k : M \to N$ is an *i-embedding* if all the above inequalities are reinforced to equalities.
- k is an isomorphism if i is an isomorhism and for all $b \in N$ there is $a \in M$ such that
$$[\![k(a) = b]\!]^{\mathfrak{N}}_{\mathsf{B}} = 1_{\mathsf{B}}.$$

Exercise 5.2.2. Show that an Id_2-morphism (for $Id_2 : 2 \to 2$ the identity map) is a morphism of Tarski models in the classical sense, and similarly for embeddings, and isomorphisms.

Consider the dense linear order $(\mathbb{R}, <)$. Recall that the theory of dense linear orders without maximum and minimum admits quantifier elimination, so if we want to study the first order properties of the models of this theory we need just to look at the quantifier free formulae. In any case even ignoring this property of the theory of dense linear orders, focusing on the analysis of the quantifier free formulae which holds in $(\mathbb{R}, <)$ gives an idea of how we can employ boolean valued models to enlarge the domain of certain given first order structures.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *analytic* in $\mathbb{R}$ if and only if for every $x_0 \in \mathbb{R}$
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^n.$$

Let $C^\omega(\mathbb{R})$ be the set of all analytic functions over $\mathbb{R}$. Let $\mathsf{B} = \mathsf{RO}(\mathbb{R})$ the boolean algebra of regular open sets of the real line. We aim to see $C^\omega(\mathbb{R})$ as a $\mathsf{RO}(\mathbb{R})$-*boolean valued extension* of $\mathbb{R}$ which naturally contains $\mathbb{R}$ as a substructure. First of all we need to say what is the boolean value that $C^\omega(\mathbb{R})$ gives to the formula $x < y$ when $x \mapsto f$ and $y \mapsto g$. A natural answer is the following:

> $b \in \mathsf{B}$ *forces* that $f < g$ if the set of $x \in b$ such that $f(x) < g(x)$ is an open dense subset of b (recall that b is a regular open subset of $\mathbb{R}$).

For example let $f(x) = sin(x)$ and $g(x) = -1$ for all x, then $\mathbb{R}$ forces that $g < f$ since the set of points $x \in \mathbb{R}$ on which $f(x) \leq g(x)$ is closed and nowhere dense. On the other hand if $f(x) = sin(x)$ and $g(x) = cos(x)$ we have that $f(x) < g(x)$ if and only if $x \in (\pi/4 + 2k \cdot \pi, 5 \cdot \pi/4 + 2k\pi)$ as k ranges in $\mathbb{Z}$. Notice that the above set is open regular and thus
$$a = \bigcup_{k \in \mathbb{Z}} (\pi/4 + 2k\pi, 5/4\pi + 2k\pi)$$

is the largest regular open subset of $\mathbb{R}$ which forces $sin(x) < cos(x)$. Notice that the complement $\neg_{\mathsf{B}} a$ in B of this set is exactly the set of points on which $cos(x) < sin(x)$ and that what is left out by $a \cup \neg_{\mathsf{B}} a$ is the closed nowhere dense set of points in which $f(x) = g(x)$ which are the extremes of the intervals defining a.

Guided by this example we can now give an interpretation of the forcing relation and a precise meaning to formulae with parameters in $C^\omega(\mathbb{R})$ as follows:

- $[\![f\ R\ g]\!]$ is the largest regular open set a such that the set of $x \in a$ on which $f(x)\ R\ g(x)$ is an open dense subset of a for R any relation among $<, =$.
- $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$.
- $[\![\neg\phi]\!] = \neg_{\mathsf{B}} [\![\phi]\!]$.

Now we are left to see that $\mathbb{R}$ can be copied inside $C^\omega(\mathbb{R})$. The natural idea is that $\mathbb{R}$ is "represented inside $C^\omega(\mathbb{R})$" by the constant functions $c_a(x) = a$ for all $a \in \mathbb{R}$. Indeed we can check that aRb holds in $\mathbb{R}$ iff $[\![c_a\ R\ c_b]\!] = \mathbb{R}$ for any binary relation R among $<, =$. So we get that essentially any of the above relations holds on two real numbers iff $\mathbb{R}$ forces the corresponding relation to hold of the corresponding constant functions. In particular the map $a \mapsto c_a$ defines an i-morphism of the 2-valued model $(\mathbb{R}, <)$ in the B-valued model $(C^\omega(\mathbb{R}), <_{\mathsf{B}})$ where $i : 2 \to \mathsf{B}$ is the unique complete homomorphism and has to map $j \mapsto j_{\mathsf{B}}$ for $j = 0, 1$.

Finally we want to show that this boolean expansion of $\mathbb{R}$ does not overlap with $\mathbb{R}$: a natural way to say this is to find some function which is forced by 1_{B} to be different from all costant functions. It is easily seen that the sinus function or any analytic function which is nowhere locally costant has this property. In particular our $\mathsf{RO}(\mathbb{R})$-boolean expansion $C^\omega(\mathbb{R})$ of $\mathbb{R}$ appears to have added lmany new elements with respect to $\mathbb{R}$.

This describes the passage from a first order structure M to an associated boolean valued model M^{B}, which in this case is given by the analytic functions on $\mathbb{R}$. However there is a disturbing issue of this boolean expansion, i.e. that we are not able to decide many basic facts, for example is $sin(x) < cos(x)$? We have already seen that the boolean value $[\![sin(x) < cos(x)]\!]$ and $[\![sin(x) > cos(x)]\!]$ are both positive while $[\![sin(x) = cos(x)]\!] = 0_{\mathsf{B}}$. In particular the boolean expansion already carries enough information to decide whether $sin(x)$ and $cos(x)$ represent different objects, but is not yet able to decide whether $sin(x) < cos(x)$ or the other way round. If we choose to restrict our attention to a small intervall like $(\pi/4, 5/4 \cdot \pi)$, this interval will *force* that $sin(x) < cos(x)$, but it will not yet be able to decide other basic relations among other functions, for example whether $sin(2x) < cos(2x)$. Making our interval smaller and smaller we end up "forcing" more and more properties regarding the mutual relationship between functions in $C^\omega(\mathbb{R})$. It seems that if we take a decreasing sequence of intervals $\{I_n : n \in \mathbb{N}\}$ with diameter converging to 0 and such that $\mathsf{Cl}(I_{n+1}) \subseteq I_n$, in the limit the unique point $x \in \bigcap_n I_n$ will be able to decide all basic relations among the analytic functions. This is not yet the case though: for example no open neighborhood of $\pi/4$ forces $sin(x) < cos(x)$ and no open neighborhood of x forces $sin(x) > cos(x)$. So actually in order to be able to decide all basic relations on the elements of $C^\omega(\mathbb{R})$ it is not enough to select a point and look at the filter of all of its regular open neighborhoods, we really need to select a ultrafilter on $\mathsf{RO}(\mathbb{R})$. In this case, since for all $f, g \in C^\omega(\mathbb{R})$ we have that

$$[\![f < g]\!] \vee [\![f > g]\!] \vee [\![f = g]\!] = 1_{\mathsf{B}},$$

we will have that any $G \in \mathrm{St}(\mathsf{B})$ will always be able to decide whether $f < g$, $f = g$ or $f > g$ for all $f, g \in C^\omega(\mathbb{R})$. Moreover any such $G \in \mathrm{St}(\mathsf{RO}(\mathbb{R}))$ will always choose a unique point $G_x \in \mathbb{R} \cup \{\pm\infty\}$ which will be the unique point in

$$\bigcap\{\mathsf{Cl}(A) : A \in G\}.$$

Note however that the same point can be associated to incompatible ultrafilters on $\mathsf{RO}(\mathbb{R})$, e.g. let G be a ultrafilter extending the regular open neighborhoods of $\pi/4$ with $(-\infty; \pi/4)$ and H extend the same filter of neighborhoods with$(\pi/4; +\infty)$.

5.3 Quotients of boolean models, fullness, Łoś theorem

This section explores the notion of quotient of a boolean valued mdoel and characterize by means of the fullness property the boolean valued modles whose semantics behaves properly with respect to quotients.

Definition 5.3.1. Let B be a boolean algebra and let $\mathcal{L} = \{R_i : i \in I, c_j : j \in J\}$ be a relational language where R_i is a m_i-ary relation symbol for every $i \in I$ and each c_j is a constant symbol. Suppose that $\mathfrak{M} = (M, R_i : i \in I, c_j : j \in J)$ is a B-model for $\mathcal{L}$. Let F be a filter on B. The *F-quotient* $\mathfrak{M}/_F = (M/_F, R_i/_F : i \in I, [c_j^{\mathfrak{M}}]_F : j \in J)$ is defined as follows:

- $M/_F = \{[h]_F : h \in M\}$ where $[h]_F = \{f \in M : [\![f = h]\!] \in F\}$;
- $R_i/_F([f_1]_F, \ldots, [f_{m_i}]_F)$ holds if and only if $[\![R_i(f_1, \ldots, f_n)]\!] \in F$ for every $i \in I$.

When G is a ultrafilter on B we say that $\mathfrak{M}/_G$ is the *Tarski quotient of* $\mathfrak{M}$ *by* G.

Exercise 5.3.2. Check that the F-quotient of a B-valued model is a well defined $\mathsf{B}/_F$-valued model; hence it is a Tarski model when F is a ultrafilter.

5.3.1 Examples of quotients

The process we described in the previous section is rather flexible and can accomodate many first order structure defined on the domain $\mathbb{R}$ (or even on many other domains, as we shall see below). For example we could repeat verbatim the same construction for the structure:

$$(\mathbb{R}, \mathbb{Z}, 0, 1, +, \cdot, <)$$

to obtain the $\mathsf{RO}(\mathbb{R})$-boolean expansion:

$$(C^\omega(\mathbb{R}), \mathbb{Z}_{\mathsf{RO}(\mathbb{R})}, c_0, c_1, +_{\mathsf{B}}, \cdot_{\mathsf{B}}, <)$$

where $+$ and $\cdot$ are interpreted by the ternary relations of their respective graphs, $+_{\mathsf{B}}$ is the ternary boolean relation

$$(f, g, h) = \mathsf{Reg}(\{x \in \mathbb{R} : f(x) + g(x) = h(x)\}),$$

and similarly for $\cdot_{\mathsf{B}}$. $\mathbb{Z}_{\mathsf{B}}$ is the predicate assigning to each $f \in C^{\omega}(\mathbb{R})$ the boolean value

$$[\![\mathbb{Z}_{\mathsf{RO}(\mathbb{R})}(f)]\!] = \mathsf{Reg}(\{x \in \mathbb{R} : \ f \restriction a \text{ is locally constant with value in } \mathbb{Z} \text{ for some open } a \ni x\}).$$

The latter predicate has either value $\mathbb{R}$ or $\emptyset$, in any case an open regular subset of $\mathbb{R}$. Here we use a specific property of analytic functions: an analyitc function is constant if and only if it is locally constant in some open set of its domain.

We can also check that for all $G \in \mathrm{St}(\mathsf{RO}(\mathbb{R}))$

$$(C^{\omega}(\mathbb{R})/_G, \mathbb{Z}_{\mathsf{B}}/_G, [c_0]_G, [c_1]_G, +_{\mathsf{B}}/_G, \cdot_{\mathsf{B}}/_G, <_{\mathsf{B}}/_G)$$

is also an ordered ring with a distinguished predicate $\mathbb{Z}_{\mathsf{B}}/_G$.

We let the map $i_G : \mathbb{R} \to C^{\omega}(\mathbb{R})/_G$ be defined by $a \mapsto [c_a]_G$. Then it is not hard to check that i_G is an injective homomorphism of rings which preseves the order relation and is also such that $i_G[\mathbb{Z}] = \mathbb{Z}_{\mathsf{B}}/_G$.

Exercise 5.3.3. Prove in detail all the above facts about the structures

$$(C^{\omega}(\mathbb{R}), \mathbb{Z}_{\mathsf{B}}, 0, 1, +_{\mathsf{B}}, \cdot_{\mathsf{B}}, <_{\mathsf{B}}).$$

and

$$(C^{\omega}(\mathbb{R})/_G, \mathbb{Z}_{\mathsf{B}}/_G G, [0]_G, [1]_G, +_{\mathsf{B}}/_G, \cdot_{\mathsf{B}}/_G, <_{\mathsf{B}}/_G).$$

More precisely let $\mathsf{RO}(\mathbb{R})$ be the complete boolean algebra of regular open subset of $\mathbb{R}$ and show that the map $i : \mathbb{R} \to C^{\omega}(\mathbb{R})$ sending $a \mapsto c_a$ is an i-embedding of B-valued models, where $i : 2 \to \mathsf{B}$ is the unique embedding of 2 into B. Show also that $i_G(a) = [a]_G$ defines an injective morphism of ordered rings such that $i_G[\mathbb{Z}] = \mathbb{Z}_{\mathsf{B}}/G$ *(HINT: notice that an analytic function is locally constant with value in $\mathbb{Z}$ iff it is everywhere constant with the same value; the left to right inclusion does not require this property the right to left inclusion does).*

Exercise 5.3.4. Let $C(\mathbb{R})$ be the family of continuous real valued functions and B be the cba given by the regular open sets of $\mathbb{R}$ with usual euclidean topology. Show that

$$(C(\mathbb{R}), \mathbb{Z}_{\mathsf{B}}, 0, 1, +_{\mathsf{B}}, \cdot_{\mathsf{B}}, <_{\mathsf{B}}).$$

is a B-valued model (where the definition of the additional predicates are the same as in the previous exercise but now apply to continuous functions rather than just analytic functions). Show also that for some (actually any) ultrafilter G on B, $i_G[\mathbb{Z}]$ is a proper subset of $\mathbb{Z}_{\mathsf{B}}/_G$ *(HINT: for this strict inclusion note that if one chooses G an ultrafilter which concentrates on $\bigcup_{n\in\mathbb{Z}}(2n; 2n+1)$ and chooses f to be locally constant on the interval $(2n; 2n+1)$ with value n and a translate of the identity on the intervals $(2n+1; 2n+2)$, then $[f]_G \in \mathbb{Z}_{\mathsf{B}}/_G$ but f is not in $i_G[\mathbb{R}]$).*

5.3.2 Counterexamples

We have no reasons to expect that a formula which is not quantifier free true in a B-valued model $\mathfrak{M}$ is also true in $\mathfrak{M}/_G$, for some $G \in \mathrm{St}(\mathsf{B})$. In general this is false, as the following example shows:

Example 5.3.5. Fix the language $\mathcal{L} = \{<, C\}$ consisting of two relation symbol, where $<$ is binary and C is unary. Let $\mathsf{B} = \mathsf{RO}(\mathbb{R})$ and consider the B-valued model for the language $\mathcal{L}$ given by $\mathfrak{M} = (C^\omega(\mathbb{R}), =, <_\mathsf{B}, C_\mathsf{B})$ with the following interpretation of the atomic formulae:

$$\begin{aligned} [\![f = g]\!] =& \mathsf{Reg}(\{x \in \mathbb{R} : f(x) = g(x)\}), \\ [\![f <_\mathsf{B} g]\!] =& \mathsf{Reg}(\{x \in \mathbb{R} : f(x) < g(x)\}), \\ [\![C_\mathsf{B}(f)]\!] =& \mathsf{Reg}(\bigcup\{U : f \restriction_U \text{ is constant}\}). \end{aligned}$$

We leave to the reader to check that $(C^\omega(\mathbb{R}), =, <_\mathsf{B}, C_\mathsf{B})$ is a B-valued model. Now, fix any $f \in C^\omega(\mathbb{R})$ and look at the formula $\phi := \exists y\, (f < y \wedge C(y))$.

$$\begin{aligned} [\![\exists y\, (f <_\mathsf{B} y \wedge C_\mathsf{B}(y))]\!] =& \bigvee_{g \in C^\omega(\mathbb{R})} [\![f <_\mathsf{B} g \wedge C_\mathsf{B}(g)]\!] \geq \\ =& \bigvee_{a \in \mathbb{R}} [\![f <_\mathsf{B} c_a]\!] \wedge [\![C_\mathsf{B}(c_a)]\!] \geq \\ & \qquad\qquad \text{where } c_a \text{ is the constant function } c_a(x) = a \\ =& \bigvee_{a \in \mathbb{R}} [\![f <_\mathsf{B} c_a]\!] \geq \\ \geq& \bigvee_{n \in \mathbb{Z}} [\![f < c_{a_n}]\!] \geq \qquad \text{where } a_n = \max(f \restriction [n-1; n]) + 1 \\ \geq& \mathsf{Reg}(\bigcup_{n \in \mathbb{Z}} (n-1; n)) = \mathbb{R}. \end{aligned}$$

Therefore, we have that $\mathfrak{M} \models \phi(f)$ and in particular

$$\mathfrak{M} \models \exists y\, (id_\mathbb{R} < y \wedge C(y)),$$

where $id_\mathbb{R}$ is the identity function $x \mapsto x$.
Now, consider $F = \{(a; +\infty) : a \in \mathbb{R}\} \subseteq \mathsf{RO}(\mathbb{R})$. Since F satisfies the finite intersection property (that is, F is closed under intersection of finite subsets), we can extend F to some $G \in \mathrm{St}(\mathsf{RO}(\mathbb{R}))$. Consider the quotient $\mathfrak{M}/_G$. The identity function $id_\mathbb{R}$ has the property that for any $a \in \mathbb{R}$

$$[\![\neg(id_\mathbb{R} <_B c_a)]\!] = (a, +\infty) \in G.$$

It follows that $\mathfrak{M}/_G \models \neg\exists y\, ([id_\mathbb{R}]_G < y \wedge C(y))$.

5.3.3 Łoś theorem for full boolean valued models

Example 5.3.5 shows that quotients of boolean valued models may not preserve validity of formulae with quantifiers. To overcome this issue we are led to the definition of full boolean valued models as those boolean valued models for which the above problem does not occur. We show that fullness characterizes the preservation of satisfiability in any quotient and we give several examples of full boolean valued models.

Definition 5.3.6. Fix a language $\mathcal{L}$ and a boolean algebra B, a well behaved B-valued model $\mathfrak{M}$ for $\mathcal{L}$ is *full* if for every formula $\phi(x, \bar{y})$ and $\bar{f} \in M^{|y|}$ there are $h_1, \dots, h_k \in M$

$$\llbracket \exists x \phi(x, \bar{f}) \rrbracket^{\mathfrak{M}}_{\mathsf{B}} = \bigvee_{i=1}^{k} \llbracket \phi(h_i, \bar{a}) \rrbracket^{\mathfrak{M}}_{\mathsf{B}}.$$

Theorem 5.3.7 (Łoś theorem)**.** *Let B be a boolean algebra. Assume $\mathfrak{M}$ is a full B-valued model. For any $G \in \mathrm{St}(\mathsf{B})$, $f_1, \dots, f_n \in M$, and for all formulae $\phi(f_1, \dots, f_n)$*

$$\mathfrak{M}/_G \models \phi([f_1]_G, \dots, [f_n]_G) \quad \textit{iff} \quad \llbracket \phi(f_1, \dots, f_n) \rrbracket^{\mathfrak{M}}_{\mathsf{B}} \in G.$$

Proof. By induction on the complexity of $\phi(f_1, \dots, f_n)$.

- If $\phi(f_1, \dots, f_n) = R(f_1, \dots, f_n)$ for some relational symbol R, then

  $$\mathfrak{M}/_G \models R([f_1]_G, \dots, [f_n]_G) \quad \text{iff} \quad \llbracket R(f_1, \dots, f_n) \rrbracket^{\mathfrak{M}}_{\mathsf{B}} \in G$$

  by definition.

- If $\phi(f_1, \dots, f_n) = \psi(f_1, \dots, f_n) \wedge \chi(f_1, \dots, f_n)$, then

  $$\begin{aligned} \mathfrak{M}/_G \models \phi([f_1]_G, \dots, [f_n]_G) \quad &\text{iff} \quad \mathfrak{M}/_G \models \psi([f_1]_G, \dots, [f_n]_G), \chi([f_1]_G, \dots, [f_n]_G) \\ &\text{iff} \quad \llbracket \psi(f_1, \dots, f_n) \rrbracket^{\mathfrak{M}}_{\mathsf{B}} \in G \text{ and } \llbracket \chi(f_1, \dots, f_n) \rrbracket^{\mathfrak{M}}_{\mathsf{B}} \in G \\ &\text{iff} \quad \llbracket \psi(f_1, \dots, f_n) \wedge \chi(f_1, \dots, f_n) \rrbracket^{\mathfrak{M}}_{\mathsf{B}} \in G \end{aligned}$$

- If $\phi(f_1, \dots, f_n) = \neg\psi(f_1, \dots, f_n)$, then

  $$\begin{aligned} \mathfrak{M}/_G \models \phi([f_1]_G, \dots, [f_n]_G) \quad &\text{iff} \quad \mathfrak{M}/_G \models \neg\psi([f_1]_G, \dots, [f_n]_G) \\ &\text{iff} \quad \mathfrak{M}/_G \not\models \psi([f_1], \dots, [f_n]) \\ &\text{iff} \quad \llbracket \psi(f_1, \dots, f_n) \rrbracket^{\mathfrak{M}}_{\mathsf{B}} \notin G \\ &\text{iff} \quad \llbracket \neg\psi(f_1, \dots, f_n) \rrbracket^{\mathfrak{M}}_{\mathsf{B}} \in G \end{aligned}$$

- If $\phi(f_1, \dots, f_n) = \exists x \psi(x, f_1, \dots, f_n)$, then

  $$\begin{aligned} \mathfrak{M}/_G \models \exists x \psi(x, [f_1]_G, \dots, [f_n]_G) \quad &\text{iff} \quad \mathfrak{M}/_G \models \psi([h], [f_1]_G, \dots, [f_n]_G) \\ &\qquad\qquad \text{for some } h \in M \\ &\text{iff} \quad \llbracket \psi(h, f_1, \dots, f_n) \rrbracket^{\mathfrak{M}}_{\mathsf{B}} \in G \\ &\qquad\qquad \text{for some } h \in M \\ &\text{which implies} \quad \llbracket \exists x \psi(x, f_1 \dots, f_n) \rrbracket^{\mathfrak{M}}_{\mathsf{B}} \in G \end{aligned}$$

The viceversa also holds, by fullness: let $h_1 \ldots, h_k \in M$ be such that

$$[\![\exists x \psi(x, f_1, \ldots, f_n)]\!]^{\mathfrak{M}}_{\mathsf{B}} = \bigvee_{i=1}^{k} [\![\phi(h_i, f_1, \ldots, f_n)]\!]^{\mathfrak{M}}_{\mathsf{B}}.$$

Assuming $[\![\exists x \psi(x, f_1, \ldots, f_n)]\!]^{\mathfrak{M}}_{\mathsf{B}} \in G$, since G is a ultrafilter, there is some $j \leq k$ such that $[\![\phi(h_j, f_1, \ldots, f_n)]\!]^{\mathfrak{M}}_{\mathsf{B}} \in G$. Then by inductive assumptions

$$\mathfrak{M}/_G \models \psi([h_j]_G, [f_1]_G, \ldots, [f_n]_G),$$

which yields that

$$\mathfrak{M}/_G \models \exists x\, \psi(x, [f_1]_G, \ldots, [f_n]_G),$$

as was to be shown.

□

5.3.4 Forcing and fullness

The following Lemma outlines a fundamental link between full B-valued models and the topological properties of $\mathrm{St}(\mathsf{B})$.

Lemma 5.3.8 (Forcing lemma I)**.** *Let* B *be a boolean algebra,* $\mathfrak{M}$ *be a full* B*-model, and* $G \in \mathrm{St}(\mathsf{B})$*. Then, for any formula* ϕ *the following statements are equivalent:*

1. $[\![\phi]\!] \geq b$.
2. $D_\phi = \{G \in \mathrm{St}(\mathsf{B}) : M/G \models \phi\}$ *is dense in* N_b.
3. $D_\phi \supseteq N_b$.

Proof. $1 \Leftrightarrow 3$ This is a straightforward consequence of Łoś' theorem.

$1 \Leftrightarrow 2$ Fix some formula ϕ. Let $b \in \mathsf{B}$ such that $[\![\phi]\!] \geq b$, and assume that D_ϕ is not dense in N_b, aiming for a contradiction. Then, by our assumption, there exists some $c \in \mathsf{B}^+$, $c \leq b$ such that $N_c \neq \varnothing$ and N_c is disjoint from D_ϕ. Now for every $G \in N_c$ $\mathfrak{M}/_G \not\models \phi$. By Łoś theorem it follows that for all $G \in N_c$, $[\![\neg\phi]\!] \in G$ and thus $[\![\neg\phi]\!] \wedge c \in G$. Therefore, we have that

$$0_{\mathsf{B}} < [\![\neg\phi]\!] \wedge c \leq [\![\neg\phi]\!] \wedge b \leq [\![\neg\phi]\!] \wedge [\![\phi]\!] = 0_{\mathsf{B}},$$

which is a contradiction.

On the other hand, assume that $[\![\phi]\!] \not\geq b$, that is $[\![\neg\phi]\!] \wedge b > 0$. So, let $d = [\![\neg\phi]\!] \wedge b$ and look at N_d, which is contained in N_b. Now, observe that if $d \in G$ then also $[\![\neg\phi]\!] \in G$ which implies that $\mathfrak{M}/_G \models \neg\phi$. It follows that for every $G \in N_d$, $\mathfrak{M}/_G \models \neg\phi$ and therefore $D_\phi \cap N_d = \varnothing$. This proves that D_ϕ is not dense in N_b.

□

According to Lemma 5.3.8, if $\mathfrak{M}$ is full, to check that a formula ϕ is valid in $\mathfrak{M}$ it suffices to show that it is valid in $\mathfrak{M}/_G$ for densely-many $G \in \mathrm{St}(\mathsf{B})$.

Lemma 5.3.9 (Forcing lemma II). *Let* B *be a boolean algebra. Given a well behaved* B*-valued model* $\mathfrak{M}$ *for* $\mathcal{L}$*,* $\phi(x_0, \dots, x_n)$ *a formula of the language* $\mathcal{L}$*,* $a_0, \dots, a_n \in M$*, define:*

$$b \Vdash \phi(a_0, \dots, a_n) \text{ (to be read as } b \text{ forces } \phi(a_0, \dots, a_n))$$

iff $b \leq [\![\phi(a_0, \dots, a_n)]\!]$.

Then the following holds:

1. $b \Vdash \phi$ *iff the set of* $G \in \mathrm{St}(\mathsf{B})$ *such that* $\mathfrak{M}/_G \models \phi$ *is dense in* N_b*,*

2. $b \Vdash \phi \wedge \psi$ *iff* $b \Vdash \phi$ *and* $b \Vdash \psi$*,*

3. $b \Vdash \neg\phi$ *iff* $c \not\Vdash \phi$ *for any* $c \leq b$*,*

4. $b \Vdash \phi \vee \psi$ *iff the set of* $c \leq b$ *such that* $c \Vdash \phi$ *or* $c \Vdash \psi$ *is dense below* b *in* B^+.

5. $b \Vdash \exists x\phi(x)$ *iff the set of* $c \leq b$ *such that* $c \Vdash \phi(\sigma)$ *for some* $\sigma \in M$ *is dense below* b.

6. *For all* $G \in \mathrm{St}(\mathsf{B})$ *and all* ϕ *formulae with parameters in* M *and no free variable,* $\mathfrak{M}/_G \models \phi$ *if and only if* $b \Vdash \phi$ *for some* $b \in G$.

7. *For all* ϕ *formulae with parameters in* M $[\![\phi]\!] = \bigvee \{b : b \Vdash \phi\}$.

Proof. Left to the reader. □

Exercise 5.3.10. Show that a well behaved B-valued model $\mathfrak{M}$ is full if and only if the conclusion of Łoś theorem holds for $\mathfrak{M}$. *(HINT: it is not so trivial; there is a compactness argument to infer that* $\mathfrak{M}/_G \models \exists x\psi$ *for all* $G \in N_{[\![\exists x\psi]\!]}$ *if and only if there are* $h_1, \dots, h_k \in M$ *such that*

$$N_{[\![\exists x\psi]\!]} = \bigcup_{i=1}^{k} N_{[\![\psi[h_i/x]]\!]}.$$

5.3.5 The mixing property and fullness

Fullness for a boolean valued model $\mathfrak{M}$ for $\mathcal{L}$ is a desirable feature of $\mathfrak{M}$ but needs to be checked on the infinitely many formulae of $\mathcal{L}$. It is oftentimes simpler and convenient to establish a property of the model which implies a strong form of fullness. The mixing property gives a sufficient condition for having the fullness property which is, usually, easier to check.

Definition 5.3.11. Let κ be a cardinal, $\mathcal{L}$ be a first order language, B a κ-complete boolean algebra, $\mathfrak{M}$ a B-valued model for $\mathcal{L}$.

- $\mathfrak{M}$ satisfies the *κ-mixing property* if for every antichain $A \subset \mathsf{B}$ of size at most κ, and for every subset $\{\tau_a : a \in A\} \subseteq M$, there exists $\tau \in M$ such that $a \leq [\![\tau = \tau_a]\!]$ for every $a \in A$.

- $\mathfrak{M}$ satisfies the *$< \kappa$-mixing property* if it satisfies the λ-mixing property for all cardinals $\lambda < \kappa$.

- $\mathfrak{M}$ satisfies the *mixing property* if it satisfies the $|\mathsf{B}|$-mixing property.

In [5] models with the $< \omega$-mixing property are called models which *admit gluing*.

Whether a B-valued model $\mathfrak{M}$ for some signature $\mathcal{L}$ has the mixing property depends only on the interpretation of the equality symbol by $[\![\cdot = \cdot]\!]^{\mathfrak{M}}_{\mathsf{B}}$.

Proposition 5.3.12. *Let B be a complete boolean algebra and let $\mathfrak{M}$ be a B-valued model for $\mathcal{L}$. Assume that $\mathfrak{M}$ satisfies the κ-mixing property for some $\kappa \geq \min\{|\mathsf{B}|, |M|\}$. Then $\mathfrak{M}$ is full.*

Proof. Fix a formula $\phi(x, y_1, \ldots, y_n)$ in $\mathcal{L}$ and $\sigma_1, \ldots, \sigma_n \in M$. Fix moreover an enumeration $\langle \tau_i : i \in \gamma \rangle$ of M. Since $[\![\exists x \phi(x, \sigma_1, \ldots, \sigma_n)]\!] = \bigvee_{i \in I} [\![\phi(\tau_i, \sigma_1, \ldots, \sigma_n)]\!]$, we can refine the family $\{[\![(\phi(\tau_i, \sigma_1, \ldots, \sigma_n)]\!] : i \in \gamma\}$ to an antichain $\{a_j : j \in J\}$ as follows: let

$$J := \gamma \setminus \left\{ i \in I : [\![\phi(\tau_i, \sigma_1, \ldots, \sigma_n)]\!] \setminus \bigvee_{j<i} [\![\phi(\tau_j, \sigma_1, \ldots, \sigma_n)]\!] = 0_{\mathsf{B}} \right\}.$$

In particular, J is well-ordered with the order induced by γ and we have that $\min J = 0$. Define

$$a_0 := [\![\phi(\tau_0, \sigma_1, \ldots, \sigma_n)]\!]$$

and, for $J \ni i > 0$,

$$a_i := [\![\phi(\tau_i, \sigma_1, \ldots, \sigma_n)]\!] \setminus \bigvee_{J \ni j < i} [\![\phi(\tau_j, \sigma_1, \ldots, \sigma_n)]\!].$$

If $A := \{a_j : j \in J\}$, it is clear that $\bigvee A = [\![\exists x \phi(x, \sigma_1, \ldots, \sigma_n)]\!]$ and $|A| \leq |M|, |\mathsf{B}| \leq \kappa$. Since $\mathfrak{M}$ satisfies the κ-mixing property, there exists $\tau \in M$ such that

$$a_i \leq [\![\tau = \tau_i]\!]$$

for every $i \in J$. In particular, since $a_i \leq [\![\phi(\tau_i, \sigma_1, \ldots, \sigma_n)]\!]$, we have that

$$a_i = a_i \wedge [\![\tau = \tau_i]\!] \leq [\![\phi(\tau_i, \sigma_1, \ldots, \sigma_n)]\!] \wedge [\![\tau = \tau_i]\!] \leq [\![\phi(\tau, \sigma_1, \ldots, \sigma_n)]\!]$$

for every $i \in J$. Hence $[\![\phi(\tau, \sigma_1, \ldots, \sigma_n)]\!] \geq \bigvee_{i \in J} a_i = \bigvee A = [\![\exists x \phi(x, \sigma_1, \ldots, \sigma_n)]\!]$ and so $\mathfrak{M}$ is full. ☐

Remark 5.3.13. The result just proven actually shows that the mixing property implies a strong version of fullness, that is: for every formula $\phi(x_0, x_1, \ldots, x_n)$ and for every $\tau_1, \ldots, \tau_n \in \mathfrak{M}$ there exists an element τ_0 such that

$$\bigvee_{\sigma \in \mathfrak{M}} [\![\phi(\sigma, \tau_1, \ldots, \tau_n)]\!] = [\![\phi(\tau_0, \tau_1, \ldots, \tau_n)]\!].$$

This is actually the definition of fullness one can find for instance in [6]. It is easy to see that this property is true in every full model $\mathfrak{M}$ satisfying the $< \omega$-mixing property.

5.4 Examples of boolean valued models with the mixing property

5.4.1 Example I: spaces of measurable functions

Let

$L^{\infty+}([0,1]) = \{f : [0,1] \to \mathbb{R} \cup \{\infty\} : f$ is Lebesgue measurable and $\mu(f^{-1}[\{\infty\}]) = 0\}$.

Recall that $\mathsf{MALG} = \mathrm{Bor}([0,1])/_{\mathrm{NULL}}$ is a complete boolean algebra (cfr Corollary 3.3.6). $L^{\infty+}([0,1])$ is a natural enlargement of $L^{\infty}([0,1])$, the space of essentially bounded measurable functions (i.e those mesurable $f : [0,1] \to \mathbb{R}$ such that $\mu(\{x : |f(x)| > C\}) = 0$ for some $C > 0$).

Proposition 5.4.1. *$(L^{\infty+}([0,1]), <_{\mathsf{B}}, C_{\mathsf{B}})$ is a MALG-valued model for $\mathcal{L} = \{<, C\}$ with the mixing property.*

Proof. Let $\mathsf{B} = \mathsf{MALG}$ in what follows. Assume $\{[X_i]_{\mathrm{NULL}} : i < \omega\}$ is a maximal antichain (recall that MALG has the CCC).

Let $\{f_i : i \in \omega\}$ be a countable family of functions in $L^{\infty+}([0,1])$. W.l.o.g. we can suppose that $\{X_i : i \in \omega\}$ consists of a partition of $\mathbb{R}$ in measurable pieces.

Set $g = \bigcup_{n \in \omega} f_n$. Then $g : [0;1] \to \mathbb{R} \cup \{\infty\}$ is meaasurable and $g^{-1}[\{\infty\}] = \bigcup_{n \in \omega} f_n^{-1}[\{\infty\}]$ has measure 0, hence $g \in L^{\infty+}([0,1])$. Clearly $[\![g = f_n]\!] \geq [X_n]$ for all $n \in \omega$. $\square$

Exercise 5.4.2. Explain what goes wrong if you try to prove the same for $L^{\infty}([0,1])$ (HINT: In this case we can only guarantee that g is measurable and $g \restriction X_i \in L^{\infty}(X_i)$ but $g \restriction \bigcup_{i \in \omega} X_i$ may not be essentially bounded in $\bigcup_{i \in \omega} X_i$, use the counterexample to the fullness of $C^{\omega}(\mathbb{R})$ replacing Id with the function $x \mapsto 1/x$).

Proposition 5.4.3. *$(L^{\infty+}([0;1]), +_{\mathsf{B}}, \cdot_{\mathsf{B}}, 0, 1)$ is a MALG-model for the theory of fields.*

Proof. We just prove that $(L^{\infty+}([0;1]), +, \cdot, 0, 1)$ satisfies the existence of the inverse for every nonzero element. Let $\mathsf{B} = \mathsf{MALG}$ for ease of notation. We must show that $[\![\forall x(x \neq 0 \to \exists y(x \cdot y = 1))]\!] = 1_{\mathsf{B}}$. Since

$$[\![\forall x(x \neq 0 \to \exists y(x \cdot y = 1))]\!] = \bigwedge_{g \in L^{\infty+}([0,1])} [\![(g \neq 0) \to \exists y(g \cdot y = 1))]\!],$$

it suffices to prove that $[\![(g \neq 0) \to \exists y(g \cdot y = 1))]\!] = 1_{\mathsf{B}}$ for all $g \in L^{\infty+}([0,1])$. Fix $g \in L^{\infty+}([0,1])$ and define

$$A_0 := \{x : g(x) = 0\}$$
$$A_1 := [0,1] \setminus A_0.$$

Consider the following function

$$g^{-1}(x) = \begin{cases} 0 & \text{if } x \in A_0, \\ \frac{1}{g(x)} & \text{otherwise.} \end{cases}$$

Now, observe that $[\![g = 0]\!] = [A_0]_{\text{NULL}}$ and therefore

$$\begin{aligned}
[\![g = 0 \vee \exists y(g \cdot y = 1)]\!] =& [A_0]_{\text{NULL}} \vee [\![\exists y(g \cdot y = 1)]\!] = \\
=& [A_0]_{\text{NULL}} \vee \bigvee_{h \in L^{\infty+}} [\![g \cdot h = 1)]\!] \geq \\
\geq& [A_0]_{\text{NULL}} \vee [\![g \cdot g^{-1} = 1]\!] = \\
=& [A_0]_{\text{NULL}} \vee [A_1]_{\text{NULL}} = 1_{\mathsf{B}}.
\end{aligned}$$

□

Exercise 5.4.4. $(L^\infty([0,1]), +_{\mathsf{MALG}}, \cdot_{\mathsf{MALG}}, 0, 1)$ is also a MALG-model for the theory of fields. (HINT: show that locally any function is invertible, i.e. given f show that one can split $\{x : f(x) \neq 0\}$ in countably many pairwise disjoint sets A_n for $n \in \omega$ such that $f \restriction A_n$ has an inverse g_n which is in $L^\infty(A_n)$. The problem is that $g = \bigcup_{n\in\omega} g_n$ is measurable but possibly not in $L^\infty(\bigcup_n A_n)$. On the other hand g will always be in $L^{\infty+}(\bigcup_n A_n)$....).

Exercise 5.4.5. Prove that the model of exercise 5.4.4 is not full *(HINT: show that the quotients $L^\infty([0,1])/_G$ by any ultrafilter G are not fields, since the germ in G of some $f \in L^\infty([0,1])$ has an inverse in $L^{\infty+}([0,1])/_G \setminus L^\infty([0,1])/_G$).*

5.4.2 Example II: standard ultraproducts

We can now sketch an argument to show that the familiar notion of ultraproduct of Tarski models is a special case of a quotient of a boolean valued model with the mixing property.

Let X be a set. Then $\mathcal{P}(X)$ is an atomic complete boolean algebra. Notice that all theorems proved so far applies equally well to atomic *complete* boolean algebras even if in the examples we focused on *atomless, complete* boolean algebras. A key observation is that $\{\{x\} : x \in X\}$ is a maximal antichain and a dense open set in $\mathcal{P}(X)^+$. Now observe that $\text{St}(\mathcal{P}(X))$ is the space of ultafilters on X and X can be identified inside $\text{St}(\mathcal{P}(X))$ as the open dense set $\{G_x : x \in X\}$ where G_x is the principal utrafilter on $\mathcal{P}(X)$ given by all supersets of $\{x\}$. Another key observation is the following:

Fact 5.4.6. *Let $(M_x : x \in X)$ be a family of Tarski-models in the first order relational language $\mathcal{L}$. Then $N = \prod_{x\in X} M_x$ is a $\mathcal{P}(X)$-model with the mixing property (letting for each n-ary relation symbol $R \in \mathcal{L}$, $[\![R(f_1,\dots,f_n)]\!]_{\mathcal{P}(X)} = \{x \in X : M_x \models R(f_1(x),\dots,f_n(x))\}$).*

Proof. We leave the proof as an instructive exercise for the reader. □

Let G be any non-principal ultrafilter on X. Then, using the notation of the previous fact, $N/_G$ is the familiar ultraproduct of the family $(M_x : x \in X)$ by G and the usual Łoś Theorem for ultraproducts of Tarski models is the specialization to the case of the full $\mathcal{P}(X)$-valued model N of Theorem 5.3.7. Notice that in this special case, if the ultraproduct is an ultrapower of a model M, the embedding $a \mapsto [c_a]_G$ (where $c_a(x) = a$ for all $x \in X$ and $a \in M$) is elementary. This is not always the case for all the other examples of full B-valued models we are giving in these notes.

5.4.3 Example III: $C(St(\mathsf{B}), 2^\omega)$

We introduce a last example of full boolean valued model, which is more in the spirit of what we are aiming for, since it can give an approach to forcing completely equivalent to the one we pursue in the next chapter.

Exercise 5.4.7. Let B be an arbitrary (complete) boolean algebra. Let $M = C(St(\mathsf{B}), 2^\omega)$ be the family of continuous functions from $\mathrm{St}(\mathsf{B})$ into 2^ω. Fix R a binary clopen relation on 2^ω. The continuity of f, g grants that the set

$$R^M(f,g) = \{G : f(G)\ R\ g(G)\} = (f \times g)^{-1}[R]$$

is clopen in $\mathrm{St}(\mathsf{B})$ (where $f \times g(G) = (f(G), g(G))$). So we can define $R^M(f,g) = [\![f\ R\ g]\!]$. Also since the diagonal is closed in $(2^\omega)^2$,

$$=^M (f,g) = \mathsf{Reg}(\{G : f(G) = g(G)\})$$

is well defined.

Check that $(C(St(\mathsf{B}), 2^\omega), =^M, R^M)$ is a full B-valued extension of the structure $(2^\omega, =, R)$ (where 2^ω is copied inside $C(St(\mathsf{B}), 2^\omega)$ as the set of constant functions). Check also that whenever G is an ultrafilter on $\mathrm{St}(\mathsf{B})$, the map $i_G : 2^\omega \to C(St(\mathsf{B}), 2^\omega)/_G$ given by $x \mapsto [c_x]_G$ (the constant function with value x) defines an injective morphism of the 2-valued structure $(2^\omega, =, R)$ into the 2-valued structure $(C(St(\mathsf{B}), 2^\omega)/_G, =^M /_G, R^M/_G)$.

The above exercise outlines a general strategy to expand many first order structure on 2^ω to extensions $C(St(\mathsf{B}), 2^\omega)/G$ indexed by $G \in \mathrm{St}(\mathsf{B})$ in such a way that the first properties of the structure $C(St(\mathsf{B}), 2^\omega)/G$ are finely controlled by the topological properties of $\mathrm{St}(\mathsf{B})$ and the algebraic properties of B via the embeddings i_G.

The general idea of forcing is to develop this technique in order to be able to replace first order structures with domain 2^ω by any first order model (M, E) of ZFC. For the sake of simplicity we assume now that M is transitive and E is $\in \restriction M$. $(M, \in, \subseteq, =)$ is expanded to a boolean extension $(M^\mathsf{B}, \in^\mathsf{B}, \subseteq^\mathsf{B}, =^\mathsf{B})$ defined by means of a boolean algebra $\mathsf{B} \in M$ in such a way to define a forcing relation which ties the logical property of the boolean structure $(M^\mathsf{B}, \in^\mathsf{B}, \subseteq^\mathsf{B}, =^\mathsf{B})$ to the topological properties of the space $\mathrm{St}(\mathsf{B})$ or, equivalently (via the Stone duality), to the algebraic properties of B. We can then pass to a natural quotient structure $M^\mathsf{B}/_G$, which is now a Tarski model for the language of M and which naturally contains an isomorphic copy of M. The definition of M^B will be done reversing the arrows and exploiting the Stone duality between 0-dimensional compact Haussdorff spaces and boolean algebras. We will develop the theory of boolean valued models for set theory defining M^B as an appropriate bunch of functions from M to B, rather than as a set of continuous functions from $\mathrm{St}(\mathsf{B})$ to (what should be) some compactification of M.

Chapter 6

Forcing

In this chapter we will present the technique of Forcing. Forcing has a crucial role in the development of modern set theory and it has had and has an immense number of applications in this field of research. For example, forcing is the standard tool to prove the consistency with the standard axioms of ZFC of a mathematical statement which can be formulated as a first order $\in$-formula ϕ in the language of set theory. Many mathematical theories can get a natural interpretation as subtheories of ZFC. In this way forcing provides an extremely powerful tool to investigate the undecidability of a given mathematical problem, since this problem can in most cases be formulated as a first order statement in the theory ZFC. The first and most celebrated example of an unexpected undecidability result is the proof of the independence of the continuum hypothesis CH from ZFC, and the aim of these notes is to develop forcing far enough in order to be able to give a complete proof of this result. The general idea of forcing is the following: we want to get a model of some first order statement ϕ in the language of set theory which we aim to show to be consistent with ZFC. To do so we enlarge the universe V which is the "standard" model of set theory to another universe of sets $N \supset V$ which is still a model of ZFC so that we are able to *force* N to be a model of ϕ. If we adopt a platonistic stance towards set theory, the statement "$N \supset V$" is nonsense since all possible sets are already elements of V, and so there cannot be a proper superuniverse N of V. To overcome this difficulty we assume that in V there is a countable transitive model $M \in V$ of ZFC and we extend M to a *generic extension* $N \supset M$ which is also in V and which is a model of $\mathsf{ZFC} + \phi$. This approach requires us to work in a theory which is slightly stronger than ZFC, since by Gödel' incompleteness theorem, ZFC cannot (unless ZFC is not consistent) derive the statement *there is a countable transitive model of* ZFC, while we will assume that V models ZFC^+, where ZFC^+ stands for ZFC plus the latter statement. Nonetheless requiring V to be a model of ZFC^+ in our eyes allows for a simpler exposition of the semantic of the forcing method and does not weaken substantially the undecidability results we are able to obtain with respect to ZFC^+ (the statement *there is a countable transitive model of* ZFC follows from the theory ZFC+*there exists an inaccessible cardinal* —see Section 7.3— and is equiconsistent with ZFC). The interested reader can find in the first pages of [7, Chapter 7] several arguments which translate the undecidability results obtained by means of forcing over the theory ZFC^+, to undecidability results obtained over ZFC.

The general strategy to prove the undecidability of ϕ by means of forcing is to start from given known countable transitive models of ZFC M_0, M_1 and to produce by means of forcing generic extensions $N_i \supseteq M_i$ such that N_0 models $\mathsf{ZFC} + \phi$ and N_1 models $\mathsf{ZFC} + \neg\phi$. In this chapter we just assume that there is one given countable transitive model of ZFC M and we will build all our generic extensions over this M.

We also need to give some intuition on the reasons why enlarging V to a larger N we can hope to be able to show that N is a model of ϕ. The strategy we will follow is that leading from a two valued logic where all statements are either true or false to a boolean valued logic where statements ϕ get evaluated as elements $[\![\phi]\!] \in \mathsf{B}$ for some boolean algebra B, we consider ϕ true if $[\![\phi]\!] = 1_{\mathsf{B}}$, false if $[\![\phi]\!] = 0_{\mathsf{B}}$, undetermined otherwise. Now observe that B corresponds to the clopen sets of $\mathrm{St}(\mathsf{B})$ its space of ultrafilters and that selecting a point $G \in \mathrm{St}(\mathsf{B})$ allows us to decide which as yet undecided statements ϕ are true or false according to G: they will be considered true by G if and only if $[\![\phi]\!] \in G$. So we are led to the consideration that boolean algebras B allow to define a B-valued logic in which we haven't yet compromised ourselves on the truth values of certain first order statements ϕ (those for which $0_{\mathsf{B}} < [\![\phi]\!] < 1_{\mathsf{B}}$), and that the points $G \in \mathrm{St}(\mathsf{B})$ will *force* us to accept ϕ as true iff $[\![\phi]\!] \in G$. Now the idea of the forcing method is to employ these boolean valued logics as follows: We start from a *transitive* model M of ZFC (where we are not able to compute ϕ), we choose in M a boolean algebra B for which we are able to calculate its combinatorial properties. We extend M to a boolean valued model M^{B} which is a definable class in M and contains an isomorphic copy of M as a B-valued substructure. M^{B} is such that for all formulae ϕ we are able to define an evalution map $[\![\phi]\!]$ which links (in a manner which is possible to compute inside M) the B-valued semantics of M^{B} to the combinatorial properties of B in M. We can pick any $G \in \mathrm{St}(\mathsf{B})$, and we get that G decides that ϕ holds iff $[\![\phi]\!] \in G$. Things can be done so properly that our heuristic assertion "G decides that ϕ holds" can be expanded in the precise statement: "The Tarski structure $M^{\mathsf{B}}/G \supset M$ is a first order model of ZFC in which ϕ holds and which properly contains M".

There are different approaches to the technique of forcing. We will follow the one through boolean algebras and boolean valued models, as in [6]. To this aim it is fundamental to exploit the theory of boolean valued models we developed in Chapter 5.

The remainder of this chapter consists of four sections:

1. In the first section we define a procedure which given any (transitive) first order structure M which is a model of ZFC and a boolean algebra $\mathsf{B} \in M$ (which M models to be complete) produces a *full* boolean valued model of set theory M^{B}, i.e. a boolean valued model of the language of set theory according to the semantic we defined in the previous chapter. We also try to give an heuristic for forcing following Cohen's original argument to introduce a new element of 2^ω to M by describing it inside M using the poset of finite strings of $0, 1$. Moreover we explain how the semantics of M^{B} is governed by means of the notion of M-genericity introduced by Cohen.

2. In the second section we develop in full details the key ideas in Cohen's development of the forcing method, which allows one to start from a *countable transitive model* M of ZFC, and to pass to the quotient M^{B}/G obtained by selecting some $G \in \mathrm{St}(\mathsf{B})$. Under the key assumption that G is an M-generic filter for B, we can prove that M^{B}/G is well founded and show that its transitive collapse, the *generic extension* $M[G]$, is countable and is the least *transitive* model of ZFC containing M and G. Cohen's forcing theorem shows that there is a fine tuning between the first order properties of the structure $(M[G], \in)$ and the combinatorial properties of B.

3. In the third section we define a boolean algebra $\mathsf{B} \in M$ in such a way that M^{B} models $\neg\mathsf{CH}$ and another boolean algebra $\mathsf{C} \in M$ in such a way that M^{C} models CH.

4. In the fourth section we show that no matter how we select M countable transitive model of ZFC and $\mathsf{B} \in M$, the axioms of ZFC all gets boolean value 1_{B} in the boolean valued model M^{B} and thus, by means of the forcing theorem, also hold in $M[G]$ for any M-generic filter G for B.

6.1 Boolean valued models for set theory

We have seen the definition of a boolean valued model for the language $\mathcal{L} = \{\in, \subseteq\}$. Now we want something more adherent to the intended meaning we have in mind for the symbols $\in, \subseteq$. To achieve this, we need to add some requests to Def. 5.1.1.

Definition 6.1.1. A ***boolean valued model for set theory*** is a boolean valued model M (with its associated cba B) for $\mathcal{L} = \{\in, \subseteq, =\}$, where, for any $\sigma, \tau, \eta \in M$:

1. $[\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!] = [\![\tau = \sigma]\!]$.
2. $[\![\tau \in \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!] \leq [\![\tau \in \eta]\!]$.

We exhibit the boolean valued models for set theory we'll be working with in the sequel of this chapter.

Definition 6.1.2. Let M be any *transitive*[1] first order model of ZFC and $\mathsf{B} \in M$ be such that M models B to be a complete boolean algebra. We let:

$$M_0^{\mathsf{B}} = \emptyset.$$

$$M_{\alpha+1}^{\mathsf{B}} = \{f : M_\alpha^{\mathsf{B}} \mapsto \mathsf{B} : f \text{ is partial}\}.$$

$$M_\beta^{\mathsf{B}} = \bigcup_{\alpha<\beta} M_\alpha^{\mathsf{B}}, \text{ where } \beta \text{ is limit.}$$

Finally:

$$M^{\mathsf{B}} = \bigcup_{\alpha \in \mathrm{Ord}^M} M_\alpha^{\mathsf{B}}.$$

[1] As we shall see below the requirement that M is transitive is redundant and we put it here just to give a clearer intuition of how the class M^{B} is generated inside M.

Remark 6.1.3.

- It is an instructive exercise to show that the class M^{B} is definable in M using the transfinite recursion theorem (applied in M) to obtain it as the extension in M of a formula in the parameter B. Formally this argument can be carried in any first order model of ZFC, thus the requirement that M is transitive in the above definition is redundant (even though if M is ill-founded it is not at all transparent what is the correct interpretation of the objects of M defined by means of the transfinite recursion theorem). However in these notes we are interested just in *transitive* well founded models of ZFC.

  For the sake of completeness, here is how the class V^{B} can be defined as the extension of a formula in the parameter B inside V: let $F : V \to V$ be defined as follows:

$$\begin{cases} F(g) = \{f : X \to \mathsf{B} : f \text{ is a partial function}\} \\ \qquad \text{if for some ordinal } \alpha,\ g : \alpha + 1 \to V \text{ is a function and } X = g(\alpha), \\ F(g) = \bigcup \operatorname{ran}(g) \text{ otherwise.} \end{cases}$$

  Then $G : \mathrm{Ord} \to V$ defined by $G(\alpha) = F(g \restriction \alpha)$ enumerates the V_α^{B} and $V^{\mathsf{B}} = \bigcup_{\alpha \in \mathrm{Ord}} G(\alpha)$.

- If V is the standard model of set theory, the definition of $V_{\alpha+1}^{\mathsf{B}}$ gives the "boolean powerset" of V_α^{B} for the boolean algebra B much in the same way as $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ is the power set of V_α for the boolean algebra $\{0, 1\}$: indeed, given a set X, $\mathcal{P}(X)$ can be identified as the set of the characteristic functions of its elements. We generalize the notion of power set using the identification of a "boolean" subset of X with its "characteristic" (partial) function $f : X \to \mathsf{B}$, with the further hidden complication that for any partial function $f : X \to \mathsf{B}$, the evaluation of how much an element of X on which f is undefined belongs to f is postponed to a later stage (i.e. Definition 6.1.25). We need to consider the family of partial functions from X to B to define the boolean power set of X for technical reasons which will become transparent in the sequel.

Every element $X \in V$ can be identified by a partial characteristic function which takes as domain X and has value constantly 1; we use these type of functions to define canonical B-names for elements of V inside V^{B}.

Definition 6.1.4. Given a boolean algebra $\mathsf{B} \in V$, for every element u of V we define by induction on its rank:

$$\check{u} = \{\langle \check{v}, 1_{\mathsf{B}} \rangle : v \in u\}.$$

For example, if $a = \{\emptyset, \{\emptyset\}\}$, then

$$\check{\emptyset} = \emptyset,$$

$$\{\check{\emptyset}\} = \{\langle \check{\emptyset}, 1_{\mathsf{B}} \rangle\} = \{\langle \emptyset, 1_{\mathsf{B}} \rangle\},$$

$$\check{a} = \{\langle \check{\emptyset}, 1_{\mathsf{B}} \rangle, \langle \{\check{\emptyset}\}, 1_{\mathsf{B}} \rangle\} = \{\langle \emptyset, 1_{\mathsf{B}} \rangle, \langle \{\langle \emptyset, 1_{\mathsf{B}} \rangle\}, 1_{\mathsf{B}} \rangle\}.$$

Definition 6.1.5. $\check{V} = \{\check{x} : x \in V\}$.

There is also another way to define V^{B} which is more convenient since it will give a Δ_1-definition in the parameter B of the class V^{B} inside V.

Definition 6.1.6. Given a set X of partial functions in V, we define:

$$\overset{*}{\bigcup} X = \bigcup \{\mathrm{dom}(z) : z \in X\}.$$

and for f a partial function

$$\mathrm{trcl}^*(f) = f \cup \bigcup \{(\overset{*}{\bigcup})^n f : n \in \omega,\, n > 0\}.$$

Exercise 6.1.7. Show that the operation $f \mapsto \mathrm{trcl}^*(f)$ is Δ_1-definable in no parameters in any model of ZFC, and thus is absolute between transitive structures which model the relevant fragment of ZFC.

We leave as an instructive exercise to check the following:

Fact 6.1.8. *$\tau \in V^{\mathsf{B}}$ if and only if*

$$\tau \textit{ is a function } \wedge\ \mathrm{ran}(\tau) \subseteq \mathsf{B}\ \wedge \forall \sigma \in \mathit{trcl}^*(\tau)[\sigma \textit{ is a function } \wedge \mathrm{ran}(\sigma) \subseteq \mathsf{B}].$$

The latter property is Δ_1-definable in the parameter B in any model of ZFC.

Remark 6.1.9. $\tau \in \check{V}$ if and only if $\tau \in V^{\mathsf{B}}$ and

$$\forall \sigma_0, \sigma_1 \in \mathrm{trcl}^*(\tau)(\sigma_0 \in \mathrm{dom}(\sigma_1) \Rightarrow \sigma_1(\sigma_0) = 1_{\mathsf{B}}).$$

Hence

$$\check{V} \subseteq V^{\mathsf{B}}.$$

The elements of V^{B} are called the family of B-***names***. Remark that in order to define the class of B-names we do not need B to be complete.

6.1.1 External definition of forcing

This section is of a rather peculiar nature: we try to give some more intuition on forcing. This forces us to mix some precise mathematical definitions with rather general considerations. Moreover at some points along our presentation we suggest some themes whose elaboration is not strictly necessary. In order to keep a straight division between the different levels of our discourse, we adopt the following typographical convention:

- The parts which introduce definitions and prove facts which will be needed also in the remainder of this chapter will maintain the usual font.
- The parts which are in our eyes of central interest to understand the basic ideas of forcing, but are not introducing mathematical definitions and results which will be needed in the remainder of this chapter will be put in font UTOPIA.
- The parts which are not of central importance will be put in a smaller font, we leave to the reader to evaluate how much effort to devote to their comprehension.

Assume we want to construct a new element r of 2^ω not in V. This is clearly not possible since all sets are in V and r is a set. Let us sidestep this problem, assuming that there is $M \in V$ countable and transitive such that $M \models \mathsf{ZFC}$ (this is the case if there is an inaccessible cardinal in V, by the results of Section 7.3). So we can assume $r \in V \setminus M$ is a "new" element r of 2^ω with respect to the ZFC-model M.

The fact that M is a transitive model of ZFC simplifies enormously the comparison of M with V: All the standard absoluteness properties established in [7, Chapter IV] and in Chapter 7 holds between between the transitive ZFC-models $M \subseteq V$, giving that most computations yields the same results when carried inside M or in V.

Exercise 6.1.10. Assume M is a transitive model of ZFC. The following notion are absolute for M and V:

- $(\mathsf{B}, \wedge, \neg, 0_\mathsf{B}, 1_\mathsf{B}) \in M$ is a boolean algebra. (HINT: the property of being a boolean algebra can be formalized as a Σ_0 property of the tuple $(\mathsf{B}, \wedge, \neg, 0_\mathsf{B}, 1_\mathsf{B})$ which requires just to quantify over B^n for a large enough n and over $\wedge, \vee \subseteq \mathsf{B}^3$ and $\neg \subseteq \mathsf{B}^2$, each of these sets is an element of M)
- $(P, \leq_P)$ is a partial order (with $P, \leq_P \in M$)..
- $i : P \to Q$ is an order and incompatibility preserving map with a dense image between the partial orders $(P, \leq_P)$ and $(Q, \leq_Q)$ (with $i, P, \leq_P, Q, \leq_Q \in M$).
- $a \in P$ is an atom of the partial order $(P, \leq_P)$ (with $a, P, \leq_P \in M$).
- $(P, \leq_P) \in M$ is an atomless partial order.
- $(P, \leq_P) \in M$ is a separative partial order.
- $G \in M$ is a filter on a partial order $(P, \leq_P) \in M$ (or a ultrafilter on a boolean algebra B).
- $b = \bigvee A$ for $A \in \mathcal{P}(\mathsf{B}) \cap M$.
- $X \subseteq P$ is predense in the partial order $(P, \leq_P)$ (with $X, P, \leq_P \in M$).
- $X \subseteq P$ is predense in the partial order P below the condition p (with $X, P, \leq_P \in M$).
- $A \subseteq P$ is a (maximal) antichain in the partial order $(P, \leq_P)$ (with $A, P, \leq_P \in M$).
- $O \subseteq P$ is open in the partial order $(P, \leq_P)$ (with $O, P, \leq_P \in M$)..

Hence for $\mathsf{B} \in M$ a boolean algebra $\mathrm{St}(\mathsf{B})^M = \mathrm{St}(\mathsf{B}) \cap M$, since:

$$\begin{aligned} \mathrm{St}(\mathsf{B})^M &= \left\{G \in \mathcal{P}(\mathsf{B})^M : M \models G \text{ is a ultrafilter on } \mathsf{B}\right\} = \\ &= \{G \in \mathcal{P}(\mathsf{B}) \cap M : V \models G \text{ is a ultrafilter on } \mathsf{B}\} = \mathrm{St}(\mathsf{B}) \cap M, \end{aligned}$$

where in the second equality, we used that $\mathcal{P}(X)^M = \mathcal{P}(X) \cap M$ for all $X \in M$ (see also [7, Lemma IV.2.9]).

Take the poset $(2^{<\omega}, \supseteq)$. By the first Item in Remark 4.3.2, its boolean completion is isomorphic to the regular open sets of 2^ω in the product topology. Notice that, by absoluteness arguments, since $M \subseteq V$ are both transitive models of ZFC, $2^{<\omega} = (2^{<\omega})^M = (2^{<\omega})^V$.

Consider in M the sets

$$D_f = \{s \in 2^{<\omega} : s \not\subseteq f\} \in M$$

for each $f \in (2^\omega)^M$, and the sets

$$E_n = \{s \in 2^{<\omega} : n \in \mathrm{dom}(s)\} \in M.$$

These are easily seen to be dense subsets of $2^{<\omega}$ (see Exercise 4.2.13) which belong to M applying the comprehension axiom in M to the formula defining them.

On the other hand in V, given $r \in 2^\omega \setminus M$ define

$$G = \{s \in 2^{<\omega} : s \subseteq r\}.$$

Then $G \in V$ and $r = \cup G$, thus $G \notin M$, else $r = \cup G \in M$ as well. G is a filter on $2^{<\omega}$ and G meets all the dense sets D_f for all $f \in (2^\omega)^M$: pick $f \in (2^\omega)^M$, since $r = \cup G \neq f$, we can find n such that $r(n) \neq f(n)$, thus $r \restriction n+1 \in G \cap D_f$. G meets also the dense sets E_n for $n \in \omega$, since $r \restriction n+1 \in G \cap E_n$ for all $n \in \omega$.

Now assume $H \in M$ is a filter on $2^{<\omega}$ meeting all the dense set E_n for all $n \in \omega$ (H exists by Lemma 4.2.9 applied in M which is a model of ZFC with $\{E_n : n \in \omega\} \in M$ which is a countable set also according to M). Then $h = \cup H \in 2^\omega$ and $H \cap D_h$ is empty, else for some $s \in H$ and $n \in \mathrm{dom}(s)$, $s(n) \neq h(n)$ with $h \supseteq s$, which is a contradiction.

Hence $G \in V \setminus M$ is a filter on $2^{<\omega}$ which meets a family[2] of dense subsets of $2^{<\omega}$ which cannot be simultaneously met by any filter $H \in M$ on $2^{<\omega}$ meeting all the dense sets E_n.

So let us make a step further, and let us assume that $r \in 2^\omega \setminus M$ is such that $G = \{s : s \subseteq r\}$ is a filter on $2^{<\omega}$ meeting ***all*** the dense subsets of $2^{<\omega}$ which belong to M (this is possible since M is countable in V). Notice that $r = \cup G$.

Before proceeding in our analysis of this specific example we need to introduce some general concepts.

M-generic ultrafilters, and the induced valuation map

Let us recall Definition 4.2.8 and specify it to the context we are interested.

Definition 6.1.11. Let M be a transitive model of ZFC and $P \in M$ be a partial order. A filter $G \subseteq P$ is M-generic for P if $G \cap D \neq \emptyset$ for all $D \in M$ predense subset of P.

For a boolean algebra $\mathsf{C} \in M$, $G \in \mathrm{St}(\mathsf{C})$ is M-generic for C if it is M-generic for C^+.

[2]The family $\{E_n : n \in \omega\} \cup \{D_f : f \in (2^\omega)^M\}$. Note that this family belongs to M but has size continuum in M: the map $f \mapsto D_f$ from $(2^\omega)^M$ into the above family is also an element of M and is an injection.

Exercise 6.1.12. Let M be a countable transitive model of ZFC and $\mathsf{B} \in M$ a boolean algebra. Given $\mathcal{D} \in M$ family of subsets of B show that

$$(\{G \in \mathrm{St}(\mathsf{B}) : \forall X \in \mathcal{D}\,(G \cap X \neq \emptyset)\})^M = \{G \in \mathrm{St}(\mathsf{B}) : \forall X \in \mathcal{D}\,(G \cap X \neq \emptyset)\} \cap M.$$

Hence letting

$$\mathcal{D} = \{X \in M \cap \mathcal{P}(\mathsf{B}) : M \models X \text{ is a predense subset of } \mathsf{B}\} \in M,$$

we have that $G \in \mathrm{St}(\mathsf{B})$ is M-generic if $G \cap X \neq \emptyset$ for all $X \in \mathcal{D}$ and

$$\{G \in \mathrm{St}(\mathsf{B}) : G \text{ is } M\text{-generic for } \mathsf{B}\} \cap M = \{G \in \mathrm{St}(\mathsf{B}) : M \models \forall X \in \mathcal{D}\,(G \cap X \neq \emptyset)\}.$$

Using Exercise 4.2.12, show that this latter set is empty if B is atomless, and consists just of the principal ultrafilters on B of the form G_a (each of which belongs to M) for a atom of B, if B is atomic. Show also that in any case the set of $G \in \mathrm{St}(\mathsf{B})$ which are M-generic is a dense subset of $\mathrm{St}(\mathsf{B})$ in V, with the property that none of the M-generic filters is in M if B is atomless.

A basic intuition on M-genericity is that dense open subsets of $\mathrm{St}(\mathsf{B})$ are the large sets and M-generic filters denote the points of $\mathrm{St}(\mathsf{B})$ which are in all large subsets of $\mathrm{St}(\mathsf{B})$ which M knows of.

Recall that any complete boolean algebra splits in the disjoint sum of an atomless cba and of an atomic cba (Lemma 3.0.2). We will see that the forcing method gains traction (i.e. it can be used to produce new interesting model of ZFC) just when it is applied to atomless cbas.

Remark 6.1.13. In the remainder of this chapter we focus on the notion of M-genericity for *atomless* cbas in M. In this case the notion of M-genericity can be used to describe inside M (by means of M^{B}) enlargements of M obtained by adding to M some $G \in \mathrm{St}(\mathsf{B})$ which is M-generic.

The following exercise briefly explains what happens if B is atomic:

Exercise 6.1.14. Assume $\mathsf{C} \in M$ is such that M models C is an *atomic* cba. Then

- $$A = \{a \in \mathsf{C} : a \text{ is an atom}\} = \bigcap \{D \subseteq \mathsf{C}^+ : D \text{ is dense}\}$$

  is open dense in C^+.
- For any atom $a \in \mathsf{C}$
  $$G_a = \{b \in \mathsf{C} : a \leq b\} \in \mathrm{St}(\mathsf{C}) \cap M$$
  is M-generic for C.
- Any $G \in \mathrm{St}(\mathsf{C})$ which belongs to V and which is M-generic for C is of the form G_a for some $a \in \mathsf{C}$ atom of C.

The following exercise shows that the notion of being a complete cba is not absolute between M and V.

Exercise 6.1.15. Assume $\mathsf{B} \in M$ is an infinite boolean algebra. Then B is not complete in V. (HINT: Assume the set $A \in M$ of atoms of B is infinite. Pick $Y \subseteq A$ with $Y \notin M$. Then Y cannot have a suprema $b \in \mathsf{B} \subseteq M$, else $Y = \{a \in A : a \leq b\} \in M$. Assume B has a finite set A of atoms. Then $c = \bigvee A \in M$ and $\mathsf{B} \restriction \neg c \in M$ is atomless. Hence we can assume that B is atomless. Enumerate in M an infinite antichain $A = \{a_n : n \in \omega\} \in M$ of B (which exists since B is atomless and infinite in M). Find $X \subseteq \omega$ such that $X \notin M$. Then $\bigvee \{a_n : n \in X\}$ does not exists: assume $a = \{a_n : n \in X\}$. Then $a \in \mathsf{B} \subseteq M$ and $\{a_n : n \in X\} = \{c \in A : c \leq a\} \in M$, giving that $X \in M$ as well).

Definition 6.1.16. Let M be a countable transitive model of ZFC and C be a complete boolean algebra in M.

For $\sigma, \tau \in M^{\mathsf{C}}$, define $\sigma\ E^{\mathsf{C}}\ \tau$ iff $\sigma \in \mathrm{dom}(\tau)$ and

$$\mathrm{rk}_{\mathsf{C}}(\tau) = \sup\left\{\mathrm{rk}_{\mathsf{C}}(\sigma) + 1 : \sigma\ E^{\mathsf{C}}\ \tau\right\}.$$

Fact 6.1.17. *Let M be a countable transitive model of ZFC and C be a complete boolean algebra in M. Then $E^{\mathsf{C}} \subseteq (M^{\mathsf{C}})^2$ is definable in M, and well founded in M and V as witnessed by the rank function*

$$\mathrm{rk}^{\mathsf{C}} : M^{\mathsf{C}} \to \mathrm{Ord} \cap M,$$

which is as well a definable class function in M.

Proof. Clearly E^{C} is definable by the Σ_0-property $\phi(x, y) \equiv x \in \mathrm{dom}(y)$. The map rk^{C} is defined by transfinite recursion inside M using the absolute function $F^M : M^{\mathsf{C}} \times M \to M$ given by

$$F^M(\sigma, h) = \cup \{h(z) + 1 : z \in \mathrm{dom}(\sigma)\}$$

and setting

$$\begin{aligned} \mathrm{rk}^{\mathsf{C}}(\sigma) = F(\sigma, \mathrm{rk}^{\mathsf{C}} \restriction \mathrm{pred}_{E^{\mathsf{C}}}(\sigma)) = \\ F^M(\sigma, \mathrm{rk}^{\mathsf{C}} \restriction \mathrm{dom}(\sigma)) = \cup\left\{\mathrm{rk}^{\mathsf{C}}(z) + 1 : z \in \mathrm{dom}(\sigma)\right\}. \end{aligned}$$

□

Definition 6.1.18. Let M be a countable transitive model of ZFC and C be a complete boolean algebra in M. Let $G \in \mathrm{St}(\mathsf{C})$.

$$\begin{aligned} \mathrm{val}_G :& M^{\mathsf{C}} \to V \\ & \sigma \mapsto \mathrm{val}_G(\sigma) = \sigma_G \end{aligned}$$

is defined in V by recursion on E^{C} for $\tau \in M^{\mathsf{C}}$ by the rule

$$\tau_G = \{\sigma_G : \tau(\sigma) \in G\}$$

for any given $\tau \in M^{\mathsf{C}}$.

For any $G \in \mathrm{St}(\mathsf{C})$

$$M[G] = \{\tau_G : \tau \in M^{\mathsf{C}}\}.$$

Remark 6.1.19. The definition of $M[G]$ is by recursion with parameters M, C, G, and can be carried in any model of ZFC to which all the relevant parameters belong:

val_G is defined by recursion on E^{C} using the function $F : M^{\mathsf{B}} \times V \to V$ given by

$$(f, g) \mapsto \{g(z) : z \in \mathrm{dom}(f) \text{ and } f(z) \in G\}$$

by the rule

$$\mathrm{val}_G(\tau) = F(\tau, \mathrm{val}_G \restriction \mathrm{pred}_E(\tau)) = F(\tau, \mathrm{val}_G \restriction \mathrm{dom}(\tau)).$$

In particular $M[G]$ can be defined in V (but a priori not in M whenever $G \notin M$).

Moreover:

Fact 6.1.20. *Following the above assumptions $M[G]$ is transitive for any M-generic ultrafilter $G \in \mathrm{St}(\mathsf{C})$ and*

$$M = \{\mathrm{val}_G(\check{a}) : a \in M\}.$$

Proof. $a \in M[G]$ entails

$$a = \sigma_G \subseteq \mathrm{val}_G[\mathrm{dom}(\sigma)] \subseteq M[G].$$

For the second fact see below (equation 6.1). □

A basic exercise which shows that $M[G]$ has nice closure properties is the following:

Exercise 6.1.21. Let M be a transitive model of ZFC and G be M-generic for some $\mathsf{B} \in M$ which M models to be a complete boolean algebra. Define $\mathrm{up} : (M^{\mathsf{B}})^2 \to M^{\mathsf{B}}$ and $\mathrm{op} : (M^{\mathsf{B}})^2 \to M^{\mathsf{B}}$ by the rules

$$\mathrm{up}(\sigma, \tau) = \{\langle \sigma, 1_{\mathsf{B}} \rangle, \langle \tau, 1_{\mathsf{B}} \rangle\},$$

$$\mathrm{op}(\sigma, \tau) = \{\langle \mathrm{up}(\sigma, \tau), 1_{\mathsf{B}} \rangle, \langle \mathrm{up}(\sigma, \sigma), 1_{\mathsf{B}} \rangle\}.$$

Show that up, op are definable class functions in M and that $\mathrm{up}(\sigma, \tau)_G = \{\sigma_G, \tau_G\}$, $\mathrm{op}(\sigma, \tau)_G = \langle \sigma_G, \tau_G \rangle$ for all $\sigma, \tau \in M^{\mathsf{B}}$.

Describing an M-generic filter G for $2^{<\omega}$ inside M.

Is it conceivable to describe the properties of an $r \notin M$ defining an M-generic filter $G = \{s : s \subseteq r\}$ for $2^{<\omega}$ reasoning just about what M can say using first order logic about itself?

The (may be surprising) answer is yes. This is what Cohen has shown with the invention of the forcing method. How can we hope to describe inside M this $r \notin M$?

First of all we can develop the notion of boolean valued model relative to M for $\mathsf{B} = \mathsf{RO}(2^{\omega})^M$ (which is a complete atomless boolean algebra in M). To this aim let

$$O_s = \left\{f \in (2^{\omega})^M : s \subseteq f\right\} = N_s \cap M$$

(i.e. O_s is what M thinks is the basic open set of $(2^{\omega})^M = 2^{\omega} \cap M$ induced by functions extending the finite string s of $0, 1$).

Then $E = \{O_s : s \in 2^{<\omega}\} \in M$ (since $M \models \mathsf{ZFC}$ and E is defined inside M as a subset of $\mathsf{RO}(2^{\omega})^M = \mathsf{B}$ obtained by applying the comprehension axiom inside M). Moreover M models that E is a dense subset of B^+, since M (being a transitive model of ZFC) models also that:

$$\left\{O_s = \left\{f \in (2^{\omega})^M : s \subseteq f\right\} : s \in 2^{<\omega}\right\}$$

is a basis consisting of clopen sets for the complete boolean algebra $\mathsf{RO}(2^{\omega})^M = \mathsf{B}$.

Also (once again because M is a transitive model of ZFC) the map

$$k : 2^{<\omega} \to \mathsf{B} \qquad\qquad s \mapsto O_s$$

belongs to M, since it is obtained as a subset of $2^{<\omega} \times \mathsf{RO}(2^\omega)^M$ (a set in M) applying an instance of the comprehension axiom in M to a formula with parameters in M. Moreover M models that the map k implements an isomorphism of $(2^{<\omega}, \supseteq)$ with $(E, \subseteq)$ since this is a Σ_0-property of this map which is true in V and thus also in M.

We can also check the following:

Fact 6.1.22. *Assume G is M-generic for $(2^{<\omega}, \supseteq)$. Then $\bar{G} = \uparrow \{O_s : s \in G\}$ is a ultrafilter on B which is M-generic for B^+.*

Proof. By assumption G meets *all* dense subsets of $2^{<\omega}$ in M. Assume we are given $D \subseteq \mathsf{B}^+$ dense open subset of B^+ and in M, we get that $D \cap E \in M$ is also a dense subset of $(E, \leq_{\mathsf{B}})$, and thus

$$\{s \in 2^{<\omega} : O_s \in D \cap E\} \in M$$

is a dense subset of $2^{<\omega}$ in M. Thus G meets this dense set, so there is some $s \in G$ such that $O_s \in D \cap E \cap \bar{G}$. □

In V, let us fix some $\bar{G}$ M-generic for B^+, and let us consider the map

$$\begin{aligned} \mathrm{val}_{\bar{G}} : & M^{\mathsf{B}} \to V \\ & \sigma \mapsto \mathrm{val}_{\bar{G}}(\sigma) = \sigma_{\bar{G}} \end{aligned}$$

given by the rule $\tau \mapsto \left\{\sigma_{\bar{G}} : \tau(\sigma) \in \bar{G}\right\}$. Let us check what this map does on the elements of $\check{M} = \check{V}^M = \check{V} \cap M$: By definition:

$$\mathrm{val}_{\bar{G}}(\check{\emptyset}) = \left\{\sigma : \sigma \in \mathrm{dom}(\check{\emptyset}) \text{ and } \check{\emptyset}(\sigma) \in \bar{G}\right\},$$

but $\check{\emptyset} = \emptyset$ is the empty function, in particular it has empty domain. We get that

$$\mathrm{val}_{\bar{G}}(\check{\emptyset}) = \left\{\sigma_{\bar{G}} : \sigma \in \mathrm{dom}(\check{\emptyset}) \text{ and } \check{\emptyset}(\sigma) \in \bar{G}\right\} = \emptyset.$$

Next:

$$\mathrm{val}_{\bar{G}}(\{\check{\emptyset}\}) = \left\{\sigma : \sigma \in \mathrm{dom}(\{\check{\emptyset}\}) \text{ and } \{\check{\emptyset}\}(\sigma) \in \bar{G}\right\},$$

but $\{\check{\emptyset}\} = \left\{\left\langle \check{\emptyset}, 1_{\mathsf{B}} \right\rangle\right\}$ and $1_{\mathsf{B}} \in \bar{G}$, thus

$$\mathrm{val}_{\bar{G}}(\{\check{\emptyset}\}) = \left\{\sigma_{\bar{G}} : \sigma \in \mathrm{dom}(\{\check{\emptyset}\}) \text{ and } \{\check{\emptyset}\}(\sigma) \in \bar{G}\right\} = \left\{\check{\emptyset}_{\bar{G}}\right\} = \{\emptyset\}$$

Now by induction on the ranks, assuming $\check{y}_{\bar{G}} = y$ for all $y \in x$, we get that

$$\begin{aligned} \check{x}_{\bar{G}} &= \left\{\sigma_{\bar{G}} : \sigma \in \mathrm{dom}(\check{x}) \text{ and } \check{x}(\sigma) \in \bar{G}\right\} = \\ &= \left\{\check{y}_{\bar{G}} : \check{y} \in \mathrm{dom}(\check{x}) \text{ and } \check{x}(\check{y}) = 1_{\mathsf{B}} \in \bar{G}\right\} = \{\check{y}_{\bar{G}} : y \in x\} = x. \end{aligned} \tag{6.1}$$

Thus $\check{M} = (\check{V})^M \subseteq M^{\mathsf{B}}$ is giving B-names for the objects of M.

Now let us take the following B-name:

$$\dot{G} = \{\langle \check{s}, O_s \rangle : s \in 2^{<\omega}\} \in M^{\mathsf{B}}.$$

$\dot{G} \in M^{\mathsf{B}}$ since it is obtained applying the replacement axiom to the function $2^{<\omega} \to M$ given by $s \mapsto \langle \check{s}, O_s \rangle$. Such a function is a definable class in M and thus its image belongs to M. It is immediate to check that $\dot{G}$ satisfies the clause for the definition of B-names in M.

We obtain:

$$\dot{G}_{\bar{G}} = \left\{ \check{s}_{\bar{G}} : \dot{G}(\check{s}) = O_s \in \bar{G} \right\} = \left\{ s : O_s \in \bar{G} \right\} = G.$$

We have the surprising fact that the object $G \in V \setminus M$ is described by an element of M^{B}! Similarly

$$\dot{H} = \left\{ \left\langle \check{b}, b \right\rangle : b \in \mathsf{B}^+ \right\} \in M^{\mathsf{B}}$$

is such that

$$\dot{H}_{\bar{G}} = \left\{ \check{b}_{\bar{G}} : \dot{H}(\check{b}) = b \in \bar{G} \right\} = \left\{ b : b \in \bar{G} \right\} = \bar{G}.$$

To get another example, let $r = \cup G$, then $G = \{s \in 2^{<\omega} : s \subseteq r\}$, we will exhibit a B-name for r in M^{B}. Consider the operations on M^{B}-names defined by

$$\text{up}(\sigma, \tau) = \{\langle \sigma, 1_{\mathsf{B}} \rangle, \langle \tau, 1_{\mathsf{B}} \rangle\}$$

and

$$\text{op}(\sigma, \tau) = \{\langle \text{up}(\sigma, \tau), 1_{\mathsf{B}} \rangle, \langle \text{up}(\sigma, \sigma), 1_{\mathsf{B}} \rangle\}$$

introduced in exercise 6.1.21. Now let

$$\dot{r} = \left\{ \left\langle \text{op}(\check{n}, \check{i}), O_{\langle n,i \rangle} \right\rangle : n < \omega, i < 2 \right\} \in M^{\mathsf{B}},$$

where $O_{\langle n,i \rangle} = \left\{ f \in (2^\omega)^M : f(n) = i \right\}$. Then (by exercise 6.1.21)

$$\begin{aligned} \dot{r}_{\bar{G}} = \left\{ \text{op}(\check{n}, \check{i})_{\bar{G}} : O_{\langle n,i \rangle} \in \bar{G},\, n < \omega, i < 2 \right\} = \\ = \left\{ \langle n, i \rangle : O_{\langle n,i \rangle} \in \bar{G},\, n < \omega, i < 2 \right\} = \\ = \left\{ \langle n, i \rangle : O_{\langle n,i \rangle} \supseteq O_s \text{ for some } s \in G,\, n < \omega, i < 2 \right\} = \\ = \{\langle n, i \rangle : \ s(n) = i \text{ for some } s \in G,\, n < \omega, i < 2\} = r. \end{aligned}$$

In particular, if we let $M[\bar{G}] \subseteq V$ be the family of objects of the form $\tau_{\bar{G}}$ for some $\tau \in M^{\mathsf{B}}$, we have that $r, G, \bar{G} \in M[\bar{G}]$, and also that whenever $a, b \in M[\bar{G}]$, then also $\{a, b\}, \langle a, b \rangle \in M[\bar{G}]$.

Our future investigations will show that M^{B} gives a family of B-names for all elements of $M[\bar{G}]$, and that all the familiar operations on sets we can conceive are reflected in corresponding operations on M^{B}. This will render M^{B} a boolean valued model for ZFC, and $M[\bar{G}]$ a transitive model of ZFC. How will we be able to control the semantic of M^{B} and that of $M[\bar{G}]$? The guiding idea will be the following:

- The B-names define in M a family of "names" for the objects of $M[\bar{G}]$ which labels elements of $M[\bar{G}]$ via the map $\mathrm{val}_{\bar{G}}$.
- The first order properties which $M[\bar{G}]$ assigns to $\mathrm{val}_{\bar{G}}(\sigma)$ for a $\sigma \in M^{\mathsf{B}}$ are conditional on the choice of $\bar{G}$.
- These properties vary continuously with respect to $\mathrm{St}(\mathsf{B})$ as $\bar{G}$ ranges among the M-generic ultrafilters.
- These properties can be described inside M by means of a natural boolean valued semantics on the class M^{B} of B-names, a semantics which is first order definable in M.

For example let $\bar{H} \in \mathrm{St}(\mathsf{B})$ be an M-generic filter on B such that $O_{\langle 0,1-r(0)\rangle} \in \bar{H}$ ($\bar{H}$ exists since there are densely many M-generic filters for B in V), $H = \left\{s \in 2^{\omega} : i(s) \in \bar{H}\right\}$, $t = \cup H$.

Since $\bar{H}$ is M-generic for B, we can also define

$$\mathrm{val}_{\bar{H}}(\tau) = \tau_{\bar{H}} = \left\{\sigma_{\bar{H}} : \tau(\sigma) \in \bar{H}\right\},$$

and we can check that $\mathrm{val}_{\bar{H}}(\check{a}) = a$ for all $a \in M$, but also that

$$\dot{r}_{\bar{H}} = t,$$
$$\dot{G}_{\bar{H}} = H,$$
$$\dot{H}_{\bar{H}} = \bar{H}.$$

This shows that certain properties of the object $\mathrm{val}_K(\tau)$ which is named by the B-name τ depend crucially on the decision an M-generic filter K-makes. In our case, if $O_{\langle 0,0\rangle} \in K$, we get that $\dot{r}_K(0) = 0$, while if $O_{\langle 0,1\rangle} \in K$, $\dot{r}_K(0) = 1$. On the other hand certain properties of τ cannot be changed by varying the M-generic filters for B. For example whichever K we choose, we will always get that $\dot{r}_K$ is a function in $(2^{\omega})^V \setminus M$.

One can introduce in V the following forcing relation for $b \in \mathsf{B}$, $\phi(x_0, \dots, x_n)$ a first order formula and $\tau_1, \dots, \tau_n \in V^{\mathsf{B}}$:

$$b \Vdash \phi(\tau_1, \dots, \tau_n)$$

if and only if

$M[K] \models \phi(\mathrm{val}_K(\tau_1), \dots, \mathrm{val}_K(\tau_n))$ for all M-generic filters K for B such that $b \in K$.

The intuition is that b decides (or "forces") certain facts (those described by ϕ) about the B-names $\tau_1, \dots, \tau_n$ to be true in $M[K]$ of the objects $(\tau_1)_K, \dots, (\tau_n)_K$, no matter how an M-generic filter $K \ni b$ evaluates $\tau_1, \dots, \tau_n$. Formally

$$b \Vdash \phi(\tau_1, \dots, \tau_n)$$

stands for:

$V \models \forall K \in \mathrm{St}(\mathsf{B})\,(K \text{ is } M\text{-generic for } \mathsf{B} \wedge b \in K) \rightarrow \mathsf{Sat}(M[K], \overline{\phi}, \langle(\tau_1)_K, \dots, (\tau_n)_K\rangle)$,

where $\mathsf{Sat}(x, y, z)$ is the satisfaction predicate for structures of the form $(N, \in)$ introduced in Section 7.2.

So far the above observations show among other things:

1. $1_{\mathsf{B}} \Vdash \dot{r} : \check{\omega} \to \check{2}$ is a function,
2. for all $f \in (2^{\omega})^M$, $1_{\mathsf{B}} \Vdash \dot{G} \cap \check{D}_f \neq \emptyset$,
3. for all $i < 2, n \in \omega$, $O_{\langle n,i\rangle} \Vdash \dot{r}(\check{n}) = \check{i}$.
4. $b \Vdash \phi \wedge \psi$ iff $b \Vdash \phi$ and $b \Vdash \psi$.
5. ...

This forcing relation tells us that when b is chosen by some G, all the properties which b assigns to a certain B-name will hold for the interpretation of that B-name by G. This is very useful and allows to compute in V what properties of a B-name τ are decided by a condition in B and in which ways. Moreover this forcing relation has the same flavor of the boolean valued semantics we met so far, and one of our main objective (i.e. Cohen's forcing theorem) amounts to show that:

> The forcing relation on $\mathsf{B}^+ \times \text{Form} \times (M^{\mathsf{B}})^{<\omega}$ defined in V obeys to the laws given by Lemma 5.3.9, if (following the notation of the Lemma) we replace all over $M/_G$ by $M[G]$, and $G \in \mathrm{St}(\mathsf{B})$ by $G \in \mathrm{St}(\mathsf{B})$ *is M-generic for* B.

However we have a great problem to match for the above forcing relation:

> The semantic for M^{B} we defined above has not been defined inside M.

To define the forcing relation $b \Vdash \phi(\tau_1, \dots, \tau_n)$, we need to be able to quantify over all G which are M-generic for B. This can be done meaningfully in V (where the above set is a dense subset of $\mathrm{St}(\mathsf{B})$), however in M the set of such G defines the empty-set (since B is atomless), and $M[G]$ cannot be defined.

This problem cause serious difficulties if our aim is to endow M^{B} of the structure of a boolean valued model definable in M.

> Can we expand the above forcing relation so to be able to give to M^{B} the structure of a B-valued model? Concretely, by means of the above forcing relation can we define inside M a class function $G_\phi : (M^{\mathsf{B}})^n \to \mathsf{B}$ which assigns a boolean value to a formula $\phi(x_1, \dots, x_n)$ evaluated in the tuple $(\tau_1, \dots, \tau_n)$ with assignment $x_i \mapsto \tau_i$?

In V we can define the set

$$A_\phi(\vec{\tau}) = \{b \in \mathsf{B}^+ : V \models b \Vdash \phi(\vec{\tau})\}.$$

$A_\phi(\vec{\tau}) \in \mathcal{P}(\mathsf{B})$ is a subset of M, since $\mathsf{B} \in M$ and M is transitive. However we have no special argument to expect that $A_\phi(\vec{\tau}) \in M$, since this set is defined using in V an instance of the comprehension axiom for a formula defining $\Vdash$ which requires to quantify over sets not in M. In particular if this set is in M, we must find some argument to be able to assert it.

Nonetheless it appears that the reasonable definition of a B-valued semantic for M^{B} is given by letting $[\![\phi(\vec{\tau})]\!] = \bigvee A_\phi(\vec{\tau})$ (as in the last item of the Forcing Lemma 5.3.9), for example this holds for:

1. $[\![\dot{r}(\check{n}) = \check{i}]\!] = \bigvee_{\mathsf{B}} \left\{b : b \Vdash \dot{r}(\check{n}) = \check{i}\right\} = \bigvee_{\mathsf{B}} \left\{b : b \leq O_{\langle n,i\rangle}\right\} = O_{\langle n,i\rangle}$,
2. $[\![\dot{r} : \check{\omega} \to \check{2}]\!] = \bigvee_{\mathsf{B}} \left\{b : b \Vdash \dot{r} : \check{\omega} \to \check{2}\right\} = \bigvee_{\mathsf{B}} \left\{b : b \leq 1_{\mathsf{B}}\right\} = 1_{\mathsf{B}}$,
3. $\dots$

In the above equalities we ended up having $A_\phi(\vec{\tau}) \in M$ for

$$\phi(\dot{r}, \check{n}, \check{i}) \equiv \dot{r}(\check{n}) = \check{i}$$

and also for

$$\phi(\dot{r}, \check{\omega}, \check{2}) \equiv \dot{r} : \check{\omega} \to \check{2} \text{ is a function}.$$

Is this a peculiarity of these formulae?

Let us work now under the assumption that $A_{x\in y}(\tau,\sigma), A_{x=y}(\tau,\sigma), A_{x\subseteq y}(\tau,\sigma)$ are in M for all $\sigma, \tau \in M^{\mathsf{B}}$.

Fact 6.1.23. *Assume M is a transitive countable model of ZFC, and $\mathsf{B} \in M$ is a complete boolean algebra such that M models B is complete. Assume further that*

$$A_R(\sigma,\tau) = \{b \in \mathsf{B} : b \Vdash \sigma\ R\ \tau\} \in M$$

for R among $\in, =, \subseteq$ and $\sigma, \tau \in M^{\mathsf{B}}$. Then we can set $[\![\sigma\ R\ \tau]\!] = \bigvee A_R(\sigma,\tau)$, and we get that

1. *For all $\tau, \sigma, \pi \in M^{\mathsf{B}}$,*

$$[\![\tau = \tau]\!] = 1_{\mathsf{B}},$$
$$[\![\tau = \sigma]\!] = [\![\sigma = \tau]\!],$$
$$[\![\tau = \sigma]\!] \wedge [\![\sigma = \pi]\!] \leq [\![\tau = \pi]\!].$$

2. *For R among $\in, =$, and for all $\langle\tau_1,\tau_2\rangle, \langle\sigma_1,\sigma_2\rangle \in (M^{\mathsf{B}})^2$,*

$$[\![\tau_1 = \sigma_1]\!] \wedge [\![\tau_2 = \sigma_2]\!] \wedge [\![\tau_1\ R\ \tau_2]\!] \leq [\![\sigma_1\ R\ \sigma_2]\!].$$

In particular, letting $R^{\mathsf{B}}(\tau,\sigma) = [\![\tau\ R\ \sigma]\!]$ for R among $\in, =$, $(M^{\mathsf{B}}, =^{\mathsf{B}}, \in^{\mathsf{B}})$ is in V a B-valued model.

Exercise 6.1.24. Prove the above inequalities (HINT: First show that it suffices to prove that $b \Vdash \phi$ entails $b \Vdash \psi$ for all the above inequalities and for all $b \in \mathsf{B}^+$ (where ϕ stands for the lefthand term of the inequality and ψ for the righthand term). Then apply the definition of the forcing relation).

We are led to the following driving questions:

1. Can we define in M class functions $A_R : (M^{\mathsf{B}})^2 \to \mathcal{P}(\mathsf{B})^M$ such that

$$A_R(\tau,\sigma) = \left\{b \in \mathsf{B}^+ : b \Vdash \sigma\ R\ \tau\right\}$$

for R among $\in, =, \subseteq$?

2. Can we prove in general, that $A_\phi(\vec{\tau}) \in M$ for all formulae $\phi(\vec{\tau})$ with parameters in M^{B}?

3. Assume both questions have a positive answer. Can we also prove that the boolean valued semantic for $\phi(\vec{\tau})$ given by $(M^{\mathsf{B}}, \in^{\mathsf{B}}, =^{\mathsf{B}}, \subseteq^{\mathsf{B}})$ (where $[\![\sigma\ R^{\mathsf{B}}\ \tau]\!] = \bigvee A_{xRy}(\sigma,\tau)$ for R among $=, \in, \subseteq$) assigns to each formula $\phi(\vec{\tau})$ the boolean value $\bigvee A_\phi(\vec{\tau})$?

4. Assume that the first question has a positive answer. Can we also prove that $(M^{\mathsf{B}}, \in^{\mathsf{B}}, =^{\mathsf{B}})$ is a *full* B-valued model in V?

We show in the next sections that all these questions have a positive answer, proving the following result:

> M can define a structure of *full* B-valued model on M^{B} which assign to any formula $\phi(x_1, \ldots, x_n)$ of the language of set theory a satisfaction class definable in M
>
> $$\left\{(\tau_1, \ldots, \tau_n, b) \in (M^{\mathsf{B}})^n \times \mathsf{B} : M \models [\![\phi(\tau_1, \ldots, \tau_n)]\!] = b\right\}$$
>
> with the feature that
>
> $$M \models [\![\phi(\tau_1, \ldots, \tau_n)]\!] \geq_{\mathsf{B}} b$$
>
> if and only if
> $$V \models b \Vdash \phi(\tau_1, \ldots, \tau_n).$$
>
> In particular the forcing relation $b \Vdash \phi(\tau_1, \ldots, \tau_n)$ and the relations $R^{\mathsf{B}}(\sigma, \tau) = [\![\sigma\ R\ \tau]\!]$ for $R \in \{=, \in\}$ can also be defined inside M making $(M^{\mathsf{B}}, =^{\mathsf{B}}, \in^{\mathsf{B}})$ a full B-valued model in V given by a triple of definable classes in M.

This will be done as follows: after having defined in M a boolean valued semantics on M^{B}, making M^{B} a full B-valued model for the first order language $\{\in, \subseteq\}$, we show that whenever G is M-generic for B, $M^{\mathsf{B}}/_G$ is isomorphic to $M[G]$ via the map $[\tau]_G \mapsto \tau_G$ (where $[\sigma]_G = \left\{\tau \in M^{\mathsf{B}} : [\![\tau = \sigma]\!] \in G\right\}$). By Łoś Theorem 5.3.7 and by the Forcing Lemmas 5.3.8 and 5.3.9, all the desired properties of M^{B} can be easily inferred, since the set of M-generic filters for $\mathrm{St}(\mathsf{B})$ is dense.

6.1.2 Internal definition of forcing: the boolean valued semantics of V^{B}

To simplify matters and notations we will assume all over this section to be working in V, the standard model of ZFC which contains all sets, however all of our definitions and results can be declined and rephrased for any arbitrary first order model of ZFC since they will be based just on the assumption that V is a model of ZFC. We will need in the next sections the relativization of many of these definitions and results to a countable transitive set $M \in V$ which is itself a model of ZFC.

The aim of this section is to define a boolean semantic on the class V^{B} making it a full boolean valued model for the language in the signature $\{\in, \subseteq\}$. In the next section we will show that this semantic, when defined in a countable transitive model M of ZFC, induces the forcing relation on M^{B} defined in the previous section.

Definition 6.1.25. Let $\tau, \sigma \in V^{\mathsf{B}}$. We define simultaneously, by induction on the pairs $(\mathrm{rk}(\tau), \mathrm{rk}(\sigma))$ well ordered in type Ord by the square order[3] on Ord^2:

1. $$[\![\tau \in \sigma]\!] = \bigvee_{\tau_0 \in \mathrm{dom}(\sigma)} ([\![\tau = \tau_0]\!] \wedge \sigma(\tau_0)).$$

2. $$[\![\tau \subseteq \sigma]\!] = \bigwedge_{\sigma_0 \in \mathrm{dom}(\tau)} (\tau(\sigma_0) \to [\![\sigma_0 \in \sigma]\!]) = \bigwedge_{\sigma_0 \in \mathrm{dom}(\tau)} (\neg\tau(\sigma_0) \vee [\![\sigma_0 \in \sigma]\!]).$$

3. $$[\![\tau = \sigma]\!] = [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!].$$

Remark 6.1.26. The definition of all three relations is by a simultaneous induction. More precisely Let $F_j : V \times V^{\mathsf{B}} \times V^{\mathsf{B}} \to \mathsf{B}$ for $j = 0, 1$ be defined by

$$\begin{cases} F_0(g, \tau, \sigma) = \bigvee_{\eta \in \mathrm{dom}(\sigma)} \sigma(\eta) \wedge g(\eta, \tau) & \text{if } \exists \alpha \ g : (V^{\mathsf{B}}_\alpha)^2 \to \mathsf{B} \text{ and } \mathrm{dom}(\sigma) \times \{\tau\} \subseteq \mathrm{dom}(g), \\ F_0(g, \tau, \sigma) = 0_{\mathsf{B}} \text{ otherwise;} \end{cases}$$

$$\begin{cases} F_1(g, \tau, \sigma) = \bigwedge_{\eta \in \mathrm{dom}(\tau)} (\tau(\eta) \to g(\eta, \sigma)) \text{ if } \exists \alpha \ g : (V^{\mathsf{B}}_\alpha)^2 \to \mathsf{B} \text{ and } \mathrm{dom}(\tau) \times \{\sigma\} \subseteq \mathrm{dom}(g), \\ F_1(g, \tau, \sigma) = 0_{\mathsf{B}} \text{ otherwise.} \end{cases}$$

Now let $G : (V^{\mathsf{B}})^2 \to \mathsf{B}^3$ be defined by transfinite recursion by the following clauses:

$$G(\tau, \sigma) = (G_0(\tau, \sigma), G_1(\tau, \sigma), G_2(\tau, \sigma))$$

where

$$\begin{aligned} G_0(\tau, \sigma) &= F_0(G_2 \restriction \mathrm{dom}(\sigma) \times \{\tau\}, \tau, \sigma), \\ G_1(\tau, \sigma) &= F_1(G_0 \restriction \mathrm{dom}(\tau) \times \{\sigma\}, \tau, \sigma), \\ G_2(\tau, \sigma) &= F_1(G_0 \restriction \mathrm{dom}(\sigma) \times \{\tau\}, \tau, \sigma) \wedge F_1(G_0 \restriction \mathrm{dom}(\sigma) \times \{\tau\}, \sigma, \tau). \end{aligned}$$

We leave to the reader to check that such a G is a definable class in V in the parameter $(\mathsf{B}, \wedge, \vee, \neg, 0_{\mathsf{B}}, 1_{\mathsf{B}})$, and that $G_0(\tau, \sigma) = [\![\tau \in \sigma]\!]$, $G_1(\tau, \sigma) = [\![\tau \subseteq \sigma]\!]$, $G_2(\tau, \sigma) = [\![\tau = \sigma]\!]$.

[3]The square order $<^2$ is given by $(\alpha, \beta) <^2 (\gamma, \delta)$ iff $\max\{\alpha, \beta\} < \max\{\gamma, \delta\}$ or $\max\{\alpha, \beta\} = \max\{\gamma, \delta\}$ and (α, β) is lexicographically below (γ, δ).

It can also be observed that the relations $[\![\tau \in \sigma]\!]$ and $[\![\tau \subseteq \sigma]\!]$ are Δ_1-definable in the parameter B and thus are absolute between M and V if M is a transitive model of ZFC to which B belongs. However this is slightly more subtle since B could be a complete boolean algebra in M while it is not such in V, thus it is less transparent why the definition of $[\![\tau \in \sigma]\!]$ and $[\![\tau \subseteq \sigma]\!]$ which are using in an essential way the completeness of B can even be formulated in V where B might not be a complete boolean algebra. We may come back to this point later on when we will need to clarify it. The key observation to solve this issue being that $M \models b = \bigvee_{\mathsf{B}} A$ iff $V \models b = \bigvee_{\mathsf{B}} A$ for all $A \in M \cap \mathcal{P}(\mathsf{B})$.

Theorem 6.1.27. *V^{B} is a boolean valued model for set theory.*

Proof. We have to check that V^{B} satisfies the four clauses of Definition 5.1.1, and the two additional items of Definition 6.1.1, i.e. we have to show that, for all $\tau, \sigma, \eta \in V^{\mathsf{B}}$:

1. $[\![\tau = \tau]\!] = 1$.
2. $[\![\tau = \sigma]\!] = [\![\sigma = \tau]\!]$.
3. $[\![\tau = \sigma]\!] \ \wedge \ [\![\sigma = \eta]\!] \leq [\![\tau = \eta]\!]$.
4. $[\![\tau = \sigma]\!] \ \wedge \ [\![\sigma \ R \ \eta]\!] \leq [\![\tau \ R \ \eta]\!]$, where $R \in \{\in, \subseteq\}$.
5. $[\![\tau = \sigma]\!] \ \wedge \ [\![\eta \ R \ \tau]\!] \leq [\![\eta \ R \ \sigma]\!]$, where $R \in \{\in, \subseteq\}$.
6. $[\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!] = [\![\tau = \sigma]\!]$.
7. $[\![\tau \in \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!] \leq [\![\tau \in \eta]\!]$.

The proof of item 1 is by induction on $\mathrm{rk}(\tau)$ and we do it rightaway. We have:

$$[\![\tau \subseteq \tau]\!] = \bigwedge_{\sigma \in \mathrm{dom}(\tau)} (\neg\tau(\sigma) \vee [\![\sigma \in \tau]\!]) =$$

$$\bigwedge_{\sigma \in \mathrm{dom}(\tau)} (\neg\tau(\sigma) \vee (\bigvee_{u \in \mathrm{dom}(\tau)} [\![\sigma = u]\!] \wedge \tau(u))) \geq \bigwedge_{\sigma \in \mathrm{dom}(\tau)} (\neg\tau(\sigma) \vee ([\![\sigma = \sigma]\!] \wedge \tau(\sigma))).$$

But $[\![\sigma = \sigma]\!] = 1$, because $\mathrm{rk}(\sigma)$ is below $\mathrm{rk}(\tau)$ and we can apply the inductive assumptions. Thus:

$$\bigwedge_{\sigma \in \mathrm{dom}(\tau)} (\neg\tau(\sigma) \vee ([\![\sigma = \sigma]\!] \wedge \tau(\sigma))) = \bigwedge_{\sigma \in \mathrm{dom}(\tau)} (\neg\tau(\sigma) \vee \tau(\sigma)) = 1,$$

i.e. $[\![\tau = \tau]\!] = 1$ as was to be shown.

Items 2 and 6 follow immediately from the definitions.

Next observe that if we can prove

$$[\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!] \leq [\![\tau \subseteq \eta]\!] \tag{6.2}$$

for all triples (τ, σ, η) we also get items 3, as well as 4, 5 for the case of R being $\subseteq$, since:

$$[\![\tau = \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!] \leq [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!] \leq [\![\tau \subseteq \eta]\!]$$

applying 6.2 to the triple (τ, σ, η) in the last inequality, which yields 4 for the case R being $\subseteq$. Similarly

$$[\![\tau = \sigma]\!] \wedge [\![\eta \subseteq \sigma]\!] \leq [\![\tau \subseteq \sigma]\!] \wedge [\![\eta \subseteq \sigma]\!] \leq [\![\eta \subseteq \tau]\!],$$

applying 6.2 to the triple (η, σ, τ) in the last inequality to infer 5 for the case R being $\subseteq$. 3 follows from 6.2 by a similar argument, left to the reader.

Next if we can prove 7 for all triples (τ, σ, η), we get 5 for the case of R being $\in$, since

$$[\![\tau = \sigma]\!] \wedge [\![\eta \in \tau]\!] \leq [\![\tau \subseteq \sigma]\!] \wedge [\![\eta \in \tau]\!] \leq [\![\eta \in \sigma]\!]$$

applying 7 to triple (η, τ, σ) in the last of the above inequalities.

Hence it suffices to prove 7, 6.2, 4 for the case of R being $\in$, i.e. the following three items:

(a) $[\![\tau \in \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!] \leq [\![\tau \in \eta]\!]$

(b) $[\![\tau = \sigma]\!] \wedge [\![\tau \in \eta]\!] \leq [\![\sigma \in \eta]\!]$

(c) $[\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!] \leq [\![\tau \subseteq \eta]\!]$

We will prove (a), (b), (c) by means of a nested induction on the triples $(\mathrm{rk}(\tau), \mathrm{rk}(\sigma), \mathrm{rk}(\eta))$ ordered by the cube well-order[4] on Ord^3; we will do the induction for (a), (b), (c) simultaneously; to prove each of these items for some triple of B-names, we will assume that all three properties (a), (b), (c) hold for all triples of B-names of lower rank in the cube ordering.

We will also use the following observation:

If (a), (b), (c) hold for all triples up to a given rank in the cube order, we get that 3, 4, 5, 6, 7 hold for all these triples.

This is the case since the arguments we gave above leading from any of (a), (b), (c) to some of 3, 4, 5, 6, 7 can be repeated verbatim for the triples at hand since no inductive assumption is needed to carry these arguments.

(a): $[\![\tau \in \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!]$ is equal to

$$\bigvee_{\sigma_0 \in \mathrm{dom}(\sigma)} ([\![\tau = \sigma_0]\!] \wedge \sigma(\sigma_0)) \wedge \bigwedge_{\sigma_1 \in \mathrm{dom}(\sigma)} (\neg\sigma(\sigma_1) \vee [\![\sigma_1 \in \eta]\!]).$$

The latter is equal to

$$\bigvee_{\sigma_0 \in \mathrm{dom}(\sigma)} \bigwedge_{\sigma_1 \in \mathrm{dom}(\sigma)} [([\![\tau = \sigma_0]\!] \wedge \sigma(\sigma_0) \wedge \neg\sigma(\sigma_1)) \vee ([\![\tau = \sigma_0]\!] \wedge \sigma(\sigma_0) \wedge [\![\sigma_1 \in \eta]\!])].$$

Now $\sigma(\sigma_0) \wedge \neg\sigma(\sigma_0)) = 0$ for any $\sigma_0 \in \mathrm{dom}(\sigma)$ and σ_0, σ_1 both range among $\mathrm{dom}(\sigma)$. Hence for all $\sigma_0 \in \mathrm{dom}(\sigma)$

$$\bigwedge_{\sigma_1 \in \mathrm{dom}(\sigma)} [([\![\tau = \sigma_0]\!] \wedge \sigma(\sigma_0) \wedge \neg\sigma(\sigma_1)) \vee ([\![\tau = \sigma_0]\!] \wedge \sigma(\sigma_0) \wedge [\![\sigma_1 \in \eta]\!])] \leq$$

$$\leq [\![\tau = \sigma_0]\!] \wedge \sigma(\sigma_0) \wedge [\![\sigma_0 \in \eta]\!] \leq$$

$$\leq [\![\tau = \sigma_0]\!] \wedge [\![\sigma_0 \in \eta]\!].$$

[4]The cube order $<^3$ is given by $(\alpha, \beta, \gamma) <^2 (\eta, \delta, \nu)$ iff $\max\{\alpha, \beta, \gamma\} < \max\{\eta, \delta, \nu\}$ or $\max\{\alpha, \beta, \gamma\} = \max\{\eta, \delta, \nu\}$ and (α, β, γ) is lexicographically below (η, δ, ν).

This gives that:

$$
\begin{aligned}
[\![\tau \in \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!] \leq & \\
& \leq \bigvee_{\sigma_0 \in \mathrm{dom}(\sigma)} [\![\tau = \sigma_0]\!] \wedge [\![\sigma_0 \in \eta]\!] \leq \\
& \leq \bigvee_{\sigma_0 \in \mathrm{dom}(\sigma)} [\![\tau \in \eta]\!] = \\
& = [\![\tau \in \eta]\!].
\end{aligned}
$$

For the last inequality we have used the inductive hypotesis (b) on the triple (τ, σ_0, η) which is below the triple (τ, σ, η) in the cube order on Ord^3.

(b): Let $t \in \mathrm{dom}(\eta)$, we have:

$$[\![\sigma = \tau]\!] \ \wedge \ [\![\tau = t]\!] \ \wedge \ \eta(t) \leq [\![\sigma = t]\!] \wedge \eta(t)$$

applying the inductive assumption on item 3 to the triple (σ, τ, t) which is below the triple (σ, τ, η) in the cube order. Thus:

$$
\begin{aligned}
[\![\tau = \sigma]\!] \wedge [\![\sigma \in \eta]\!] = & \\
& = [\![\sigma = \tau]\!] \wedge [\![\sigma \in \eta]\!] = \\
& = [\![\sigma = \tau]\!] \wedge (\bigvee_{t \in \mathrm{dom}(\eta)} [\![\tau = t]\!] \ \wedge \ \eta(t)) = \\
& = \bigvee_{t \in \mathrm{dom}(\eta)} ([\![\sigma = \tau]\!] \wedge [\![\tau = t]\!] \ \wedge \ \eta(t)) \leq \\
& \leq \bigvee_{t \in \mathrm{dom}(\eta)} ([\![\sigma = t]\!] \ \wedge \eta(t)) = \\
& = [\![\sigma \in \eta]\!].
\end{aligned}
$$

(c): Let $t \in \mathrm{dom}(\tau)$. We can apply the inductive assumption (a) on the triple (t, σ, η) which is below the triple (τ, σ, η) in the cube order on Ord^3 to get

$$[\![t \in \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!] \leq [\![t \in \eta]\!].$$

This gives that

$$[\![\tau \subseteq \eta]\!] = \bigwedge_{t \in \mathrm{dom}(\tau)} \tau(t) \to [\![t \in \eta]\!] \geq \bigwedge_{t \in \mathrm{dom}(\tau)} \tau(t) \to ([\![t \in \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!]).$$

The latter is equal to

$$(\bigwedge_{t \in \mathrm{dom}(\tau)} \tau(t) \to [\![t \in \sigma]\!]) \wedge (\bigwedge_{t \in \mathrm{dom}(\tau)} \tau(t) \to [\![\sigma \subseteq \eta]\!]).$$

Now observe that

$$(\bigwedge_{t \in \mathrm{dom}(\tau)} \tau(t) \to [\![\sigma \subseteq \eta]\!]) = [\![\sigma \subseteq \eta]\!] \vee (\bigwedge_{t \in \mathrm{dom}(\tau)} \neg\tau(t)) \geq [\![\sigma \subseteq \eta]\!],$$

while

$$\bigwedge_{t \in \mathrm{dom}(\tau)} \tau(t) \to [\![t \in \sigma]\!] = [\![\tau \subseteq \sigma]\!].$$

We conclude that

$$[\![\tau \subseteq \eta]\!] \geq [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \eta]\!],$$

as was to be shown.

The proof is complete. □

Remark 6.1.28. The above proof can be formalized in the following manner: letting $G : (V^{\mathsf{B}})^2 \to \mathsf{B}^3$ the class function defining the relations $[\![\tau = \sigma]\!] = G_0(\tau, \sigma)$, $[\![\tau \subseteq \sigma]\!] = G_1(\tau, \sigma)$, $[\![\tau = \sigma]\!] = G_2(\tau, \sigma)$, one shows by recursion on the appropriate (pair or triple of) rank(s) that

1. $G_1(\tau, \tau) = G_2(\tau, \tau) = 1_{\mathsf{B}}$,
2. $G_2(\tau, \sigma) = G_2(\sigma, \tau)$,
3. $G_1(\tau, \sigma) \wedge G_1(\sigma, \tau) = G_2(\sigma, \tau)$,
4. $G_2(\tau, \sigma) \wedge G_2(\sigma, \eta) \leq G_2(\eta, \tau)$,
5. $G_2(\tau, \sigma) \wedge G_j(\sigma, \eta) \leq G_j(\tau, \eta)$ for $j = 0, 1$,
6. $G_2(\tau, \sigma) \wedge G_j(\tau, \eta) \leq G_j(\sigma, \eta)$ for $j = 0, 1$,
7. $G_0(\tau, \sigma) \wedge G_1(\sigma, \eta) \leq G_1(\tau, \eta)$.

Once we have shown that in V the class V^{B} with the classes G_0, G_1, G_2 for $\in^{\mathsf{B}}, \subseteq^{\mathsf{B}}, =^{\mathsf{B}}$ satisfies the clauses for a boolean valued model given in Def. 5.1.1, we can give an interpretation to all formulae of the first order language $\{\in, \subseteq, =\}$ assigning by recursion to each formula ϕ its boolean satisfaction class G_ϕ as follows:

Definition 6.1.29. For each formula $\phi(x_0, \ldots, x_n)$ in the first order language $\{\in, \subseteq, =\}$, we let

$$G_\phi : (V^{\mathsf{B}})^n \to \mathsf{B} \qquad (\tau_0, \ldots, \tau_n) \mapsto [\![\phi(\tau_0, \ldots, \tau_n)]\!]_{\mathsf{B}}$$

be the class defined by the requirements:

$$[\![\psi(\tau_0, \ldots \tau_n)]\!]_{\mathsf{B}} \wedge_{\mathsf{B}} [\![\theta(\tau_0, \ldots, \tau_m)]\!]_{\mathsf{B}} = [\![\psi(\tau_0, \ldots \tau_n) \wedge \theta(\tau_0, \ldots, \tau_m)]\!]_{\mathsf{B}}.$$

$$[\![\psi(\tau_0, \ldots \tau_n)]\!]_{\mathsf{B}} \vee_{\mathsf{B}} [\![\theta(\tau_0, \ldots, \tau_m)]\!]_{\mathsf{B}} = [\![\psi(\tau_0, \ldots \tau_n) \vee \theta(\tau_0, \ldots, \tau_m)]\!]_{\mathsf{B}}.$$

$$\neg_{\mathsf{B}} [\![\psi(\tau_0, \ldots \tau_n)]\!]_{\mathsf{B}} = [\![\neg\psi(\tau_0, \ldots \tau_n)]\!]_{\mathsf{B}}.$$

$$[\![\exists x_j \psi(\tau_0, \ldots, \tau_{j-1}, x, \tau_j, \ldots, \tau_n)]\!]_{\mathsf{B}} = \bigvee_{\sigma \in V^{\mathsf{B}}} [\![\psi(\tau_0, \ldots, \tau_{j-1}, \sigma, \tau_j, \ldots, \tau_n)]\!]_{\mathsf{B}}$$

Remark 6.1.30. More formally for each each formula $\phi(x_0, \dots, x_n)$ in the first order language $\{\in, \subseteq, =\}$, we let:

$\phi \equiv x\ R\ y$ for R among $\in, \subseteq, =$:

$$G_\phi = \{(\tau_0, \tau_1, b) : V \models [\![\tau_0\ R\ \tau_1]\!]_{\mathsf{B}} = b\}.$$

$\phi \equiv \psi(x_0, \dots, x_n) \wedge \theta(x_0, \dots, x_m)$: We let $l = \max\{m, n\}$ and set

$$G_\phi = \{(\tau_0, \dots, \tau_l, b) : V \models \exists c, d \in \mathsf{B}\,(b = c \wedge_{\mathsf{B}} d) \wedge G_\psi(\tau_0, \dots, \tau_n, c) \wedge G_\theta(\tau_0, \dots, \tau_m, d)\}.$$

I.e.:

$$b = [\![\psi(\tau_0, \dots \tau_n)]\!]_{\mathsf{B}} \wedge_{\mathsf{B}} [\![\theta(\tau_0, \dots, \tau_m)]\!]_{\mathsf{B}}.$$

$\phi \equiv \psi(x_0, \dots, x_n) \vee \theta(x_0, \dots, x_m)$: We let $l = \max\{m, n\}$ and set

$$G_\phi = \{(\tau_0, \dots, \tau_l, b) : V \models \exists c, d \in \mathsf{B}\,(b = c \vee_{\mathsf{B}} d) \wedge G_\psi(\tau_0, \dots, \tau_n, c) \wedge G_\theta(\tau_0, \dots, \tau_m, d)\}.$$

I.e.:

$$b = [\![\psi(\tau_0, \dots \tau_n)]\!]_{\mathsf{B}} \vee_{\mathsf{B}} [\![\theta(\tau_0, \dots, \tau_m)]\!]_{\mathsf{B}}.$$

$\phi \equiv \neg\psi(x_0, \dots, x_n)$:

$$G_\phi = \{(\tau_0, \dots, \tau_l, b) : V \models \exists c \in \mathsf{B}\,(b = \neg c) \wedge G_\psi(\tau_0, \dots, \tau_n, c)\}.$$

I.e.:

$$b = \neg\, [\![\psi(\tau_0, \dots \tau_n)]\!]_{\mathsf{B}}.$$

$\phi \equiv \exists x_j \psi(x_0, \dots, x_n)$: Letting

$$\theta(x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_n, z) \equiv \forall x_j[\exists y(G_\psi(x_0, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n, y)] \to z \geq_{\mathsf{B}} y,$$

set G_ϕ to be the class of $(\tau_0, \dots, \tau_{j-1}, \tau_{j+1} \dots, \tau_n, c)$ such that

$$\begin{aligned} V \models \exists\sigma\, G_\psi(\tau_0, \dots, \tau_{j-1}, \sigma, \tau_{j+1}, \dots, \tau_n, c) \\ \wedge\theta(\tau_0, \dots \tau_{j-1}, \tau_{j+1}, \dots, \tau_n, c). \end{aligned}$$

I.e. $(\tau_0, \dots, \tau_{j-1}, \tau_{j+1} \dots, \tau_n, c) \in G_\phi$ iff

$$c = \bigvee_{\sigma \in V^{\mathsf{B}}} [\![\psi(\tau_0, \dots \tau_{j-1}, \sigma, \tau_{j+1}, \dots, \tau_n)]\!]_{\mathsf{B}}$$

With these definitions we have that $\left\langle V^{\mathsf{B}}, \in^{\mathsf{B}}, \subseteq^{\mathsf{B}}, =^{\mathsf{B}} \right\rangle$ is a B-valued model. Note also that each class G_ϕ is a definable class in V, but (it can be shown that) the collection of classes $\{G_\phi : \phi \text{ a formula of } \mathcal{L}\}$ cannot be represented as a definable class in V. We now show that $\left\langle V^{\mathsf{B}}, \in^{\mathsf{B}}, \subseteq^{\mathsf{B}}, =^{\mathsf{B}} \right\rangle$ is full, which formally amounts to show that

For all formula $\phi \equiv \exists x_j \psi(x_0, \ldots, x_n)$ and all $(\tau_0, \ldots, \tau_{j-1}, \tau_{j+1}, \ldots \tau_n, b) \in G_\phi$, there exists $\sigma \in V^{\mathsf{B}}$ such that

$$(\tau_0, \ldots, \tau_{j-1}, \sigma, \tau_{j+1}, \ldots \tau_n, b) \in G_\psi,$$

and

$$b \geq d$$

for all d such that for some $\tau \in V^{\mathsf{B}}$

$$(\tau_0, \ldots, \tau_{j-1}, \tau, \tau_{j+1}, \ldots \tau_n, d) \in G_\psi.$$

and informally to assert that

$$\bigvee_{\tau \in V^{\mathsf{B}}} [\![\psi(\tau_0, \ldots, \tau_{j-1}, \tau, \tau_{j+1}, \ldots \tau_n)]\!] = [\![\psi(\tau_0, \ldots, \tau_{j-1}, \sigma, \tau_{j+1}, \ldots \tau_n)]\!]$$

for some $\sigma \in V^{\mathsf{B}}$.

Towards this aim, we need the following basic properties of boolean algebras:

Exercise 6.1.31. In a boolean algebra B, for any $a, b, c \in \mathsf{B}$ we have:

$$c \wedge a \leq c \wedge b \Leftrightarrow c \leq a \rightarrow b.$$

Lemma 6.1.32 (Mixing Lemma)**.** *Let B be a boolean algebra. Let A be an antichain of B and for any $a \in A$ let τ_a be an element of V^{B}. Then there exists some $\tau \in V^{\mathsf{B}}$ such that $a \leq [\![\tau = \tau_a]\!]$ for all $a \in A$.*

Proof. Let $D = \bigcup_{a \in A} \mathrm{dom}(\tau_a)$ and, for every $t \in D$, let

$$\tau(t) = \bigvee \{a \wedge \tau_a(t) : a \in A \wedge t \in \mathrm{dom}(\tau_a)\}.$$

Since A is an antichain and by the definition of $\tau(t)$, we have that

$$a \wedge \tau(t) = a \wedge \tau_a(t)$$

for any $a \in A$ and any $t \in \mathrm{dom}(\tau_a)$. So, by exercise 6.1.31, for any $a \in A$,

$$\text{For all } t \in \mathrm{dom}(\tau_a)\ (a \leq \tau_a(t) \leftrightarrow \tau(t)). \tag{6.3}$$

On the other hand

$$\text{For all } t \in D \setminus \mathrm{dom}(\tau_a)\ (a \wedge \tau(t) = 0) \tag{6.4}$$

holds since for such elements t

$$\tau(t) = \bigvee \{\tau_b(t) \wedge b : b \neq a,\ b \in A,\ t \in \mathrm{dom}(\tau_b)\} \leq (\bigvee A) \setminus a.$$

Now we use 6.3 and 6.4 to obtain that:

Claim 6.1.32.1.

$$a \leq [\![\tau_a \subseteq \tau]\!]$$

$$a \leq [\![\tau \subseteq \tau_a]\!].$$

Proof. We use equation 6.3 to infer that $a \leq [\![\tau_a \subseteq \tau]\!]$ as follows: First of all

$$[\![t \in \tau]\!] \geq \tau(t)$$

for all $t \in D = \mathrm{dom}(\tau)$, so we have:

$$a \leq \tau_a(t) \to \tau(t) \leq \tau_a(t) \to [\![t \in \tau]\!]$$

for any $a \in A$ and any $t \in \mathrm{dom}(\tau_a) \subseteq D$. So we have:

$$a \leq \bigwedge_{t \in \mathrm{dom}(\tau_a)} \tau_a(t) \to \tau(t) \leq \bigwedge_{t \in \mathrm{dom}(\tau_a)} \tau_a(t) \to [\![t \in \tau]\!] = [\![\tau_a \subseteq \tau]\!].$$

This proves the first inequality of the claim.

To prove the second inequality of the claim it is enough to show that

$$a \leq \tau(t) \to [\![t \in \tau_a]\!]$$

for all $t \in D$. We prove it using 6.4 as follows:

- If $t \in D \setminus \mathrm{dom}(\tau_a)$, then 6.4 gives that

  $$a \wedge \tau(t) = 0.$$

  If we combine it with exercise 6.1.31 we get that

  $$a \leq \tau(t) \to b$$

  for any $b \in \mathsf{B}$ and $t \in D \setminus \mathrm{dom}(\tau_a)$. In particular

  $$\text{For all } t \in D \setminus \mathrm{dom}(\tau_a)\ a \leq \tau(t) \to [\![t \in \tau_a]\!]. \tag{6.5}$$

- If $t \in \mathrm{dom}(\tau_a)$ we can follow the pattern we have seen in the proof of the first inequality to get using 6.3:

  $$\text{For all } t \in \mathrm{dom}(\tau_a)\ (a \leq \tau(t) \to [\![t \in \tau_a]\!]). \tag{6.6}$$

Thus by 6.5,6.6 we get

$$a \leq \bigwedge_{t \in \mathrm{dom}(\tau)} \tau(t) \to [\![t \in \tau_a]\!] = [\![\tau \subseteq \tau_a]\!].$$

The second inequality of the Claim is proved. □

The proof of the Mixing lemma is completed. □

The following would be an instance of Proposition 5.3.12 if V^{B} were a set. We include a proof since V^{B} is not a set.

Theorem 6.1.33 (Maximum Principle)**.** *For any boolean algebra* B*,* V^{B} *is full, i.e. for all formulae* $\varphi(x, \bar{y})$ *and* $\bar{\tau} \in (V^{\mathsf{B}})^{<\omega}$

$$[\![\exists x \varphi(x, \bar{\tau})]\!] = [\![\varphi(\sigma, \bar{\tau})]\!]$$

for some $\sigma \in V^{\mathsf{B}}$*.*

Proof.

$$[\![\exists x \varphi(x, \bar{\tau})]\!] \geq [\![\varphi(\sigma, \bar{\tau})]\!]$$

holds always. So we want to show that

$$[\![\varphi(\sigma, \bar{\tau})]\!] \geq [\![\exists x \varphi(x, \bar{\tau})]\!]$$

for some $\sigma \in V^{\mathsf{B}}$. Let

$$u_0 = [\![\exists x \varphi(x, \bar{\tau})]\!] > 0_{\mathsf{B}}.$$

Let

$$D = \{u \in \mathsf{B}^+ : \text{ there is some } \sigma_u \in V^{\mathsf{B}} \text{ such that } u \leq [\![\varphi(\sigma_u, \bar{\tau})]\!]\}.$$

D is dense and open below u_0 in B^+. Let A be a maximal antichain of D (A exists applying exercise 4.2.2 to the boolean algebra $\mathsf{B} \restriction u_0$); clearly

$$\bigvee\{u : u \in A\} = u_0.$$

Now we can appeal to the Mixing lemma to find $\sigma \in V^{\mathsf{B}}$ such that $[\![\sigma = \sigma_u]\!] \geq u$ for any $u \in A$. Thus for each $u \in A$ we have

$$u \leq [\![\sigma = \sigma_u]\!] \wedge [\![\varphi(\sigma_u, \bar{\tau})]\!] \leq [\![\varphi(\sigma, \bar{\tau})]\!].$$

Therefore

$$[\![\exists x \varphi(x, \bar{\tau})]\!] = u_0 = \bigvee A \leq [\![\varphi(\sigma, \bar{\tau})]\!].$$

The proof is complete. □

The remainder of this section is not of key importance for the development of our core results.

Fact 6.1.34. V^{B} *satisfies the Axiom of extensionality.*

Proof. Let $\tau, \sigma \in V^{\mathsf{B}}$. We want to prove that:

$$[\![\forall u(u \in \tau \leftrightarrow u \in \sigma) \to \tau = \sigma]\!] = 1.$$

By lemma 5.1.7, it is enough to show that:

$$[\![\forall u(u \in \tau \leftrightarrow u \in \sigma)]\!] \leq [\![\tau = \sigma]\!].$$

We observe that if $a \leq a'$ then $(a' \to b) \leq (a \to b)$. Thus for any $u \in \mathrm{dom}(\tau)$ we have

$$([\![u \in \tau]\!] \to [\![u \in \sigma]\!]) \leq (\tau(u) \to [\![u \in \sigma]\!]),$$

and therefore

$$\bigwedge_{u\in V^{\mathsf{B}}}(\llbracket u\in\tau\rrbracket\to\llbracket u\in\sigma\rrbracket)\leq\bigwedge_{u\in\mathrm{dom}(\tau)}(\llbracket u\in\tau\rrbracket\to\llbracket u\in\sigma\rrbracket)\leq$$
$$\leq\bigwedge_{u\in\mathrm{dom}(\tau)}(\tau(u)\to\llbracket u\in\sigma\rrbracket)$$

The left-hand side of the above equation is equal to $\llbracket\forall u(u\in\tau\to u\in\sigma)\rrbracket$, while the right-hand side is the definition of $\llbracket\tau\subseteq\sigma\rrbracket$. Consequently:

$$\llbracket\forall u(u\in\tau\leftrightarrow u\in\sigma)\rrbracket\leq\llbracket\tau=\sigma\rrbracket.$$

The proof is completed. □

Summing up, so far we have proved that V^{B} is a full boolean valued model for set theory which satisfies the Axiom of Extensionality. We will later see that it satisfies all the axioms of ZFC.

We now connect the B-names for elements of 2^ω with the boolean valued model $C(St(\mathsf{B}),2^\omega)$ we examined in Section 5.4.3.

Definition 6.1.35. Assume $\left\llbracket\tau:\check{\lambda}\to\check{2}\right\rrbracket_{\mathsf{B}}=1_{\mathsf{B}}$. We define $f_\tau:\mathrm{St}(\mathsf{B})\to 2^\lambda$ as

$$f_\tau(G)(\alpha)=i\iff\left\llbracket\tau(\check{\alpha})=\check{i}\right\rrbracket\in G.$$

Now assume $f:St(\mathsf{B})\to 2^\lambda$ is a continuous function, then we define

$$\tau_f=\{\langle\check{(\alpha,i)},\{G:f(G)(\alpha)=i\}\rangle:\alpha<\lambda,i<2\}\in M^{\mathsf{B}}.$$

Note that we consider 2^λ as a topological space with the product topology. Observe also that

$$\{G:f(G)(\alpha)=i\}=f^{-1}[N_{\alpha,i}],$$

where $N_{\alpha,i}=\{g\in 2^\lambda:g(\alpha)=i\}$. Since f is continuous then $f^{-1}[N_{\alpha,i}]$ is clopen and so it is an element of the Boolean algebra.

Proposition 6.1.36. *Assume* $\left\llbracket\tau:\check{\lambda}\to\check{2}\right\rrbracket_{\mathsf{B}}=1_{\mathsf{B}}$ *and* $f:\mathrm{St}(\mathsf{B})\to 2^\lambda$ *is continuous. Then*

1. $\tau_f\in V^{\mathsf{B}}$;
2. $f_\tau:\mathrm{St}(\mathsf{B})\to 2^\lambda$ *is continuous;*
3. $\llbracket\tau_{f_\tau}=\tau\rrbracket_{\mathsf{B}}=1_{\mathsf{B}}$;
4. $f_{\tau_f}=f$.

Proof. 1. By definition.

2. We need just to check that the preimage of a basic open set is a basic open set. Fix α,i,
$$f_\tau^{-1}[N_{\alpha,i}]=\{G:\left\llbracket\tau(\check{\alpha})=\check{i}\right\rrbracket\in G\}\in\mathsf{Cl}(()St(\mathsf{B})),$$
since by definition
$$\llbracket\tau_f(\alpha)=i\rrbracket\in G\iff G\in\{H:f(H)(\alpha)=i\}\iff f(G)(\alpha)=i.$$
3. By definition for any G
$$\llbracket\tau_{f_\tau}(\alpha)=i\rrbracket\in G\iff f_\tau(G)(\alpha)=i\iff\llbracket\tau(\alpha)=i\rrbracket\in G.$$
Then for any α,i
$$\left\llbracket\tau_{f_\tau}(\check{\alpha})=\check{i}\right\rrbracket=\left\llbracket\tau(\check{\alpha})=\check{i}\right\rrbracket.$$
Therefore $\llbracket\tau_{f_\tau}=\tau\rrbracket_{\mathsf{B}}=1_{\mathsf{B}}$.
4. As in the proof of point 2 we can observe that
$$\llbracket\tau_f(\alpha)=i\rrbracket\in G\iff G\in\{H:f(H)(\alpha)=i\}\iff f(G)(\alpha)=i.$$
Moreover we have that
$$f_{\tau_f}(G)(\alpha)=i\iff\llbracket\tau_f(\alpha)=i\rrbracket\in G,$$
hence we are done.

□

6.2 Cohen's forcing theorem

The goal of this section is to give a positive answer to the questions of section 6.1.1 regarding the definability inside M of the forcing relation. We show that for a countable transitive model M of ZFC and a $\mathsf{B} \in M$ which is in M a cba, the forcing relation defined externally in Section 6.1.1 is induced by the boolean valued semantic defined internally on M^{B} (by relativizing to M all the results of Section 6.1.2 for V^{B}) and allows to control the theory of the models $M[G]$ we introduced in Section 6.1.1. We will also give examples of how this identification can greatly simplify several computations.

Lemma 6.2.1. *Assume $M \in V$ is a transitive set such that $M \models \mathsf{ZFC}$ and $\mathsf{B} \in M$ is such that M models* B is a complete boolean algebra*. Then V models that*

$$\langle M^{\mathsf{B}} = (V^{\mathsf{B}})^M, (\in^{\mathsf{B}})^M, (=^{\mathsf{B}})^M, (\subseteq^{\mathsf{B}})^M \rangle$$

is a full B-valued model.

Notice that this occurs regardless of the fact that B is a complete boolean algebra in V, which is never the case if B is infinite and M is countable (see exercise 6.1.15).

Proof. Since M is a model of ZFC, Theorem 6.1.27 applied in M shows that M models that $\in^{\mathsf{B}}, =^{\mathsf{B}}, \subseteq^{\mathsf{B}}$ are binary relations on the class M^{B} satisfying the clauses of 5.1.1. In particular V models that

$$\langle M^{\mathsf{B}}, (\in^{\mathsf{B}})^M, (=^{\mathsf{B}})^M, (\subseteq^{\mathsf{B}})^M \rangle$$

is a B-valued model. Moreover by the Mixing Lemma 6.1.32 and the Maximum Principle 6.1.33 applied in M (which is a model of ZFC), we get that for all formula $\phi(x_0, \dots, x_n)$ and all $\tau_1, \dots, \tau_n \in M$, there exists a $\tau_0 \in M$ such that

$$M \models [\![\phi(\tau_0, \tau_1, \dots, \tau_n)]\!]_{\mathsf{B}} = \bigvee_{\sigma \in M^{\mathsf{B}}} [\![\phi(\sigma, \tau_1, \dots, \tau_n)]\!]_{\mathsf{B}},$$

i.e.

$$V \models [\![\phi(\tau_0, \tau_1, \dots, \tau_n)]\!]_{\mathsf{B}}^M = \bigvee_{\sigma \in M^{\mathsf{B}}} [\![\phi(\sigma, \tau_1, \dots, \tau_n)]\!]_{\mathsf{B}}^M.$$

In particular V models that

$$\langle M^{\mathsf{B}}, (\in^{\mathsf{B}})^M, (=^{\mathsf{B}})^M, (\subseteq^{\mathsf{B}})^M \rangle$$

is a full B-valued model. □

Notation 6.2.2. Let $M \models \mathsf{ZFC}$ be transitive and $\mathsf{B} \in M$ be such that M models B *is a complete boolean algebra*. Let $G \subseteq \mathsf{B}$ be any ultrafilter in V. Denote by R_G the binary relation $(R^{\mathsf{B}})^M/_G$ on M^{B}/G for R among $=, \in, \subseteq$.

Assuming M, B, G are as in the Notation above, since $\langle M^{\mathsf{B}}, (R^{\mathsf{B}})^M : R \in \{=, \subseteq, \in\} \rangle$ is a full B-valued model in V, and by Łoś Theorem 5.3.7, we get that $(M^{\mathsf{B}}/_G, \in_G, \subseteq_G)$ is a Tarski model for the language $\mathcal{L} = \{\in, \subseteq\}$ such that

$$(M^{\mathsf{B}}/G, \in_G, \subseteq_G) \models \phi$$

if and only if $[\![\phi]\!]_{\mathsf{B}}^M \in G$.

A pair of comments:

- In the definition of M^{B}/G, it is possible that $G \notin M$. However, also in this case the definition makes sense.

- There is no reason why the classes $[x]_G$ should be definable in M if $G \notin M$, thus M^{B}/G in general is not a definable class in M. We shall see that this is exactly what occurs if G is M-generic for an atomless boolean algebra.

- If $G \in M$, then the classes $[x]_G$ are definable in M and in M one can define classes which define an isomorphic copy of the structure $(M^{\mathsf{B}}/G, \in_G, \subseteq_G)$ as the extension of a formula in the parameter B.

- We have a clear meaning of what is the boolean valued model
$$\left\langle M^{\mathsf{B}}, (\in^{\mathsf{B}})^M, (=^{\mathsf{B}})^M, (\subseteq^{\mathsf{B}})^M \right\rangle$$
in V, since all the relevant objects are now sets in V. On the other hand inside M, we can speak of (i.e. formalize) the satisfaction predicate for each single formula of the language (as a class definable in M), but we cannot speak simultaneously inside M of the family of classes of M given by the satisfaction predicates for formulae of the language.

- From now on for the sake of simplicity, we denote by $[\![\phi]\!]$ the boolean value $[\![\phi]\!]^M_{\mathsf{B}}$ as M ranges over countable transitive models of ZFC and $\mathsf{B} \in M$ among the (complete in M) boolean algebras of M.

We now start to plug in some of the material developed in Chapter 4. First of all, by Lemma 4.2.9, if M is a countable transitive model of ZFC and $\mathsf{B} \in M$ is a boolean algebra, there exists an ultrafilter G M-generic for B.

We want to study which is the relationship between $M[G]$ and M^{B}/G. These observations were made in Section 6.1.1:

> Assume M is a countable transitive model of ZFC and $\mathsf{B} \in M$ is an atomless boolean algebra which M models to be complete. Then:
>
> - The family of M-generic filters in $\mathrm{St}(\mathsf{B})$ forms a dense subset of $\mathrm{St}(\mathsf{B})$, i.e. for any $b \in \mathsf{B}^+$, there exists an ultrafilter G M-generic, with $b \in G$ (i.e. $G \in N_b$), and each such $G \notin M$.
> - For any G M-generic of B we can define in V by recursion on M^{B}:
> $$\tau_G = \{\sigma_G : \tau(\sigma) \in G\}.$$
> for any given $\tau \in M^{\mathsf{B}}$.
> We also define:
> $$M[G] = \{\tau_G : \tau \in M^{\mathsf{B}}\}.$$
> $M[G]$ is transitive, $M \subseteq M[G]$, $G \in M[G]$.

We will now proceed to identify the models $(M[G], \in)$ and $(M^{\mathsf{B}}/_G, \in_G)$ under the assumption that G is M-generic for some complete boolean algebra $\mathsf{B} \in M$. This is the content of Theorem 6.2.4 below. In case $G \in \mathrm{St}(\mathsf{B})$ is not M-generic, $M[G]$ is still a transitive set and $(M^{\mathsf{B}}/_G, \in_G)$ is a well defined Tarski model, but it can be shown that the two structures cannot be isomorphic.

Lemma 6.2.3. *Let $M \vDash$ ZFC, $M \in V$ and transitive. Let B be a cba in M. Assume $G \in V$ is an M-generic filter for B. The following holds:*

(a) Assume $b \in G$, $X \in M$, and $X \subseteq \mathsf{B}$ is predense below b. Then $G \cap X \neq \emptyset$.

(b) Assume $A \in M$ and $A \subseteq G$. Then $\bigwedge A \in G$.

Proof. We proceed as follows:

(a): Let

$$E = \{c \in B : M \models c \leq \neg b \vee (c \leq b \wedge c \in\downarrow X)\}.$$

We leave to the reader to check that $E \in M$, M models that E is dense and that $c \in G \cap E$ if $c \leq b$ and $c \in\downarrow X$.

(b) Suppose $A \in M$, $A \subseteq G$ and $M \vDash \bigwedge A = c$. Since M satisfies the Axiom of choice, we can write $A = \{a_\xi : \xi < \gamma\} \in M$ for some $\gamma \in M$. Let $b_\xi = \bigwedge_{\alpha<\xi} a_\alpha$. The sequence $\langle b_\xi : \xi \leq \gamma\rangle \in M$ is decreasing, and $c = b_\gamma$. Let $\xi \leq \gamma$ be the least ordinal such that $b_\xi \notin G$. Then $\neg b_\xi \in G$. Set $c_\alpha = \neg b_\xi \wedge b_\alpha$. Then $c_\alpha \in G$ for all $\alpha < \xi$, $\{c_\alpha : \alpha < \xi\} \in M$ and $\bigwedge_{\alpha<\xi} c_\alpha = 0_G$. Thus $\bigvee_{\alpha<\xi} \neg c_\alpha = 1_{\mathsf{B}}$ and $\{\neg c_\alpha : \alpha < \xi\} \in M$ as well. Since G is M-generic, $\neg c_\alpha \in G$ for some $\alpha < \xi$. Then $c_\alpha \wedge \neg c_\alpha = 0_{\mathsf{B}} \in G$ a contradiction which proves the lemma.

□

Theorem 6.2.4. *Let M be a transitive model of ZFC, B be a complete boolean algebra in M, and G be an M-generic filter for B. Then*

$$\begin{aligned} \pi^M_G :& M^{\mathsf{B}}/_G \to M[G] \\ & [\tau]_G \mapsto \tau_G \end{aligned}$$

defines an isomorphism between the structures $(M^{\mathsf{B}}/_G, \in_G)$ and $(M[G], \in)$. In particular π^M_G is the Mostowski collapse of the well-founded extensional relation $\in_G$ on $M^{\mathsf{B}}/_G$.

Proof. It suffices to prove:

1. $[\![\tau \in \sigma]\!] \in G \Leftrightarrow \tau_G \in \sigma_G$.
2. $[\![\tau = \sigma]\!] \in G \Leftrightarrow \tau_G = \sigma_G$.

We prove both items by induction on $(\mathrm{rk}^{\mathsf{B}}(\tau), \mathrm{rk}^{\mathsf{B}}(\sigma))$ with respect to the square order on Ord^2.

1. ($\Rightarrow$) Suppose $[\![\tau \in \sigma]\!] \in G$. By definition

$$[\![\tau \in \sigma]\!] = \bigvee_{u \in \mathrm{dom}(\sigma)} \sigma(u) \wedge [\![u = \tau]\!].$$

Let $b_u = \sigma(u) \wedge [\![\tau = u]\!]$. Notice that $\{b_u : u \in \mathrm{dom}(\sigma)\} \in M$ is pre-dense under $[\![\tau \in \sigma]\!] \in G$. So[5] we can appeal to (a) of lemma 6.2.3 to find

[5]Everywhere in this proof we appeal to Lemma 6.2.3, we are crucially using the assumption that G is M-generic.

$u \in \mathrm{dom}(\sigma)$ such that $b_u = [\![\tau = u]\!] \wedge \sigma(u) \in G$. Thus $[\![\tau = u]\!] \in G$, so we can apply the inductive assumption 2 on the pair (τ, u) which has lower rank than (τ, σ) in the square order on Ord^2. We conclude that $\tau_G = u_G$. Now observe that $u_G \in \sigma_G = \{v_G : \sigma(v) \in G\}$ since $u \in \mathrm{dom}(\sigma)$ and $\sigma(u) \in G$.

($\Leftarrow$) Suppose $\tau_G \in \sigma_G$. Then there is $u \in \mathrm{dom}(\sigma)$ such that $\sigma(u) \in G$ and $\tau_G = u_G$. Therefore, applying again 2 on the pair (τ, u) which has lower rank than the pair (τ, σ), we get that $[\![\tau = u]\!] \in G$. So

$$[\![\tau = u]\!] \wedge \sigma(u) \in G$$

as well. Now we can observe that

$$[\![\tau = u]\!] \wedge \sigma(u) \leq \bigvee_{v \in \mathrm{dom}(\sigma)} [\![\tau = v]\!] \wedge \sigma(v) = [\![\tau \in \sigma]\!].$$

Therefore $[\![\tau \in \sigma]\!] \in G$.

2. ($\Rightarrow$) Suppose $[\![\tau = \sigma]\!] \in G$. Observe that $[\![\tau \subseteq \sigma]\!] \in G$ gives that

$$\neg\tau(u) \vee [\![u \in \sigma]\!] \geq [\![\tau \subseteq \sigma]\!]$$

is also in G for all $u \in \mathrm{dom}(\tau)$. This gives that $[\![u \in \sigma]\!] \in G$ for all $u \in \mathrm{dom}(\tau)$ such that $\tau(u) \in G$. Since the pairs (u, σ) are of lower rank than the pair (τ, σ) for all such u we can apply the first item to these pairs to get that $u_G \in \sigma_G$ for all $u \in \mathrm{dom}(\tau)$ such that $\tau(u) \in G$. This gives that $\tau_G \subseteq \sigma_G$. The other inclusion is proved in exactly the same manner.

($\Leftarrow$) Suppose $[\![\tau \neq \sigma]\!] \in G$. W.l.o.g. we can suppose that $[\![\tau \not\subseteq \sigma]\!] \in G$. But

$$[\![\tau \not\subseteq \sigma]\!] = \bigvee_{u \in \mathrm{dom}(\tau)} \tau(u) \wedge \neg[\![u \in \sigma]\!].$$

Since G is M-generic we can appeal to (a) of lemma 6.2.3 to find $u \in \mathrm{dom}(\tau)$ such that

$$\tau(u) \wedge [\![u \notin \sigma]\!] \in G.$$

Applying the first item on the pair (u, σ) which has lower rank then (τ, σ) we get that $u_G \notin \sigma_G$, while $u_G \in \tau_G$ since $\tau(u) \in G$. Hence $\tau_G \not\subseteq \sigma_G$, which also gives that $\tau_G \neq \sigma_G$.

The proof is complete. □

Summing up, we can prove:

Theorem 6.2.5 (***Cohen's forcing theorem***)**.** *Let M be a countable transitive model of ZFC and $\mathsf{B} \in M$ be a boolean algebra which M models to be complete. Then for all formulae $\phi(x_1, \ldots, x_n)$ in the free variables $x_1, \ldots, x_n$ and all $\tau_1, \ldots, \tau_n \in M^{\mathsf{B}}$:*

1. *$[\![\varphi(\tau_1, \ldots, \tau_n)]\!] \geq b$ if and only if for all G M-generic filter for* B *with $b \in G$ we have*
$$M[G] \vDash \varphi((\tau_1)_G, \ldots, (\tau_n)_G).$$
2. *$M[G] \vDash \varphi((\tau_1)_G, \ldots, (\tau_n)_G)$ for some G M-generic filter for* B *if and only if $[\![\varphi(\tau_1, \ldots, \tau_n)]\!] \in G$.*

Proof. We sketch just some parts of the proof leaving the others as an instructive exercise for the reader.

1. ($\Rightarrow$) Suppose $b \leq [\![\phi]\!]$. Let G be M-generic for B with $b \in G$. Then $[\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$ as well, thus
$$M^{\mathsf{B}}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$$
by Theorem 5.3.7. Since the map $[\tau]_G \mapsto \tau_G$ is an isomorphism, we also get that
$$M[G] \models \phi((\tau_1)_G, \ldots, (\tau_n)_G).$$
($\Leftarrow$) is left to the reader.
2. It is an immediate consequence of the isomorphism of the structures M^{B}/G and $M[G]$ and of Theorem 5.3.7 applied in V to the full B-valued model M^{B}.

The proof is complete. $\square$

Definition 6.2.6 (Cohen's forcing relation)**.** Let M be a countable transitive model of ZFC and $\mathsf{B} \in M$ be a complete boolean algebra. For each formula $\phi(x_0, \ldots, x_n)$, $\tau_0, \ldots, \tau_n \in M^{\mathsf{B}}$ and $b \in \mathsf{B}^+$ the forcing relation is defined in V by
$$b \Vdash \phi(\tau_0, \ldots, \tau_n) \text{ (} b \text{ forces } \phi(\tau_0, \ldots, \tau_n))$$
if and only if
$$M[K] \models \phi((\tau_0)_K, \ldots, (\tau_n)_K) \text{ for all } M\text{-generic filters } K \text{ for } \mathsf{B} \text{ such that } b \in K.$$

Lemma 6.2.7. *Let M be a countable transitive model of* ZFC *and $\mathsf{B} \in M$ be a complete boolean algebra. For each formula $\phi(x_0, \ldots, x_n)$, $\tau_0, \ldots, \tau_n \in M^{\mathsf{B}}$ and $b \in \mathsf{B}^+$ the following are equivalent:*

1. *$b \Vdash \phi(\tau_0, \ldots, \tau_n)$.*
2. *The set of K M-generic for* B *such that $b \in K$ and*
$$M[K] \models \phi((\tau_0)_K, \ldots, (\tau_n)_K)$$
is dense in N_b.
3. *$M \models b \leq_{\mathsf{B}} [\![\phi(\tau_0, \ldots, \tau_n)]\!]$.*

Proof. A useful exercise for the reader. $\square$

We are almost ready to prove that every axiom of ZFC is valid in M^{B} and that CH is independent from the ZFC-axioms (more precisely from the theory ZFC+*there exists a countable transitive model of ZFC*. To do this we will often appeal to Cohen's forcing theorem. So let us explain how we are going to use it.

6.2.1 How to use Cohen's forcing theorem

Recurring examples of how we will use of the Cohen forcing theorem is the following:

Example 6.2.8. Consider a forcing statement of the following form:

$$\left[\!\left[\dot{f}:\check{\gamma}\to\check{\beta} \text{ is a function}\right]\!\right] > 0_{\mathsf{B}}.$$

We define for each $\alpha < \gamma$ a set in M

$$A_\alpha = \left\{b_\eta \in \mathsf{B} : M \models b_\eta = \left[\!\left[\dot{f}:\check{\gamma}\to\check{\beta} \text{ is a function}\right]\!\right] \wedge \left[\!\left[\dot{f}(\check{\alpha}) = \check{\eta}\right]\!\right]\right\} \in M.$$

We want to argue that M models that A_α is an antichain, proving the stronger assertion stating that $\eta \neq \nu$ entails that $b_\eta \wedge b_\nu = 0_{\mathsf{B}}$.

Towards this aim we proceed as follows: we assume by contradiction that we can find $\eta \neq \nu < \beta$ such that $c = b_\eta \wedge b_\nu > 0_{\mathsf{B}}$. We pick G M-generic with $c \in G$ and we get that $\left[\!\left[\dot{f}:\check{\gamma}\to\check{\beta} \text{ is a function}\right]\!\right]$, $\left[\!\left[\dot{f}(\check{\alpha}) = \check{\eta}\right]\!\right]$, $\left[\!\left[\dot{f}(\check{\alpha}) = \check{\nu}\right]\!\right]$ are all in G. Then by the forcing theorem

$$M[G] \models \dot{f}_G : \gamma \to \beta \text{ is a function},$$
$$M[G] \models \dot{f}_G(\alpha) = \eta,$$
$$M[G] \models \dot{f}_G(\alpha) = \nu.$$

This gives that

$$M[G] \models \nu = \eta,$$

and thus that $\nu = \eta$, which contradicts our assumpion that $\nu \neq \eta$. A direct argument carried entirely in M (without ever appealing to the forcing theorem) yielding that A_α is an antichain can also be found, but is much more convoluted.

Lemma 6.2.9. *Assume $M \models$ ZFC and is countable, and $\mathsf{B} \in M$ is a cba in M. Then for all $\tau, \sigma \in M^{\mathsf{B}}$, there exists $\eta : \mathrm{dom}(\sigma) \to \mathsf{B}$ in M^{B} such that*

$$[\![\tau \subseteq \sigma]\!] \leq [\![\tau = \eta]\!].$$

Proof. Given $\tau, \sigma \in M^{\mathsf{B}}$, set

$$\eta = \{\langle u, \sigma(u) \wedge [\![u \in \sigma \leftrightarrow u \in \tau]\!]\rangle : u \in \mathrm{dom}(\sigma)\} \in M^{\mathsf{B}}.$$

First of all we prove that

$$[\![\eta \subseteq \sigma]\!] = 1_{\mathsf{B}} :$$

Fix G M-generic for B, pick $a \in \eta_G$, then $a = u_G$ for some $u \in \mathrm{dom}(\eta) = \mathrm{dom}(\sigma)$ with $[\![u \in \sigma \leftrightarrow u \in \tau]\!] \wedge \sigma(u) \in G$. Hence $\sigma(u) \in G$ as well, giving that $u_G \in \sigma_G$. Since this holds for all $a \in \eta_G$, we conclude that $M[G] \models \eta_G \subseteq \sigma_G$ for all G M-generic for B. We conclude by the forcing theorem.

Now assume G is M-generic for B with $[\![\tau \subseteq \sigma]\!] \in G$. We show that $\eta_G = \tau_G$. By the forcing theorem this suffices to prove the Lemma.

$\eta_G \subseteq \tau_G$ Assume $a \in \eta_G$. Then $a = u_G$ for some $u \in \mathrm{dom}(\eta) = \mathrm{dom}(\sigma)$ and $\eta(u) = [\![u \in \sigma \leftrightarrow u \in \tau]\!] \wedge \sigma(u) \in G$. Since $\sigma(u) \in G$ and

$$[\![u \in \sigma]\!] = \bigvee \{v \in \mathrm{dom}(\sigma) : [\![u = v]\!] \wedge \sigma(v)\} \geq [\![u = u]\!] \wedge \sigma(u) = \sigma(u) \in G,$$

we get that $[\![u \in \sigma]\!] \in G$. Since $[\![u \in \sigma \leftrightarrow u \in \tau]\!] \in G$ as well, we conclude that $[\![u \in \tau]\!] \in G$ as well, giving that $u_G \in \tau_G$ by the forcing theorem.

$\eta_G \supseteq \tau_G$ assume $a \in \tau_G$. Then $a = v_G$ for some $v \in \mathrm{dom}(\tau)$ with $\tau(v) \in G$. Since $[\![\tau \subseteq \sigma]\!] \in G$, we get that $M[G] \models \tau_G \subseteq \sigma_G$, hence we also get that $a = u_G$ for some $u \in \mathrm{dom}(\sigma)$ with $\sigma(u) \in G$. We get that $\sigma(u) \wedge \tau(v) \wedge [\![u = v]\!] \wedge [\![\tau \subseteq \sigma]\!] \in G$. Now:

- $\tau(v) \leq \bigvee_{t \in \mathrm{dom}(\tau)} [\![v = t]\!] \wedge \tau(t) = [\![v \in \tau]\!]$. Hence $[\![v \in \tau]\!] \in G$.
- $[\![u = v]\!] \in G$. Therefore $[\![u = v]\!] \wedge [\![v \in \tau]\!] \in G$ as well, and

  $$[\![u \in \tau]\!] \geq [\![u = v]\!] \wedge [\![v \in \tau]\!] \in G.$$

  Hence $[\![u \in \tau]\!] \in G$.
- $\sigma(u) \leq \bigvee_{t \in \mathrm{dom}(\sigma)} [\![u = t]\!] \wedge \sigma(t) = [\![u \in \sigma]\!]$. Hence $[\![u \in \sigma]\!] \in G$.

We conclude that

$$[\![u \in \tau]\!] \wedge [\![u \in \sigma]\!] \in G.$$

But

$$[\![u \in \tau]\!] \wedge [\![u \in \sigma]\!] \leq [\![u \in \tau \leftrightarrow u \in \sigma]\!].$$

Hence

$$[\![u \in \tau \leftrightarrow u \in \sigma]\!] \in G$$

as well. Then $\sigma(u) \wedge [\![u \in \tau \leftrightarrow u \in \sigma]\!] \in G$, yielding that

$$a = u_G \in \eta_G = \{t_G : t \in \mathrm{dom}(\sigma) \text{ and } \sigma(t) \wedge [\![t \in \tau \leftrightarrow t \in \sigma]\!] \in G\},$$

as was to be shown.

□

6.3 Independence of CH

In this section we prove the independence of the Continuum Hypothesis from the axioms of ZFC using the forcing method over a countable tansitive model M of ZFC. We assume throughout this section that $M[G]$ models ZFC whenever G is M-generic for some $\mathsf{B} \in M$ which M models to be a complete boolean algebra. This will be proved in full details in the next section (cfr.: 6.4). In order to appreciate the full power of the forcing theorem, we believe it is more instructive to understand how this theorem allows us to compute the truth value of specific statements in forcing extensions of M (i.e. models of the form $M[G]$ with G M-generic for a cba $\mathsf{B} \in M$). This is what we do in this section using the forcing theorem to compute the truth values of CH in two distinct forcing extensions of M.

We first show that if G is M-generic for $\mathsf{RO}(2^{\omega_2\times\omega})^M$ then CH fails in $M[G]$. Next we show that there is $\mathsf{B}\in M$ such that M^{B} models CH. Combined with the results of Section 6.4, these two proofs will give the indipendence of CH with respect to the theory ZFC over the theory ZFC+*there is a countable transitive model of ZFC*. It is possible to convert these proofs in a proof of the independence of CH from ZFC rightaway from ZFC using arguments rooted in the reflection properties of V (see [7, Sections IV.7, VII.1]).

6.3.1 A model of ¬CH

Lemma 6.3.1. *Assume M is a countable transitive model of ZFC and M models that B is a CCC complete boolean algebra. Then*

$$M[G]\models \mathrm{cf}(\kappa)=\alpha \Leftrightarrow M\models \mathrm{cf}(\kappa)=\alpha$$

for all M generic filters G for B and $\alpha\leq\kappa\in\mathrm{Ord}^M$.

We limit ourselves to prove the following weak form of the Lemma which is sufficient for our aims:

Lemma 6.3.2. *Assume M is a countable transitive model of ZFC and M models that B is a CCC complete boolean algebra. Then for all M generic filters G for B and $i=0,1,2$,*

$$M[G]\models (\omega_i)^M \textit{is the i-th infinite regular cardinal},$$

where $(\omega_i)^M$ is in M the (countable in V) ordinal which M models to be the i-th infinite cardinal.

Proof. The case $i=0$ follows from the absoluteness of the statement ω *is the least infinite ordinal* for transitive models of ZFC.

For the cases $i=1,2$ is enough to show the following:

Fact 6.3.3. *Assume G is M-generic for B. Then $M[G]$ models that every function $\sigma_G\in M[G]$ from $\lambda=(\omega_j)^M$ to $\kappa=(\omega_{j+1})^M$ is bounded (i.e has range contained in some $\beta<(\omega_{j+1})^M$) for each $j=0,1$.*

For if this fact holds $(\omega_1)^M$ is the first uncountable ordinal of $M[G]$ and $(\omega_2)^M$ is the first cardinal larger than $(\omega_1)^M$ of $M[G]$, whenever G is M-generic for B.

Proof. We prove in detail the case $j=0$ and leave to the reader to prove the case $j=1$ or more generally the strong version of the Lemma.

Let $\sigma\in M^{\mathsf{B}}$, and $b\in G$ be such that M models

$$[\![\sigma \text{ is a function from } \check{\omega} \text{ to } \check{\kappa}]\!]_{\mathsf{B}}=b.$$

For every $n<\omega$ consider the set:

$$A_n=\{\beta<\kappa : M\models [\![\sigma(\check{n})=\check{\beta}]\!]>0_{\mathsf{B}}\}.$$

Notice that the sequence $\{A_n : n < \omega\} \in M$, since it is the extension in M of a formula with parameters in M: Let

$$X = \left\{ \langle n, \beta, c\rangle : M \models \llbracket \sigma(\check{n}) = \check{\beta} \rrbracket = c > 0_{\mathsf{B}} \right\}.$$

Then $X \in M$, since it is a subset of $\omega \times \kappa \times \mathsf{B}$ defined by an application of the comprehension axiom in M. Therefore for each $n < \omega$, $A_n \in M$ as well, since

$$A_n = \{\beta : \exists c \, \langle n, \beta, c\rangle \in X\},$$

and $(A_n : n < \omega) \in M$ since

$$(A_n : n < \omega) = \{\langle n, u\rangle : M \models \beta \in u \leftrightarrow \exists b \, \langle n, \beta, b\rangle \in X\}.$$

Thus $\{A_n : n < \omega\} = \mathrm{ran}(\langle A_n : n < \omega\rangle) \in M$ as well.

Claim 6.3.3.1. *M models that every A_n is at most countable.*

Moreover $\sigma_H(n) \in A_n$ for all $n < \omega$, and any H M-generic for B with

$$b = \llbracket \sigma : \check{\omega} \to \check{\kappa} \textit{ is a function} \rrbracket \in H.$$

Proof. Let

$$W_n = \{b^n_\beta = \llbracket \sigma(\check{n}) = \check{\beta} \rrbracket \wedge b : \beta \in A_n\}.$$

Then:

Subclaim 6.3.3.1. *The sequence $\{W_n : n < \omega\} \in M$, and M models that each W_n is a countable antichain in M.*

Proof. The first part of the subclaim is a useful exercise for the reader. For the second part it is enough to show that M models that each W_n is an antichain. Since M models B is CCC, we conclude that M models that each W_n is a countable antichain.

Assume W_n is not an antichain and find $\gamma \neq \beta \in A_n$ with $c = b^n_\gamma \wedge b^n_\beta > 0_{\mathsf{B}}$. Let H be M generic for B with $c \in H$. Since

$$c \leq b = \llbracket \sigma : \check{\omega} \to \check{\kappa} \text{ is a function} \rrbracket,$$

$M[H]$ models that $\sigma_H : \omega \to \kappa$ is a function by the forcing theorem. On the other hand since

$$c \leq b^n_\beta \leq \llbracket \sigma(\check{n}) = \check{\beta} \rrbracket, b^n_\gamma \leq \llbracket \sigma(\check{n}) = \check{\gamma} \rrbracket$$

again by the forcing theorem, we get that $\sigma_H(n) = \beta$ and $\sigma_H(n) = \gamma$. Since $M[H]$ models that σ_H is a function we get that $\beta = \gamma$, a contradiction. □

To complete the proof of the Claim observe the following:

- For each $n < \omega$ the map $\phi_n : A_n \to W_n$ given by $\beta \mapsto b^n_\beta$ is in M, since

  $$\phi_n = \{\langle \beta, b\rangle : \langle n, \beta, b\rangle \in X\},$$

  and is injective: assume $c = b^n_\gamma = b^n_\beta$, then pick H M-generic for B with $c \in M$ to get that in $M[H]$, $\sigma_H : \omega \to \kappa$ is a function and $\gamma = \sigma_H(n) = \beta$. In particular $M \models \gamma = \beta$ as well.

  Hence M models that A_n is countable, since it is mapped injectively in a countable set by a map in M.

- For each $n < \omega$ and H M-generic for B with
$$b = [\![\sigma : \check{\omega} \to \check{\kappa} \text{ is a function}]\!] \in H$$
we have that $\sigma_H(n) \in A_n$, since $\sigma_H(n) = \eta$ iff $[\![\sigma(\check{n}) = \check{\eta}]\!] \wedge b \in H \cap W_n$, giving that $\eta \in A_n$.

The proof of the Claim is completed. □

To conclude the proof of the Fact, observe that M models that $\{A_n : n < \omega\} \in M$ is for M a countable family of countable subsets of $(\omega_1)^M$ which is the least uncountable ordinal for M. Hence M models that the union of the A_n is a countable subset of $(\omega_1)^M$.

We conclude that for some $\beta < (\omega_1)^M$,
$$M \models \bigcup \{A_n : n < \omega\} \subseteq \beta < (\omega_1)^M.$$

The Fact for the case $j = 0$ is proved since we get that in $M[G]$
$$\sigma_G[\omega] \subseteq A \subseteq \beta < (\omega_1)^M.$$

The proof of the Fact for the case $j = 1$ is obtained repeating verbatim the proof of the Fact for $j = 0$, setting $\kappa = (\omega_2)^M$, and replacing all over ω with $\lambda = (\omega_1)^M$, $(A_n : n < \omega)$ with $(A_\alpha : \alpha < \lambda)$ and $(W_n : n < \omega)$ with $(W_\alpha : \alpha < \lambda)$, to argue that each W_α is a countable antichain for M, and hence also that each $A_\alpha \in M$ is a countable subset of κ for M. We can therefore conclude that M models that $A = \bigcup \{A_\alpha : \alpha < \lambda\}$ is a subset of $(\omega_2)^M$ of size at most $(\omega_1)^M$, since it is the union indexed by $(\omega_1)^M$ of sets which are countable for M. This gives that M models that $A \subseteq \beta$ for some $\beta < (\omega_2)^M$ and that $M[H]$ models that $\sigma_H(\lambda) \subseteq \beta < \kappa$ cannot be a bijection of $(\omega_1)^M$ onto $(\omega_2)^M$, concluding the proof of the Fact for $j = 1$ as well. □

The proof of the Lemma is completed. □

For the sake of completeness we add a proof of Lemma 6.3.1.

Proof. The statement $\mathrm{cf}(\kappa) \leq \alpha$ is a Σ_1-property in the parameters $\alpha \leq \kappa$:
$$\exists f \ (f \subseteq \alpha \times \kappa \text{ is a function }) \wedge (\sup(f[\alpha]) = \kappa).$$

If M models $\mathrm{cf}(\kappa) \leq \alpha$, then $M[G]$ models that $\mathrm{cf}(\kappa) \leq \alpha$, since the witness in M of this property is in $M[G]$ as well. So it is enough to show that if $M[G]$ models that $\mathrm{cf}(\kappa) \leq \alpha$, then also M models $\mathrm{cf}(\kappa) \leq \alpha$. If this is the case we can easily conclude that
$$\mathrm{cf}(\kappa)^M = \min\{\alpha : M \models \mathrm{cf}(\kappa) \leq \alpha\} = \min\{\alpha : M[G] \models \mathrm{cf}(\kappa) \leq \alpha\} = \mathrm{cf}(\kappa)^{M[G]}.$$

Towards a contradiction assume that κ is the least such that $\mathrm{cf}(\kappa)^M \neq \mathrm{cf}(\kappa)^{M[G]}$.

First of all we claim that M models that $\mathrm{cf}(\kappa)$ is an uncountable regular cardinal. Else if M models that $\mathrm{cf}(\kappa) = \omega$, we get that $M[G]$ models that $\mathrm{cf}(\kappa)^{M[G]} \leq \omega$. But ω is the least possible value for the cofinality of a limit ordinal, hence
$$M[G] \models \mathrm{cf}(\kappa)^{M[G]} = \omega$$
as well, contradicting our assumption that κ is the least on which $\mathrm{cf}(\kappa)^M \neq \mathrm{cf}(\kappa)^{M[G]}$.

Now in $M[G]$ there is $\lambda < \mathrm{cf}(\kappa)^M = \eta$ and $\sigma_G : \lambda \to \kappa$ with $\sigma \in M^{\mathsf{B}}$ such that $\sigma_G[\lambda]$ is unbounded in κ.

It is enough to show the following:

Fact 6.3.4. *$M[G]$ models that every function $\sigma_G \in M[G]$ from some $\lambda < \mathrm{cf}(\kappa)^M$ into κ is bounded (i.e has range contained in some $\beta < \kappa$).*

Proof. Let $\sigma \in M^{\mathsf{B}}$, and $b \in G$ be such that M models

$$\llbracket \sigma \text{ is a function from } \check{\lambda} \text{ to } \check{\kappa} \rrbracket_{\mathsf{B}} = b.$$

For every $\alpha < \lambda$ consider the set:

$$A_\alpha = \{\beta < \kappa : M \models \llbracket \sigma(\check{\alpha}) = \check{\beta} \rrbracket > 0_{\mathsf{B}}\}.$$

As in the proof of Lemma 6.3.2 we can argue that the following holds:

- $\{A_\alpha : \alpha < \lambda\} \in M$.
- M models that every A_α is an at most countable subset of κ.
- For any H M-generic for B, and $\alpha < \lambda$, $\sigma_H(\alpha) \in A_\alpha$.

Since $\lambda < \mathrm{cf}(\kappa)^M$ and $\mathrm{cf}(\kappa)^M$ is a regular uncountable cardinal in M we get that

$$M \models |\bigcup\{A_\alpha : \alpha < \lambda\}| < \mathrm{cf}(\kappa)^M$$

In particular (since M is a model of ZFC and all relevant objects are in M) we can carry the following reasoning inside M: a subset A of κ of size smaller than the cofinality of κ cannot be unbounded in κ, else a bijection of A with its size gives that $\mathrm{cf}(\kappa)^M \leq |A| < \mathrm{cf}(\kappa)^M$ a contradiction.

We conclude that for some $\beta < \kappa$,

$$M \models \bigcup\{A_\alpha : \alpha < \kappa\} \subseteq \beta < \kappa.$$

We get that in $M[G]$

$$\sigma_G[\lambda] \subseteq A \subseteq \beta < \kappa.$$

The Fact is proved. □

The proof of the Lemma is completed. □

Now let us choose in M the poset $\mathsf{RO}(2^{\omega_2 \times \omega})^M$ (which M models to be CCC by Proposition 4.4.8 applied in M). We can use the facts proved so far to check the following:

Theorem 6.3.5. *Assume M is a countable transitive model of ZFC and let*

$$\mathsf{B} = \mathsf{RO}(2^{\omega_2 \times \omega})^M.$$

Then $M[G] \models \neg\mathsf{CH}$ for all M generic filters G for B.

Proof. Set:

$$\tau = \{\langle \mathrm{op}(\mathrm{op}(\check{\alpha}, \check{n}), \check{i}), N_{\langle \alpha,n,i\rangle}\rangle : \alpha < \omega_2^M, n \in \omega, i < 2\} \in M^{\mathsf{B}},$$

where

$$N_{\langle \alpha,n,i\rangle} = \left\{f \in 2^{\omega_2 \times \omega} \cap M : f(\alpha, n) = i\right\}.$$

Let G be M-generic for B. Then

$$g = \tau_G = \{\langle\langle \alpha, n\rangle, i\rangle : N_{\langle \alpha,n,i\rangle} \in G\}.$$

For all $\alpha < \omega_2^M$ and $n < \omega$, define $g_\alpha(n) = g(\alpha, n)$.

Claim 6.3.5.1. *$M[G]$ models that $g : \omega_2^M \times \omega \to 2$ is a total function. Moreover $g_\alpha \neq g_\beta$ are distinct element of $2^\omega \cap M[G]$ for all $\alpha < \beta < \omega_2^M$.*

Assume the Claim is proved. Since M models that $\mathsf{RO}(2^{\omega_2 \times \omega})^M$ has the CCC (applying Corollary 4.4.10 inside M), by Lemma 6.3.2 we get that $M[G]$ models that ω_2^M is the second uncountable cardinal. By the Claim

$$M[G] \models |2^\omega| \geq \omega_2^M,$$

thus CH fails in $M[G]$.

We are left with proof of the Claim.

Proof. Let for any $s \in Fn(\omega_2 \times \omega, 2)^M$

$$N_s = \left\{ f \in 2^{\omega_2 \times \omega} \cap M : s \subseteq f \right\}.$$

We can apply exercise 4.3.7 in M to get that the sets

- $D_{n,\alpha} = \{N_s : s \in Fn(\omega_2 \times \omega, 2)^M,\ (\alpha, n) \in \mathrm{dom}(s)\}$
- $E_{\alpha,\beta} = \{N_s : s \in Fn(\omega_2 \times \omega, 2),\ \exists n\ s(\alpha, n) \neq s(\beta, n)\}$

are dense in $\mathsf{RO}(2^{\omega_2 \times \omega})^M$ for all $\alpha \neq \beta < \omega_2^M$ and $n < \omega$.

Our definitions now give that:

- $M[G] \models (\alpha, n) \in \mathrm{dom}(g)$ for all $\alpha < \omega_2^M$ and $n < \omega$ since $D_{n,\alpha} \cap G \neq \emptyset$ for all such n, α.
- $M[G] \models (\alpha, n, i), (\alpha, n, j) \in g$ iff $i = j$: on the one hand $(\alpha, n, i) \in g$ iff $N_{\langle \alpha,n,i \rangle} \in G$, on the other hand $N_{\langle \alpha,n,i \rangle}$ and $N_{\langle \alpha,n,j \rangle}$ are compatible conditions in $\mathsf{RO}(2^{\omega_2 \times \omega})^M$ iff $i = j$ for all $\alpha < \omega_2^M, n < \omega$.
- $M[G] \models g(\alpha, n) \neq g(\beta, n)$ for some n, since $E_{\alpha,\beta} \cap G \neq \emptyset$ for all $\alpha < \beta < \omega_2^M$.

The Claim follows immediately from the above observations. □

The Theorem is proved. □

6.3.2 A model of CH

In this section we prove that $\mathsf{CH} + \mathsf{ZFC}$ is coherent relative to the theory $\mathsf{ZFC}+$*there is a countable transitive model of ZFC*.

Definition 6.3.6. A boolean algebra B is $< \lambda$ distributive if for all collections $\{D_\alpha : \alpha < \gamma\}$ of γ-many dense open sets in B^+ with $\gamma < \lambda$, we have that

$$D = \bigcap_{\alpha<\gamma} D_\alpha \text{ is an open dense subset of } \mathsf{B}^+.$$

Definition 6.3.7. Let λ be an infinite cardinal. A pre-order $(P, <)$ is $< \lambda$-*closed* if for every $\gamma < \lambda$, every decreasing sequence $(p_\alpha)_{\alpha<\gamma}$ contained in P has a lower bound in P.

$< \omega_1$-closed posets are said to be countably closed and $< \omega_1$-distributive boolean algebra are said to be countably distributive.

Lemma 6.3.8. *Assume $(P, \leq_P)$ is a separative $< \lambda$-closed poset. Then $\mathsf{RO}(P)$ is $< \lambda$-distributive.*

The assumption that P is separative is redundant, but the proof without this assumption is slightly more intricate, and we wiil use the Lemma just for separative posets P, thus we prove the lemma using this assumption.

Proof. Let $i : P \to \mathsf{RO}(P) = \mathsf{B}$ be the dense embedding of P into its boolean completion provided by Theorem 3.2.5. Since P is separative, i is injective and $i(p) \leq_{\mathsf{B}} i(q)$ if and only if $p \leq_P q$, by Corollary 3.2.8.

Assume

$$\{D_\alpha : \alpha < \gamma\}$$

is a family of dense open subsets of $\mathsf{RO}(P)$ for some $\gamma < \lambda$. It is immediate to check that

$$D = \bigcap\{D_\alpha : \alpha < \gamma\}$$

is open. We need to show that D is dense i.e. given $b \in \mathsf{B}^+$, we need to find $q \leq_{\mathsf{B}} b$ in D.

Build $\{p_\alpha : \alpha \leq \gamma\} \subseteq P$ by recursion as follows:

- Choose p_0 such that $i(p_0) \leq b$ and $p_0 \in D_0$ (which is possible since $i[P]$ is a dense subset of B^+),
- Given $p_\alpha \in P$, let $s \in D_{\alpha+1}$ be such that $s \leq_{\mathsf{B}} i(p_\alpha)$ and find $p_{\alpha+1} \in P$ such that $i(p_{\alpha+1}) \leq_{\mathsf{B}} s$ (which is possible since $i[P]$ is a dense subset of B^+). Then $p_{\alpha+1} \leq_P p_\alpha$ (since $i(p_{\alpha+1}) \leq_{\mathsf{B}} i(p_\alpha)$, and P is separative), and $i(p_{\alpha+1}) \in D_{\alpha+1}$.
- Given $\langle p_\beta : \beta < \alpha\rangle \subseteq P$ with $\beta < \gamma$ limit, first of all we notice that, by our construction, $\langle p_\beta : \beta < \alpha\rangle$ is a descending sequence in P. Since P is $< \lambda$-closed, we have that $\langle p_\beta : \beta < \alpha\rangle$ has a lower bound $r \in P$ refining each p_β. Now refine $i(r)$ to some $s \in D_\alpha$ and find $p_\alpha \in P$ such that $i(p_\alpha) \leq_{\mathsf{B}} s$ (which is possible since $i[P]$ is a dense subset of B^+). Then $i(p_\alpha) \in D_\alpha$ and p_α is a lower bound for the chain $\{p_\xi : \xi < \alpha\}$, since $i(p_\alpha) \leq_P s \leq_P i(p_\beta)$ for all $\beta < \alpha$, and P is separative.
- Let u be a lower bound for the descending sequence $\langle p_\beta : \beta < \gamma\rangle \subseteq P$.

Then

$$q = i(u) \in D = \bigcap\{D_\alpha : \alpha < \gamma\}$$

since $0_{\mathsf{B}} < i(u) \leq i(p_\alpha) \in D_\alpha$ for all $\alpha < \gamma$.

Since $b \geq_{\mathsf{B}} u$ is arbitrary, the proof is completed. ☐

Definition 6.3.9. Given an uncountable cardinal κ, let

$$P_\kappa = \{f : \alpha \to \kappa : f \text{ is an injection and } \alpha < \omega_1\}$$

ordered by $f \leq_{P_\kappa} g$ iff $f \supseteq g$.

Fact 6.3.10. *$(P_\kappa, \leq_{P_\kappa})$ is $< \omega_1$-closed and separative.*

Proof. First of all notice that f, g are incompatible in P_κ if and only if they disagree on some j in $\mathrm{dom}(g) \cap \mathrm{dom}(f)$, else their union is a common refinement.

We prove both properties of P as follows:

P_κ **is separative:** Assume $f \not\leq g$, then $g \not\subseteq f$. In particular, either $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$, in which case f and g are already incompatible, or there is $i \in \mathrm{dom}(g) \setminus \mathrm{dom}(f)$. In this case we let $h : i+1 \to \kappa$ be defined by the requirements:

- $h \supseteq f$,
- $h \restriction (i+1 \setminus \mathrm{dom}(f)) \to (\kappa \setminus (\mathrm{ran}(f) \cup \mathrm{ran}(g)))$ is injective.

Since κ is an uncountable cardinal and $\mathrm{ran}(f) \cup \mathrm{ran}(g)$ is a countable subset of κ, $\kappa \setminus (\mathrm{ran}(f) \cup \mathrm{ran}(g))$ has size κ, thus the at most countable set $i+1 \setminus \mathrm{dom}(f)$ can be injected inside it.

We conclude that $h \supseteq f$, and $h \in P_\kappa$ since it is an injective function with domain a countable ordinal, moreover h is incompatible with g, since $h(i) \notin \mathrm{ran}(g)$, thus $h(i) \neq g(i)$.

P_κ **is countably closed:** Assume we have a decreasing sequence

$$\{f_\alpha : \alpha < \gamma\}$$

of elements of P_κ indexed by some countable ordinal γ. Let $f = \bigcup_{\alpha<\gamma} f_\alpha$, we show that $f \in P$ is a lower bound for all the f_α: It is enough to show that f is also an element of P_κ and this is the case since its domain is a countable ordinal (a countable union of countable ordinals is a countable ordinal), and f is injective, since it is the coherent union of injective functions.

□

Let M be a countable transitive model of ZFC such that $M \models 2^{\aleph_0} = \kappa$ and consider the partial order $P = (P_\kappa)^M$ in M. Then M models that P is countably closed and separative.

Let $\mathsf{B} = \mathsf{RO}(P)^M$ and $i : P \to \mathsf{B}$ in M be a canonical injection of P in its boolean completion. Then M models that B is countably distributive, applying Lemma 6.3.8 inside M to P and B. We will show the following:

Theorem 6.3.11. *M models that $[\![\mathsf{CH}]\!] = 1_{\mathsf{B}}$.*

The theorem will be an immediate consequence of the following proposition:

Proposition 6.3.12. *Assume G is M-generic for P. Then:*

1. $(\omega_1)^M = (\omega_1)^{M[G]}$.
2. $M[G] \cap 2^\omega = M \cap 2^\omega$.
3. *$M[G]$ models that there is a bijection of $(\omega_1)^M$ with κ.*

Assume the proposition has been proved. Let $h : 2^\omega \cap M \to \kappa$ in M be a bijection of $(2^\omega)^M = 2^\omega \cap M = (2^\omega)^{M[G]}$ with κ, then $g \circ h : (2^\omega)^{M[G]} \to (\omega_1)^M$ is a bijection and $(\omega_1)^M = (\omega_1)^{M[G]}$, i.e. $M[G]$ models h is a bijection of the powerset of ω with the first uncountable cardinal, as was to be shown.

We first prove item 2 of the above proposition:

Proof. Let $\dot{r} \in M^{\mathsf{B}}$ be a B-name such that $[\![\dot{r} : \check{\omega} \to \check{2}]\!] = b > 0_{\mathsf{B}}$. Define:

$$D_n = \{f \in \mathsf{B} : M \models \exists i < 2 \; f \leq [\![\dot{r}(\check{n}) = \check{i}]\!]\}.$$

By an application of the forcing theorem, we can prove that each $D_n \in M$ is an open dense subset of B^+ below b as follows: for each $q \in \mathsf{B}^+$ refining b, pick $G \in M$ such that $q \in G$. Then $b \in G$ gives that

$$M[G] \models \dot{r}_G : \omega \to 2,$$

thus for some $i < 2$

$$M[G] \models \dot{r}_G(n) = i,$$

yielding that

$$s = [\![\dot{r}(\check{n}) = \check{i}]\!] \wedge q \in G$$

and thus $0_{\mathsf{B}} < s \in D_n$ refines q.

This gives that M models that for all $q \leq b$ there exists $0_{\mathsf{B}} < s \leq_{\mathsf{B}} q$ in D_n. Thus M models that each D_n is dense. Notice also that the sequence

$$\{D_n : n \in \omega\} \in M.$$

Towards this aim observe that

$$X = \left\{\langle n, q, i\rangle \in \omega \times \mathsf{B} \times 2 : 0_{\mathsf{B}} < q \leq_{\mathsf{B}} b \wedge \left[\!\!\left[\dot{f}(\check{n}) = \check{i}\right]\!\!\right]\right\} \in M,$$

since it is obtained applying comprehension in M to define a subset of $\omega \times \mathsf{B} \times 2$. Now

$$\langle D_n : n \in \omega\rangle = (\langle n, c\rangle : n \in \omega, \forall x(x \in c \leftrightarrow \exists i < 2 \,\langle n, x, i\rangle \in X) \in M$$

is obtained as a subset of $(\omega \times \mathcal{P}(\mathsf{B}))^M$, applying comprehension in M once again.

Since M models that B is countably distributive, we get that

$$D_{\dot{r}} = \bigcap\{D_n : n \in \omega\}$$

is also in M, and is open dense below b. We claim the following:

$$D_{\dot{r}} = \{q \leq_{\mathsf{B}} b : \exists s \in 2^\omega \cap M \text{ such that } g \Vdash \dot{r} = \check{s}\}. \tag{6.7}$$

To this aim choose $r \leq_{\mathsf{B}} b$ arbitrarily. Find $q \leq_{\mathsf{B}} r$ in $D_{\dot{r}}$, which is possible since $D_{\dot{r}}$ is open dense. Then $q \in D_n$ for all $n \in \omega$. In M, we can let for each $q \in D_{\dot{r}}$

$$f_q = \left\{\langle n, i\rangle \in \omega \times 2 : \; M \models [\![\dot{r}(\check{n}) = \check{i}]\!] \geq q\right\}.$$

Then $f_q \in M$ applying the comprehension axiom in M to isolate f_q as a subset of $\omega \times 2$, defined by a property in the parameters $B, \dot{r}, q$.

We claim that $q \leq_{\mathsf{B}} [\![\dot{r} = \check{f}_q]\!]$. To this aim let G be M-generic with $q \in G$. Then $b \in G$ yields that

$$M[G] \models \dot{r}_G : \omega \to 2.$$

Let $i_n = \dot{r}_G(n)$ for each $n \in \omega$. Then $[\![\dot{r}(\check{n}) = \check{i}_n]\!] \in G$ for all $n \in \omega$. Now $q \in D_n \cap G$ for all n, and for each n, $q \in D_n$ entails that $q \leq_{\mathsf{B}} [\![\dot{r}(\check{n}) = \check{i}]\!]$ if and only if $f_q(n) = i$, by definition of f_q. We get that

$$M[G] \models \dot{r}_G = f_q.$$

Since this occurs for all M-generic filters G for B to which q belongs, we get that $q \leq_{\mathsf{B}} [\![\dot{r} = \check{f}_q]\!]$. This concludes the proof that equation 6.7 holds.

In particular we get that:

- $D_{\dot{r}}$ is open dense below b for all $\dot{r} \in M^{\mathsf{B}}$ such that $[\![\dot{r} : \omega \to 2]\!] = b$,
- for any condition q in $D_{\dot{r}}$ $q \leq_{\mathsf{B}} [\![\dot{r} = \check{f}_q]\!]$.

This gives that for any $r = \dot{r}_G \in 2^\omega \cap M[G]$, we have that $[\![\dot{r} : \check{\omega} \to \check{2}]\!] = b \in G$, thus $G \cap D_{\dot{r}}$ is non-empty, giving that $r = \dot{r}_G = f_q \in 2^\omega \cap M$ for some $q \in G \cap D_{\dot{r}}$. Item 2 of the proposition is proved. □

Exercise 6.3.13. Prove item 1 of the proposition. (HINT: follow the pattern of the proof of item 2 of the proposition. Now start from $\dot{r}$ a B-name for a function from ω into ω_1, and argue once again that $\dot{r}_G \in M$ whenever G is M-generic for P).

We now prove item 3 of the proposition:

Proof. Let $\dot{g} \in M^{\mathsf{B}}$ be such that

$$\dot{g} = \{\left\langle \mathrm{op}(\check{j}, \check{f(j)}), i(f) \right\rangle : f \in P,\, j \in \mathrm{dom}(f)\}.$$

We claim that $g = \dot{g}_G$ is a bijection of $(\omega_1)^M$ into κ. To this aim observe that

$$f \subseteq g \leftrightarrow i(f) \in G$$

for all $f \in P$, since

$$\langle j, \alpha \rangle \in g$$

if and only if there is some $i(f) \in G$ such that

$$\left\langle \mathrm{op}(\check{j}, \check{\alpha}), i(f) \right\rangle \in \dot{g}$$

if and only if

$$f(j) = \alpha \text{ for some (any) } i(f) \in G \text{ with } j \in \mathrm{dom}(f).$$

This gives immediately that g is an injective function since it is the coherent union of injective functions. Moreover for all $\alpha < \kappa$ and all $\xi < (\omega_1)^M$ the following sets are easily seen to be dense and in M:

$$D_\alpha = \{i(f) : f \in P,\, \alpha \in \mathrm{ran}(f)\},$$

$$E_\xi = \{i(f) : f \in P,\, \xi \in \mathrm{dom}(f)\}.$$

Observe that whenever G is an M-generic filter for P, $G \cap D_\alpha \neq \emptyset$ iff $\alpha \in \mathrm{ran}(\dot{g}_G)$ and $G \cap E_\xi \neq \emptyset$ iff $\xi \in \mathrm{dom}(\dot{g}_G)$.

This gives that $M[G]$ models that $\dot{g}_G$ is a bijection of $(\omega_1)^M$ with κ. The proof of item 3 of the proposition is completed. $\square$

The relative coherence of CH with respect to T is established.

6.4 M^{B} models ZFC

In this section we prove that the axioms of ZFC are valid in any model M^{B}, whenever M is a countable transitive model of ZFC and B is a boolean algebra that belongs to M and which M models to be complete.

Theorem 6.4.1. *Assume M is a transitive countable model of ZFC in V and $\mathsf{B} \in M$ is such that M models B is a complete boolean algebra. Then*

$$M \models [\![\phi]\!]_{\mathsf{B}} = 1_{\mathsf{B}}$$

for every axiom ϕ of ZFC.

Remark 6.4.2. We can actually prove a stronger result stating that whenever M is *any* model of ZFC and $\mathsf{B} \in M$ is a boolean algebra which M models to be complete, then

$$M \models [\![\phi]\!]_{\mathsf{B}} = 1_{\mathsf{B}}$$

for every axiom ϕ of ZFC. I.e. we can remove the assuption that M is countable and transitive in the above theorem. However the proof of this latter result is slightly more involved since we cannot appeal to the forcing theorem to obtain it. As we will see below the forcing theorem plays a crucial role in most of the arguments to follow.

Proof. We show that $M[G]$ satisfies all ZFC-axioms whenever G is M-generic for B. The proof can be completed appealing to Theorem 6.2.5. From now on we assume that G is an M-generic filter for B.

Extensionality. $M[G]$ is transitive, hence it models the Extensionality Axiom by [7, Lemma IV.2.4].

Foundation. $M[G]$ is a transitive set contained in V, so $M[G]$ models the Axiom of Foundation, by [7, Theorems III.3.6, III.4.1].

Infinity. $\omega = \check{\omega}_G \in M[G]$.

Pairing. Let $\sigma_G, \tau_G \in M[G]$. Given $\sigma, \tau \in M^{\mathsf{B}}$, let

$$\mathrm{up}(\sigma,\tau) = \{\langle \sigma, 1\rangle, \langle \tau, 1\rangle\}.$$

Then $\mathrm{up}(\sigma,\tau) = \{\sigma_G, \tau_G\}$ (since $\rho(\sigma) = \rho(\tau) = 1_{\mathsf{B}} \in G$) is a witness for the pairing axiom (see also exercise 6.1.21).

Union. Given $\sigma_G \in M[G]$, we let

$$\tau = \{\langle \rho, 1_{\mathsf{B}}\rangle : \exists u \in \operatorname{dom}(\sigma)(\rho \in \operatorname{dom}(u))\}.$$

Then τ_G is a witness for the union axiom for σ_G, since:

$$\begin{aligned}
\bigcup \sigma_G = \{a : \exists u_G \in \sigma_G(a \in u_G)\} = \\
= \{a : \exists u \in \operatorname{dom}(\sigma)(\sigma(u) \in G \wedge\ a \in u_G)\} \subseteq \\
\subseteq \{a : \exists u \in \operatorname{dom}(\sigma)(a \in u_G)\} = \\
= \{\rho_G : \exists u \in \operatorname{dom}(\sigma)(\rho \in \operatorname{dom}(u) \wedge\ u(\rho) \in G)\} \subseteq \\
\subseteq \{\rho_G : \exists u \in \operatorname{dom}(\sigma)(\rho \in \operatorname{dom}(u))\} = \tau_G.
\end{aligned}$$

Power set. Let $\sigma_G \in M[G]$. Set

$$\mathcal{P}^{\mathsf{B}}(\sigma) = \{\langle \tau, [\![\tau \subseteq \sigma]\!]\rangle : \operatorname{dom}(\tau) = \operatorname{dom}(\sigma)\} \in M,$$

applying the Comprehension axiom in M to the set

$$\left\{\langle \tau, b\rangle \in \mathcal{P}(\operatorname{dom}(\sigma) \times \mathsf{B})^M \times \mathsf{B}\right\} \in M$$

and the formula $\phi(z, \sigma, \mathsf{B})$ stating that *"$z = \langle x, y\rangle$ with $x : \operatorname{dom}(\sigma) \to \mathsf{B}$ a B-name and $y = [\![x \subseteq \sigma]\!]_{\mathsf{B}}$"*.

We claim that

$$M[G] \models (\mathcal{P}^{\mathsf{B}}(\sigma))_G = \{a \in M[G] : a \subseteq \sigma_G\}$$

for all G M-generic for B. Both inclusions follow by Lemma 6.2.9:

- $\subseteq$: $\eta_G \in (\mathcal{P}^{\mathsf{B}}(\sigma))_G$ for some $\eta \in \operatorname{dom}(\mathcal{P}^{\mathsf{B}}(\sigma))$ iff $[\![\eta \subseteq \sigma]\!] = \mathcal{P}^{\mathsf{B}}(\sigma)(\eta) \in G$, giving that $\eta_G \subseteq \sigma_G$ by the forcing theorem.
- $\supseteq$: Assume $a \subseteq \sigma_G$ for some $a \in M[G]$. Then $a = \tau_G$ with $[\![\tau \subseteq \sigma]\!] \in G$, by the forcing theorem. By Lemma 6.2.9 we get that there is $\eta \in M^{\mathsf{B}}$ with $\operatorname{dom}(\eta) = \operatorname{dom}(\sigma)$ such that $[\![\tau = \eta]\!] \geq [\![\tau \subseteq \sigma]\!]$. Since $[\![\tau \subseteq \sigma]\!] \in G$ we get that $[\![\tau = \eta]\!] \in G$ as well, hence $\tau_G = \eta_G$. However $\eta_G \in (\mathcal{P}^{\mathsf{B}}(\sigma))_G$, since $\operatorname{dom}(\eta) = \operatorname{dom}(\sigma)$ and

$$[\![\eta \subseteq \sigma]\!] \geq [\![\tau \subseteq \sigma]\!] \wedge [\![\tau = \eta]\!] \in G.$$

Comprehension. Let $\varphi(x, z)$ be a formula, $a \in M[G]$, $\sigma_G = a$ and $\bar{\tau}_G = \bar{d}$, where $\bar{\tau} = (\tau_1, \ldots, \tau_n)$ and $\bar{\tau}_G = ((\tau_1)_G, \ldots, (\tau_n)_G)$. Let

$$b = \{c \in a : M[G] \vDash \varphi(c, \bar{d})\}.$$

We must show that the definable class b of $M[G]$ is an element of $M[G]$.

Let

$$\eta = \{\langle \tau, [\![\varphi(\tau, \bar{\tau})]\!] \wedge [\![\tau \in \sigma]\!]\rangle\rangle : \tau \in \operatorname{dom}(\sigma)\} \in M^{\mathsf{B}}.$$

We claim that $\eta_G = b$.

$\eta_G \subseteq b$**:** We have that

$$\eta_G = \{\tau_G : \tau \in \text{dom}(\sigma) \text{ and } [\![\varphi(\tau, \bar{\tau})]\!] \wedge [\![\tau \in \sigma]\!] \in G\}$$

Hence $\tau_G \in b$ for all $\tau_G \in \eta_G$, since

$$M[G] \models \varphi(\tau_G, \bar{\tau}_G) \wedge \tau_G \in \sigma_G$$

by the forcing theorem.

$\eta_G \supseteq b$**:** $\tau_G \in b$ iff $\tau_G \in \sigma_G$ and

$$M[G] \models \phi(\tau_G, \bar{\tau}_G).$$

First observe that $\tau_G \in \sigma_G$ iff $\tau_G = \nu_G$ for some $\nu \in \text{dom}(\sigma)$ with $\sigma(\nu) \in G$. This gives that $[\![\tau = \nu]\!], \sigma(\nu) \in G$ by the forcing theorem.
Moreover

$$\sigma(\nu) \leq \bigvee_{u \in \text{dom}(\sigma)} [\![u = \nu]\!] \wedge \sigma(u) = [\![\nu \in \sigma]\!],$$

hence $[\![\nu \in \sigma]\!] \wedge [\![\tau = \nu]\!] \in G$.
The forcing theorem also gives that $[\![\phi(\tau, \bar{\tau})]\!] \in G$. Hence

$$[\![\phi(\nu, \bar{\tau})]\!] \geq [\![\phi(\tau, \bar{\tau})]\!] \wedge [\![\tau = \nu]\!] \in G.$$

We conclude that

$$\eta(\nu) = [\![\phi(\nu, \bar{\tau})]\!] \wedge [\![\nu \in \sigma]\!] \in G.$$

Hence $\tau_G = \nu_G \in \eta_G$, concluding the proof.

Replacement. Let $F : M[G] \to M[G]$ be a functional class in $M[G]$ and let $\psi(x, y, \bar{\tau}_G)$ be the formula such that $F(u) = v$ iff $M[G] \models \psi(u, v, \bar{\tau}_G)$ and

$$M[G] \models \forall x \exists! y \phi(x, y, \bar{\tau}_G),$$

where $\bar{\tau}_G = ((\tau_1)_G, \ldots, (\tau_n)_G)$. By the forcing theorem we have that

$$[\![\forall x \exists! y \phi(x, y, \bar{\tau})]\!] \in G.$$

Fix $\sigma_G \in M[G]$ and in M consider the function

$$F^* : M^{\mathsf{B}} \to M^{\mathsf{B}} \times \mathsf{B} \qquad\qquad \eta \mapsto (\nu_\eta, b_\eta),$$

where

$$b_\eta = [\![\exists y \phi(\eta, y)]\!]$$

and ν_η is provided by the Fullness Lemma applied in M to M^{B}, to find a $\nu \in M^{\mathsf{B}}$ such that

$$b_\eta = [\![\exists y \phi(\eta, y, \bar{\tau})]\!] = [\![\phi(\eta, \nu, \bar{\tau})]\!].$$

Then F^* is a definable class in M (useful exercise for the reader). Hence we can apply replacement in M to find $\tau \subseteq M^{\mathsf{B}} \times \mathsf{B}$ with $\tau \in M$ such that

$$\tau = \{\langle \nu_\eta, b_\eta \rangle : \eta \in \text{dom}(\sigma)\}.$$

Since $\tau \in M$ and (it can be checked that) τ is a function, we get that $\tau \in M^{\mathsf{B}}$. We claim that $\tau_G = F[\sigma_G]$.

$\tau_G \supseteq F[\sigma_G]$**:** Assume $a = \nu_G \in F[\sigma_G]$. Then for some $\eta \in \mathrm{dom}(\sigma)$ with $\sigma(\eta) \in G$ we have that $F(\eta_G) = \nu_G$ holds in $M[G]$.

By the forcing theorem we get that $[\![\phi(\eta, \nu, \bar{\tau})]\!] \in G$. This gives that

$$b_\eta = [\![\phi(\eta, \nu_\eta, \bar{\tau})]\!] = [\![\exists y \phi(\eta, y, \bar{\tau})]\!] \geq [\![\phi(\eta, \nu, \bar{\tau})]\!] \in G$$

as well. Hence

$$M[G] \models \phi(\eta_G, (\nu_\eta)_G, \bar{\tau}_G),$$

i.e. $F(\eta_G) = (\nu_\eta)_G$. This gives that $(\nu_\eta)_G = a = \nu_G$. We conclude that $F(\eta_G) = (\nu_\eta)_G \in \tau_G$ since $\nu_\eta \in \mathrm{dom}(\tau)$ and $\tau(\nu_\eta) = b_\eta \in G$.

$\tau_G \subseteq F[\sigma_G]$**:** Assume $\nu_\eta \in \mathrm{dom}(\tau)$ and $b_\eta \in G$. Then

$$b_\eta = [\![\exists y \phi(\eta, y, \bar{\tau})]\!] = [\![\phi(\eta, \nu_\eta, \bar{\tau})]\!].$$

Since $[\![\forall x \exists! y \phi(x, y, \bar{\tau})]\!] \in G$ as well, we get that

$$M[G] \models \exists! y \phi(\eta_G, y, \bar{\tau}_G)$$

and

$$M[G] \models \phi(\eta_G, (\nu_\eta)_G, \bar{\tau}_G).$$

Hence $F(\eta_G) = (\nu_\eta)_G$. Since this holds for all $\eta_G \in \sigma_G$ we conclude that $\tau_G \subseteq F[\sigma_G]$.

Choice. We prove that $M[G]$ satisfies that every set can be well-ordered. More precisely, we show that $\forall X \in M[G]$, there exists an injection $f : X \to \mathrm{Ord}$ belonging to $M[G]$.

Let $\tau_G = X \in M[G]$. Since $M \models \mathsf{ZFC}$, there exists $f \in M$ such that $M \models f : \tau \to \beta$ is a bijection. Recall the operation op defined in exercise 6.1.21 which gives a canonical B-name for an ordered pair. Let

$$\tau^* = \{\langle \mathrm{op}(\sigma, \check{\alpha}), 1_{\mathsf{B}} \rangle : \sigma \in \mathrm{dom}(\tau),\ f(\sigma) = \alpha\} \in M^{\mathsf{B}}.$$

Then:

$$\begin{aligned} \tau^*_G = \{\mathrm{op}(\sigma, \check{\alpha})_G : \sigma \in \mathrm{dom}(\tau),\ f(\sigma) = \alpha\} = \\ = \{(\sigma_G, f(\sigma)) : \sigma \in \mathrm{dom}(\tau)\} = R. \end{aligned}$$

Notice that

$$R \subseteq \{\sigma_G : \sigma \in \mathrm{dom}(\tau)\} \times \beta$$

may not be a functional relation (there could be distinct $\sigma, \sigma' \in \mathrm{dom}(\tau)$ such that $\sigma_G = \sigma'_G$, with $f(\sigma) \neq f(\sigma')$). Notice also that $\tau_G \subseteq \mathrm{dom}(R)$: indeed

$$\tau_G = \{\sigma_G : \tau(\sigma) \in G\} \subseteq \{\sigma_G : \langle \sigma_G, f(\sigma) \rangle \in R : \sigma \in \mathrm{dom}(\tau)\} = \mathrm{dom}(R).$$

By what we have shown so far $M[G] \models \mathsf{ZF}$. In particular we can use the comprehension axiom in $M[G]$ to refine R to a functional relation g with the same domain letting:

$$g = \{(\sigma_G, \xi) \in R : \forall \gamma \in \beta (\sigma_G, \gamma) \in R \Rightarrow \xi \leq \gamma\}.$$

Clearly $R \in M[G]$ implies that $g \in M[G]$ by Comprehension applied in $M[G]$. We leave to the reader to check that $g : \mathrm{dom}(R) \to \mathrm{Ord}$ is an injective function. Hence, g witnesses that $X = \tau_G$ can be well ordered in $M[G]$.

The proof that all axioms of ZFC hold in $M[G]$ is complete. $\square$

Corollary 6.4.3. *Assume M is a transitive countable model of ZFC and G is M-generic for a $\mathsf{B} \in M$ which M models to be a complete boolean algebra. Then $M[G]$ is the smallest transitive model N of ZFC with $N \supseteq M$ and $G \in N$. Moreover $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$*

Proof. $G = \dot{G}_G \in M[G]$, where $\dot{G} \in M^{\mathsf{B}}$ is the B-name $\left\{\left\langle \check{b}, b\right\rangle : b \in \mathsf{B}^+\right\}$, and $M \subseteq M[G]$ since $\check{a}_G = a$ for all $a \in M$. In particular $M[G] \subseteq M$ for all M transitive model of ZFC containing M and with $G \in M$. Since $M[G] \models \mathsf{ZFC}$ and is transitive we are done.

For the second part of the Corollary (i.e. the assertion that $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$, we proceed as follows: since $M[G] \models \mathsf{ZFC}$, we get that

$$\mathrm{Ord} \cap M[G] = \left\{\mathrm{rk}(\tau_G) : \tau \in M^{\mathsf{B}}\right\}.$$

An easy induction show that $\mathrm{rk}(\tau_G) \leq \mathrm{rk}(\tau)$ for all $\tau \in M[G]$, moreover $\mathrm{rk}(\check{\alpha}_G) = \mathrm{rk}(\alpha) = \alpha$ for all limit $\alpha \in M \cap \mathrm{Ord}$. We get that

$$M \cap \mathrm{Ord} = \{\check{\alpha}_G : \alpha \in M \cap \mathrm{Ord}\} \subseteq \mathrm{Ord} \cap M[G] = \left\{\mathrm{rk}(\tau_G) : \tau \in M^{\mathsf{B}}\right\} \subseteq M \cap \mathrm{Ord}.$$

$\square$

Chapter 7

Appendix A: Absoluteness

This appendix is meant as an integration to [7, Chapters III, IV, V] for the parts of this book which we cover in this course. [7, Chapters III, IV] is our basic reference and the results in this appendix provides some more details on what is shown there.

Section 7.1 is an integration to [7, IV.1,IV.2,IV.3,IV.5]. Section 7.2 provide a different approach to the results of [7, IV.8,IV.9,IV.10,V.1] and covers the fragmnt of these results which we need to provide a solid metamathematical foundation for our treatment of forcing in chapter 6.

7.1 Absoluteness

Definition 7.1.1.

- $R \subseteq V^n$ is a definable class if there exists a formula $\phi_R(x_1, \dots, x_n, y_1, \dots, y_{m_R})$ and $b_1^R, \dots, b_{m_R}^R \in V$ (with the number m_R of parameters depending on R) such that

$$R = \left\{(a_1, \dots, a_n) : (V, \in, =) \models \phi_R(a_1, \dots, a_n, b_1^R, \dots, b_{m_R}^R)\right\}.$$

- $R \subseteq A^2$ for some $A \subseteq V$ is set-like if for all $a \in A$

$$\text{pred}_R(a) = \{b \in A : R(a, b)\} \in V.$$

- $R \subseteq A^2$ for some $A \subseteq V$ is well-founded if for all $Z \subseteq A$ non-empty, there exists $b \in Z$ such that $\text{pred}_R(b) \cap Z$ is empty.

- Given some $M \subseteq N \subseteq V$, a definable $R \subseteq V^n$ is absolute between M and N iff $b_1^R, \dots, b_{m_R}^R \in M$ and for all $a_1, \dots, a_n \in M$

$$\begin{aligned}&(N, \in, =) \models \phi_R(a_1, \dots, a_n, b_1^R, \dots, b_{m_R}^R)\\ &\text{ if and only if}\\ &(M, \in, =) \models \phi_R(a_1, \dots, a_n, b_1^R, \dots, b_{m_R}^R).\end{aligned}$$

- Given some $M \subseteq N \subseteq V$, a definable $G : V^n \to V$ is absolute for M, N if the graph of G is absolute for M and N and both M and N models the formula

$$\forall x_1 \dots \forall x_n \exists! y \phi_G(x_1, \dots, x_n, y, a_1^G, \dots, a_{m_G}^G).$$

- A relation R is absolute for M if it is absolute for M and V. Similarly we define the notion of a class function G being absolute for M.

- Given a definable class $M \subseteq V$ and a formula $\phi(x_1, \ldots, x_m)$, $\phi^M(x_1, \ldots, x_m)$ is the formula obtained from $\phi(x_1, \ldots, x_m)$ replacing all its universal quantifiers $\forall x$ by the block of symbols $\forall x \phi_M(x, b_1^M, \ldots, b_{m_M}^M) \rightarrow$ and all its existential quantifiers $\exists x$ by the block of symbols $\exists x \phi_M(x, b_1^M, \ldots, b_{m_M}^M) \wedge$. I.e. $\phi^M(x_1, \ldots, x_m)$ is the formula obtained restricting all its quantifers to range over elements of M.

- Given a definable relation $R \subseteq V^n$ and some $M \subseteq V$ with $b_1^R, \ldots, b_{m_R}^R \in M$,

$$R^M = \{(a_1, \ldots, a_n) \in M : (M, \in, =) \models \phi_R(a_1, \ldots, a_n, b_1^R, \ldots, b_{m_R}^R)\} = \\ = \{(a_1, \ldots, a_n) \in M : (V, \in, =) \models \phi_R^M(a_1, \ldots, a_n, b_1^R, \ldots, b_{m_R}^R)\}.$$

Fact 7.1.2. *Assume $R \subseteq A^n$, $M, A \subseteq V$ are all definable classes, with A and R defined by formulae with parameters in M and M transitive.*
Then:

- *$R^M = R \cap M^n$ if and only if R is absolute for M.*

- *If $R \subseteq A^2$ and $b \in A$,*

$$\mathrm{pred}_R(b) = \{a \in V : V \models \phi_A(a, b_1^A, \ldots, b_{m_A}^A) \wedge \phi_R(a, b, b_1^R, \ldots, b_{m_R}^R)\}$$

  is a definable class in the parameters $b_1^A, \ldots, b_{m_A}^A, b, b_1^R, \ldots, b_{m_R}^R$.

- *$R \subseteq A^2$ is well-founded if and only if*

$$V \models \forall z(\exists x \phi_A(x, b_1^A, \ldots, b_{m_A}^A) \wedge x \in z) \rightarrow \\ \rightarrow [\exists x \phi_A(x, b_1^A, \ldots, b_{m_A}^A) \wedge x \in z \wedge \forall y(\phi_R(y, x, b_1^R, \ldots, b_{m_R}^R) \rightarrow y \notin z)].$$

- *$R \subseteq A^2$ is set-like if and only if*

$$V \models \forall x[\phi_A(x, b_1^A, \ldots, b_{m_A}^A) \rightarrow \exists y \forall z[z \in y \leftrightarrow (\phi_A(z, b_1^A, \ldots, b_{m_A}^A) \wedge \phi_R(z, x, b_1^R, \ldots, b_{m_R}^R))]].$$

- *If $R \subseteq A^2$ is set-like, $\mathrm{pred}_R : A \rightarrow V$ is a definable class given by*

$$\mathrm{pred}_R = \{(a, b) : V \models \forall z[z \in b \leftrightarrow (\phi_A(z, b_1^A, \ldots, b_{m_A}^A) \wedge \phi_R(z, a, b_1^R, \ldots, b_{m_R}^R))]$$

Proof. Left to the reader. □

Lemma 7.1.3. *Assume $G_i : V^{n_i} \rightarrow V$ for $i = 1, \ldots, m$ and $R \subseteq V^m$ are absolute between transitive sets or classes $M \subseteq N$. Then so is $R(G_1(x_1, \ldots, x_{n_1}), \ldots, G_1(x_1, \ldots, x_{n_m}))$.*

Proof. [7, IV.3.10]. □

Definition 7.1.4. We enrich the first order syntax with bounded quantifiers $\forall x \in y$, $\exists x \in y$ with the provision that

$$\exists x \in y\, \phi(x, y, x_1, \dots, x_n) \equiv \exists x\, \phi(x, y, x_1, \dots, x_n) \wedge x \in y$$

and

$$\forall x \in y\, \phi(x, y, x_1, \dots, x_n) \equiv \forall x\, (x \in y \to \phi(x, y, x_1, \dots, x_n)).$$

$\phi(x_1, \dots, x_n)$ is a Δ_0-formula in this expanded language if all the quantifiers in ϕ are bounded.

Lemma 7.1.5. *Assume $R \subseteq V^n$ is defined by means of a Δ_0-formula (i.e. $\phi_R(x_1, \dots, x_n, a_1^R, \dots, a_{m_R}^R)$ is a Δ_0-formula). Then R is absolute between transitive models $M \subseteq N$ of ZF − power-set axiom.*

The same holds for any $G : V^n \to V$ such that $\phi_G(x_1, \dots, x_n, y, a_1^R, \dots, a_{m_G}^G)$ is a Δ_0-formula and M, N both model that

$$\forall x_1 \dots \forall x_n \exists! y \phi_G(x_1, \dots, x_n, y, a_1^G, \dots, a_{m_G}^G).$$

Proof. [7, IV.3.6]. □

Lemma 7.1.6. *Assume $M \subseteq N$ and $\phi(y, x_1, \dots, x_n, a_1, \dots, a_k)$, $\psi(y, x_1, \dots, x_n, a_1, \dots, a_k)$ define properties which are absolute for M, N. Assume moreover M, N are both models of some theory T and that T proves*

$$\forall z_1 \dots \forall z_k \forall x_1 \dots \forall x_n [\forall y \phi(y, x_1, \dots, x_n, z_1, \dots, z_k) \leftrightarrow \exists y \psi(y, x_1, \dots, x_n, z_1, \dots, z_k)].$$

Then

$$\begin{aligned} R = \{(b_1, \dots, b_n) : (V, \in, =) \models \forall y \phi(y, b_1, \dots, b_n, a_1, \dots, a_k)\} = \\ = \{(b_1, \dots, b_n) : (V, \in, =) \models \exists y \psi(y, b_1, \dots, b_n, a_1, \dots, a_k)\} \end{aligned}$$

is absolute for M, N.

Proof. Assume $R^M(b_1, \dots, b_n)$. Then for some $b \in M$

$$(M, \in, =) \models \psi(b, b_1, \dots, b_n, a_1, \dots, a_k)$$

But $\psi(b, b_1, \dots, b_n, a_1, \dots, a_k)$ is absolute between M, N, yielding that

$$(N, \in, =) \models \exists x \psi(x, b_1, \dots, b_n, a_1, \dots, a_k)$$

as witnessed by $b \in M$. Thus we get that $R^N(b_1, \dots, b_n)$ holds.

Conversely assume $R^N(b_1, \dots, b_n)$. Then for all $b \in N$

$$(N, \in, =) \models \phi(b, b_1, \dots, b_n, a_1, \dots, a_k)$$

But $\phi(b, b_1, \dots, b_n, a_1, \dots, a_k)$ is absolute between M, N for all $b \in M$, yielding that

$$(M, \in, =) \models \phi(b, b_1, \dots, b_n, a_1, \dots, a_k)$$

for all $b \in M$. Thus we get that

$$(M, \in, =) \models \forall y \phi(y, b_1, \dots, b_n, a_1, \dots, a_k)$$

and also that $R^M(b_1, \dots, b_n)$ holds. □

Definition 7.1.7. A relation $R \subseteq V^n$ is Δ_1 over a theory T in the language of set theory if for every transitive model M of T we have that for Δ_0-formulae ϕ, ψ and $b_1, \dots, b_m \in M$

$$\begin{aligned} R^M &= \{(a_1, \dots, a_n) \in M^n : (M, \in, =) \models \exists x \phi(x, a_1, \dots, a_n, b_1, \dots, b_m)\} = \\ &= \{(a_1, \dots, a_n) \in M^n : (M, \in, =) \models \forall x \psi(x, a_1, \dots, a_n, b_1, \dots, b_m)\}. \end{aligned}$$

The following is an immediate corollary of Lemmas 7.1.5, 7.1.6:

Lemma 7.1.8. *Assume R is Δ_1 over a theory T extending* ZF−Power-set axiom. *Then R is absolute for M, N if both are transitive models of T.*

Lemma 7.1.9. *Assume M is transitive and is a model of ZF − Power-Set Axiom. Assume $A \subseteq V$ and $R \subseteq A^2$ are definable classes defined by parameters in M. Assume M models that R^M is set-like and well founded. Then M models that there is a unique definable class function*

$$\begin{aligned} \mathrm{rk}_R^M :& A \to \mathrm{Ord} \\ & a \mapsto \mathrm{rk}_R^M(a) = \sup\{\mathrm{rk}_R^M(b) + 1 : b \in \mathrm{pred}_R(a)\} \end{aligned}$$

Proof. Let $\phi_{\mathrm{rk}_R}(z, b_1^A, \dots, b_{m_A}^A, b_1^R, \dots, b_{m_R}^R)$ be the following formula:

$$\begin{aligned} (z \text{ is a function }) \wedge (\mathrm{ran}(z) \text{ is an ordinal }) \wedge (\mathrm{dom}(z) \subseteq A) \wedge \\ \wedge \forall t \in \mathrm{dom}(z)(\mathrm{pred}_R(t) \subseteq \mathrm{dom}(z)) \wedge \\ \wedge \forall t \in \mathrm{dom}(z)(z(t) = \bigcup \{z(u) \cup \{u\} : u \in \mathrm{pred}_R(t)\}) \end{aligned}$$

We leave to the reader to check that the above expression can be meant as a short-hand for a formula in the parameters $b_1^A, \dots, b_{m_A}^A, b_1^R, \dots, b_{m_R}^R$ which are needed to define the class-function pred_R, the formula $(\mathrm{dom}(z) \subseteq A)$, and the formula $(\mathrm{pred}_R(t) \subseteq \mathrm{dom}(z))$. Now we can use the transfinite recursion theorem on the well-founded relation R^M inside M, to check that

$$\begin{aligned} \mathrm{rk}_R^M &= \{(a, b) \in M^2 : (M, \in, =) \models \exists z\, [\langle a, b \rangle \in z \wedge \phi_{\mathrm{rk}_R}(z, b_1^A, \dots, b_{m_A}^A, b_1^R, \dots, b_{m_R}^R)]\} = \\ &= \{(a, b) \in M^2 : (M, \in, =) \models \forall z \phi_{\mathrm{rk}_R}(z, b_1^A, \dots, b_{m_A}^A, b_1^R, \dots, b_{m_R}^R) \wedge a \in \mathrm{dom}(z) \to z(a) = b\}. \end{aligned}$$

□

Lemma 7.1.10. *Assume $A \subseteq V$ and $R \subseteq A^2$ are definable classes and are absolute between M and V. Assume R is set-like and such that* pred_R *is absolute between M and V. Assume further that M is transitive and is a model of* ZF − *power-set axiom.*

Then $R \cap M^2 = R^M$ is well-founded in V if and only if R^M is well-founded in M.

Proof. Assume $R^M = R \cap M^2$ is well-founded in V. Pick a non-empty $Z \in M$ such that $M \models Z \subseteq A^M$. Since $Z \in M$ and M is transitive, we have that $Z \subseteq M$ and since A is absolute for M, we have that $A^M = A \cap M$. In particular $V \models \emptyset \neq Z \subseteq A^M = A \cap M$. Since R^M is well-founded in V, Z has an R^M-minimal element a in V. Since $Z \subseteq M$, $a \in M$ and since $R^M = R \cap M^2$, we have that M

models that a is R^M-minimal for Z. Since this holds for all $Z \in M$, we conclude that M models that R^M is well-founded.

Conversely assume that $R^M = R \cap M^2$ is well-founded in M. Towards a contradiction assume $R \cap M^2$ is not well-founded in V. Then there exists a sequence $(a_n : n \in \omega) \in V$ such that $a_j \in M$ and $R(a_{j+1}, a_j)$ holds in V for all $j \in \omega$.

Now $M \models R^M$ is well-founded and M is a transitive model of $\mathsf{ZF}-$power-set axiom. Thus in M we can define $\mathrm{rk}_R^M : A^M \to \mathrm{Ord}^M$ such that

$$\mathrm{rk}_R^M(a) = \sup\{\mathrm{rk}_R^M(b) + 1 : b \in \mathrm{pred}_R^M(a)\}.$$

Observe that $R(a,b)$ entails that $\mathrm{rk}_R^M(a) \in \mathrm{rk}_R^M(b)$. Since $a_j \in M$ for all $j \in \omega$, we get that for all $j \in \omega$

$$M \models \mathrm{rk}_R^M(a_j) > \mathrm{rk}_R^M(a_{j+1}).$$

But $\mathrm{rk}_R^M(a_j)$ is an ordinal of M and thus really an ordinal in V, since M is transitive. This gives that $(\alpha_j = \mathrm{rk}_R^M(a_j) : j < \omega) \in V$ is a strictly decreasing sequence in $\mathrm{Ord}^M = \mathrm{Ord} \cap M$, which contradicts the validity of the Foundation axiom in V. $\square$

One can check that many notions which are defined using transifinite recursion over well-founded relation are absolute between transitive models of $\mathsf{ZF}-$power-set axiom.

Lemma 7.1.11. *Assume $A \subseteq V$ and $R \subseteq A^2$ are definable classes and are absolute between M and V. Assume R is set-like and such that* pred_R *is absolute between M and V. Assume further that M is transitive and is a model of* ZF *– power-set axiom.*

Assume R is well-founded in V and $F : A \times V \to V$ is a definable class-function which is absolute for M.

Then $G : A \to V$ given by $G(a) = F(a, G \restriction \mathrm{pred}_R(a))$ is absolute between M and V.

Proof. Assume for some $a \in M \cap A = A^M$ we have that $G^M(a) \neq G(a)$. Let a be R-minimal in V in the non-empty sub-class of $A^M = A \cap M$ definable in V as:

$$\{a \in M \cap A : (V, \in, =) \models \exists y \exists z (y \neq z) \wedge \phi_G(a, y, a_1^G, \ldots, a_{m_G}^G) \wedge \\ \wedge \phi_M(z, a_1^M, \ldots, a_{m_M}^M) \wedge \phi_G^M(a, z, a_1^G, \ldots, a_{m_G}^G)\}.$$

Then $G(a) = F(G \restriction a) = F(G^M \restriction a) = F^M(G^M \restriction a) = G^M(a)$, a contradiction. $\square$

Examples on how to employ the above Lemma are given by the following:

Lemma 7.1.12. *Let* $\mathrm{trcl} : V \to V$ *be the class function mapping a set a to its transitive closure* $\mathrm{trcl}(a)$*, i.e. the intersection of all transitive sets $b \supseteq a$. Then* trcl *is absolute for any M which is a transitive model of* ZF *– power-set axiom.*

Proof. It can be checked that $\mathrm{trcl}(a) = \bigcup\{\bigcup^n(x) : n \in \omega\}$ where $\bigcup^0(x) = x$ and $\bigcup^{n+1}(x) = \bigcup(\bigcup^n(x))$. We show that this definition of trcl can be given by applying the transfinite recursion theorem on the well founded order $(\omega, \in)$. Such an order relation is well-founded and absolute for transitive models of ZF – power-set axiom. To define $\mathrm{trcl}(a)$ consider the function $F(x, y, z)$ defined as follows:

$$\begin{cases} F(x,y,z) = \bigcup y(x-1, z) \text{ if } \emptyset \neq x \in \omega \text{ and } y \text{ is a function and } \mathrm{dom}(y) = x \times \{z\}, \\ F(x,y,z) = z \text{ otherwise.} \end{cases}$$

We leave to the reader to check that $F(x, y, z) = w$ can be defined by means of a Δ_0-formula.

Now apply the transfinite recursion theorem in M and in V to $(\omega, \in)$, to get that $G(n, a) = F(n, G \restriction n \times \{a\}, a)$ is absolute between M and V and $G(n, a) = \bigcup^n a$. Finally apply replacement in M and in V to get that $\mathrm{trcl}(a) = \bigcup G[\omega \times \{a\}]$. Check that the formula $\phi(z, y, w, t)$ stating that

$$\begin{aligned} \phi(z, y, w, t) \equiv &(t = \omega) \wedge (w \textit{ is a function}) \wedge (\mathrm{dom}(w) = t \times \{z\}) \wedge \\ \wedge &(y = \bigcup \mathrm{ran}(w)) \wedge (\forall n \in t(w(n, z) = F(n, w \restriction n \times \{z\}, z)) \end{aligned}$$

is expressible by a Δ_0-formula and that $\mathrm{trcl}(a) = b$ if and only if

$$V \models \forall w \forall y\, \phi(a, y, w, \omega) \rightarrow y = b$$

if and only if

$$V \models \exists w \phi(a, b, w, \omega).$$

Moreover this checking amounts to give a proof in ZF − power-set axiom of the formula:

$$\forall t \forall u \forall z(z = \omega) \rightarrow [(\forall w \forall y \phi(t, y, w, z) \rightarrow y = u) \leftrightarrow \exists w \phi(t, u, w, z)].$$

In particular $\exists w \phi(t, u, w, \omega) \equiv \forall w \forall y \phi(t, y, w, \omega) \rightarrow y = u$ is a provably Δ_1-property in models of ZF − power-set axiom. Thus

$$\exists w \phi(t, u, w, \omega) \equiv \forall w \forall y \phi(t, y, w, \omega) \rightarrow y = u$$

defines a property which is absolute for transitive models of ZF − power-set axiom by Lemma 7.1.8. □

We now give a second example in which the recursion is done on a more complex relation which one can prove that it is well-founded in M, and then argue that it remains well-founded in V using Lemma 7.1.10.

Lemma 7.1.13. *Let M be a transitive model of ZF − Power-set axiom and $R \in M$ be a well founded relation in M on some set $A \in M$. Then R is well-founded in V and the Mostowski collapsing map $\pi_R : A \to V$ given by $\pi_R(a) = \pi_R[\mathrm{pred}_R(a)]$ is absolute between M and V.*

Proof. Since R is a set, $R^M = R$ is set-like and such that the class-function $\mathrm{pred}_R = \mathrm{pred}_R^M \in M$ is absolute between M and V. Since M models that R is well-founded, V models that R^M is well-founded by Lemma 7.1.10. Now π_R is defined by induction on $R = R^M$ using the class function $F : A \times V \to V$ defined by $(a, g) \mapsto g[\mathrm{pred}_R(a)]$, since

$$\pi_R(a) = \pi_R[\mathrm{pred}_R(a)] = F(a, \pi_R).$$

It can be checked by means of the standard methods that $F(a, g) = c$ is definable by a formula $\phi_F(x, y, z)$ which is absolute for M. By Lemma 7.1.11, we get that $\pi_R = \pi_R^M$.

□

Lemma 7.1.14. *Let $\mathrm{rk} : V \to \mathrm{Ord}$ be the class function mapping a set a to its rank. Then rk is absolute for any M which is a transitive model of ZF − Power-set axiom.*

Proof. Left to the reader (hint: use the definition $\mathrm{rk}(a) = \sup\{\mathrm{rk}(b)+1 : b \in a\}$). □

7.2 Syntax and semantics inside V

We can code the syntax and the semantics of any first order language inside V in an absolute manner. We limit ourselves to describe how to code the syntax and the semantics of the language for ZFC with two binary relation predicates for $=$ and $\in$.

7.2.1 Syntax

- The set of natural numbers ω stands for the set of free variables $\{x_n : n \in \omega\}$ of the language.
- $\langle i, j, 0\rangle$ with $i, j \in \omega$ stands for the formula $x_i \in x_j$.
- $\langle i, j, 1\rangle$ with $i, j \in \omega$ stands for the formula $x_i = x_j$.
- Given formulae ϕ, ψ:
  - $\langle \phi, \psi, 0\rangle$ stands for the formula $\phi \vee \psi$.
  - $\langle \phi, \psi, 1\rangle$ stands for the formula $\phi \wedge \psi$.
  - $\langle \phi, \psi, 2\rangle$ stands for the formula $\neg\phi$.
  - $\langle \phi, \psi, 2n+3\rangle$ stands for the formula $\exists x_n \phi$.
  - $\langle \phi, \psi, 2n+4\rangle$ stands for the formula $\forall x_n \phi$.

Formally we define the set Form $\subseteq V_\omega$ by recursion on ω, letting $F : V^2 \times \omega \to V$ be defined by $F(x, y, w) = \langle x, y, w\rangle$. Now we can let

$$\mathrm{AtForm} = \{\langle i, j, k\rangle : i \in \omega \wedge j \in \omega \wedge k \in 2\}$$

represent in V the set of atomic formulae and

$$\mathrm{Form} = \bigcap \{Z \subseteq V_\omega : \mathrm{AtForm} \subseteq Z \wedge \forall x \in Z\, \forall y \in Z\, \forall w \in \omega\, F(z, y, w) \in Z\}$$

represent in V the set of formulae.

Clearly AtForm is absolute for transitive models of $\mathsf{ZF} - \{\text{Power-set axioms}\}$ and by the same methods by which one can prove that ω is absolute for transitive models of $\mathsf{ZF} - \{\text{Power-set axioms}\}$, one can also prove that Form is absolute for transitive models of $\mathsf{ZF} - \{\text{Power-set axioms}\}$.

Moreover the functions:

- Subform : Form $\to [\mathrm{Form}]^{<\omega}$ recognizing which are the proper subformulae of a formula,
- Freevar : Form $\to [\omega]^{<\omega}$ recognizing which are the free variables of a formula

can also be shown to be absolute for transitive models of $\mathsf{ZF} - \{\text{Power-set axioms}\}$. We leave the details to the reader.

7.2.2 Semantics

As of now we have just defined certain subsets of V_ω which are absolute for transitive models of $\mathsf{ZF} - \{\text{Power-set axioms}\}$. In order to show that they can really represent the concept of formula as the extension of a set in V, we need to define a semantics inside V which given a Tarski structure $(M, \in, =) \in V$ shows that our definition in the meta-language of $(M, \in, =) \models \phi(a_1, \dots, a_n)$ given according to Tarski truth rules can be described as a definable property in V of the triple $(M, \overline{\phi}, \langle a_1, \dots, a_n \rangle)$ where $\overline{\phi}$ is the triple $\langle z, y, j \rangle$ in Form which codes the formula ϕ as an element of Form.

Definition 7.2.1. The satisfaction predicate

$$\text{Sat} : V \times \text{Form} \times V^{<\omega} \to 3$$

(where 0 stands for false, 1 for true, 2 for meaningless) is defined by the following rules:

$$\begin{cases} \text{Sat}(Z, \overline{\phi}, \vec{a}) = 2 \text{ if } \text{Freevar}(\overline{\phi}) \not\subseteq \text{dom}(\vec{a}) \\ \text{(i.e. } \vec{a} \text{ does not give an assignment to some of the free variables of } \phi), \\ \text{or if } \vec{a} = \langle a_1, \dots, a_n \rangle \notin Z^{<\omega}, \text{ otherwise:} \\ \\ \text{Sat}(Z, \overline{x_i \in x_j}, \langle a_1, \dots, a_n \rangle) = 1 \text{ if } a_i \in a_j \text{ and } 0 \text{ otherwise },\\ \text{Sat}(Z, \overline{x_i = x_j}, \langle a_1, \dots, a_n \rangle) = 1 \text{ if } a_i = a_j \text{ and } 0 \text{ otherwise },\\ \\ \text{Sat}(Z, \overline{\psi \wedge \phi}, \langle a_1, \dots, a_n \rangle) = \text{Sat}(Z, \overline{\psi}, \langle a_1, \dots, a_n \rangle) \cdot \text{Sat}(Z, \overline{\phi}, \langle a_1, \dots, a_n \rangle), \\ \text{Sat}(Z, \overline{\psi \vee \phi}, \langle a_1, \dots, a_n \rangle) = \max\{\text{Sat}(Z, \overline{\psi}, \langle a_1, \dots, a_n \rangle), \text{Sat}(Z, \overline{\phi}, \langle a_1, \dots, a_n \rangle)\}, \\ \text{Sat}(Z, \overline{\neg\psi}, \langle a_1, \dots, a_n \rangle) = 1 - \text{Sat}(Z, \overline{\psi}, \langle a_1, \dots, a_n \rangle), \\ \\ \text{Sat}(Z, \overline{\exists x_{i_j} \psi(x_{i_1}, \dots, x_{i_k})}, \langle a_1, \dots, a_n \rangle) = \\ = \sup\{\text{Sat}(Z, \overline{\psi(x_{i_1}, \dots, x_{i_k})}, \langle a_1, \dots a_{i_{j-1}}, a, a_{i_{j+1}}, \dots, a_n \rangle) : a \in Z\}, \\ \text{Sat}(Z, \overline{\forall x_{i_j} \psi(x_{i_1}, \dots, x_{i_k})}, \langle a_1, \dots, a_n \rangle) = \\ = \inf\{\text{Sat}(Z, \overline{\psi(x_{i_1}, \dots, x_{i_k})}, \langle a_1, \dots a_{i_{j-1}}, a, a_{i_{j+1}}, \dots, a_n \rangle) : a \in Z\}, \end{cases}$$

Lemma 7.2.2. *Sat* : $V^3 \to 2$ *is a definable class-function which is absolute for transitive models* M *of* ZF − *power-set axiom with* $Z \in M$.

Proof. We leave to the reader to check this property of Sat by means of the methods developed in the first section of this chapter. □

Moreover:

Lemma 7.2.3. *For any formula* $\phi(x_1, \dots, x_n)$ *and any* $Z \in V$ *and* $(a_1, \dots, a_n) \in Z^{<\omega}$

$$(Z, \in, =) \models \phi(a_1, \dots, a_n)$$

if and only if

$$(V, \in, =) \models \mathit{Sat}(Z, \overline{\phi}, (a_1, \dots, a_n))$$

Proof. The proof is a straightforward induction on the complexity of ϕ and is left to the reader. $\square$

Lemma 7.2.4 (Downward Lowenheim-Skolem Theorem)**.** *Assume $X \subseteq Z$ are sets in V. Then there is a set $W \in V$ with $X \subseteq W \subseteq Z$, such that $|W| = |X| + \aleph_0$ and*

$$V \models \forall \vec{a} \in W^{<\omega} \, \forall \overline{\phi} \in Form \, [Sat(Z, \overline{\phi}, \vec{a}) = Sat(W, \overline{\phi}, \vec{a})]$$

Proof. Since V is a model of ZFC we can run inside V the proof of the Downward Lowenheim-Skolem Theorem where we replace the notion of formula by elements of Form and the notion of Tarski truth is interpreted by means of the class function Sat. $\square$

7.3 Getting countable transitive models of ZFC and Levy absoluteness

7.3.1 Getting countable transitive models of ZFC

We use the previous results to argue the following:

Lemma 7.3.1. *Assume there is a strongly inaccessible cardinal $\kappa \in V$. Then there is a countable transitive $M \in V$ which is a model of ZFC.*

Proof. By [7, Theorem IV.6.6], $V_\kappa \models$ ZFC. Apply the previous Lemma to $X = \emptyset$ to get some countable W such that

$$V \models \forall \vec{a} \in W^{<\omega} \, \forall \overline{\phi} \in \mathrm{Form} \, [\mathrm{Sat}(V_\kappa, \overline{\phi}, \vec{a}) = \mathrm{Sat}(W, \overline{\phi}, \vec{a})].$$

Then $\in \cap W^2$ is extensional and well founded since $(W, \in, =)$ models the axiom of extensionality, being a model of ZFC.

This gives that the Mostowski collapsing map $\pi_W : W \to V$ of the well-founded relation $\in \cap W^2$ on W^2 is an isomorphism with its image $M = \pi_W[W]$ and that $M \in V$ is transitive and countable, being the image of the set $W \in V$. Since $\pi_W \in V$ we get that

$$V \models \pi_W : W \to M \text{ is an isomorphism of } (W, \in, =) \text{ with } (M, \in, =).$$

This gives that for all $\overline{\phi} \in$ Form and $\langle a_1, \ldots, a_n \rangle \in W^{<\omega}$

$$V \models \mathrm{Sat}(W, \overline{\phi}, \langle a_1, \ldots, a_n \rangle) = \mathrm{Sat}(M, \overline{\phi}, \langle \pi_W(a_1), \ldots, \pi_W(a_n) \rangle).$$

But this gives that $(M, \in, =)$ is a model of ZFC since for all axioms ϕ of ZFC

$$(V, \in, =) \models \mathrm{Sat}(M, \overline{\phi}, \emptyset)$$

and by Lemma 7.2.3 this occurs only if

$$(M, \in, =) \models \phi.$$

$\square$

Da rivedere – M

7.3.2 Levy absoluteness

– *M*

We introduce the basic signatures and fragments of set theory we will always include in any signature of interest to us.

Notation 7.3.2. We let $\in_{\Delta_0}$ be $\in_D$ for $D \subseteq \mathsf{Form}_\in \times 2$ extending the set $\Delta_0 \times \{0\}$ with the pairs $(\phi, 1)$ as ϕ ranges over the following Δ_0-formulae:

- The Δ_0-formulae $\phi_\omega(x)$, $\phi_\emptyset(x)$ defining $\emptyset$ and ω in any model of ZF^-, where the latter includes all axioms of ZF with the exception of power-set axiom (also we denote by ω and $\emptyset$ the constants $f_{\phi_\emptyset}$, f_{ϕ_ω}).
- The Δ_0-formulae $\phi_i(\vec{x}, y)$ as G_i ranges over the Goedel operations $G_1, \ldots, G_{10}$ as defined in [**?**, Def. 13.6] and $\phi_i(\vec{x}, y)$ is the Δ_0-formula defining the graph of G_i in any $\in$-model of [1] ZF^-.

We let T_{Δ_0} be given by the axioms:

$$\forall \vec{x}\,(R_{\forall z \in y \phi}(y, z, \vec{x}) \leftrightarrow \forall z(z \in y \to R_\phi(y, z, \vec{x})), \tag{7.1}$$

$$\forall \vec{x}\,[R_{\phi \wedge \psi}(\vec{x}) \leftrightarrow (R_\phi(\vec{x}) \wedge R_\psi(\vec{x}))], \tag{7.2}$$

$$\forall \vec{x}\,[R_{\neg\phi}(\vec{x}) \leftrightarrow \neg R_\phi(\vec{x})] \tag{7.3}$$

$$\forall x\,(x \notin \emptyset) \tag{7.4}$$

$$\omega \textit{ is a non-empty ordinal all whose elements are successor ordinals or } \emptyset. \tag{7.5}$$

$$\forall \vec{x}\,\exists! y\,(y = G_i(\vec{x})) \tag{7.6}$$

$$\forall \vec{x}\,\forall y\,[y = G_i(\vec{x}) \leftrightarrow R_{\phi_i}(\vec{x}, y)] \tag{7.7}$$

for the Goedel operations $G_1, \ldots, G_{10}$.

We axiomatize suitable fragments of the $\in$-theory $\mathsf{ZFC} + T_{\Delta_0}$ as follows:

- $\mathsf{Z}^-_{\Delta_0}$ stands for the $\in_{\Delta_0}$-theory given by:
  (a) the Extensionality Axiom
  $$\forall x, y, z\;[(z \in x \leftrightarrow z \in y) \to x = y]\,,$$
  (b) the Foundation Axiom
  $$\forall x\,[x = \emptyset \vee \exists y \in x\,\forall z \in x\,(z \notin y)]\,,$$
  (c) T_{Δ_0}.
- Z_{Δ_0} enriches $\mathsf{Z}^-_{\Delta_0}$ adding the power-set axiom
  $$\forall x \exists y\;[\forall z\,(z \subseteq x \leftrightarrow z \in y]\,.$$

[1]In models of ZF^- the Goedel operations $G_1, \ldots, G_{10}$ as listed and defined in [**?**, Def. 13.6] and their compositions have as graph the extension of a Δ_0-formula (by [**?**, Lemma 13.7]).

- $\mathsf{ZC}^-_{\Delta_0}$ enriches $\mathsf{Z}^-_{\Delta_0}$ adding the axiom of choice AC

$$\forall x \exists f\ [(f \text{ is a bijection}) \wedge \operatorname{dom}(f) = x \wedge (\operatorname{ran}(f) \text{ is an ordinal})].$$

- $\mathsf{ZF}^-_{\Delta_0}$ enriches $\mathsf{Z}^-_{\Delta_0}$ adding the replacement axiom for all $\in_{\Delta_0}$-formulae.

- $\mathsf{ZFC}^-_{\Delta_0}$, ZF_{Δ_0}, ZFC_{Δ_0} are defined as expected.

Remark 7.3.3. We took the pain of giving an explicit axiomatization of $\mathsf{Z}^-_{\Delta_0}$ using Extensionality, Foundation, and axioms 7.1,...,7.7 because this axiomatizion is given by Π_2-sentences of $\in_{\Delta_0}$, hence it is preserved by Σ_1-substructures. Note that AC is a Π_2-axiom of $\in_{\Delta_0}$ while the power-set axiom and the replacement schema for a quantifier free $\in_{\Delta_0}$-formula are both Π_3.

A simple inductive argument shows that $\mathsf{ZF}^- + T_{\in,D}$ (where D is the subset of $\mathsf{Form}_\in \times 2$ used in Not. 7.3.2 to define $\in_{\Delta_0}$) is logically equivalent to ZF^- enriched with axioms 7.1,...,7.7 (with $\emptyset$ taking the place of $c_\in$ and ω being the constant of $\in_{\Delta_0}$ associated to the Δ_0-formula defining it). We skip the details.

We now introduce the terminology to handle set theory formalized in signatures richer than $\in_{\Delta_0}$.

Notation 7.3.4. Let $\tau \supseteq \in_{\Delta_0}$. For a τ-formula $\phi(\vec{x}, \vec{y}, \vec{z})$:

- The *Replacement Axiom* for ϕ ($\mathsf{Rep}(\phi)$) states:

  $$\forall \vec{z} \forall X\ [(\forall x \in X \exists! y\, \phi(x, y, \vec{z})) \rightarrow \exists F\, (F \text{ is a function} \wedge \operatorname{dom}(F) = X \wedge \forall x \in X\, \phi(x, F(x), \vec{z}))];$$

  Rep_τ holds if $\mathsf{Rep}(\phi)$ holds for all τ-formulae ϕ.

- ZF^-_τ is $\mathsf{Z}^-_{\Delta_0} + \mathsf{Rep}_\tau$.

- Accordingly we define ZFC_τ, ZFC^-_τ, ZF_τ, ZFC_τ,...

- We write ZFC_{Δ_0} rather than ZFC_τ when $\tau = \in_{\Delta_0}$, etc.

- If $A \subseteq \mathsf{Form}_\in \times 2$ is such that $\in_{\Delta_0} \subseteq \in_A$, we write ZFC^-_A rather than $\mathsf{ZFC}^- + T_{\in,A}$,...

Clearly (the suitable fragment of) $\mathsf{ZFC} + T_{\in,A}$ is logically equivalent to (the suitable fragment of) ZFC_A.

We state and prove the Lemma under the assumption that the model of ZFC we work in is transitive; but this assumption is unnecessary. Here and in other places of this paper we just need that the models in question satisfy ZFC^- or slightly more.

Lemma 7.3.5. *Let $(V, \in_{\Delta_0})$ be a model of ZFC_{Δ_0} and $\lambda > \kappa$ be infinite cardinals for V with λ regular. Assume $\phi_1(\vec{x}_1), \ldots, \phi_k(\vec{x}_k), \psi_1(\vec{x}_1, y), \ldots, \psi_n(\vec{x}_n, y)$ are $\in$-formulae which are in $\Delta_1(\mathsf{ZFC}^-)$ and*

$$\mathsf{ZFC}^- \models \forall \vec{x} \exists! y\, \psi_i(\vec{x}_i, y)$$

for $i = 1, \dots, n$. *Then the structure*

$$(H_\lambda, \in_{\Delta_0}^{H_\lambda}, R_{\phi_j}^{H_\lambda} : j = 1, \dots, k,\ f_{\phi_l}^{H_\lambda} : l = 1, \dots, n,\ A : A \subseteq \text{pow}(\kappa)^k,\ k \in \mathbb{N})$$

is Σ_1*-elementary in*

$$(V, \in_{\Delta_0}^{V}, R_{\phi_j}^{V} : j = 1, \dots, k,\ f_{\psi_l}^{V} : l = 1, \dots, n,\ A : A \subseteq \text{pow}(\kappa)^k,\ k \in \mathbb{N}),$$

where R_{ϕ_j} *and* f_{ψ_l} *are interpreted by means of axioms* $\mathsf{Ax}^0_{\phi_j}$ *and* $\mathsf{Ax}^1_{\psi_l}$ *for* $j = 1, \dots, k$, $l = 1, \dots, n$ *in both structures.*

Its proof is a variant of the classical result of Levy (which is the above theorem stated just for the signature $\in_{\Delta_0}$); it is a slight expansion of [**?**, Lemma 5.3]; we include it here since it is not literally the same:

Proof. Let τ be the signature $\in_{\Delta_0} \cup \{R_{\phi_j} : j = 1, \dots, k\} \cup \{f_{\psi_l} : l = 1, \dots, n\}$, $\phi(\vec{x}, y)$ be a quantifier free formula for the signature under consideration where only predicates $A_1, \dots, A_k$ appears[2], and $\vec{a} \in H_\lambda$ be such that

$$(V, \tau^V, A_1, \dots, A_k) \models \exists y \phi(\vec{a}, y).$$

Let $\alpha > \kappa$ be large enough so that for some $b \in V_\alpha$

$$(V, \tau^V, A_1, \dots, A_k) \models \phi(\vec{a}, b).$$

Then

$$(V_\alpha, \tau^{V_\alpha}, A_1, \dots, A_k) \models \phi(\vec{a}, b)$$

(since $(V_\alpha, \tau^{V_\alpha}, A_1, \dots, A_k) \sqsubseteq (V, \tau^V, A_1, \dots, A_k)$ by Fact **??**). By the downward Lowenheim-Skolem theorem, we can find $X \subseteq V_\alpha$ which is the domain of a $\tau \cup \{A_1, \dots, A_k\}$-elementary substructure of

$$(V_\alpha, \tau^{V_\alpha}, A_1, \dots, A_k)$$

such that X is a set of size κ containing κ and such that $A_1, \dots, A_k, \kappa, b, \vec{a} \in X$. Since $|X| = \kappa \subseteq X$, a standard argument shows that $H_\lambda \cap X$ is a transitive set, and that κ^+ is the least ordinal in X which is not contained in X. Let M be the transitive collapse of X via the Mostowski collapsing map π_X.

We have that the first ordinal moved by π_X is κ^+ and π_X is the identity on $H_{\kappa^+} \cap X$. Therefore $\pi_X(a) = a$ for all $a \in H_{\kappa^+} \cap X$. Moreover for $A \subseteq \text{pow}(\kappa)^n$ in X

$$\pi_X(A) = A \cap M. \tag{7.8}$$

We prove equation (7.8):

Proof. Since $X \cap V_{\kappa+1} \subseteq X \cap H_{\kappa^+}$, π_X is the identity on $X \cap H_{\kappa^+}$, and $A \subseteq \text{pow}(\kappa) \subseteq V_{\kappa+1}$, we get that

$$\pi_X(A) = \pi_X[A \cap X] = \pi_X[A \cap X \cap V_{\kappa+1}] = A \cap M \cap V_{\kappa+1} = A \cap M.$$

□

[2] Note that $\exists x \in y A(y)$ is not a quantifier free formula, and is actually equivalent to the Σ_1-formula $\exists x (x \in y) \wedge A(y)$.

It suffices now to show that

$$(M, \tau^M, \pi_X(A_1), \dots, \pi_X(A_k)) \sqsubseteq (H_\lambda, \tau^{H_\lambda}, A_1, \dots, A_k). \tag{7.9}$$

Assume 7.9 holds; since π_X is an isomorphism and $\pi_X(A_j) = \pi_X[A_j \cap X]$, we get that

$$(M, \tau^M, \pi_X(A_1), \dots, \pi_X(A_k)) \models \phi(\pi_X(b), \vec{a})$$

since

$$(X, \tau^V, A_1 \cap X, \dots, A_k \cap X) \models \phi(b, \vec{a}).$$

By (7.9) we get that

$$(H_\lambda, \tau^{H_\lambda}, A_1, \dots, A_k) \models \phi(\pi_X(b), \vec{a})$$

and we are done.

We prove (7.9):

Proof. since $(M, \in)$ is a transitive model of ZFC^- with $M \subseteq H_\lambda$, any atomic τ-formula holds true in (M, τ^M) if and only if it holds in $(H_\lambda, \tau^{H_\lambda})$ (again by Fact **??**). It remains to argue that the same occurs for the formulae of type $A_j(x)$, i.e. that $A_j \cap M = \pi_X(A_j)$ for all $j = 1, \dots, n$; which is the case by (7.8). □

□

Chapter 8

Appendix B: Orders and topology

8.0.1 Topological spaces

A *topology* on a given set X is a family $\tau \subseteq \mathcal{P}(X)$ with $\emptyset, X \in \tau$ which is closed under arbitrary unions and finite intersections. We call the pair (X, τ) a *topological space.*

The elements of τ are the *open sets* for the topology τ. Complements of open sets are called *closed sets*, we denote by τ^c the family of closed sets (the family of closed sets of a topological space is closed under arbitrary intersections and finite unions). When a set A is both open and closed, we call it a *clopen set* of τ and we denote this family by $\mathsf{CLOP}(X, \tau)$ (or just $\mathsf{CLOP}(X)$ if τ is clear from the context).

A *basis* σ for a topological space (X, τ) is a subfamily of τ with the property that every open set in τ can be written as an union of elements of σ. We say that τ is *generated* by σ. Notice that if σ is a basis for τ any intersection of finitely many elements of σ contains an element of σ (i.e. $\sigma \setminus \{\emptyset\}$ is a prefilter on $\mathcal{P}(X)$).

A *semibasis* σ for a topological space (X, τ) is a subfamily of τ with the property that the set of finite intersections of elements of σ is a basis. σ is a semibasis for (X, τ) if and only if τ is the weakest (i.e. smallest) topology on X containing σ. If σ is a semibasis for τ, we say that τ is generated by σ.

We say that $U \subset X$ is a *neighborhood* of some $x \in X$ if $x \in U$.

A *Hausdorff* space (X, τ) is a topological space (X, τ) in which any two distinct points x and y can be *separated* by two open sets U and V in τ, that is x is in U, y is in V and U and V are disjoint. Recall that in a Hausdorff space X points are closed (i.e. $\{x\}$ is closed for all $x \in X$).

We say that (X, τ) is 0-dimensional if τ admits a basis of clopen sets.

$x \in X$ is an *isolated* point if $\{x\}$ is open and closed.

Given a topological space (X, τ) and an arbitrary subset A of X, we denote by $\mathsf{Cl}(A)$ (the *closure* of A) the smallest closed set containing A. We denote by $\mathsf{Int}(A)$ (the *interior* of A) the biggest open set that is contained in A. An open set A is *regular open* if $A = \mathsf{Int}(\mathsf{Cl}(A))$. For any $A \subseteq X$ $\mathsf{Reg}(A) = \mathsf{Int}(\mathsf{Cl}(A))$ denotes the *regularization* of the set A.

Example 8.0.1. Let τ be the euclidean topology on $\mathbb{R}$; then any interval is a regular open set.

If $a < b < c$, we have that $(a;b)$, $(b;c)$ are regular open while $(a;b) \cup (b;c)$ is not with its regularization being $(a;c)$.

In general regular open sets are those open sets which can be written in the form $\bigcup_{j \in J}(a_j;b_j)$ with the family $\{(a_j;b_j) : j \in J\}$ consisting of pairwise disjoint open intervals such that $a_i \neq b_j$ for any $i, j \in J$.

Given $B \subseteq A$, B is *dense* in A if $\mathsf{Cl}(B) = \mathsf{Cl}(A)$. Remark that if B is dense in A and $C \subseteq A$ is open, then $B \cap C$ is dense in C.

A map $f : X \to Y$ between topological spaces (X, τ) and (Y, σ) is *continuous* if the preimage by f of any open set of Y is open, *open* if the (direct) image of an open set of X is open in Y, a *homeomorphism* if it is an open and continuous bijection.

Given a topological space (X, τ) and $Y \subseteq X$ the restriction $\tau \restriction Y$ of τ to Y is given by the family $\{A \cap Y : A \in \tau\}$ and is a topology on Y.

Product topologies

Let I be a set of indexes and for all $i \in I$, let (X_i, τ_i) be a topological space and $X = \prod_{i \in I} X_i$ be the cartesian product of the sets X_i. The product topology τ on X is the weakest topology making all the projections maps $\pi_i : f \mapsto f(i)$ continuous. It is generated by the family of sets of the form $\prod_{i \in I} A_i$, where each A_i is open in X_i and $A_i \neq X_i$ only for finitely many i.

Compactness

A topological space (X, τ) is compact if any of the following equivalent conditions are met:

- every family $\mathcal{F}$ of closed sets with the finite intersection property[1] has a non-empty intersection.
- Every open covering of X has a finite subcovering.

We emphasize the following two statements:

- *We focus on either Hausdorff compact spaces or on order topologies.*
- *We often interplay between the topological notion of density and the notion of dense subset of a partial order.*

8.1 Stone-Cech compactifications

Recall that the compactification of a space (X, τ) is a compact space (K, σ) together with a topological embedding $i : X \to K$ (i.e. a continuous injective map such that (X, τ) is homeomorphic to $(i[X], \sigma_{\restriction i[X]})$).

The aim of this section is to characterize the Hausdorff spaces which admit at least one compactification. These are the Tychonoff spaces. We will show that for these spaces it is always possible to build the largest possible compactification.

[1] $\mathcal{F}$ has the finite intersection property if any finite subfamily of $\mathcal{F}$ has a non-empty intersection. A family $\mathcal{A}$ of subsets of X such that $\bigcup \mathcal{A} = X$ is a covering of X.

Definition 8.1.1. Let (X, τ) be a topological space.
$C \subseteq X$ is a 0-set if there exists $f : X \to [0;1]$ continuous such that $f^{-1}[0] = C$.

Notation 8.1.2. Let (X, τ) be a topological space. τ^0 denotes the family of 0-sets of (X, τ).

Remark that clopen sets are 0-sets (as witnessed by the characteristic function of their complement) and 0-sets are closed. The basic geometric picture captured by these definitions is that 0-sets are those closed sets which can be approximated from above continuously and with great precision by open supersets. On the other hand in general closed sets may not be 0-sets.

Definition 8.1.3. A space (X, τ) is Tychonoff if singletons of points are closed sets and for all $x \in X$ and C closed with $x \notin C$, we can find $f : X \to [0;1]$ continuous and such that $f(x) = 0$, $f \restriction C = 1$.

The following is fundamental in the arguments to follow:

Proposition 8.1.4. *Let (X, τ) be a topological space and C_0, C_1 be closed subsets of X.*

1. *Assume C_0, C_1 are 0-sets. Then $C_0 \cap C_1$ and $C_0 \cup C_1$ are also 0-set.*
2. *Assume C_0, C_1 are disjoint 0-sets. Then there are open sets $V_i \supseteq C_i$ for $i = 0, 1$ with disjoint closures, and which are the complement of 0-sets.*

Proof. Let f_i witness that C_i is a 0-set for $i = 0, 1$. Then:

1. $h = \frac{f_1+f_0}{2}$ and $k = f_1 \cdot f_0$ are continuous and witness that $C_0 \cap C_1$ and $C_1 \cup C_2$ are 0-sets.
2. $g = \frac{f_0}{f_0+f_1}$ is continous and such that $g^{-1}[\{i\}] = C_i$. $V_0 = g^{-1}[[0;1/3)]$ $V_1 = g^{-1}[(2/3;1]]$ are the complements of 0-sets (as witnessed by $g_0(x) = 1 - \min\{1, 3g(x)\}$ for V_0 and $g_1(x) = \max\{0, 3g(x) - 2\}$ for V_1 which are both continuous) such that $C_i \subseteq V_i$ and $\mathsf{Cl}(V_0) \cap \mathsf{Cl}(V_1)$ is empty.

□

The notion of 0-set has been introduced to get the separation property given by the second item above: in general for an Hausdorff topological space it is not true that disjoint closed sets can be separated by disjoint open sets, on the other hand for disjoint 0-sets this is always possible. This separation property of 0-sets will be used to define βX and to prove that it is Hausdorff.

Definition 8.1.5. (X, τ) is normal if for every pair of closed disjoint sets C_0, C_1 there is $f : X \to [0, 1]$ continuous such that $f^{-1}[\{i\}] \supseteq C_i$ for $i = 0, 1$.

Lemma 8.1.6 (Urysohn Lemma)**.** (X, τ) *is normal if and only if any two closed disjoint sets can be separated by disjoint open sets.*

Proof. See [10, Thm. 1.5.6] □

Exercise 8.1.7. Assume (X,τ) is normal (i.e. any two closed disjoint sets can be separated by open disjoint sets). Then any closed set in X is a 0-set.

The outcome is that for spaces which are not normal, the 0-sets define a large collection of closed sets Γ which is closed under finite unions and intersections, contains the clopen sets, and satysfies the property that any two disjoint sets in Γ can be separated by disjoint open sets whose complement are in Γ. As we will see below, these are the key properties one needs to prove that the space of maximal filters on Γ is Hausdorff and compact.

Fact 8.1.8. *Locally compact Hausdorff spaces are Tychonoff.*

Proof. Left to the reader. $\square$

Definition 8.1.9. Given a topological space (X,τ), βX is the family of maximal filters of 0-sets in the partial order $(\tau^0 \setminus \{\emptyset\}, \supseteq)$.

The following is a fundamental easy outcome of Proposition 8.1.4.

Proposition 8.1.10. *Let (X,τ) be a topological space.*

- *$C, D \in \tau^0$ are compatible in the partial order $(\tau^0 \setminus \{\emptyset\}, \supseteq)$ if and only if $C \cap D$ is non-empty.*
- *If $\mathcal{F}$ is a filter in the partial order $(\tau^0 \setminus \{\emptyset\}, \supseteq)$ and $C_1, \dots, C_n \in \mathcal{F}$, $\bigcap_{i=1}^n C_i \in \mathcal{F}$, hence is non-empty.*
- *If $\mathcal{F}$ is a maximal filter in the partial order $(\tau^0 \setminus \{\emptyset\}, \supseteq)$, $C \in \mathcal{F}$, and $C = C_1 \cup \dots \cup C_n$ with each C_i a 0-set, at least one C_i is in $\mathcal{F}$.*
- *Assume X is normal. Then βX coincides with the family of maximal filters of closed sets.*
- *Assume X is Tychonoff. Then for all $x \in X$ the set*

$$\mathcal{F}_x = \{F : \ F \subseteq X \text{ is a 0-set with } x \in F\}$$

  is a non-empty maximal filter in $(\tau^N \setminus \{\emptyset\}, \supseteq)$.

Proof. We prove just the last assertion: Assume $C \cap D \neq \emptyset$ for all $D \in \mathcal{F}_x$ but $x \notin C$. Then there exists $f : X \to [0;1]$ continuous such that $f(x) = 0$ and $f \restriction C = 1$. Hence $C \cap f^{-1}[\{0\}] = \emptyset$, but $f^{-1}[\{0\}] \in \mathcal{F}_x$ since $x \in f^{-1}[\{0\}]$, a contradiction. $\square$

Definition 8.1.11. Given a topological space (X,τ) and $Y \subseteq X$

$$\beta^c Y = \{\mathcal{F} \in \beta X : \exists C \in \mathcal{F}\,(C \subseteq Y)\}.$$

$$\beta^o Y = \{\mathcal{F} \in \beta X : \forall C \in \mathcal{F}\,(C \cap Y \neq \emptyset)\}.$$

Proposition 8.1.12. *Assume (X,τ) is a Topological space. Then for all $Y_1, \dots, Y_n \subseteq X$ the following holds:*

1. *For any 0-set E $\beta^c E = \emptyset$ if and only if $E = \emptyset$,*

2. $\beta^c Y_1 \subseteq \beta^o Y_1$,
3. $\beta^o Y_1 = \beta X \setminus \beta^c(X \setminus Y_1)$,
4. $\beta^c Y_1 = \beta X \setminus \beta^o(X \setminus Y_1)$,
5. $\beta^c Y_1 \cap \cdots \cap \beta^c Y_n = \beta^c(Y_1 \cap \cdots \cap Y_n)$,
6. $\beta^o Y_1 \cup \cdots \cup \beta^o Y_n = \beta^o(Y_1 \cup \cdots \cup Y_n)$,
7. *If E is a* 0*-set, $\beta^o E = \beta^c E$. Hence $\mathcal{F} \notin \beta^c E$ if and only if some $D \in \mathcal{F}$ is disjoint from E, and $\beta^o(X \setminus E) = \beta^c(X \setminus E)$.*

Proof.

1. Given $E \in \tau^0$ and non-empty, extend $\{E\}$ to a maximal filter.
2. Trivial by definition.
3. Unravelling the definitions

$$\begin{aligned}\beta X \setminus \beta^c(X \setminus Y_1) &= \beta X \setminus \{\mathcal{F} : \exists C \in \mathcal{F}\, C \subseteq (X \setminus Y_1)\} = \\ = \{\mathcal{F} : \forall C \in \mathcal{F}\, C \not\subseteq (X \setminus Y_1)\} &= \{\mathcal{F} : \forall C \in \mathcal{F}\, C \cap Y_1 \neq \emptyset\} = \beta^o(Y_1).\end{aligned}$$

4. Again unravelling the definitions

$$\begin{aligned}\beta X \setminus \beta^o(X \setminus Y_1) &= \beta X \setminus \{\mathcal{F} : \forall C \in \mathcal{F}\, C \cap (X \setminus Y_1) \neq \emptyset\} = \\ = \{\mathcal{F} : \exists C \in \mathcal{F}\, C \cap (X \setminus Y_1) = \emptyset\} &= \{\mathcal{F} : \exists C \in \mathcal{F}\, C \subseteq Y_1\} = \beta^c(Y_1).\end{aligned}$$

5. $\mathcal{F} \in \beta^c Y_1 \cap \cdots \cap \beta^c Y_n$ if and only if there are $C_i \in \mathcal{F}$ such that $C_i \subseteq U_i$ for all $i = 1, \ldots, n$, which gives that $C = \bigcap_{i=1}^n C_i \subseteq Y_1 \cap \cdots \cap Y_n$ is a 0-set in $\mathcal{F}$. We conclude that $\beta^c Y_1 \cap \cdots \cap \beta^c Y_n \subseteq \beta^c(Y_1 \cap \cdots \cap Y_n)$. The converse inclusion is trivial.
6. By the previous items:

$$\begin{aligned}\beta X \setminus (\beta^o Y_1 \cup \cdots \cup \beta^o Y_n) &= (\beta X \setminus \beta^o Y_1) \cap \cdots \cap (\beta X \setminus \beta^o Y_n) = \\ &= \beta^c(X \setminus Y_1) \cap \cdots \cap \beta^c(X \setminus Y_n) = \\ &= \beta^c((X \setminus Y_1) \cap \cdots \cap (X \setminus Y_n)) = \\ &= \beta^c(X \setminus (Y_1 \cup \cdots \cup Y_n)) = \\ &= \beta X \setminus \beta^o(Y_1 \cup \cdots \cup Y_n).\end{aligned}$$

   Hence the thesis.
7. Assume $\mathcal{F} \in \beta^o C$. Observe that

$$\mathcal{G} = \left\{E \in \tau^N : \exists D \in \mathcal{F}\, E \supseteq D \cap C\right\}$$

   is a filter on $(\tau^0 \setminus \{\emptyset\}, \supseteq)$ containing $\mathcal{F} \cup \{C\}$. By maximality of $\mathcal{F}$, $C \in \mathcal{F}$, hence $\mathcal{F} \in \beta^c C$.

□

Definition 8.1.13. Given a topological space (X, τ) We let β_τ be the topology on βX generated by the family

$$\left\{\beta^o U : X \setminus U \in \tau^0\right\}$$

(i.e. β_τ is the weakest topology containing all $\beta^o U$ with $X \setminus U \in \tau^0$).

By the previous propositions $\emptyset = \beta^o\emptyset$, $\beta X = \beta^o X$, $\beta^o U = \beta^c U$ for all $X \setminus U \in \tau^0$ and

$$\beta^o(U_1 \cap \cdots \cap U_n) = \beta^c(U_1 \cap \cdots \cap U_n) = \beta^c U_1 \cap \cdots \cap \beta^c U_n = \beta^o U_1 \cap \cdots \cap \beta^o U_n.$$

Therefore:

Fact 8.1.14. *Let (X, τ) be a topological space. Then $\{\beta^o U : X \setminus U \in \tau^0\}$ is a base for β_τ and any closed set for β_τ is the intersection of a family of basic closed sets of the form $\beta^c E$ with $E \in \tau^0$.*

We are ready to prove the main properties of the Stone-Cech compactification of a topological space.

Theorem 8.1.15. *Assume (X, τ) is a topological space. Then $(\beta X, \beta_\tau)$ is a compact Hausdorff space. Moreover assume (X, τ) is a Tychonoff space, then:*

- *The map*
$$i_X : X \mapsto \mathcal{F}_x = \left\{C \in \tau^0 : x \in C\right\}$$
is a topological embedding.

- *Any continuous $f : X \to K$ with K compact Hausdorff admits a unique continuous extension to a $\beta f : \beta X \to K$ such that $f = \beta f \circ i_X$.*

- *$(\beta X, \beta_\tau)$ is unique up to homeomorphisms with these properties. In particular any compactification of (X, τ) is the continuous image of $(\beta X, \beta_\tau)$.*

- *(X, τ) is locally compact and normal if and only if $i_X[X]$ is a dense open subset of βX.*

Proof. We divide the proof of the theorem in several distinct steps:

βX **is Hausdorff.** Pick $\mathcal{F}_1 \neq \mathcal{F}_0 \in \beta X$, let $C \in \mathcal{F}_1 \setminus \mathcal{F}_0$. By Proposition 8.1.12 applied to $\mathcal{F}_0$ and C, there is $D \in \mathcal{F}_0$ such that $C \cap D$ is empty. By Proposition 8.1.4 find U and V complement of 0-sets and disjoint such that $C \subseteq U$ and $D \subseteq V$. Then $\mathcal{F}_1 \in \beta^o U$, $\mathcal{F}_0 \in \beta^o V$ and $\beta^o U \cap \beta^o V = \beta^c U \cap \beta^c V = \beta^c(U \cap V) = \emptyset$.

βX **is compact.** Fix a family $\mathcal{H}$ of closed sets of βX with the finite intersection property. We can assume $\mathcal{H}$ consists of basic closed sets of the form $\beta^c E$ with E a 0-set (by the same argument we used in the proof of the compactness of

St(B) in 2.8.2, since any closed set is the intersection of a family of sets of type $\beta^c E$ with $E \in \tau^0$).

Consider the family $\mathcal{H}_0$ given by the 0-sets E such that $\beta^c E \in \mathcal{H}$. Then $\mathcal{H}_0$ is non empty (since $X \in \mathcal{H}_0$) and has the finite intersection property: fix $C_1, \dots, C_n \in \mathcal{H}_0$. Then

$$\beta^c(C_1 \cap \dots \cap C_n) = \beta^c(C_1) \cap \dots \cap \beta^c(C_n) \neq \emptyset,$$

hence $C_1 \cap \dots \cap C_n \neq \emptyset$.

Find $\mathcal{F}$ maximal filter of 0-sets extending $\mathcal{H}_0$. Then $\mathcal{F} \in \bigcap \mathcal{H}$: pick $\beta^c E \in \mathcal{H}$, then $E \in \mathcal{F}$, hence $\mathcal{F} \in \beta^c E$.

i_X is a topological embedding if (X, τ) is Tychonoff. i_X is well defined since $\mathcal{F}_x$ is a maximal filter of 0-sets for all $x \in X$ by Proposition 8.1.10.

i_X is continous, and open on its target $i_X[X]$ (seen as a subspace of βX with the inherited topology), with a dense image in βX: for all U complement of a 0-set, $\mathcal{F}_x \in \beta^c U$ if and only if $x \in U$, hence

- $i_X[X]$ is a dense subset of βX, since it has non-empty intersection with all basic open sets;
- i_X is open and continuous, since $i_X[X] \cap \beta^c U = i_X[U]$ for all basic open sets $\beta^c U$.

i_X is injective: if $x \neq y$, find $f : X \to [0;1]$ continuous with $f(x) = 0$ and $f(y) = 1$, then we can separate $\mathcal{F}_x, \mathcal{F}_y$ with the basic open sets $\beta^o(f^{-1}[[0;1/3)])$, $\beta^o(f^{-1}[(2/3;1]])$.

Unique extension property. We show that any continuous $f : X \to K$ with K compact Hausdorff extends uniquely to a continuous $\beta f : \beta X \to K$ such that $\beta f \circ i_X = f$. Let $\mathcal{F} \in \beta X$. Choose a net $(x_C)_{C \in \mathcal{F}}$ with $x_C \in C$ for all $C \in \mathcal{F}$. By [10, Theorem 1.3.8] there is a universal subnet $(x_\lambda)_{\lambda \in \Lambda}$ of $(x_C)_{C \in \mathcal{F}}$. Since $(i_X(x_C))_{C \in \mathcal{F}}$ is eventually in any open neighborhood of $\mathcal{F}$ of the form βU, we have that $(i_X(x_\lambda))_{\lambda \in \Lambda}$ converges to $\mathcal{F}$. Now the image of any universal net under *any* function is again a universal net. Since K is compact Hausdorff, we have that the image net $(f(x_\lambda))_{\lambda \in \Lambda}$ converges to some unique point $\beta f(\mathcal{F})$ (see [10, Prop 1.5.2, Thm 1.6.2]). Now if f is *continuous* $\beta f(\mathcal{F})$ does not depend on the choice of the net with values in X converging to $\mathcal{F}$ (we leave to the reader to check this property). In particular βf is well defined. The uniqueness and continuity of βf follows from the fact that continuous functions on Hausdorff spaces are determined by their restriction to a dense subset; $i_X[X]$ is a dense subset of βX on which βf is continuous; on $\beta X \setminus i_X[X]$ βf is defined exactly in the unique way to make it continuous (see [10, Prop. 1.4.3]).

Uniqueness up to homeomorphism of βX. We now show that any compact space (Y, σ) satisfying the above extension property for (X, τ) is homeomorphic to $(\beta X, \beta_\tau)$. So assume that (Y, σ) is a Hausdorff compactification of X via a topological embedding $j : X \to Y$ such that any continuous map $f : X \to K$

with K compact Hausdorff admits a unique extension continuous extension $f^* : Y \to K$ such that $f^* \circ j = f$.

Now consider $\beta j : \beta X \to Y$. This map is surjective since any point y in Y is the limit of a net $(j(x_\lambda))_{\lambda\in\Lambda}$, hence $y = \beta j(\mathcal{F})$ where $\mathcal{F}$ is the limit in βX of the net $(i_X(x_\lambda))_{\lambda\in\Lambda}$.

By the universal property of Y find $i^*_X : Y \to \beta X$ extending i_X. Then $i^*_X \circ \beta j \restriction i_X[X]$ is the identity map on $i_X[X]$ (since $i^*_X \circ \beta j(i_X(x)) = i^*_X(j(x)) = i_X(x)$). The identity map on βX is a continuous extension of $i^*_X \circ \beta j \restriction i_X[X]$, hence by the uniqueness property of βX, we get that $i^*_X \circ \beta j$ is the identity map on βX. By a symmetric argument we get that $\beta j \circ i^*_X$ is the identity map on Y. Hence βj and i^*_X are homeomorphisms which invert one another.

$i_X[X]$ is open in βX if and ony if X is locally compact and normal: In case X is normal, we have that βX is the family of maximal filters of closed sets (all closed sets are 0-sets since X is normal). If some $C \in \mathcal{F}$ is a compact subset of X, $\mathcal{F}$ has a non-empty intersection in X. Since X is Hausdorff and $\mathcal{F}$ maximal, this intersection must be a singleton $\{x\}$. Hence $\mathcal{F} = \mathcal{F}_x$ for some $x \in X$ if and only if some $C \in \mathcal{F}$ is compact in X. We get that $i_X[X]$ is the union of $\beta^o U$ such that $\mathsf{Cl}(U)$ is compact in X, hence $i_X[X]$ is open in βX. Conversely any open subset of βX is locally compact.

□

Remark 8.1.16. Notice that normality is not preserved for subspaces: given K compact, $X \subseteq K$ dense subset of K, and C_1, C_2 closed subsets of K with $C_i \cap X \neq \emptyset$ for both $i = 0, 1$, it is well possible that $C_1 \cap C_2 \cap X$ is empty while $C_1 \cap C_2$ is not. If this were the case, in K these two closed sets cannot be separated by disjoint open sets, hence also in X (being it a dense subset of K). However this potential counterexample cannot occur if C_1 is the singleton of a point in X. This is one of the reason why one introduces the weaker Tychonoff property which uses the characterization of normality given by Urysohn Lemma. Remark that the separation property for disjoint closed sets C_0, C_1 given by the existence of a continuous $f : X \to [0; 1]$ such that $f \restriction C_i = i$, when predicated for disjoint closed sets of which one is the singleton of a point, is strictly stronger than the assertion that closed sets can be separated from points by disjoint open sets.

Remark 8.1.17. One resorts to the introduction of the notion of 0-sets to grant that βX is Hausdorff. If one defines $\beta^* X$ as the set of maximal filters of closed sets with the topology given by the corresponding definition of $\beta^o U$ for U open in X, we would run into trouble in proving the Hausdorff property for βX. It actually fails if X is not Tychonoff. The compactness part of the proof survives with these new definitions. The problem in the proof of the Hausdorff property is the separation of arbitrary disjoint closed sets C, D by means of disjoint open sets (to establish the Hausdorff property of βX we used that two disjoint 0-sets can be separated by open disjoint sets which are the complements of 0-sets). If C and D are closed but not 0-sets, this cannot always be done.

It is somewhat peculiar that one has to introduce the space $[0; 1]$ to describe a family of closed sets which are then used with almost no reference to the properties of real numbers. One explanation is given by the following post on math.stackexchange where it is argued that $[0;1]$ can be described in purely topological terms in a variety of ways. These topological characterizations of $[0;1]$ are essential in the arguments we sketched above.

We conclude with the following characterization of extremally disconnected compact Hausdorff spaces:

Proposition 8.1.18. *Let (X, τ) be a compact Hausdorff space. Then X is extremally disconnected if and only if it is the Stone-Čech compactification of any of its dense subsets.*

The Proposition follows by a combination of [12, Prop. Pag. 284 Section 10.47, Thm. Pag. 25 Section 1.46].

Proof. Assume (X, τ) is compact Hausdorff but not extremally disconnected. Then there is $U \in \tau$ which is regular open but not closed. This gives that (letting $\neg U = X \setminus \overline{U}$) $U \cup \neg U$ is open dense in X. Fix $x_0 \in U$ and $x_1 \in \neg U$. Then $f(x) = x_0$ if $x \in U$ and $f(x) = x_1$ if $x \in \neg U$ is continuous on $U \cup \neg U$ but cannot be extended to a continuous map on the whole of X: note that $\overline{U} \cap \overline{\neg U}$ is non-empty (else U is closed); if y belongs to this set and g extends continuously f to X, $g(y)$ is either x_0 or x_1, but it can be neither of them since it is an accumulation point both for $U = f^{-1}[\{x_0\}]$ and for $\neg U = f^{-1}[\{x_1\}]$.

The converse direction is given by the next Lemma. □

Lemma 8.1.19. *Let X be a compact Hausdorff extremally disconnected space. Then for every dense subset $W \subseteq X$, every compact Hausdorff space K and every continuous function $f : W \to K$, there exists an unique continuous map $\beta(f) : X \to K$ such that $\beta(f) \restriction W = f$.*

Proof. We can assume that $X = \mathrm{St}(\mathsf{B})$ where $\mathsf{RO}(\mathrm{St}(\mathsf{B})) = \mathsf{B}$ up to isomorphism. In particular we identify any G in X with the ultrafilter on $\mathsf{RO}(X)$ given by its regular open (clopen) neighboorhoods. Let $G \in X$. Since W is dense, $W \cap V \neq \emptyset$ for every open neighborhood V of G. In particular, the family $\{V \cap W : V \in G\}$ has the finite intersection property. Consequently the same holds for $\{f[V \cap W] : V \in G\}$ and for $\{\overline{f[V \cap W]} : V \in G\}$. Being K compact, we have

$$\bigcap_{V \in G} \overline{f[V \cap W]} \neq \emptyset.$$

Claim 8.1.19.1. *Let U be an open subset of K. Then $U \cap \bigcap_{V \in G} \overline{f[V \cap W]} \neq \emptyset$ if and only if $\mathsf{Reg}(f^{-1}[U]) \in G$.*

Proof.

⇒ Since U is open in K, $U \cap f[V \cap W] \neq \emptyset$ for every $V \in G$. Thus (in X) $f^{-1}[U] \cap V \neq \emptyset$ for every $V \in G$. This entails that (in X) $\mathsf{Cl}(f^{-1}[U]) \cap V \neq \emptyset$ for every $V \in G$. Now (in X) $\mathsf{Reg}(f^{-1}[U])$ is a dense subset of $\mathsf{Cl}(f^{-1}[U])$, and (in $\mathsf{Cl}(f^{-1}[U])$) $V \cap \mathsf{Cl}(f^{-1}[U])$ is a non-empty open subset of $\mathsf{Cl}(f^{-1}[U])$ for any $V \in G$. Therefore $\mathsf{Reg}(f^{-1}[U]) \cap V \neq \emptyset$ for all $V \in G$, which occurs if and only if $\mathsf{Reg}(f^{-1}[U]) \in G$.

$\Leftarrow$ Conversely assume that $\mathsf{Reg}(f^{-1}[U]) \in G$. Then $\mathsf{Reg}(f^{-1}[U]) \cap Z \neq \emptyset$ for all $Z \in G$. Since $\mathsf{Reg}(f^{-1}[U])$ is dense in $\mathsf{Cl}(f^{-1}[U])$, we get that $\mathsf{Cl}(f^{-1}[U]) \cap Z \neq \emptyset$ for all $Z \in G$. Since $f^{-1}[U]$ is dense in $\mathsf{Cl}(f^{-1}[U])$ and $\mathsf{Cl}(f^{-1}[U]) \cap Z \neq \emptyset$ is open in $\mathsf{Cl}(f^{-1}[U])$ for all $Z \in G$, we get that $f^{-1}[U] \cap Z \neq \emptyset$ for all $Z \in G$. This gives that $U \cap f[Z \cap W] \neq \emptyset$ for all $Z \in G$, as desired.

□

Claim 8.1.19.2. *For any $G \in X$, there is exactly one point in $\bigcap_{V \in G} \overline{f[V \cap W]}$.*

Proof. By contradiction, assume that in $\bigcap_{V \in G} \overline{f[V \cap W]}$ there are two distinct points y_1, y_2. Being K Hausdorff, there are disjoint open sets U_1, U_2 such that $y_i \in U_i$, $i = 1, 2$. By Claim 8.1.19.1, $\mathsf{Reg}(f^{-1}[U_1]), \mathsf{Reg}(f^{-1}[U_2]) \in G$. Hence $\mathsf{Reg}(f^{-1}[U_1]) \cap \mathsf{Reg}(f^{-1}[U_2])$ is non-empty clopen. By density of W in X, $\mathsf{Reg}(f^{-1}[U_1]) \cap \mathsf{Reg}(f^{-1}[U_2]) \cap W$ is non-empty and clopen in W. Since $f^{-1}[U_1]$ is dense open in $\mathsf{Reg}(f^{-1}[U_1]) \cap W$ we get that $f^{-1}[U_1]$ has non-empty intersection with $\mathsf{Reg}(f^{-1}[U_2]) \cap W$ and is relatively open in it. Since $f^{-1}[U_2]$ is dense in $\mathsf{Reg}(f^{-1}[U_2]) \cap W$, we get that $f^{-1}[U_1] \cap f^{-1}[U_2]$ is non-empty, which is a contradiction. □

Thus we can define $\beta(f)(G)$ as the unique point y_G in $\bigcap\{\overline{f[V \cap W]} : V \in G\}$. We leave to the reader to check that $\beta(f) \restriction W = f$.

Finally, we prove the continuity of $\beta(f)$.

Claim 8.1.19.3. *Assume U is an open set of K. Then $\beta(f)[\mathsf{Reg}(f^{-1}[U])] \subseteq \mathsf{Cl}(U)$.*

Proof. Assume $G \in \mathsf{Reg}(f^{-1}[U])$. Note that $G \in \mathsf{Reg}(f^{-1}[V])$ for any V open neighborhood of $\beta(f)(G)$ (by Claim 8.1.19.1). Hence $\mathsf{Reg}(f^{-1}[V]) \cap \mathsf{Reg}(f^{-1}[U])$ is non-empty for all such V. As in the proof of Claim 8.1.19.2, we obtain that $f^{-1}[V] \cap f^{-1}[U]$ is non-empty for all such V. Therefore so is $U \cap V$ for all V open neighborhood of $\beta(f)(G)$.

We conclude that $\beta(f)(G)$ is an accumulation point of U. □

Now, fix $G \in X$ and take an open neighborhood A of $\beta(f)(G)$. We prove that there exists an open neighborhood U of G such that $\beta(f)[U] \subseteq A$: by normality of K (being it compact Hausdorff), we can find an open neighborhood V of $\beta(f)(G)$ such that $\mathsf{Cl}(V) \subseteq A$. By Claim 8.1.19.1, $\mathsf{Reg}(f^{-1}[V])$ is an open neighborhood of G. By Claim 8.1.19.3 its image is contained in $\mathsf{Cl}(V) \subseteq A$. □

8.1.1 Topological spaces constructed from posets

From now on our focus will be on posets $(P, \leq)$ with a maximum 1_P and no minimum, guiding examples are $\tau \setminus \{\emptyset\}$ for (X, τ) a topological space, or B^+ for B a boolean algebra. We characterize separation and compactness properties of a topological space (X, τ) by means of combinatorial properties of the poset given by some base P for τ consisting of non-empty sets. We are doing below something parallel to the theory of locales: we identify posets P with bases for topological spaces X whose points p are determined by the filter F of basic open neighborhoods of p which are in P, i.e. by filters F on P. However the parallel is loose: in locale theory a point $p \in X$ is identified by the maximal ideal of open sets disjoint from the closure of $\{p\}$, here we identify a point with a filter coinitial in its family of open neighborhoods. While the notion of point on a frame/locale as in [9, Section IX.2] can be seen as an instance of the notion of point we analyze below, the converse does not hold in general.

Definition 8.1.20. Let $(P, \leq)$ be a preorder with a maximum 1_P and no minimum.

A family $\mathcal{A}$ of filters on P is *dense in* P if for all $p \in P$ there is $F \in \mathcal{A}$ with $p \in F$.[2]

Given $\mathcal{A}$ family of filters on P, $(\mathcal{A}, \tau_{P,\mathcal{A}})$ is the topological space whose points are the filters in A and whose topology is generated by the sets[3]

$$N_p^{\mathcal{A}} = \{F \in \mathcal{A} : p \in F\}.$$

$C_p^{\mathcal{A}}$ denotes the closure of $N_p^{\mathcal{A}}$ in $(\mathcal{A}, \tau_{P,\mathcal{A}})$.

Proposition 8.1.21. *Let $(P, \leq)$ be a preorder with a maximum 1_P and no minimum, and $\mathcal{A}$ be a* dense *family of filters on P. Then:*

1. *$(\mathcal{A}, \tau_{P,\mathcal{A}})$ is a T_0-topological space.*
2. *The family $\left\{N_p^{\mathcal{A}} : p \in P\right\}$ is a base for $(\mathcal{A}, \tau_{P,\mathcal{A}})$.*
3. *For each $F \in \mathcal{A}$, $\left\{N_p^{\mathcal{A}} : p \in F\right\}$ is a neighborhood base for F.*
4. *$\mathcal{A}$ is an antichain in $(\mathrm{pow}(P), \subseteq)$ (i.e. for any $F, G \in \mathcal{A}$, neither $F \subseteq G$ nor $G \subseteq F$) if and only if $(\mathcal{A}, \tau_{P,\mathcal{A}})$ is T_1.*
5. *For each $F \in \mathcal{A}$ and $p \in P$, $F \in C_p^{\mathcal{A}}$ if and only if r and p are compatible in P for all $r \in F$, i.e. if $F \cup \{p\}$ is a prefilter on P. In particular $F \in \mathcal{A} \setminus C_p^{\mathcal{A}}$ if and only if p and r are incompatible for some $r \in F$.*
6. *$(\mathcal{A}, \tau_{P,\mathcal{A}})$ is Hausdorff if and only if for any $F, G \in \mathcal{A}$, $F \cup G$ is not a prefilter on P.*
7. *$\mathsf{RO}(\mathcal{A}, \tau_{P,\mathcal{A}})$ is isomorphic to $\mathsf{RO}(P, \leq)$ and the map $P \to \mathsf{RO}(\mathcal{A}, \tau_{P,\mathcal{A}})$, $p \mapsto \mathsf{Reg}(N_p^{\mathcal{A}})$ is a complete embedding with a dense target.*
8. *$\mathsf{Reg}(N_p^{\mathcal{A}})$ is given by those $G \in \mathcal{A}$ having non empty intersection with the family*
$$\{q \in P : \downarrow q \cap \downarrow p \text{ is dense below } q\}.$$

The proof of the first four items given below does not require that $\mathcal{A}$ is dense.

Proof.

1. If $F \neq G$ are in $\mathcal{A}$ and $p \in F \setminus G$, $F \in N_p^{\mathcal{A}}$ while $G \notin N_p^{\mathcal{A}}$.

[2] In case P is a locale and the family $\mathcal{A}$ consists of points of P according to [9, Sections IX.1, IX.2,IX.3], our notion of density overlaps with that of having *enough points* or being *spatial* as in [9, pag. 480].

[3] $N_p^{\mathcal{A}}$ is empty exactly when p witnesses that $\mathcal{A}$ is not dense.

2. Assume $N^{\mathcal{A}}_{p_1} \cap \dots \cap N^{\mathcal{A}}_{p_k}$ is non-empty. Let $F \in N^{\mathcal{A}}_{p_1} \cap \dots \cap N^{\mathcal{A}}_{p_k}$. Since F is a filter and $p_1, \dots, p_k \in F$ there is some $r \in F$ refining all p_js. Then $N^{\mathcal{A}}_{r} \subseteq N^{\mathcal{A}}_{p_1} \cap \dots \cap N^{\mathcal{A}}_{p_k}$.

3. Given $F \in \mathcal{A}$ and A open with $F \in A$, by the preceding item we have that for some $R \subseteq P$,
$$A = \bigcup \left\{ N^{\mathcal{A}}_{r} : r \in R \right\}.$$
Then for some $r \in R$, $F \in N^{\mathcal{A}}_{r}$ and $N^{\mathcal{A}}_{r} \subseteq A$.

4. $F \subseteq G$ if and only if for all $p \in F$, $G \in N^{\mathcal{A}}_{p}$.

5. Since $\left\{ N^{\mathcal{A}}_{q} : q \in F \right\}$ is a neighborhood base for F, the following are equivalent:
   - F is in the closure of $N^{\mathcal{A}}_{p}$;
   - $N^{\mathcal{A}}_{q} \cap N^{\mathcal{A}}_{p}$ is non-empty for all $q \in F$;
   - for all $q \in F$ there is some filter $H_q \in \mathcal{A}$ such that $q, p \in H_q$;
   - q, p are compatible in P for all $q \in F$.

   The upward direction of the last equivalence uses the density of $\mathcal{A}$.

6. Assume $F \neq G$ in $\mathcal{A}$. Now $F \cup G$ is not a prefilter on P if and only if p and q are incompatible in P for some $p \in F$ and $q \in G$, if and only if for some $p \in F$ and $q \in G$ $N^{\mathcal{A}}_{q} \cap N^{\mathcal{A}}_{p}$ is empty. We use density of $\mathcal{A}$ to infer that $N^{\mathcal{A}}_{q} \cap N^{\mathcal{A}}_{p}$ is empty entails p and q are incompatible in P.

7. The argument which shows it is standard and is the one used in the proof of the existence and uniqueness of the boolean completion.

8. Use Lemma 3.1.6.

□

The above family of topological spaces describes exactly the class of T_0-spaces.

Fact 8.1.22. *Let (X, τ) be a topological space and $P \subseteq \tau$ be a base for it consisting of non-empty sets. Then the map $i_{X,\tau,P} : x \mapsto F_x = \{p \in P : x \in p\}$ is open and continuous onto $(\mathcal{A}_X, \tau_{\mathcal{A}_X,P})$, where $\mathcal{A}_X = \{F_x : x \in X\}$. $i_{X,\tau,P}$ is an homeomorphism if (X, τ) is T_0.*

Proof. Note that $i_{X,\tau,P}[p] = N^{\mathcal{A}_X}_{p}$ for all $p \in P$ and that (X, τ) is T_0 exactly and only if $i_{X,\tau,P}$ is injective. □

We now want to characterize which families of filters on a poset $(P, \leq)$ give compact Hausdorff spaces. We note that if (X, τ) is compact Hausdorff and P is a base of regular open neighboorhoods for X, P generates a smallest subalgebra of regular open sets in which it sits as a dense subposet, i.e. the algebra given by finite boolean combination of sets $U \in \tau$ such that $U \in P$ or $X \setminus \mathsf{Cl}(U) \in P$. We therefore may assume w.l.o.g. that given a compact Hausdorff space our focus is on some base P which forms a dense subalgebra of its regular open sets. In case of $[0; 1]$ with euclidean topology, P could be the family of finite unions of intervals with rational endpoints and disjoint closures. More generally:

Lemma 8.1.23. *Let* B *be a boolean algebra,* $P = \mathsf{B}^+$*,* $\mathcal{A}$ *be a dense family of filters on* P*. Then* $N_p^{\mathcal{A}}$ *is regular open in* $(\mathcal{A}, \tau_{P,\mathcal{A}})$ *for all* $p \in P$*.*

Proof. Assume $F \in \mathsf{Reg}(N_p^{\mathcal{A}})$. Find $q \in F$ such that $N_q^{\mathcal{A}} \cap N_p^{\mathcal{A}}$ is dense in $N_q^{\mathcal{A}}$. If $q \leq p$, the inclusion of the former in the latter is an equality and $F \in N_p^{\mathcal{A}}$. Otherwise we would have $q \wedge \neg p > 0_{\mathsf{B}}$. By the density of $\mathcal{A}$, we get that $N_{q\wedge\neg p}^{\mathcal{A}}$ is a non-empty open subset of $N_q^{\mathcal{A}}$ with which $N_p^{\mathcal{A}} \cap N_q^{\mathcal{A}}$ clearly has empty intersection; this contradicts the density of $N_p^{\mathcal{A}} \cap N_q^{\mathcal{A}}$ in $N_q^{\mathcal{A}}$. □

Definition 8.1.24. A topological space (X, τ) is completely regular if there is a base $\mathcal{B}$ such that:

- If $x \in X$ and $U \in \mathcal{B}$ is such that $x \in U$, there is $V \in \mathcal{B}$ with $x \in \mathsf{Cl}(V) \subseteq U$.
- If C, D are complements of sets in $\mathcal{B}$ which are disjoint, there are $U, V \in \mathcal{B}$ disjoint and such that $C \subseteq U$ and $D \subseteq V$.

Fact 8.1.25. *Assume* (X, τ) *is completely regular as witnessed by a base* $\mathcal{B}$ *which is a boolean subalgebra of the regular open sets. Then:*

- *the closure of any set in* $\mathcal{B}$ *is the complement of a set in* $\mathcal{B}$ *and conversely;*
- *for all* C *complement of a set in* $\mathcal{B}$ *and* $U \in \mathcal{B}$ *containing* C*, there is* V *in* $\mathcal{B}$ *with* $C \subseteq V$ *and* $\mathsf{Cl}(V) \subseteq U$*.*

Proof. Note that if $U \in \mathcal{B}$ and $V = X \setminus \mathsf{Cl}(U)$, we have that $V \in \mathcal{B}$ and $\mathsf{Cl}(U) = X \setminus V$ is the complement of an element in $\mathcal{B}$.

Now assume $C \subseteq U$ with $U \in \mathcal{B}$ and C the closure of some $A \in \mathcal{B}$ with $C \subseteq U$. Let $D = X \setminus U$. By complete regularity we can find $V \supseteq C$ and $V' \supseteq D$ disjoint and in $\mathcal{B}$. Then $\mathsf{Cl}(V)$ and D are disjoint; hence $C \subseteq V$, $\mathsf{Cl}(V) \subseteq U$. □

Fact 8.1.26. *Assume* (X, τ) *is Tychonoff. Then its regular open sets contain a base which witnesses its complete regularity and is a boolean algebra.*

Proof. The complements of 0-sets in a Tychonoff space form a base which witness the complete regularity of (X, τ). Furthermore this base satisfies the strong form of complete regularity given in 8.1.25. Now note that if A is the complement of a 0-set, $\mathsf{Cl}(A)$ is a 0-set, hence $A \subseteq \mathsf{Reg}(A) \subseteq \mathsf{Cl}(A)$ and $X \setminus \mathsf{Cl}(A)$ is the complement of a 0-set.

Let $\mathcal{B}$ consist of the regular open sets $\mathsf{Reg}(C)$ for C a 0-set. Consequently if C, D are disjoint closures of elements of $\mathcal{B}$ they are 0-sets, hence by 8.1.25, they can be separated by disjoint 0-sets E, F such that $C \subseteq U \subseteq \mathsf{Reg}(E)$ and $D \subseteq V \subseteq \mathsf{Reg}(F)$ with U, V, complements of 0 sets whose closure is respectively E and F.

This gives that $\mathcal{B}$ form a base which witnesses complete regularity of (X, τ). Furthermore the complement V of a $\mathsf{Cl}(()U)$ for $U \in \mathcal{B}$ is regular open and is in $\mathcal{B}$: V being the complement of a 0-set, we have that $\mathsf{Cl}(V)$ is a 0-set and $\mathsf{Reg}(\mathsf{Cl}(V)) = V$. □

In particular the Tychonoff spaces (X, τ) are completely regular as witnessed by a base of regular open sets which form a boolean subalgebra of $\mathsf{RO}(X, \tau)$. [3] shows that completely regular spaces are Tychonoff. Here we show the weaker result that so are the completely regular spaces as witnessed by a base which form a subalgebra of their regular opens.

Lemma 8.1.27. *Let* B *be a boolean algebra, and* $\mathcal{A}$ *be a dense family of filters on* $P = \mathsf{B}^+$ *such that* $(\mathcal{A}, \tau_{P,\mathcal{A}})$ *is an Hausdorff space. Assume* $\left\{N_p^{\mathcal{A}} : p \in P\right\}$ *witnesses that* $(\mathcal{A}, \tau_{P,\mathcal{A}})$ *is completely regular.*

Then $\mathcal{A}$ *is compact if and only if any maximal filter on* P *extends some element of* $\mathcal{A}$.

A guiding example catching all the subtleties of the proof to follow is given by $[0; 1]$ with euclidean topology presented as the family $\mathcal{A}$ of maximal filters on the poset given by open intervals with rational endpoints and P being the boolean algebra given by the regular open subsets of $(0; 1)$ which are finite unions of such intervals.

Proof. By Lemma 8.1.23 $(\mathcal{A}, \tau_{P,\mathcal{A}})$ admits as a base of regular opens the sets $N_p^{\mathcal{A}}$ for $p \in P$. Observe also that $C_p^{\mathcal{A}} = \mathcal{A} \setminus N_{\neg p}^{\mathcal{A}}$, and $N_p^{\mathcal{A}} \cup N_{\neg p}^{\mathcal{A}}$ is open dense in $\mathcal{A}$ for all p. In particular we can infer the following:

1. For all[4] $p \in P$
$$N_p^{\mathcal{A}} = \mathsf{Int}(C_p^{\mathcal{A}}) = \mathcal{A} \setminus C_{\neg p}^{\mathcal{A}} = N_{\neg\neg p}^{\mathcal{A}}.$$
2. The closed sets $C_p^{\mathcal{A}}$ form a base for the closed sets of $(\mathcal{A}, \tau_{P,\mathcal{A}})$; furthermore $C_q^{\mathcal{A}} \supseteq C_p^{\mathcal{A}}$ if and only if $q \geq p$.
3. For all $F \in \mathcal{A}$ and $p \in F$ there is $q_p \in F$ such that $C_{q_p}^{\mathcal{A}} \subseteq N_p^{\mathcal{A}}$.
4. For all closed sets C
$$C = \bigcap \left\{N_q^{\mathcal{A}} : N_q^{\mathcal{A}} \supseteq C\right\}.$$

Proof. The first two items follow almost immediately by the fact that P consists of the positive elements of a boolean algebra and that the $N_p^{\mathcal{A}}$s are regular open.

The third item follows by our assumptions on $\mathcal{A}$ and Fact 8.1.25.

For the last one: Assume $F \notin C$. By the third item there is q such that $F \in C_q^{\mathcal{A}}$ and $C_q^{\mathcal{A}} \cap C$ is empty. Hence $C \subseteq N_{\neg q}^{\mathcal{A}}$ and $F \notin N_{\neg q}^{\mathcal{A}}$. □

Let $\mathcal{F}$ be a maximal family of closed sets with the finite intersection property and $Z = \left\{p : C_p^{\mathcal{A}} \in \mathcal{F}\right\}$.

We claim that for all $r \in P$ either $r \in Z$ or $\neg r \in Z$ (if not both), i.e. that Z is ultra. Otherwise there are $C, D \in \mathcal{F}$ disjoint from $C_r^{\mathcal{A}}$ and $C_{\neg r}^{\mathcal{A}}$. Then $\emptyset \neq C \cap D \subseteq \mathcal{A}$ is disjoint from $\mathcal{A} = C_r^{\mathcal{A}} \cup C_{\neg r}^{\mathcal{A}}$, a contradiction.

However Z may not be a filter[5]. Now let G be a maximal filter in the partial order Z seen as a subset of $P = \mathsf{B}^+$. Since Z is ultra, G is a ultrafilter on B. Then

[4] Here we use essentially that $N_p^{\mathcal{A}}$ is regular open, else the first equality might be a strict inclusion. This would then cause troubles in the proof of the second and third item below.

[5] For example considering the presentation of $[0; 1]$ we suggested, $\mathcal{F}$ could be the family of closed subsets of $[0; 1]$ such that $1/2$ belongs to them. Then $(0; 1/2)$ and $(1/2; 1)$ are both in Z.

(by the density of $\mathcal{A}$) $\left\{N_p^{\mathcal{A}} : p \in G\right\}$ is a family of regular open sets with the finite intersection property. Furthermore for any $C \in \mathcal{F}$ and $p \in G$, $C \cap C_p^{\mathcal{A}}$ is non-empty, by the very definition of Z.

By assumption there is a unique $F \in \mathcal{A}$ which is extended by G; this gives that $C_p^{\mathcal{A}} \in \mathcal{F}$ for all $p \in F$. Furthermore note that for all $r \in G$ and $p \in F$ we have that $N_p^{\mathcal{A}} \cap C_r^{\mathcal{A}}$ is non-empty. Since $C_r^{\mathcal{A}}$ is closed, we conclude that $F \in C_r^{\mathcal{A}}$ for all $r \in G$. Clearly $F \in N_p^{\mathcal{A}}$ for all $p \in F$.

We get that

$$\{F\} = \bigcap \left\{N_r^{\mathcal{A}} : r \in F\right\} = \bigcap \left\{C_{q_r}^{\mathcal{A}} : r \in F\right\},$$

where each $q_r \in F$ is such that $C_{q_r}^{\mathcal{A}} \subseteq N_r^{\mathcal{A}}$.

Now note that for any $C \in \mathcal{F}$, F is a limit point of the net $\{F_r : r \in F\}$ with each $F_r \in C \cap C_{q_r}^{\mathcal{A}} \subseteq C \cap N_r^{\mathcal{A}}$ (since $q_r \in F \subseteq G$ for all $r \in F$), henceforth F is in C, being the latter closed. This shows that $\{F\} \in \mathcal{F}$ and that $\{F\} = \bigcap \mathcal{F}$ is non-empty.

Now assume $(\mathcal{A}, \tau_{P,\mathcal{A}})$ is compact Hausdorff and let G be a maximal filter on P. We must find $F \in \mathcal{A}$ extended by G.

Remark that $\mathcal{G} = \left\{C_p^{\mathcal{A}} : p \in G\right\}$ has the finite intersection property, and therefore it has non-empty intersection. Let $F \in \mathcal{A}$ be in this intersection.

Now for $q \in G$ we get $F \in C_q^{\mathcal{A}}$, yielding that $F \cup \{q\}$ is a prefilter for all $q \in G$. This gives that $F \cup G$ is a prefilter. Since G is an ultrafilter this can occur only if $G \supseteq F$.

□

We have the following characterization of Tychonoff spaces:

Proposition 8.1.28. *Let* B *be a boolean algebra and* $\mathcal{A}$ *be a dense family of filters on* $P = \mathsf{B}^+$ *such that* $(\mathcal{A}, \tau_{P,\mathcal{A}})$ *is Hausdorff and completely regular as witnessed by* $\{N_p : p \in P\}$*. Then* $(\mathcal{A}, \tau_{P,\mathcal{A}})$ *is Tychonoff.*

Proof. To simplify notation we assume B is a subalgera of $\mathsf{RO}(P)$.

Let $\mathcal{B}$ be the family of ultrafilters G on $\mathsf{RO}(X, \tau)$ such that G does not extend any $F \in \mathcal{A}$ and $\mathcal{A}^*$ be the family consisting of the upward closure in $\mathsf{RO}(P)$ of the filters on P which are in $\mathcal{A}$. Set $\mathcal{C} = \mathcal{A}^* \cup \mathcal{B}$. Note that $\mathcal{A}^*$ sits inside $(\mathcal{C}, \tau_{\mathcal{C},\mathsf{RO}(P)^+})$ as a dense subspace, and that $N_p^{\mathcal{A}^*} = N_p^{\mathcal{C}} \cap \mathcal{A}^*$ for all $p \in P$. Furthermore:

Claim 8.1.28.1. *The family of* $N_p^{\mathcal{C}}$ *as* p *ranges in* P *is a base of regular non-empty open sets of* $(\mathcal{C}, \tau_{\mathcal{C},\mathsf{RO}(P)^+})$ *witnessing complete regularity for it.*

Proof. We first show that the family of $N_p^{\mathcal{C}}$ as p ranges in P is a base: for any $A \in \mathsf{RO}(P)^+$ $N_A^{\mathcal{C}} \supseteq N_p^{\mathcal{C}}$ for any $p \leq_{\mathsf{RO}(P)} A$. Furthermore: if $G \in \mathcal{B}$, we have that $G \cap \mathsf{B} \subseteq P$ is a ultrafilter on B; then for $p \in G$, $G \in N_p^{\mathcal{C}}$; the case for $G \in \mathcal{A}^*$ is even simpler.

The family of $N_p^{\mathcal{C}}$ as p ranges in P consists of regular open sets of in view of Lemma 8.1.23.

Now assume $C_p^{\mathcal{C}}$ and $C_q^{\mathcal{C}}$ are disjoint. Then so are $C_p^{\mathcal{A}^*} = C_p^{\mathcal{C}} \cap \mathcal{A}^*$ and $C_q^{\mathcal{A}^*} = C_q^{\mathcal{C}} \cap \mathcal{A}^*$, where the required equalities hold in view of Fact 8.1.21(5). Since $(\mathcal{A}^*, \tau_{\mathcal{A}^*,\mathsf{RO}(P)})$ is homeomorphic to $(\mathcal{A}, \tau_{\mathcal{A},P})$ via the map $F \mapsto F \cap P$, we get that for some $r, s \in P$

with $r \wedge s = 0_{\mathsf{B}}$, $C_q^{\mathcal{A}^*} \subseteq N_r^{\mathcal{A}^*}$ and $C_p^{\mathcal{A}^*} \subseteq N_s^{\mathcal{A}^*}$. Note that $r \geq q$ and $s \geq p$ as $N_q^{\mathcal{A}^*} \subseteq N_r^{\mathcal{A}^*}$ and $N_p^{\mathcal{A}^*} \subseteq N_s^{\mathcal{A}^*}$.

Then we easily conclude that $C_q^{\mathcal{C}} \subseteq N_r^{\mathcal{C}}$ and $C_p^{\mathcal{C}} \subseteq N_s^{\mathcal{C}}$: if $G \in C_q^{\mathcal{C}} \cap \mathcal{B}$, G is an ultrafilter on $\mathsf{RO}(P)$, hence $q \in G$ since $G \cup \{q\}$ is a prefilter; this gives that $s \in G$ as well, since $q \geq s$.

□

We now show that $(\mathcal{C}, \tau_{\mathcal{C},\mathsf{RO}(P)^+})$ is Hausdorff. Let $G, H \in \mathcal{C}$. Then:

- If both of them are ultrafilters on $\mathrm{St}(\mathsf{RO}(P)) \setminus \mathcal{A}^*$, they are separated by some $N_A^{\mathcal{C}}$, $N_{\neg A}^{\mathcal{C}}$ for some $A \in G \setminus H$ (with A possibly in $\mathsf{RO}(P)^+ \setminus P$).

- If both are in $\mathcal{A}^*$, by Hausdorffness of $(\mathcal{A}, \tau_{\mathcal{A},P})$ we could find $r \in G \cap P, s \in H \cap P$ such that $N_r^{\mathcal{A}} \cap N_s^{\mathcal{A}}$ is empty. Note that by density of $\mathcal{A}$ in P this yields $r \wedge_{\mathsf{B}} s = 0_{\mathsf{B}}$, hence $r \wedge_{\mathsf{RO}(P)} s = 0_{\mathsf{RO}(P)}$ holds as well, yielding that $N_r^{\mathcal{C}} \cap N_s^{\mathcal{C}}$ is also empty.

- If $G \in \mathcal{B}$ and $H \in \mathcal{A}^*$, and $N_r^{\mathcal{C}}$ has non-empty intersection with $N_q^{\mathcal{C}}$ for all $q \in G \cap P$ and $r \in H \cap P$, we would get that $q \wedge r > 0_{\mathsf{B}}$ for all $q \in G \cap P$ and $r \in H \cap P$, which occurs if and only if $G \supseteq H$, since G is a ultrafilter on $\mathsf{RO}(P) \supseteq P \supseteq H \cap P$ and $H \cap P$ is a prefilter on $\mathsf{RO}(P)$. This cannot be the case by definition of $\mathcal{B}$. Therefore we can find $q \in G \cap P$ and $r \in H \cap P$ such that $N_q^{\mathcal{C}}$ and $N_r^{\mathcal{C}}$ have empty intersection.

By Lemma 8.1.27 $(\mathcal{C}, \tau_{\mathcal{C},\mathsf{RO}(P)^+})$ is compact.

We conclude that $(\mathcal{A}, \tau_{\mathcal{A},P})$ is Tychonoff being homeomorphic to the subspace $(\mathcal{A}^*, \tau_{\mathcal{A}^*,\mathsf{RO}(P)^+})$ of the compact Hausdorff space $(\mathcal{C}, \tau_{\mathcal{C},\mathsf{RO}(P)^+})$. □

It can be checked that $(\mathcal{C}, \tau_{\mathcal{C},\mathsf{RO}(P)^+})$ gives another preentation of the Stone-Cech compactification of $(\mathcal{A}, \tau_{\mathcal{A},P})$.

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