The fixed point property on tree-like Banach spaces

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Definitions.(1) Let $(X, \|.\|)$ be an infinite dimensional Banach space, let K be a weakly compact and convex subset of X and let $T : K \to K$ be a map such that $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in K$. Such a map T is called *non-expansive*.
(2)We say that X has the *fixed point property* (f.p.p.) if for every K and every $T : K \to K$ as above, the map T has a fixed point (i.e. there is $x \in K$ such that $Tx = x$).

Banach's fixed point theorem: If $T : K \to K$ is a contraction (i.e. there is $0 < L < 1$ such that $\|Tx - Ty\| \leq L\|x - y\|$ $\forall x, y \in K$), then T has a unique fixed point.
Schauder's fixed point theorem: If K is compact and convex, then any continuous map $T : K \to K$ has a fixed point.

General problem: "Characterize" the spaces X which have the f.p.p.

- Any Hilbert space has the f.p.p
  (F.E. Browder, 1965)
- Any uniformly convex Banach space has the f.p.p
  (F. E. Browder, 1965) $[\ell_p,\ L_p,\ 1 < p < \infty]$
- Any Banach space with normal structure has the f.p.p
  (W. A. Kirk, 1965)

Minimal Sets

In the following K will always be a convex, weakly compact set and $T : K \to K$ a non-expansive map.

Definition. Let C be a convex, weakly compact subset of K such that $T(C) \subseteq C$. We say that C is *minimal for T* if there is no strictly smaller subset of C with the same properties.
(i.e. $E \subseteq C$ convex, weakly compact, $T(E) \subseteq E \Rightarrow E = C$)

T has a fixed point $x \Leftrightarrow$ the set $C = \{x\}$ is minimal

Proposition. There always are subsets C of K which are minimal for the map T.

Standard technique:
Assume that K is minimal and then show that $diam(K) = 0$

Proposition. Suppose that K is a weakly compact, convex set and $T : K \to K$ is non-expansive. Then there is a sequence (x_n) in K such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.
(x_n): *approximate fixed point sequence* for the map T

Theorem. [Karlovitz, 1976] Suppose that K is minimal for the non-expansive map T and let (x_n) be an approximate fixed point sequence in K. Then, for all $x \in K$,

$$\lim_{n\to\infty} \|x - x_n\| = diam(K).$$

Definition. The space X has *normal structure* if every weakly compact, convex subset K with $diam(K) > 0$ contains a non-diametral point, i.e. a point $x_0 \in K$ such that

$$\sup\{\|x - x_0\| \mid x \in K\} < diam(K).$$

Theorem. If X has normal structure, then X has the f.p.p.

Proof. Let K be weakly compact, convex and minimal for the non-expansive T. Assume that $diam(K) > 0$. Then K contains a non-diametral point x_0.
If (x_n) is an approximate fixed point sequence in K, then

$$\lim_{n\to\infty} \|x_n - x_0\| = diam(K).$$

On the other hand,

$$\lim_{n\to\infty} \|x_n - x_0\| \leq \sup\{\|x - x_0\| \mid x \in K\} < diam(K)$$

and we have a contradiction. Therefore $diam(K) = 0$.

Theorem.[Alspach, 1981] The space L_1 fails the f.p.p.

Theorem.[Maurey, 1981] The space c_0 has the f.p.p.

Theorem.[Maurey, 1981] Let (x_n) and (y_n) be approximate fixed point sequences for the map T such that $\lim_{n\to\infty} \|x_n - y_n\|$ exists. Then there is an approximate fixed point sequence (z_n) in K such that

$$\lim_{n\to\infty} \|z_n - x_n\| = \lim_{n\to\infty} \|z_n - y_n\| = \frac{1}{2} \lim_{n\to\infty} \|x_n - y_n\|.$$

Tree-like Banach spaces

ℓ_1: separable, $\ell_1^* = \ell_\infty$: non-separable

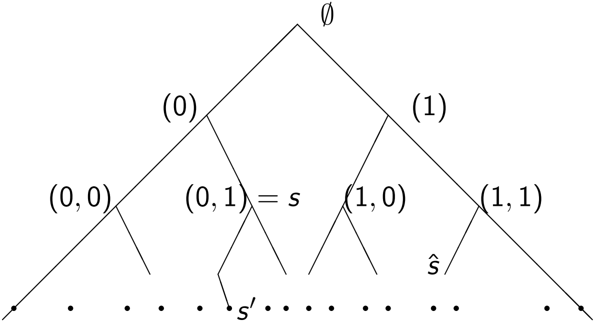
X separable, $\ell_1 \subset X \Rightarrow X^*$: non-separable

Problem. Is ℓ_1 the "only" separable space with non-separable dual?

Answer: Negative. There are separable spaces with non-separable dual, which do not contain any isomorphic copy of ℓ_1.

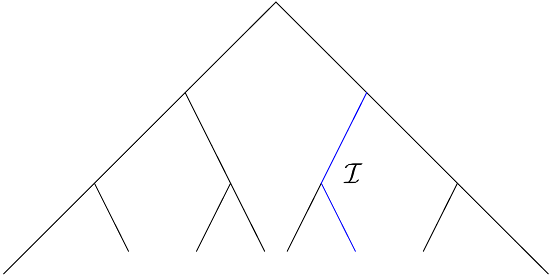
The James Tree space (JT)

Consider the dyadic tree $\mathcal{D} = \cup_{n=0}^{\infty}\{0,1\}^n$, that is the set of all finite sequences s in $\{0,1\}$.



The elements of the set $\mathcal{D}$ are called *nodes*.
In this tree we have a partial order:
$s < s'$ and $s, \hat{s}$ are non-comparable.

Definition. Let $\mathcal{I}$ be a finite subset of $\mathcal{D}$ such that $\mathcal{I}$ is linearly ordered and if $s, s' \in \mathcal{I}$ and $s < t < s'$ then $t \in \mathcal{I}$. The set $\mathcal{I}$ is called a segment on the tree $\mathcal{D}$.



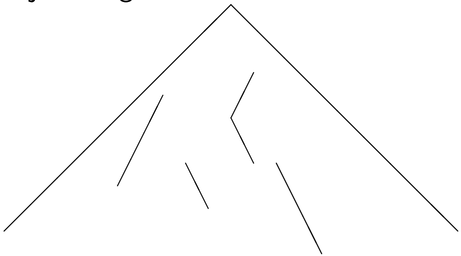
Let $c_{00}(\mathcal{D}) = \{x : \mathcal{D} \to \mathbb{R} \mid x \text{ has finite support}\}$.
If $\mathcal{I}$ is a segment, then we set

$$\mathcal{I}^*(x) = \sum_{s \in \mathcal{I}} x(s).$$

For any $x \in c_{00}(\mathcal{D})$ we define the norm

$$\|x\| = \max \Big(\sum_{k=1}^{r} (\mathcal{I}_k^*(x))^2 \Big)^{1/2}$$

where the maximum is taken over all finite families $\mathcal{S} = \{\mathcal{I}\}_{k=1}^{r}$ of pairwise disjoint segments.



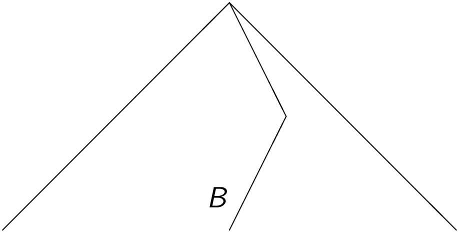
The space JT is the completion of $(c_{00}(\mathcal{D}), \|.\|)$

- *JT is separable:* For any node s we define $e_s : \mathcal{D} \to \mathbb{R}$ with

$$e_s(t) = \begin{cases} 0, & \text{for any } t \neq s; \\ 1, & \text{if } t = s. \end{cases}$$

Then $JT = \overline{span}\{e_s \mid s \in \mathcal{D}\}$ and $\mathcal{D}$ is a countable set.

• *JT^* is non-separable:* For any branch B we define the functional $B^* : JT \to \mathbb{R}$ such that $B^*(x) = \sum_{s \in B} x(s)$.



If $B_1 \neq B_2$ then $\|B_1^* - B_2^*\| = 1$ and we have $2^{\aleph_0}$ branches.

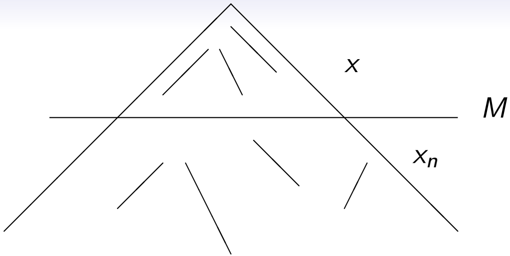
• JT does not contain any isomorphic copy of ℓ_1.

Definition. A Banach space X satisfies the *Opial condition* if whenever a sequence (x_n) in X converges weakly to 0 and $\liminf \|x_n\| = 1$, then

$$\liminf \|x_n + x\| > 1 \quad \text{for all } x \neq 0.$$

Theorem. If X satisfies the Opial condition, then X possesses normal structure.

Theorem.[Khamsi 1989, Kuczumow and Reich 1994] The space JT satisfies the Opial condition.
Proof. Let (x_n) be a sequence in JT such that (x_n) converges weakly to 0 and $\liminf \|x_n\| = 1$ and let $x \in JT$, $x \neq 0$.



There is a level M such that $x(s)$ is (almost) zero for any node s with $lev(s) > M$.

(x_n) converges weakly to 0. Therefore $x_n(s) \to 0$ for every s. If n is quite large, then $x_n(s)$ is (almost) zero for every node s with $lev(s) \leq M$.

$$\|x\| = \Big(\sum_k (\mathcal{I}_k^*(x))^2 \Big)^{1/2} \quad \|x_n\| = \Big(\sum_\ell (\mathcal{J}_\ell^*(x_n))^2 \Big)^{1/2}$$

Set $\mathcal{S} = \{\mathcal{I}_k\}_k \cup \{\mathcal{J}_\ell\}_\ell$.

Using the family $\mathcal{S}$ we estimate the norm of $x_n + x$:

$$\begin{aligned} \|x_n + x\| &\geq (\|x_n\|^2 + \|x\|^2)^{1/2} \\ \liminf \|x_n + x\| &\geq (1 + \|x\|^2)^{1/2} > 1. \end{aligned}$$

Definition. Let $\mathcal{S}$ be a finite family of pairwise disjoint segments of the dyadic tree. The family $\mathcal{S}$ is called *admissible* if for every segment $\mathcal{I} \in \mathcal{S}$ there is at most one segment $\mathcal{I}' \in \mathcal{S}$ such that $\min \mathcal{I} < \min \mathcal{I}'$

Consider the space

$$c_{00}(\mathcal{D}) = \{x : \mathcal{D} \to \mathbb{R} \mid x \text{ has finite support}\}$$

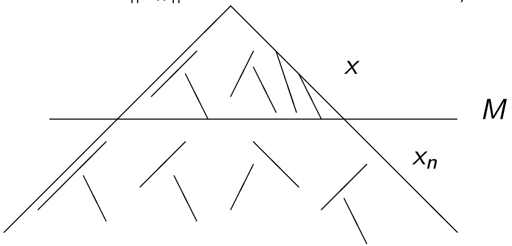
For any $x \in c_{00}(\mathcal{D})$ we define

$$\|x\| = \max \Big(\sum_{k=1}^{r} (\mathcal{I}_k^*(x))^2 \Big)^{1/2}$$

where the maximum is taken over all finite *admissible* families $\mathcal{S} = \{\mathcal{I}\}_{k=1}^{r}$ of pairwise disjoint segments.

The space X is the completion of $(c_{00}(\mathcal{D}), \|.\|)$.

Let (x_n) be a sequence in the space X such that (x_n) converges weakly to 0, $\liminf \|x_n\| = 1$ and let $x \in X$, $x \neq 0$.



$$\|x\| = \Big(\sum (\mathcal{I}_k^*(x))^2 \Big)^{1/2} \quad \|x_n\| = \Big(\sum (\mathcal{J}_\ell^*(x+n))^2 \Big)^{1/2}$$

$$\mathcal{S} = \{\mathcal{I}_k\}_k \cup \{\mathcal{J}_\ell\}_\ell$$

Theorem. The space X does not possess normal structure.

Theorem. The space X has the fixed point property.

Proof. Let K be weakly compact, convex and minimal for the non-expansive map $T : K \to K$.

Suppose that $diam(K) > 0$. By multiplication with some positive constant we may assume that $diam(K) = 1$.

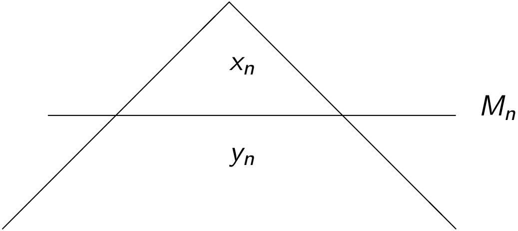
Let (x_n) be an approximate fixed point sequence for the map T in the set K, i.e. $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Since K is weakly compact, we may assume that (x_n) converges weakly to some point $x \in K$. By a translation, we may also assume that $0 \in K$ and (x_n) converges weakly to 0.

Since K is minimal, we know that $\lim_{n\to\infty} \|x_n - x\| = diam(K) = 1$ for every $x \in K$. Therefore $\lim_{n\to\infty} \|x_n\| = 1$.

$diam(K) = 1$, (x_n): a.f.p.s., $x_n \xrightarrow{w} 0$, $\lim_{n\to\infty} \|x_n\| = 1$

Choose a subsequence (y_n) of (x_n) as follows:
Fix $n \in \mathbb{N}$. There is a level M_n such that $x_n(s) = 0$ for every node s with $lev(s) \geq M_n$.
(x_k) converges weakly to 0. Hence, $x_k(s) \to 0$ for every $s \in \mathcal{D}$. We find $k_n \in \mathbb{N}$ such that $x_{k_n}(s) = 0$ for every s with $lev(s) \leq M_n$. Let $y_n = x_{k_n}$.

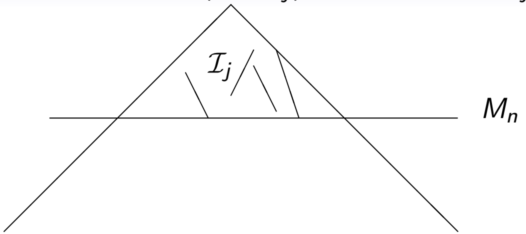


(x_n), (y_n) are a.f.p.s.'s and $\lim_{n\to\infty} \|x_n - y_n\| = 1$.

Fix $N \in \mathbb{N}$ and let $\delta = \frac{1}{2^N}$. By Maurey's theorem we find a sequence (z_n) in the set K such that
(1) (z_n) is an a.f.p.s. Therefore $\lim_{n\to\infty} \|z_n\| = 1$.
(2) $\lim \|z_n - y_n\| = \frac{1}{2^N} \lim \|x_n - y_n\|$, $\lim \|z_n - x_n\| = 1 - \frac{1}{2^N}$
$\lim \|z_n - y_n\| = \delta \lim \|z_n - x_n\| = 1 - \delta$
For every n there is an admissible family $\mathcal{S} = \{\mathcal{I}_j\}$ of pairwise disjoint segments on the dyadic tree, such that

$$\|z_n\|^2 = \sum_{\mathcal{I}_j \in \mathcal{S}} (\mathcal{I}_j^*(z_n))^2.$$

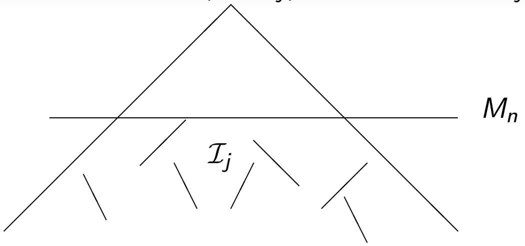
Case 1. Assume that $lev(\max \mathcal{I}_j) \leq M_n$ for every $\mathcal{I}_j \in \mathcal{S}$



$$
\begin{aligned}
\|z_n\|^2 = \sum (\mathcal{I}_j^*(z_n))^2 &= \sum (\mathcal{I}_j^*(z_n) - \mathcal{I}_j^*(y_n))^2 \\
&= \sum (\mathcal{I}_j^*(z_n - y_n))^2 \leq \|z_n - y_n\|^2
\end{aligned}
$$

$$
\begin{gathered}
\lim \|z_n\|^2 \leq \lim \|z_n - y_n\|^2 \\
1 \leq \delta^2
\end{gathered}
$$

Case 2. Assume that $lev(\min \mathcal{I}_j) \geq M_n$ for every $\mathcal{I}_j \in \mathcal{S}$



$$\|z_n\|^2 = \sum (\mathcal{I}_j^*(z_n))^2 = \sum (\mathcal{I}_j^*(z_n) - \mathcal{I}_j^*(x_n))^2$$
$$= \sum (\mathcal{I}_j^*(z_n - x_n))^2 \leq \|z_n - x_n\|^2$$

$$\lim \|z_n\|^2 \leq \lim \|z_n - x_n\|^2$$
$$1 \leq (1-\delta)^2$$

Case 3. Assume that $lev(\max \mathcal{I}_j) \leq M_n$ for every $\mathcal{I}_j \in \mathcal{S}_1 \subseteq \mathcal{S}$ and $lev(\min \mathcal{I}_j) \geq M_n$ for every $\mathcal{I}_j \in \mathcal{S} \setminus \mathcal{S}_1$.

$$
\begin{aligned}
\|z_n\|^2 = \sum (\mathcal{I}_j^*(z_n))^2 &= \sum_{\mathcal{I}_j \in \mathcal{S}_1} (\mathcal{I}_j^*(z_n))^2 + \sum_{\mathcal{I}_j \in \mathcal{S} \setminus \mathcal{S}_1} (\mathcal{I}_j^*(z_n))^2 \\
&\leq \|z_n - y_n\|^2 + \|z_n - x_n\|^2
\end{aligned}
$$

$$
\begin{aligned}
\lim \|z_n\|^2 \leq \lim \|z_n - y_n\|^2 + \lim \|z_n - x_n\|^2 &\Rightarrow \\
1 \leq \delta^2 + (1-\delta)^2 &\Rightarrow \\
1 \leq 1 - 2\delta(1+\delta).
\end{aligned}
$$

Result: The family $\mathcal{S}$ contains segments which pass through the level M_n, that is

$$lev(\min \mathcal{I}_j) < M_n < lev(\max \mathcal{I}_j)$$

$$\begin{aligned} \mathcal{S}_1 &= \{\mathcal{I}_j \in \mathcal{S} \mid lev(\max \mathcal{I}_j) \leq M_n\} \\ \mathcal{S}_2 &= \{\mathcal{I}_j \in \mathcal{S} \mid lev(\min \mathcal{I}_j) \geq M_n\} \\ \mathcal{S}_3 &= \{\mathcal{I}_j \in \mathcal{S} \mid \mathcal{I}_j \text{ pass through the level } M_n\} \neq \emptyset \end{aligned}$$

Each $\mathcal{I}_j \in \mathcal{S}_3$ is divided into two parts $\mathcal{I}_j = E_j \cup K_j$:

$$\begin{aligned} E_j &= \mathcal{I}_j \cap \{s \mid lev(s) < M_n\} \\ K_j &= \mathcal{I}_j \cap \{s \mid lev(s) \geq M_n\} \end{aligned}$$

For all sufficiently large n we have

$$1 \approx \|z_n\|^2 = \sum_{\mathcal{I}_j \in \mathcal{S}_1} (\mathcal{I}_j^*(z_n))^2 + \sum_{\mathcal{I}_j \in \mathcal{S}_2} (\mathcal{I}_j^*(z_n))^2 + \sum_{\mathcal{I}_j \in \mathcal{S}_3} (E_j^*(z_n) + K_j^*(z_n))^2$$

$$(a+b)^2 \leq (1+\frac{1}{\epsilon})a^2 + (1+\epsilon)b^2$$

$$(E_j^*(z_n) + K_j^*(z_n))^2 \leq (1+\frac{1}{\epsilon})(E_j^*(z_n))^2 + (1+\epsilon)(K_j^*(z_n))^2$$

$$\sum_{\mathcal{I}_j \in \mathcal{S}_3} (E_j^*(z_n)+K_j^*(z_n))^2 \leq (1+\frac{1}{\epsilon}) \sum_{\mathcal{I}_j \in \mathcal{S}_3} (E_j^*(z_n))^2 + (1+\epsilon) \sum_{\mathcal{I}_j \in \mathcal{S}_3} (K_j^*(z_n))^2$$

$$1 \approx \|z_n\|^2 \leq \sum_{\mathcal{I}_j \in \mathcal{S}_1} (\mathcal{I}_j^*(z_n))^2 + \sum_{\mathcal{I}_j \in \mathcal{S}_2} (\mathcal{I}_j^*(z_n))^2 +$$

$$(1 + \frac{1}{\epsilon}) \sum (E_j^*(z_n))^2 + (1+\epsilon) \sum (K_j^*(z_n))^2$$

$$= \Big(\sum_{\mathcal{I}_j \in \mathcal{S}_1} (\mathcal{I}_j^*(z_n))^2 + \sum (E_j^*(z_n))^2 \Big) + \frac{1}{\epsilon} \sum (E_j^*(z_n))^2$$

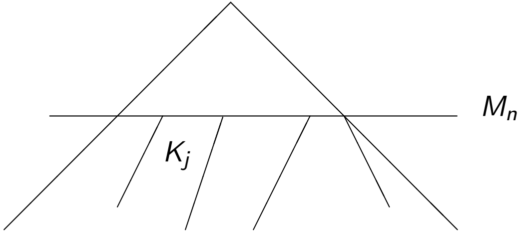
$$\Big(\sum_{\mathcal{I}_j \in \mathcal{S}_2} (\mathcal{I}_j^*(z_n))^2 + \sum (K_j^*(z_n))^2 \Big) + \epsilon \sum (K_j^*(z_n))^2$$

$$\leq \|z_n - y_n\|^2 + \frac{1}{\epsilon}\|z_n - y_n\|^2 + \|z_n - x_n\|^2 + \epsilon\|z_n - x_n\|^2$$

$$\approx \delta^2 + \frac{1}{\epsilon}\delta^2 + (1-\delta)^2 + \epsilon(1-\delta)^2$$

$$= 1 \quad \text{for} \quad \epsilon = \frac{\delta}{1-\delta}$$

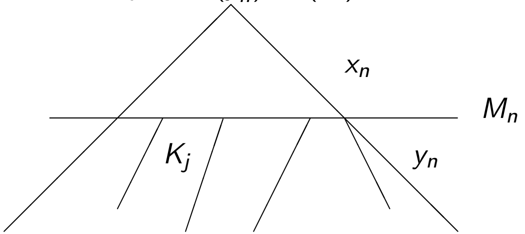
It follows that $\left(\sum(K_j^*(z_n))^2\right)^{1/2} \approx 1-\delta.$



Since $\|z_n - y_n\| \approx \delta$, we have that

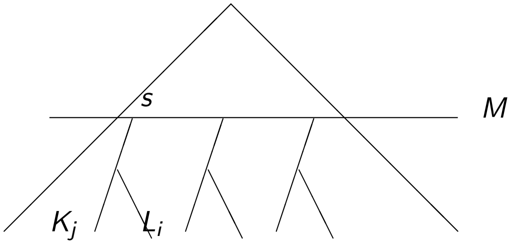
$$\left(\sum(K_j^*(y_n))^2\right)^{1/2} \approx 1-2\delta$$

We choose a subsequence (y'_n) of (x_n) as follows: Fix $n \in \mathbb{N}$



There is $\ell_n \in \mathbb{N}$ such that:
(i) $x_{\ell_n}(s) = 0$ for every s with $lev(s) \leq M_n$
(ii) $x_{\ell_n}(s) = 0$ for every $s \in \cup K_j$.
We set $y'_n = x_{\ell_n}$.

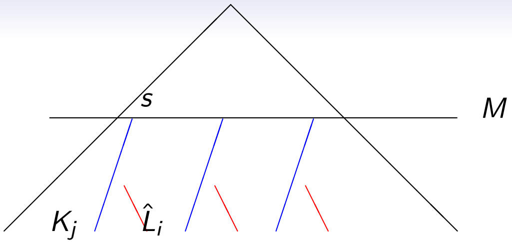
If we repeat the previous part of the proof, we find segments $\{L_i\}$ such that $\left(\sum(L_i^*(y'_n))^2\right)^{1/2} \approx 1 - 2\delta$ and for any i the minimum node of L_i lies on the level M_n.



There may be nodes s on the level M_n such that:
s is the minimum node of one K_j
s is the minimum node of one L_i

$y_n'(s) = 0$ for every $s \in K_j \cap L_i$. We set $\hat{L}_i = L_i \setminus (K_j \cap L_i)$ then we have

$$\Big(\sum (\hat{L}_i^*(y_n'))^2\Big)^{1/2} \approx 1 - 2\delta$$



$$\mathcal{S} = \{K_j\} \cup \{\hat{L}_i\}$$

Using the *admissible* family $\mathcal{S}$, we have

$$\|y_n - y_n'\| \approx (1 - 2\delta) + (1 - 2\delta) = 2 - 4\delta$$

On the other hand

$$\|y_n - y_n'\| \leq diam(K) = 1$$

and we have the final contradiction.

REMARKS

Proposition. For any $M > 0$, there is a subspace Y_M of X such that Y_M is isomorphic to c_0 and $d(Y_M, c_0) > M$.

$$d(Y_M, c_0) = \inf\{\|T\| \cdot \|T^{-1}\| \ : \ T : Y_M \to c_0 \text{ isomorphism, onto } c_0\}$$

Corollary. For any $M > 0$, there is a Banach space Y isomorphic to c_0 such that Y has the fixed point property and $d(Y, c_0) > M$.

Problem. Find a non-trivial class of Banach spaces such that the members of this class are isomorphic to each other and each member has the f.p.p.
(Trivial example: the Banach spaces isomorphic to ℓ_1)

Question. Let $M > 0$. Is there a subspace Y of c_0 such that Y is isomorphic to c_0 and $d(Y, c_0) > M$?

The Hagler Tree space (HT)

Question. Does HT have the fixed point property?

THANK YOU!