The descriptive set-theoretic complexity of the set of points of continuity of a multi-valued function

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A *multi-valued* (total) function from a set X to another set Y is a function $F : X \to \mathcal{P}(Y) \setminus \{\emptyset\}$ i.e., F maps points to non-empty sets. Such a function F will be denoted by $F : X \rightrightarrows Y$.

From now on we assume that X and Y are metric spaces.

Definition. Let (X, p) and (Y, d) be metric spaces; a multi-valued function $F : X \rightrightarrows Y$ is *continuous at x* if

$$(\exists y \in F(x))(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x' \in B_p(x, \delta))[F(x') \cap B_d(y, \varepsilon) \neq \emptyset]$$

Question (Martin Ziegler). We know that the set of points of continuity of a usual function is a Π^0_2 set. Assume that F is a multi-valued function such $F(x)$ is closed for all x, what can be said about the complexity of the set of points of continuity of F?

Theorem. Let (X, p) and (Y, d) be metric spaces with (Y, d) being separable and let $F : X \rightrightarrows Y$ be a multi-valued function.

(a) If the set $F(x)$ is compact for all $x \in X$ then the set of points of continuity of F is Π^0_2.

(b) If $Y = \cup_m K_m$ where K_m is compact with $K_m \subseteq K^\circ_{m+1}$ for all m and the set $F(x)$ is closed for all $x \in X$, then the set of points of continuity of F is Σ^0_3.

Corollary. Suppose that X is a metric space and that $F : X \rightrightarrows \mathbb{R}^m$ is a multi-valued function such that the set $F(x)$ is closed for all $x \in X$.

(a) The set of the points of continuity of F is Σ^0_3.

(b) If moreover the set $F(x)$ is bounded for all $x \in X$ then the set of points of continuity of F is Π^0_2.

Sketch of the proof. In the single valued case we know that a function $f : X \to Y$ is continuous at x if and only if

$$(\forall n) \inf\Big\{ \sup\{d(f(x), f(x')) \mid x' \in B_p(x, \delta)\} \mid \delta > 0\Big\} < \frac{1}{n+1}$$

For (a) let $\{y_s \mid s = 0, 1, \ldots\}$ be dense in Y; we have that:

$$F \text{ is continuous at } x \iff$$
$$(\forall n)(\exists s)\inf\Big\{ \sup\{d(y_s, F(x')) \mid x' \in B_p(x, \delta)\} \mid \delta > 0\Big\} < \frac{1}{n+1}$$

For fixed n and s the relation
$P_{n,s}(x) \iff \inf\{\sup\{d(y_s, F(x')) \mid x' \in B_p(x, \delta)\} \mid \delta > 0\} < \frac{1}{n+1}$
defines an open subset of X.
For (b) we replace $F(x')$ with $F(x') \cap K_m$ and we start our condition as follows $(\exists m)(\forall n)(\exists s)$.

The previous results are optimum.

Theorem. There is a multi-valued function $F : [0, 1] \rightrightarrows \mathbb{R}$ such that the set $F(x)$ is closed for all x and the set of points of continuity of F is not Π^0_3. Therefore the Σ^0_3-answer is the best possible for a multi-valued function F from $[0, 1]$ to $\mathbb{R}$.

Lemma (almost obvious). Suppose that X_1 and Y are metric spaces and that X_0 is a closed subset of X_1. Given a multi-valued function $F : X_0 \rightrightarrows Y$ we define the multi-valued function $\tilde{F} : X_1 \rightrightarrows Y$ as follows:

$$\tilde{F}(x) = F(x) \quad \text{if } x \in X_0 \text{ and} \quad \tilde{F}(x) = Y \quad \text{if } x \in X_1 \setminus X_0.$$

Denote by $C_{\tilde{F}}$ and C_F the set of points of continuity of the corresponding multi-valued function. Then

$$C_{\tilde{F}} = C_F \cup (X_1 \setminus X_0).$$

Sketch of the proof of the Theorem. From the previous lemma it is enough to define the multi-valued function on $2^{\omega\times\omega}$. A typical example of a Σ^0_3 set which is not Π^0_3 is the following:

$$R = \{\gamma \in 2^{\omega\times\omega} \mid (\exists m)(\forall n)(\exists s \geq n)[\gamma(m,s) = 1]\}.$$

We denote by R_m the m-section of R. Define $F : 2^{\omega\times\omega} \rightrightarrows \mathbb{R}$ as follows

$$F(\gamma) = \{m \mid \gamma \in R_m\} \cup \{m + \frac{1}{n(\gamma, m) + 2} \mid \gamma \notin R_m\},$$

where

$$n(\gamma, m) = \textit{ the least } n \left\{\textit{for all } s \geq n \textit{ we have that } \gamma(m,s) = 0\right\}.$$

for $\gamma \notin R_m$. Then F is continuous at γ exactly when $\gamma \in R$.

Corollary. Define $\mathcal{F}(Y) = \{C \subseteq Y \mid C \text{ is closed}\}$. We can view every multi-valued function $F : X \rightrightarrows Y$ with closed images as a usual function $F : X \to \mathcal{F}(Y)$. It is not true in general that if $F : X \rightrightarrows Y$ there is a metrizable topology on $\mathcal{F}(Y)$ such that for all $x \in X$, F is continuous at $x \in X$ in the sense of multi-valued functions exactly when $F : X \to \mathcal{F}(Y)$ is continuous at x in the usual sense.

One can ask what is the best that we can say about the set of points of continuity of F without any additional topological assumptions for Y or for $F(x)$.

Proposition. Let (X, p) and (Y, d) be complete and separable metric spaces and let $F : X \rightrightarrows Y$ be a multi-valued function such that the set $F \subseteq X \times Y$ is *analytic*. Then the set of points of continuity of F is *analytic* as well.

We will show that this result is optimum.

Theorem. There is a multi-valued function $F : \mathcal{C} \Rightarrow \mathcal{N}$ such that the set $F(x)$ is closed for all $x \in \mathcal{C}$ and the set of points of continuity of F is analytic and not Borel. Moreover the set F is a Borel subset of $\mathcal{C} \times \mathcal{N}$.

Idea of the Proof. A set of finite sequences of naturals T is a *tree on the naturals* if it is closed under initial segments. The set *Tr* of all trees on the naturals can be viewed as a closed subset of $\mathcal{C}$. From the previous lemma it is enough to define F on *Tr*. The set of all *ill-founded* trees i.e., the set of trees which have an infinite branch is analytic and not Borel. The idea is to define F in such a way so that for a tree T we have that

$$F \text{ is continuous at } T \iff T \text{ is ill founded.}$$

We define $F : Tr \Rightarrow \mathcal{N}$ as follows

$$F(T) = [T^{+1}] \cup \{v\hat{}(0,0,0,\dots) \mid v \text{ terminal in } T^{+1}\}$$

Remark. There is a multi-valued function $F : [0, 1] \Rightarrow [0, 1]$ for which the set of the points of continuity of F is analytic and not Borel. Moreover the set F is a Borel subset of $[0, 1] \times [0, 1]$.

Definition. Let (X, p) and (Y, d) be metric spaces; a multi-valued function $F : X \Rightarrow Y$ is *strongly continuous at x* if

$$(\forall y \in F(x))(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x' \in B_p(x, \delta))[F(x') \cap B_d(y, \varepsilon) \neq \emptyset].$$

Remark. Let A be a dense subset of $[0, 1]$; define the multi-valued function $F : [0, 1] \Rightarrow \{0, 1\}$ as follows

$$F(x) = \{0\}, \text{ if } x \in A \text{ and } F(x) = \{0, 1\} \text{ if } x \notin A,$$

for all $x \in [0, 1]$. Then the set of points of strong continuity of F is exactly the set A.

Theorem. Let (X, p) and (Y, d) be metric spaces with (Y, d) being separable and let $F : X \rightrightarrows Y$ be a multi-valued function such that *F is a Σ^0_2* subset of $X \times Y$.

(a) If Y is compact and the set $F(x)$ is closed for all $x \in X$ then the set of points of strong continuity of F is Π^0_2.

(b) If $Y = \cup_m K_m$ where K_m is compact with $K_m \subseteq K^\circ_{m+1}$ for all m and the set $F(x)$ is closed for all $x \in X$, then the set of points of strong continuity of F is Σ^0_3.

Sketch of the proof. Recall the basic equivalence in the proof of the first theorem: F is continuous at x exactly when

$$(\forall n)(\exists s) \inf\{\sup\{d(y_s, F(x')) \mid x' \in B_p(x, \delta)\} \mid \delta > 0\} < \frac{1}{n+1}$$

In the case of strong continuity one replaces $(\forall n)(\exists s)$ with $(\forall n)(\forall s \text{ with } d(y_s, F(x)) \leq \frac{1}{3(n+1)})$. This is exactly where we need the assumption about the graph of F.

Proposition.
Let (X, p) and (Y, d) be complete and separable metric spaces and let $F : X \rightrightarrows Y$ be a multi-valued function such that the set $F \subseteq X \times Y$ is analytic. Then the set of points of strong continuity of F is the *coanalytic* set.

Question. Are the results about strong continuity optimum?

The *Fell topology* on $\mathcal{F}(Y)$ is the topology which has as basis the family of all sets of the form

$$\begin{aligned} \mathcal{W} &\equiv \mathcal{W}(K, U_1, \ldots, U_n) \\ &= \{C \in \mathcal{F}(Y) \mid C \cap K = \emptyset \;\&\; (\forall i \leq n)[C \cap U_i \neq \emptyset]\}, \end{aligned}$$

where K is a compact subset of Y and $U_1, \ldots, U_n$ are open subsets of Y. If Y is a locally compact Polish space then the Fell topology is compact metrizable.

Proposition. Consider a multi-valued function $F : X \rightrightarrows Y$, with Y Polish and suppose that F is a closed subset of $X \times Y$. Then the multi-valued function $F : X \rightrightarrows Y$ is strongly continuous at $x \in X$ exactly when the function $F : X \to \mathcal{F}(Y)$ is continuous at x with respect to the Fell topology.
It follows that in the case of multi-valued functions with closed graph and range a locally compact Polish space, the notion of strong continuity is metrizable.

THANK YOU!