Turning Borel sets into Clopen effectively

Vassilis Gregoriades
TU Darmstadt
gregoriades@mathematik.tu-darmstadt.de

Trends in set theory
Warsaw Poland
10^{th} July, 2012

Theorem. If A is a Borel subset of a Polish space $(\mathcal{X}, \mathcal{T})$ there exists a Polish topology $\mathcal{T}_\infty$ on $\mathcal{X}$ which extends $\mathcal{T}$, and thus has the same Borel sets as $\mathcal{T}$ such that A is $\mathcal{T}_\infty$-clopen.

Theorem. (Lusin-Suslin) Every Borel subset of a Polish space is the continuous injective image of a *closed* subset of the Baire space $\mathcal{N} = {}^\omega\omega$.

We consider the family of all recursive functions from ω^k to ω^n. A set $P \subseteq \omega^k$ is *recursive* when the characteristic function χ_p is recursive.

Relativization. For every $\varepsilon \in \mathcal{N}$ one defines the *relativized* family of *ε-recursive* functions. Similarly one defines the family of ε-recursive subsets of ω^k.

Definition. (Moschovakis) Suppose that $\mathcal{X}$ is a Polish space, d is compatible distance function for $\mathcal{X}$ and $(x_n)_{n\in\omega}$ is a sequence in $\mathcal{X}$. Define the relation $P_<$ of ω^4 as follows $P_<(i,j,k,m) \iff d(x_i,x_j) < \frac{k}{m+1}$. Similarly we define the relation $P_\leq$.

The sequence $(x_n)_{n\in\omega}$ is a *recursive presentation* of $\mathcal{X}$, if
(1) it is a dense sequence and
(2) the relations $P_<$ and $P_\leq$ are recursive.

The spaces $\mathbb{R}$, $\mathcal{N}$ and ω^k admit a recursive presentation i.e., they are *recursively presented*. Some other examples: $\mathbb{R}\times\omega$, $\mathbb{R}\times\mathcal{N}$. However not all Polish spaces are recursively presented.

Every Polish space admits an ε-recursive presentation for some suitable ε.

Definition. (Moschovakis) Suppose that $\mathcal{X}$ is a Polish space, d is compatible distance function for $\mathcal{X}$ and $(x_n)_{n\in\omega}$ is a sequence in $\mathcal{X}$. Define the relation $P_<$ of ω^4 as follows
$P_<(i,j,k,m) \iff d(x_i,x_j) < \frac{k}{m+1}$. Similarly we define the relation $P_\leq$.

The sequence $(x_n)_{n\in\omega}$ is a *recursive presentation* of $\mathcal{X}$, if
(1) it is a dense sequence and
(2) the relations $P_<$ and $P_\leq$ are recursive.

The spaces $\mathbb{R}$, $\mathcal{N}$ and ω^k admit a recursive presentation i.e., they are *recursively presented*. Some other examples: $\mathbb{R}\times\omega$, $\mathbb{R}\times\mathcal{N}$. However not all Polish spaces are recursively presented.

Every Polish space admits an ε-recursive presentation for some suitable ε.

Definition. (Moschovakis) Suppose that $\mathcal{X}$ is a Polish space, d is compatible distance function for $\mathcal{X}$ and $(x_n)_{n\in\omega}$ is a sequence in $\mathcal{X}$. Define the relation $P_<$ of ω^4 as follows $P_<(i,j,k,m) \Longleftrightarrow d(x_i,x_j) < \frac{k}{m+1}$. Similarly we define the relation $P_\leq$.

The sequence $(x_n)_{n\in\omega}$ is a *recursive presentation* of $\mathcal{X}$, if
(1) it is a dense sequence and
(2) the relations $P_<$ and $P_\leq$ are recursive.

The spaces $\mathbb{R}$, $\mathcal{N}$ and ω^k admit a recursive presentation i.e., they are *recursively presented*. Some other examples: $\mathbb{R}\times\omega$, $\mathbb{R}\times\mathcal{N}$. However not all Polish spaces are recursively presented.

Every Polish space admits an ε-recursive presentation for some suitable ε.

Definition. (Moschovakis) Suppose that $\mathcal{X}$ is a Polish space, d is compatible distance function for $\mathcal{X}$ and $(x_n)_{n\in\omega}$ is a sequence in $\mathcal{X}$. Define the relation $P_<$ of ω^4 as follows
$P_<(i,j,k,m) \iff d(x_i,x_j) < \frac{k}{m+1}$. Similarly we define the relation $P_\leq$.

The sequence $(x_n)_{n\in\omega}$ is an *ε-recursive presentation* of $\mathcal{X}$, if
(1) it is a dense sequence and
(2) the relations $P_<$ and $P_\leq$ are ε-recursive.

The spaces $\mathbb{R}$, $\mathcal{N}$ and ω^k admit a recursive presentation i.e., they are *recursively presented*. Some other examples: $\mathbb{R}\times\omega$, $\mathbb{R}\times\mathcal{N}$. However not all Polish spaces are recursively presented.

Every Polish space admits an ε-recursive presentation for some suitable ε.

$N(\mathcal{X}, s) =$ the ball with center $x_{(s)_0}$ and radius $\frac{(s)_1}{(s)_2+1}$.

A set $P \subseteq \mathcal{X}$ is *semirecursive* if $P = \bigcup_{i\in\omega} N(\mathcal{X}, \alpha(i))$ where α is a recursive function from ω to ω.

$\Sigma^0_1 =$ all semirecursive sets

$\rightsquigarrow$ effective open sets.

$\Pi^0_1 =$ the complements of semirecursive sets

$\rightsquigarrow$ effective closed sets.

Similarly one defines the class Δ^1_1 of *effective Borel* sets, Σ^1_1 of *effective analytic* and so on.

A function $f : \mathcal{X} \to \mathcal{Y}$ is *Σ^0_1-recursive* if and only if the set $R^f \subseteq \mathcal{X} \times \omega$, $R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$, is Σ^0_1.

A point $x \in \mathcal{X}$ is *Δ^1_1 point* if the relation $U \subseteq \omega$ which is defined by $s \in U \iff x \in N(\mathcal{X}, s)$, is in Δ^1_1.

Similarly one defines the *relativized* pointclasses with respect to some parameter ε.

$N(\mathcal{X}, s) =$ the ball with center $x_{(s)_0}$ and radius $\frac{(s)_1}{(s)_2+1}$.

A set $P \subseteq \mathcal{X}$ is *semirecursive* if $P = \bigcup_{i\in\omega} N(\mathcal{X}, \alpha(i))$ where α is a recursive function from ω to ω.

$\Sigma^0_1 =$ all semirecursive sets
$\rightsquigarrow$ effective open sets.

$\Pi^0_1 =$ the complements of semirecursive sets
$\rightsquigarrow$ effective closed sets.

Similarly one defines the class Δ^1_1 of *effective Borel* sets, Σ^1_1 of *effective analytic* and so on.

A function $f : \mathcal{X} \to \mathcal{Y}$ is *Σ^0_1-recursive* if and only if the set $R^f \subseteq \mathcal{X} \times \omega$, $R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$, is Σ^0_1.

A point $x \in \mathcal{X}$ is Δ^1_1 *point* if the relation $U \subseteq \omega$ which is defined by $s \in U \iff x \in N(\mathcal{X}, s)$, is in Δ^1_1.

Similarly one defines the *relativized* pointclasses with respect to some parameter ε.

$N(\mathcal{X}, s) =$ the ball with center $x_{(s)_0}$ and radius $\frac{(s)_1}{(s)_2+1}$.

A set $P \subseteq \mathcal{X}$ is *semirecursive* if $P = \bigcup_{i\in\omega} N(\mathcal{X}, \alpha(i))$ where α is a recursive function from ω to ω.

$\Sigma^0_1 =$ all semirecursive sets
$\quad\rightsquigarrow$ effective open sets.

$\Pi^0_1 =$ the complements of semirecursive sets
$\quad\rightsquigarrow$ effective closed sets.

Similarly one defines the class Δ^1_1 of *effective Borel* sets, Σ^1_1 of *effective analytic* and so on.

A function $f : \mathcal{X} \to \mathcal{Y}$ is Σ^0_1*-recursive* if and only if the set $R^f \subseteq \mathcal{X} \times \omega$, $R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$, is Σ^0_1.

A point $x \in \mathcal{X}$ is Δ^1_1 *point* if the relation $U \subseteq \omega$ which is defined by $s \in U \iff x \in N(\mathcal{X}, s)$, is in Δ^1_1.

Similarly one defines the *relativized* pointclasses with respect to some parameter ε.

$N(\mathcal{X}, s) =$ the ball with center $x_{(s)_0}$ and radius $\frac{(s)_1}{(s)_2+1}$.

A set $P \subseteq \mathcal{X}$ is *semirecursive* if $P = \bigcup_{i\in\omega} N(\mathcal{X}, \alpha(i))$ where α is a recursive function from ω to ω.

$\Sigma^0_1 =$ all semirecursive sets
$\rightsquigarrow$ effective open sets.

$\Pi^0_1 =$ the complements of semirecursive sets
$\rightsquigarrow$ effective closed sets.

Similarly one defines the class Δ^1_1 of *effective Borel* sets, Σ^1_1 of *effective analytic* and so on.

A function $f : \mathcal{X} \to \mathcal{Y}$ is *Σ^0_1-recursive* if and only if the set $R^f \subseteq \mathcal{X} \times \omega$, $R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$, is Σ^0_1.

A point $x \in \mathcal{X}$ is *Δ^1_1 point* if the relation $U \subseteq \omega$ which is defined by $s \in U \iff x \in N(\mathcal{X}, s)$, is in Δ^1_1.

Similarly one defines the *relativized* pointclasses with respect to some parameter ε.

$N(\mathcal{X}, s) =$ the ball with center $x_{(s)_0}$ and radius $\frac{(s)_1}{(s)_2+1}$.

A set $P \subseteq \mathcal{X}$ is *semirecursive* if $P = \bigcup_{i\in\omega} N(\mathcal{X}, \alpha(i))$ where α is a recursive function from ω to ω.

$\Sigma^0_1 =$ all semirecursive sets
$\rightsquigarrow$ effective open sets.

$\Pi^0_1 =$ the complements of semirecursive sets
$\rightsquigarrow$ effective closed sets.

Similarly one defines the class Δ^1_1 of *effective Borel* sets, Σ^1_1 of *effective analytic* and so on.

A function $f : \mathcal{X} \to \mathcal{Y}$ is Σ^0_1*-recursive* if and only if the set $R^f \subseteq \mathcal{X} \times \omega$, $R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$, is Σ^0_1.

A point $x \in \mathcal{X}$ is Δ^1_1 *point* if the relation $U \subseteq \omega$ which is defined by $s \in U \iff x \in N(\mathcal{X}, s)$, is in Δ^1_1.

Similarly one defines the *relativized* pointclasses with respect to some parameter ε.

$N(\mathcal{X}, s) =$ the ball with center $x_{(s)_0}$ and radius $\frac{(s)_1}{(s)_2+1}$.

A set $P \subseteq \mathcal{X}$ is *semirecursive* if $P = \bigcup_{i\in\omega} N(\mathcal{X}, \alpha(i))$ where α is a recursive function from ω to ω.

$\Sigma^0_1 =$ all semirecursive sets
$\rightsquigarrow$ effective open sets.

$\Pi^0_1 =$ the complements of semirecursive sets
$\rightsquigarrow$ effective closed sets.

Similarly one defines the class Δ^1_1 of *effective Borel* sets, Σ^1_1 of *effective analytic* and so on.

A function $f : \mathcal{X} \to \mathcal{Y}$ is Δ^1_1*-recursive* if and only if the set $R^f \subseteq \mathcal{X} \times \omega$, $R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$, is Δ^1_1.

A point $x \in \mathcal{X}$ is Δ^1_1 *point* if the relation $U \subseteq \omega$ which is defined by $s \in U \iff x \in N(\mathcal{X}, s)$, is in Δ^1_1.

Similarly one defines the *relativized* pointclasses with respect to some parameter ε.

Theorem. Every Δ^1_1 subset of a recursively presented Polish space is the recursive injective image of a Π^0_1 subset of $\mathcal{N}$.

Theorem. (G.) Suppose that $(\mathcal{X}, \mathcal{T})$ is a recursively presented Polish space, d is a suitable distance function for $(\mathcal{X}, \mathcal{T})$ and A is a Δ^1_1 subset of $\mathcal{X}$. There exists an $\varepsilon_A \in \mathcal{N}$, which is recursive in Kleene's O and a Polish topology $\mathcal{T}_\infty$ with suitable distance function d_∞, which extends $\mathcal{T}$ and has the following properties:

(1) The Polish space $(\mathcal{X}, \mathcal{T}_\infty)$ is ε_A-recursively presented.

(2) The set A is a $\Delta^0_1(\varepsilon_A)$ subset of $(\mathcal{X}, d_\infty)$.

(3) If $B \subseteq \mathcal{X}$ is a $\Delta^1_1(\alpha)$ subset of $(\mathcal{X}, d)$, where $\alpha \in \mathcal{N}$, then B is a $\Delta^1_1(\varepsilon_A, \alpha)$ subset of $(\mathcal{X}, d_\infty)$.

(4) If $B \subseteq \mathcal{X}$ is a $\Delta^1_1(\varepsilon_A, \alpha)$ subset of $(\mathcal{X}, d_\infty)$, where $\alpha \in \mathcal{N}$, then B is a $\Delta^1_1(\varepsilon_A, \alpha)$ subset of $(\mathcal{X}, d)$.

Remark. If the inverse function in the Lusin-Suslin Theorem is continuous, then the set A that we start with is G_δ.

Lemma. (G.) For every $A \subseteq \mathcal{N}$ in Π^0_2 there is a set $F \subseteq \mathcal{N}$ in Π^0_1 and a recursive function $\pi : \mathcal{N} \to \mathcal{N}$ which is injective on A such that $\pi[F] = A$ and the inverse π^{-1} is continuous.

Corollary. (G.) Suppose that A is a Δ^1_1 subset of $\mathcal{N}$, which is also in $\underset{\sim}{\boldsymbol{\Delta}}{}^0_2$ and assume moreover that the class Δ^1_1 is dense in A and $\mathcal{N} \setminus A$. Then one can choose the previous parameter ε_A in Δ^1_1.

Sketch of the proof. It's just a sketch - really! From of a theorem of Louveau the set A is in $\Delta^0_2(\varepsilon)$ for some $\varepsilon \in \Delta^1_1$. Apply the previous lemma and proceed as usual.

Theorem (The Strong Δ-Selection Principal). Suppose that $\mathcal{Z}$ and $\mathcal{Y}$ are recursively presented Polish spaces and that $P \subseteq \mathcal{Z} \times \mathcal{Y}$ is in Π^1_1 and such that for all $z \in \mathcal{Z}$ there exists $y \in \Delta^1_1(z)$ such that $(z, y) \in P$. Then there exists a Δ^1_1-recursive function $f : \mathcal{Z} \to \mathcal{Y}$ such that $(z, f(z)) \in P$ for all $z \in \mathcal{Z}$.

Corollary. (G.) Suppose that $\mathcal{Z}$ is a Polish space, $\mathcal{X}$ is a closed subset of $\mathcal{N}$ and that P is a Borel subset of $\mathcal{Z} \times \mathcal{X}$ such that the sets P_z and $\mathcal{X} \setminus P_z$ are infinite for all $z \in \mathcal{Z}$. Assume moreover that $(*)$ $\Delta^1_1(z)$ is dense in both P_z and $\mathcal{X} \setminus P_z$ for all $z \in \mathcal{Z}$.

Then there is a Borel-measurable function $f : \mathcal{Z} \to \mathcal{N}$ such that $f(z)$ "encodes" a distance function d_z on $\mathcal{X}$ such that: (1) the space $(\mathcal{X}, d_z)$ is complete and separable, (2) the topology $\mathcal{T}_{d_z}$ extends $\mathcal{T}$ and (3) P_z is d_z-clopen, for all $z \in \mathcal{Z}$.

Thanks to results of Tanaka, Sacks, Thomason and Hinman, we may replace the effective condition $(*)$ with one of the following classical conditions:

(1) there is a "reasonable" Borel measure μ on $\mathcal{X}$ such that for all open V and for all $z \in \mathcal{Z}$ if $P_z \cap V \neq \emptyset$ we have that $P_z \cap V$ is countable or $\mu(P_z \cap V) > 0$. Similarly for $\mathcal{X} \setminus P_z$;

(2) P_z is countable or co-countable for all $z \in \mathcal{Z}$.