Classes of Polish spaces under effective Borel isomorphism

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The motivation

It is essential for the development of effective descriptive set theory to deal with spaces which are either a finite product of the naturals or are perfect, i.e., they have no isolated points.

Problem. For all recursive Polish spaces $\mathcal{X}$ and $\mathcal{Y}$ there exists a Σ^1_1 subset of $\mathcal{Y} \times \mathcal{X}$, which is universal for $\underset{\sim}{\mathbf{\Sigma}}^1_1 \upharpoonright \mathcal{X}$.

The preceding statement is correct for a *perfect* $\mathcal{Y}$ but it is open whether it is true for an arbitrary uncountable $\mathcal{Y}$.

The good news. Most of the effective theory on perfect spaces can be carried out to the general case. For example $\Sigma^1_1 \upharpoonright \mathcal{X} \neq \Pi^1_1 \upharpoonright \mathcal{X}$.

The interesting news. Some very few but basic results cannot be carried out, and they all have a common cause: an arbitrary recursive Polish space does not seem to contain (in fact does not contain) the recursive image of 2^ω.

Basic notions of effective theory

Suppose that $(\mathcal{X}, d)$ is a complete separable metric space and that $(x_n)_{n\in\omega}$ is a sequence in $\mathcal{X}$. We define $P_< \subseteq \omega^4$ as follows

$$P_<(i,j,k,n) \iff d(x_i, x_j) < k/(n+1).$$

Similarly we define the relation $P_\leq$.

The sequence $(x_n)_{n\in\omega}$ is a *recursive presentation* of $(\mathcal{X}, d)$, if
(1) it is a dense sequence and
(2) the relations $P_<$ and $P_\leq$ are recursive.

The space of reals $\mathbb{R}$, the Baire space $\mathcal{N} = \omega^\omega$ and ω^k admit a recursive presentation in other words they are *recursively presented*. Some other examples: $\mathbb{R} \times \omega$, $\mathbb{R} \times \mathcal{N}$. However not all complete separable metric spaces are recursively presented.

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Without loss of generality we will deal with recursively presented metric spaces.

Polish spaces $\leftrightsquigarrow$ complete separable metric spaces

recursive Polish spaces $\leftrightsquigarrow$ recursively presented (complete separable) metric spaces

$N(\mathcal{X}, s)$ = the ball with center $x_{(s)_0}$ and radius $\frac{(s)_1}{(s)_2+1}$.

A set $P \subseteq \mathcal{X}$ is *semirecursive* if $P = \bigcup_{i\in\omega} N(\mathcal{X}, \alpha(i))$ where α is a recursive function from ω to ω.

$\Sigma^0_1 =$ all semirecursive sets
$\rightsquigarrow$ effective open sets.

$\Pi^0_1 =$ the complements of semirecursive sets
$\rightsquigarrow$ effective closed sets.

$\Sigma^1_1 =$ projections of Π^0_1 sets
$\rightsquigarrow$ effective analytic sets.

$\Pi^1_1 =$ the complements of Σ^1_1 sets
$\rightsquigarrow$ effective coanalytic sets.

$\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1 =$ effective Borel sets (Kleene).

Relativization. If Γ is one of the preceding pointclasses and $y \in \mathcal{Y}$ we define the class

$$\Gamma(y) = \{P_y \subseteq \mathcal{X} \mid P \subseteq \mathcal{Y} \times \mathcal{X} \textit{ is in } \Gamma, \mathcal{X} \textit{ is recursive Polish}\}.$$

Γ-recursive functions. A function $f : \mathcal{X} \to \mathcal{Y}$ is Γ-recursive if the set $R^f \subseteq \mathcal{X} \times \omega$ defined by

$$R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s),$$

is in Γ. We are mostly interested in the case $\Gamma = \Delta^1_1$ (effective Borel measurable functions).

Points in Γ. A point $x \in \mathcal{X}$ is a Γ *point* if the relation $U \subseteq \omega$ defined by

$$U(s) \iff x \in N(\mathcal{X}, s)$$

is in Γ. We are mostly interested in the cases $\Gamma = \Delta^1_1, \Delta^1_1(y)$.

We write $x =_{\mathrm{h}} y$ if $x \in \Delta^1_1(y)$ and $y \in \Delta^1_1(x)$. In this case we say that x and y have the same *hyperdegree*.

Fact. If $f : \mathcal{X} \to \mathcal{Y}$ is Δ^1_1-recursive then $f(x) \in \Delta^1_1(x)$, and if f is injective then $f(x) =_{\mathrm{h}} x$ for all x.

The problem of Δ^1_1-isomorphism

Every perfect recursive Polish space is Δ^1_1-isomorphic to the Baire space.

Question. Does there exist an uncountable recursive Polish space which is not Δ^1_1-isomorphic to the Baire space? And if yes what else can be said about this kind of spaces?

Notation. We write $\mathcal{X} \preceq_{\Delta^1_1} \mathcal{Y}$ if there exists a Δ^1_1-injection $f : \mathcal{X} \rightarrowtail \mathcal{Y}$ and $\mathcal{X} \simeq_{\Delta^1_1} \mathcal{Y}$ if there exists a Δ^1_1-bijection $f : \mathcal{X} \rightarrowtail\!\!\!\!\rightarrow \mathcal{Y}$. It holds

$$\mathcal{X} \simeq_{\Delta^1_1} \mathcal{Y} \iff \mathcal{X} \preceq_{\Delta^1_1} \mathcal{Y} \textit{ and } \mathcal{Y} \preceq_{\Delta^1_1} \mathcal{X}.$$

It also holds

$$\omega \preceq_{\Delta^1_1} \mathcal{X} \preceq_{\Delta^1_1} \mathcal{N}$$

for all recursive Polish $\mathcal{X}$.

Definition and main properties

Definition

For every tree T on ω we define the space $(\mathcal{N}^T, d^T)$ as follows

$$\mathcal{N}^T = T \cup [T]$$

and

$$d^T(x, y) = (\text{least } n[x(n) \neq y(n)] + 1)^{-1}$$

for $x, y \in \mathcal{N}^T$. It is easy to verify that d^T is a metric on $\mathcal{N}^T$.

There are various ways to view these spaces.

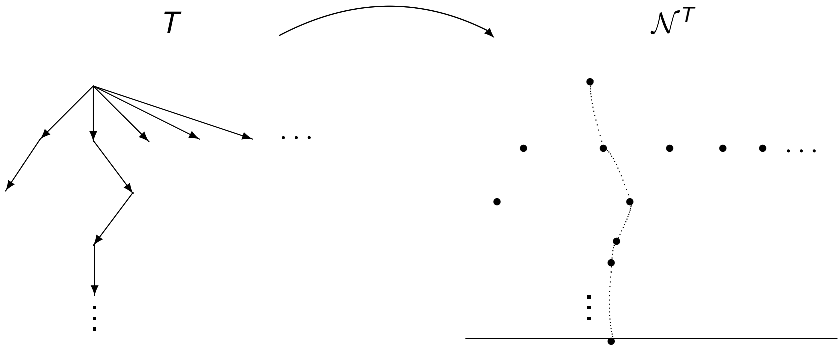


Figure: The space $\mathcal{N}^T$ as a subset of the plane.

Theorem

Suppose that T is a tree on ω.

(1) *Every point of T is an isolated point of $\mathcal{N}^T$.*

(2) *The tree T is dense in $\mathcal{N}^T$.*

(3) *If T is a recursive tree the space $(\mathcal{N}^T, d^T)$ is recursively presented and is isometric to a Π^0_1 subset of $\mathcal{N}$.*

(4) *If T is recursive then T is a Σ^0_1 subset of $\mathcal{N}^T$ and so $[T]$ is a Π^0_1 subset of $\mathcal{N}^T$.*

(5) *For all $x \in [T]$ we have that*

$$x \in \Delta^1_1 \cap \mathcal{N}^T \iff x \in \Delta^1_1 \cap \mathcal{N}.$$

(6) *Every recursively presented metric space $\mathcal{X}$ is Δ^1_1-isomorphic to a space of the form $\mathcal{N}^T$ for a recursive T. From this it follows that $\Sigma^1_1 \upharpoonright \mathcal{X} \neq \Pi^1_1 \upharpoonright \mathcal{X}$ for all $\mathcal{X}$.*

Let us write

$$[T] \preceq_{\Delta^1_1} [S]$$

if there exists a Δ^1_1-recursive function $f : \mathcal{N} \to \mathcal{N}$ which is injective on $[T]$ and $f[[T]] \subseteq [S]$.

It is easy to see that $[T] \preceq_{\Delta^1_1} [S]$ implies $\mathcal{N}^T \preceq_{\Delta^1_1} \mathcal{N}^S$. The converse in general not true, however it does hold in the categories of spaces that we will focus on.

A counterexample

Suppose that T is a recursive tree on ω with non-empty body such that $\alpha \notin [T]$ for all $\alpha \in \Delta^1_1 \cap \mathcal{N}$. (Kleene)

If x is a Δ^1_1 point of $\mathcal{N}^T = T \cup [T]$ then x cannot belong to $[T]$, for otherwise x would be a Δ^1_1 point of $\mathcal{N}$ belonging to $[T]$. Hence

$$\Delta^1_1 \cap \mathcal{N}^T = T$$

in particular the set of all Δ^1_1 points of $\mathcal{N}^T$ is semirecursive.

Fact. For every perfect $\mathcal{X}$ the set $\Delta^1_1 \cap \mathcal{X}$ is a proper Π^1_1 set.

It follows that the preceding space $\mathcal{N}^T$ is not Δ^1_1-isomorphic to the Baire space.

Definition and basic facts

Definition

A recursive tree T on ω is a *Kleene tree* if the body $[T]$ is non-empty and does not contain Δ^1_1-members. A space of the form $\mathcal{N}^T$ is a *Kleene space* if T is a Kleene tree.

A Kleene space is an uncountable set and the set of all of its Δ^1_1 points is a semirecursive set, so no Kleene space is Δ^1_1-isomorphic to the Baire space. In fact Kleene spaces characterize the class of all spaces for which the set of their Δ^1_1 points is Δ^1_1.

Theorem

An uncountable recursively presented metric space $\mathcal{X}$ is Δ^1_1-isomorphic to a Kleene space if and only if $\Delta^1_1 \cap \mathcal{X}$ is Δ^1_1.

Corollary

If $\mathcal{Y}$ is a Kleene space, $\mathcal{X}$ is uncountable and $\mathcal{X} \preceq_{\Delta^1_1} \mathcal{Y}$ then $\mathcal{X}$ is Δ^1_1-isomorphic to a Kleene space.

Lemma

For Kleene trees T and S we have that

$$\mathcal{N}^T \preceq_{\Delta^1_1} \mathcal{N}^S \iff [T] \preceq_{\Delta^1_1} [S].$$

Moving downwards

Theorem

For every Kleene tree T, there is some initial segment u of the leftmost infinite branch of T such that

$$\mathcal{N}^{T_u} \prec_{\Delta^1_1} \mathcal{N}^T.$$

It follows that every Kleene space $\mathcal{X}$ is the top of an infinite sequence of Kleene spaces which is strictly decreasing under $\preceq_{\Delta^1_1}$,

$$\mathcal{X} \succ_{\Delta^1_1} \mathcal{X}_1 \succ_{\Delta^1_1} \mathcal{X}_2 \succ_{\Delta^1_1} \dots.$$

Sketch of the proof.

There exists some $\gamma \in [T]$ such that $\alpha_L \notin \Delta^1_1(\gamma)$, where α_L is the leftmost infinite branch of T (Gandy Basis Theorem).

If we had $\Delta^1_1(\gamma) \cap [T_u] \neq \emptyset$ for all initial segments u of α_L, then α_L would be the unique infinite branch which lies on the left of every $\beta \in \Delta^1_1(\gamma) \cap [T]$. The latter implies that $\{\alpha_L\}$ is in $\Sigma^1_1(\gamma)$, from which it follows $\alpha_L \in \Delta^1_1(\gamma)$, a contradiction.

We pick some $u \sqsubseteq \alpha_L$ such that $[T_u] \cap \Delta^1_1(\gamma) = \emptyset$. Any Δ^1_1 function

$$f : [T] \to [T_u]$$

would carry γ to a $\Delta^1_1(\gamma)$ point inside $[T_u]$ contradicting the choice of u. $\dashv$

Moving sideways

Theorem (Fokina-Friedman-Törnquist)

There exists a sequence of recursive trees $(T_i)_{i\in\omega}$ and a sequence $(\alpha_i)_{i\in\omega}$ in $\mathcal{N}$ such that $\alpha_i \in [T_i]$ and $\Delta^1_1(\alpha_i) \cap [T_j] = \emptyset$ for all $i \neq j$.

In particular there does not exist a Δ^1_1-recursive function $f : \mathcal{N} \to \mathcal{N}$ which carries $[T_i]$ inside $[T_j]$ for all $i \neq j$.

The proof of the preceding theorem suggests the following idea: we verify that arbitrarily large "portions" of the required incomparability condition are satisfied, and using some compactness argument (Barwise compactness) we infer that this condition is satisfied as well.

Kreisel compactness

Theorem (Kreisel compactness)

Suppose that $\mathcal{X}$ is a recursively presented metric space, $D \subseteq \omega \times \mathcal{X}$ is Σ^1_1 and $P \subseteq \omega$ is Π^1_1. Suppose that for all Δ^1_1 sets $H \subseteq P$ the intersection $\cap_{n \in H} D_n$ is non-empty. Then

$$\cap_{n \in P} D_n \neq \emptyset.$$

Example (Does anyone know how to turn this blue?)

We consider the space Tr of all trees on ω and we define $D \subseteq \omega \times \mathcal{N} \times \mathrm{Tr}$ by

$$\begin{aligned} D(i, \alpha, T) \iff & T \text{ is a recursive tree and } \alpha \in [T] \text{ and} \\ & (\forall \beta \in \Delta^1_1)[\text{"}\beta \leq_{\mathrm{T}} \emptyset^i\text{"} \longrightarrow \beta \notin [T]], \end{aligned}$$

where "$\beta \leq \emptyset^i$" is a Π^1_1 condition on (i, β) which is equivalent to $\beta \leq_{\mathrm{T}} \emptyset^\xi$, whenever i is a notation for the (recursive) ordinal ξ in some system of ordinal notations, say Kleene's O.

The set D is Σ^1_1 and it is relatively easy to show that $\cap_{n \in H} D_n \neq \emptyset$ for all Δ^1_1 sets $H \subseteq O$. Hence from Kreisel compactness there exists some (α, T) in the intersection $\cap_{n \in O} D_n \neq \emptyset$.

It follows that T is a Kleene tree.

The incomparability Lemma in Kleene spaces

Lemma

For all Kleene trees $T_1, \dots, T_n$ there exists a Kleene tree S and $\alpha_1, \dots, \alpha_n, \beta$ in $[T_1], \dots, [T_n]$ and $[S]$ respectively such that

$$\Delta^1_1(\beta) \cap [T_k] = \emptyset \quad \textit{and} \quad \gamma \neq_{\mathrm{h}} \alpha_k$$

for all $\gamma \in [S]$ and all $k = 1, \dots, n$. In particular we have $[T_k] \not\preceq_{\Delta^1_1} [S]$ and $[S] \not\preceq_{\Delta^1_1} [T_k]$ for all $k = 1, \dots, n$.

Theorem

For every finite sequence $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ of Kleene spaces, there is a Kleene space $\mathcal{Y}$ which is $\preceq_{\Delta^1_1}$-incomparable with each $\mathcal{X}_i$. It follows that every Kleene space is the member of an infinite sequence of Kleene spaces which are pairwise incomparable under $\preceq_{\Delta^1_1}$.

Idea of the proof of the Incomparability Lemma.

We assume that have just one Kleene tree T. We say that $D(i, \alpha, \beta, S)$ holds exactly when

$$\begin{aligned} &\alpha \in [T] \ \&\ S \text{ is a recursive tree } \&\ \beta \in [S] \\ &\&\ (\forall \gamma \in \Delta^1_1)[\text{"}\gamma \leq \emptyset^i\text{"} \longrightarrow \gamma \notin [S]] \\ &\&\ (\forall \delta \in \Delta^1_1(\beta))[\delta \notin [T]] \\ &\&\ (\forall \gamma \in \Delta^1_1(\alpha))[\alpha \in \Delta^1_1(\gamma) \longrightarrow \gamma \notin [S]]. \end{aligned}$$

$\dashv$

Moving upwards

Theorem

For every Kleene tree T there exists a Kleene tree K such that $\mathcal{N}^T \prec_{\Delta^1_1} \mathcal{N}^K$.

It follows that every Kleene space $\mathcal{X}$ is the bottom of an infinite sequence of Kleene spaces which is strictly increasing under $\preceq_{\Delta^1_1}$,

$$\mathcal{X} \prec_{\Delta^1_1} \mathcal{X}_1 \prec_{\Delta^1_1} \mathcal{X}_2 \prec_{\Delta^1_1} \dots .$$

Idea of the proof.

We choose a Kleene tree T_1 such that the spaces $\mathcal{N}^T, \mathcal{N}^{T_1}$ are $\preceq_{\Delta^1_1}$-incomparable and we take the topological sum $\mathcal{N}^T \oplus \mathcal{N}^{T_1}$. We continue similarly. $\dashv$

The (strong form) of the Spector-Gandy Theorem

Theorem (Spector-Gandy)

For every Π^1_1 set $P \subseteq \omega$ there exists a recursive tree T on ω such that

$$\begin{aligned} P(n) &\iff (\exists \alpha \in \Delta^1_1)[(n)\hat{}\alpha \in [T]] \\ &\iff (\exists ! \alpha)[(n)\hat{}\alpha \in [T]]. \end{aligned}$$

Definition

A recursive tree T on ω is a *Spector-Gandy tree* if

$$(\exists x \in \Delta^1_1)[x \in [T] \ \&\ x(0) = n] \iff (\exists! x)[x \in [T] \ \&\ x(0) = n]$$

for all $n \in \omega$ and the Π^1_1 set

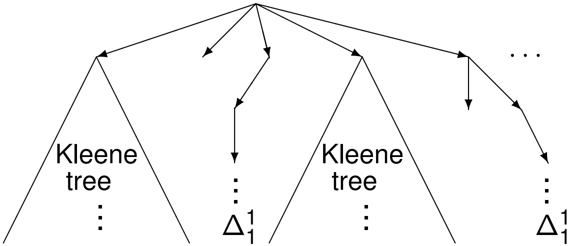
$$P(n) \iff (\exists x \in \Delta^1_1)[x \in [T] \ \&\ x(0) = n]$$

is not Δ^1_1.

A space of the form $\mathcal{N}^T$ is a *Spector-Gandy space* if T is a Spector-Gandy tree.

The set P from above is the *companion set* of the Spector-Gandy space $\mathcal{N}^T$.

A typical Spector-Gandy tree



Basic facts

Remark - this is blue, this is blue, this is blue . . .

1. The set of Δ^1_1-points of a Spector-Gandy space is not Δ^1_1, so no Spector-Gandy space is Δ^1_1-isomorphic to a Kleene space.
2. Every Spector-Gandy space contains a Kleene space.
3. Every Kleene space is contained in a Spector-Gandy space.
4. No Spector-Gandy space is Δ^1_1-isomorphic to the Baire space.
5. For every Spector-Gandy trees T and S we have that

$$\mathcal{N}^T \preceq_{\Delta^1_1} \mathcal{N}^S \iff [T] \preceq_{\Delta^1_1} [S].$$

Kreisel compactness applied to Spector-Gandy spaces

Removing Δ^1_1 points arbitrarily guarantees that we will end up with a Kleene tree. In order to make sure that we will get a Spector-Gandy tree at the end, we will use a given Spector-Gandy tree as a pilot.

Lemma (The pilot Lemma)

Suppose that K is a Spector-Gandy tree with a companion set P and that T is a recursive tree such that $T_{(n)} = K_{(n)}$, whenever $n \in P$ or $[T_{(n)}]$ has a Δ^1_1 member. Then T is also a Spector-Gandy tree with the same companion set P.

Moving downwards

We say that a Spector-Gandy space $\mathcal{N}^T$ is *nice* if its companion set is the well-founded part of a recursive linear ordering.

Lemma

For every nice Spector-Gandy tree T there exists a Spector-Gandy tree $S \subseteq T$ with the same companion set and some $\gamma \in [T]$ not in Δ^1_1, such that $\alpha \neq_{\mathrm{h}} \gamma$ for all $\alpha \in [S]$.

Theorem

Every nice Spector-Gandy space is the top of a strictly decreasing sequence of nice Spector-Gandy trees under $\preceq_{\Delta^1_1}$.

Idea of the proof of the Lemma.

We define the set $D \subseteq \omega \times \mathcal{N} \times \mathrm{Tr}$ by saying that $D(i, \gamma, S)$ holds exactly when

$$\begin{aligned}
& S \text{ is a recursive tree and } S \subseteq T \\
\& \ & \gamma \in [T] \ \& \ \gamma \notin \Delta^1_1 \\
\& \ & (\forall j \preceq i)[S_{(j)} = T_{(j)}] \\
\& \ & (\forall n)(\forall \alpha^* \in \Delta^1_1)[(n)^\frown\alpha^* \in [S] \longrightarrow S_{(n)} = T_{(n)}] \\
\& \ & (\forall \alpha \in \Delta^1_1(\gamma))[\gamma \in \Delta^1_1(\alpha) \longrightarrow \alpha \notin [S]].
\end{aligned}$$

$\dashv$

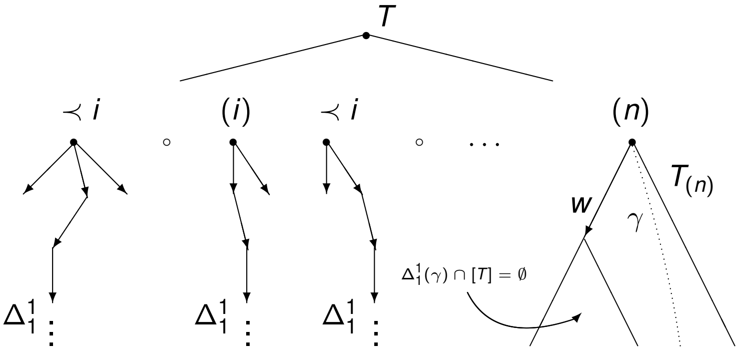


Figure: The approximation at the stage i.

Moving sideways and upwards

One can get similar results for nice Spector-Gandy spaces by applying the Kreisel compactness technique and the Incomparability Lemma in Kleene spaces.

Also we can move upwards using topological sums.

Thank you for your attention!