J. T. MOORE. *Set mapping reflection*. ***Journal of Mathematical Logic***, vol. 5 (2005), pp. 87–97.

J. T. MOORE. *A five element basis for the uncountable linear orders*. ***Annals of Mathematics***, vol. 163 (2006), pp. 669–688.

In[1] the last twenty years an impressive amount of results in combinatorial set theory ranging from cardinal arithmetics to general topology and model theory has been obtained by means of forcing axioms. These axioms are technical hypothesis which can be formulated as a strengthening of the Baire category theorem but require a great familiarity with forcing and set theory in order to be grasped. Nonetheless almost all of these results can be proved by interpolation using a short list of *Black box*-principles each of which is known to follow from the appropriate forcing axiom and some of which will imply the desired combinatorial result. One advantage of this approach is that most of these *"Black boxes"* can be understood and used by anyone with a modest familiarity with set theory and with no knowledge of the forcing method. For this reason also mathematicians which are not specialists in set theory may be interested in these type of hypotheses. Much in the same way as principles which holds in L like *diamond* or CH have been used to obtain the consistency of a certain solution to problems and conjectures coming from various branches of mathematics, among which general topology (the failure of Suslin's hypothesis using *diamond*, a classical result by Jensen), group theory (positive answer to Whitehead problem on free groups using *diamond*, a result of Shelah) and functional analysis (existence of outer automorphisms of the Calkin algebra using CH, a recent result by Phillips and Weaver *There are outer automorphisms of the Calkin algebra*, ***Duke Mathematical Journal***, vol. 139 (2007), pp. 185–202), there are already several applications of these "Black boxes" which provide the consistency of the opposite solutions to the same problems (in our list of examples the classical result of Solovay and Tennebaum for Suslin's hypothesis, Shelah's surprising negative solution to Whitehead's problem and Farah (unpublished) for the latter problem on the automorphisms of the Calkin algebra.

Last but not least in this list of "Black Boxes" is the *mapping reflection principle* MRP introduced by Moore in the first reviewed item as a consequence of the *proper forcing axiom* PFA. There is a long list of combinatorial problems which have been solved by Moore and others by means of this principle. We mention, among others, the proof by means of MRP that PFA implies the *five elements basis*-conjecture[2] presented in the second reviewed item, the proof by means of MRP that[3] the *bounded proper forcing axiom* BPFA decides $2^{\aleph_0} = \aleph_2$ presented in the first reviewed item and the proof that MRP implies the *singular cardinal hypothesis*[4] SCH. It has also to be noted that MRP has a large consistency strength as it implies the failure of the *square principle* as it is shown in the first reviewed item and thus, for example, it implies the existence of inner models for Woodin cardinals and that the *axiom of determinacy* holds in $L(\mathbb{R})$.

We now formulate MRP.

[1] A premise is in order, I decided to give precise references for results which are recent or which I consider essential for the discussion. The lack of a reference to a journal does not mean that the result I mention is unpublished, if this is the case I will explicitly state it.

[2] This conjecture asserts that there are five uncountablee linear orders such that any other uncountable linear order contains an isomorphic copy of one of these five. This proof has been later refined by König, Moore, Larson and Veličković in *Bounding the consistency strength of a five element linear basis*, ***Israel Journal of Mathematics***, vol. 164, (2008), pp. 1–18, reducing the large cardinal strength of the assumptions needed to obtain the consistency of the conjecture to a very weak large cardinal hypothesis (the existence of a Mahlo cardinal suffices).

[3] Subsequently Caicedo and Veličković refined this result and showed that MRP (or BPFA) implies that there is a Δ_1-definable well ordering of $\mathbb{R}$ in the structure $H(\aleph_2)$ (see *Bounded proper forcing axiom and well orderings of the reals*, ***Mathematical Research Letters***, vol. 13 (2006), pp. 393–408).

[4] SCH holds if $\kappa^{\mathrm{cf}(\kappa)} = 2^{\mathrm{cf}(\kappa)} + \kappa^+$ for all singular cardinals κ.

Given a countable set A and s a finite subset of A, let $[s; A]$ be the family of all Y infinite subsets of A such that $s \subseteq Y$.

Let X be an uncountable set. The Ellentuck topology on $[X]^{\aleph_0}$, the set of countable subsets of X, is defined declaring the sets $[s; A]$ to be clopen for all $A \in [X]^{\aleph_0}$ and finite $s \subseteq A$. $\mathcal{C}$ is a club subset of $[X]^{\aleph_0}$ if there is some $f\colon X^{<\omega} \to X$ such that $A \in \mathcal{C}$ whenever $f[A] \subseteq A$.

Let $\theta > |X|$ be a regular cardinal. Let $H(\theta)$ denote the set of all sets whose transitive closure have size less than θ and M be a subset of $H(\theta)$. $\Sigma \subseteq P([X]^{\omega}) \cap M$ is M-stationary if for every $\mathcal{C} \in M$ club subset of $[X]^{\aleph_0}$, there is $Y \in \mathcal{C} \cap M \cap \Sigma$.

$$\Sigma\colon \mathcal{E} \to P([X]^{\aleph_0})$$

is an open and stationary mapping if $\mathcal{E}$ is a club subset of $[H(\theta)]^{\aleph_0}$ and $\Sigma(M)$ is open in the Ellentuck topology and M-stationary for all $M \in \mathcal{E}$.

> MRP holds if for all open and stationary maps Σ with domain $\mathcal{E}$ there is a continuous $\in$-chain[5] $\{M_\xi : \xi < \omega_1\}$ of elements of $\mathcal{E}$ such that for all non-zero limit ordinals $\xi < \omega_1$ there is $\eta < \xi$ such that $M_\gamma \cap X \in \Sigma(M_\xi)$ for $\gamma \in (\eta, \xi)$.

A sequence satisfying the conclusion of MRP for a mapping Σ is called a reflecting sequence for Σ. Here are some considerations by Moore which may help the reader to clarify the content of the above definition.

1. Let M be a countable elementary submodel of $H(\theta)$ for some large enough regular θ and c be a subset of $\delta = M \cap \omega_1$ of order type smaller than δ. The standard example of an M-stationary subset of $[\omega_1]^{\aleph_0} \cap M$ is the set $(\delta \cap M) \setminus c$.
2. Let f be a regressive map on ω_1 and Σ_f be the stationary map which sends a countable model M in $\delta \setminus f(\delta)$ where $\delta = M \cap \omega_1$. Notice that any continuous $\in$-sequence $\{M_\xi : \xi < \omega_1\}$ is a reflecting sequence for Σ_f. MRP is an attempt to maximize the set of mappings Σ for which there is a reflecting sequence.

The introduction of the first reviewed item and recent unpublished seminal works of Moore present at least two arguments to motivate the success of MRP: on one side this principle resembles closely the reflection principles which follow from *Martin's maximum* MM, a forcing axiom stronger than PFA. Thus MRP provides also in the context of PFA a type of argument which before was just peculiar to the setting of MM. This is exemplified by the main results of the first reviewed item where exactly the same pattern devised by Woodin and employed by Todorčević[6] in his proof that $2^{\aleph_0} = \aleph_2$ follows from BMM is used by Moore to obtain the same conclusion assuming the weaker BPFA. In Moore's proof MRP takes the place that the strong reflection principle SRP has in Todorčević's proof.

The second argument devised by Moore to motivate MRP is rather technical and carves deeply into the meaning of properness. Moore argues that MRP incorporates a schema of arguments to prove properness which is peculiar to a class of proper posets which are not *(weakly) ω-proper* and is **essential** to prove many interesting consequences of PFA.

It is outside of the scope of this review to define the notion[7] of (weak) ω-properness, it has to be noted however that any poset which has the countable chain condition or is countably closed is ω-proper, that most of the known applications of PFA are obtained by means of a weakly ω-proper poset[8] and that both properties are preserved under countable support

[5] $\{M_\xi : \xi < \lambda\}$ is a continuous $\in$-chain if $\{M_\xi : \xi \leq \eta\} \in M_{\eta+1}$ and $\bigcup_{\xi<\eta} M_\xi = M_\eta$ for all $\eta < \lambda$.

[6] See *Generic absoluteness and the continuum*, ***Mathematical Research Letters***, vol. 9 (2002), pp. 465–471.

[7] Eisworth and Nyikos in *First countable, countably compact spaces and the Continuum Hypothesis*, ***Transactions of the American Mathematical Society***, vol. 357 (2005), pp. 4329–4347, introduce and analyze the class of weakly ω-proper posets.

[8] For example the *Black box*-principles PID and OCA can be forced by weakly ω-proper posets.

iterations. Moore points out that there is a neat dividing line between the consequences of PFA that can be obtained by means of weakly ω-proper posets and those that cannot, MRP and the *five element basis*-conjecture being in this latter category. In particular Shelah's work shows that a (weak) club guessing sequence[9] on ω_1 is preserved by a countable support iteration of (weakly) ω-proper posets. While a standard application of MRP shows that for any $\mathcal{C}$-sequence[10] $(C_\delta : \delta < \omega_1)$, there is C club subset of ω_1 such that $C \cap C_\delta$ is finite[11] for all $\delta < \omega_1$ and thus (assuming MRP) no $\mathcal{C}$-sequence can be a weak club guessing sequence on ω_1. Now it is not hard to see that *diamond* implies that there is a club guessing sequence on ω_1 and Moore has shown in *Aronszajn lines and the club filter*, JSL LXXIII 1029, that the *five element basis*-conjecture does not hold in any model of set theory where a club guessing sequence on ω_1 exists. Thus Moore's proof that PFA implies the *five element basis*-conjecture cannot be established with the use of ω-proper posets[12] since the above discussion plus standard arguments on iterated forcing yields that the forcing axiom for ω-proper posets is consistent with the existence of a club guessing sequence on ω_1.

There are yet more arguments to motivate MRP (and in the large forcing axioms): the spectacular solution given by Moore to the *five elements basis*-conjecture follows the standard pattern for applications of BPFA: a certain poset P living in the ground model universe V is devised; an easy density argument shows that the desired conclusion holds in the generic extension V^P by P; an hard technical argument shows that P is proper; since the conclusion is a Σ_1-statement with parameters in $H(\aleph_2)$, an application of BPFA yields that the desired conclusion holds also in the ground model V. However in this proof of Moore, MRP is crucially used in the proof that P is proper: i.e., the assumption that the ground model V satisfies MRP is necessary to establish that the required poset P is proper and it is not yet known whether the properness of P can be established rightaway in ZFC without any further assumption. This is the first known example of an application of a forcing axiom which require the forcing axiom even in the proof that the devised partial order has the required properties.

Let me close this note with a consideration linking this application of MRP to Woodin's program for the solution of the *continuum's problem*. The large number of consistency results obtained by means of forcing axioms are undoubtably due to the fact that different models of the strongest forcing axioms like MM have a strong degree of similarity. Woodin has been able to give a precise mathematical meaning to these considerations. In a few words,[13] Woodin has devised a non-constructive "proof system" $\vdash_\Omega$ and a strengthened version of BMM which we may call Woodin's maximum WM which (in the presence of large cardinals) have three remarkable properties:

1. $\vdash_\Omega$ is generically absolute i.e., $V \models$ "$T \vdash_\Omega \phi$" iff $V^{\mathbb{B}} \models$ "$T \vdash_\Omega \phi$" for any complete boolean algebra $\mathbb{B}$, thus the notion $\vdash_\Omega$ of Ω-deduction is not affected by forcing.
2. WM decides in the "proof system" $\vdash_\Omega$ the theory of $H(\aleph_2)$, i.e., for all sentences ϕ, $\mathsf{ZFC} + \mathsf{WM} \vdash_\Omega$ "$H(\aleph_2) \models \phi$" or $\mathsf{ZFC} + \mathsf{WM} \vdash_\Omega$ "$H(\aleph_2) \models \neg\phi$".
3. Any "axiom" Ψ which satisfies item 2 above denies CH.

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[9] $(C_\delta : \delta < \omega_1)$ is a weak club guessing sequence if C_δ has type ω for all limit δ and for every C club subset of ω_1 there is a δ such that $C_\delta \cap C$ is infinite. $(C_\delta : \delta < \omega_1)$ is a club guessing sequence if for all clubs C there is a δ such that $C_\delta \subseteq C$.

[10] $(C_\delta : \delta < \omega_1)$ is a $\mathcal{C}$-sequence if C_δ is cofinal in δ of type ω for all limit ordinals δ.

[11] Given a $\mathcal{C}$-sequence apply MRP to the mapping $\Sigma_{\mathcal{C}}(M) = [\delta]^{\aleph_0} \setminus C_\delta$ where $\delta = M \cap \omega_1$ and $M \prec H(\aleph_2)$ is a countable model such that $\mathcal{C} \in M$.

[12] Moore can refine this argument also to the case of weakly ω-proper posets.

[13] An introduction to this work of Woodin can be found in his expository articles *The continuum hypothesis, I*, ***Notices of the American Mathematical Society***, vol. 48, pp. 567–576 and *The continuum hypothesis, II*, ***Notices of the American Mathematical Society***, vol. 48, pp. 681–690.

The Ω-conjecture asserts that $\vdash_\Omega$ is complete for Π_2-sentences[14] valid in Ω-logic, where the latter is the set of sentences realized by all boolean valued model $V_\alpha^{\mathbb{B}}$ such that $V_\alpha \models$ "$\mathbb{B}$ *is a complete boolean algebra*".[15] Item 1 above is the main reason for which $\vdash_\Omega$ can be considered a proof system. Since it is a delicate matter to show that it is consistent that WM does not follow from forcing axioms,[16] item 2 above can be considered as a "proof" that virtually any concrete mathematical problem which can be formulated in the structure $H(\aleph_2)$ has a solution assuming forcing axioms. Moore's result on the *five element basis*-conjecture can be seen as another small brick added to Woodin's program as it gives another concrete and surprising example of how perspicuous can be the analysis of the structure $H(\aleph_2)$ in a universe where forcing axioms hold.

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[14]Notice that $\neg$CH is a Π_2-sentence.

[15]For α a limit ordinal, $V_\alpha^{\mathbb{B}}$ is the set of boolean valued terms in $V^{\mathbb{B}}$ of rank less than α. In order to avoid complications in the definition of $V_{\alpha+1}^{\mathbb{B}}$ only boolean valued models of the type $V_\alpha^{\mathbb{B}}$ with α a limit ordinal can be considered. This does not cause any loss of generality in the definition of Ω-truth.

[16]Nonetheless this has been shown by Larson in *Martin's Maximum and the Pmax axiom (*)*, ***Annals of Pure and Applied Logic***, vol. 106 (2000), pp. 135–149.