Effectivity and Polish group actions

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Spector's W

For all $e \in \omega$ we define the partial ordering $\leq_e$ on ω by

$$i \in \mathrm{Domain}(\leq_e) \iff \{e\}(\langle i, i\rangle) = 1$$
$$i \leq_e j \iff i, j \in \mathrm{Domain}(\leq_e) \ \& \ \{e\}(\langle i, j\rangle) = 1.$$

Spector's W is the set

$$\{e \in \omega \mid \leq_e \text{ is a well-ordering}\}.$$

Relativized version. For every x in a recursive Polish space $\mathcal{X}$ we define $\leq_e^x$ by replacing the term "recursive" with "x-recursive" and we put

$$\mathrm{W}^x = \{e \in \omega \mid \leq_e^x \text{ is a well-ordering}\}.$$

We say that W^x is the hyperjump of x.

Well-known facts

Theorem (Kleene Basis Theorem)

Every non-empty $\Sigma^1_1(x)$ set contains a member which is recursive in W^x.

Notation: ω_1^{CK} is the least non-recursive ordinal, ω_1^x is the least non-x-recursive ordinal, by $x \leq_{\mathrm{h}} y$ we mean that $x \in \Delta^1_1(y)$.

Definition. Given recursive Polish spaces $\mathcal{X}$, $\mathcal{Y}$ and $x \in \mathcal{X}$ we consider the sets

$$\mathrm{M}^{\mathcal{Y}} = \{y \in \mathcal{Y} \mid \omega_1^y = \omega_1^{\mathrm{CK}}\} \quad \text{and} \quad \mathrm{M}^{\mathcal{Y}}(x) = \{y \in \mathcal{Y} \mid \omega_1^{(x,y)} = \omega_1^x\}.$$

It is well known that the set $\mathrm{M}^{\mathcal{Y}}$ is Σ^1_1, Borel, and comeager. Similarly for $\mathrm{M}^{\mathcal{Y}}(x)$.

Theorem (Spector)

For all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ we have the following:

1. $\mathrm{W}^x \leq_{\mathrm{h}} y$ *implies* $\omega_1^x < \omega_1^y$.
2. $\omega_1^x < \omega_1^y$ *and* $x \leq_{\mathrm{h}} y$ *implies* $\mathrm{W}^x \leq_{\mathrm{h}} y$.

Corollary

For all $y \in \mathrm{M}^{\mathcal{Y}}(x)$ we have $\mathrm{W}^{(x,y)} \leq_{\mathrm{h}} (\mathrm{W}^x, y)$, and similarly for all $y \in \mathrm{M}^{\mathcal{Y}}$ we have that $\mathrm{W}^y \leq_{\mathrm{h}} (\mathrm{W}, y)$.

Proof.

Given $\omega_1^{(x,y)} = \omega_1^x$ we apply (2) of the preceding theorem of Spector with $x' = (x, y)$ and $y' = (\mathrm{W}^x, y)$. $\dashv$

The latter is related to

Theorem (Sacks)

For Cohen-generic (tree) T we have that $\mathrm{W}^T \leq_{\mathrm{h}} (\mathrm{W}, T)$.

Corollary

The hyperjump function $y \in \mathcal{Y} \mapsto \mathrm{W}^{(x,y)}$ *is Borel-measurable on* $\mathrm{M}^{\mathcal{Y}}(x)$.

An example

Suppose that $\mathcal{X}$ is (w.l.o.g. recursive) Polish space and that P is analytic, say P is $\Sigma^1_1(\alpha)$ for $\alpha \in \omega^\omega$. Then there exists some $\Pi^0_1(\alpha)$ set F such that $P(x) \iff (\exists \gamma \in \omega^\omega) F(x, \alpha)$. We compute

$$x \in P \cap \mathrm{M}^{\mathcal{X}}(\alpha) \iff (\exists \gamma \leq_{\mathrm{T}} \mathrm{W}^{(\alpha, x)}) F(x, \gamma) \ \& \ x \in \mathrm{M}^{\mathcal{X}}(\alpha)$$

(Kleene Basis)

$$\iff (\exists \gamma \in \Delta^1_1(\mathrm{W}^\alpha, x)) F(x, \gamma) \ \& \ x \in \mathrm{M}^{\mathcal{X}}(\alpha).$$

The relation $Q(x) \iff (\exists \gamma \in \Delta^1_1(\mathrm{W}^\alpha, x)) F(x, \gamma)$ is $\Pi^1_1(\mathrm{W}^\alpha)$ and hence coanalytic. It follows that set $P' := P \cap \mathrm{M}^{\mathcal{X}}(\alpha)$ is coanalytic, and therefore Borel. Moreover $P \triangle P' \subseteq \mathcal{X} \setminus \mathrm{M}^{\mathcal{X}}(\alpha)$, and the latter set is meager.

So we just proved that *analytic sets have the Baire property*.

Suppose that we have a group G acting continuously on a Polish space $\mathcal{X}$, and we assume that everything is recursively presented (recursive), cf. Becker-Kechris. In this case we say that $(\mathcal{X}, G, \cdot)$ is recursive Polish G-space. By E_G we denote the induced equivalence relation

$$xE_G y \iff (\exists g \in G)[y = g \cdot x] \iff y \text{ belongs to the orbit of } x.$$

From the preceding if follows that the set $E_G \cap \{(x, y) \mid \omega_1^{(x,y)} = \omega_1^{\mathrm{CK}}\}$ is Borel. This is because the g above can be chosen to be recursive in $\mathrm{W}^{(x,y)} \leq_{\mathrm{h}} (\mathrm{W}, x, y)$.

Lemma

Suppose that $(\mathcal{X}, G, \cdot)$ is a recursive Polish G-space, then there exists some $\alpha \in \omega^\omega$ such that

$$xE_G y \iff (\exists g \in \Delta_1^1(\mathrm{W}^{(\alpha,x)}, y))[y = g \cdot x].$$

The preceding lemma is derived by analyzing the proof of

Theorem (D. E. Miller)

Orbits of Borel actions of Polish groups on Polish spaces are Borel sets.

Now we can get the following (known?) result.

Theorem

For every Polish G-space $(\mathcal{X}, G, \cdot)$ there is some $\alpha \in \omega^\omega$ such that the set $E_G \cap \mathrm{M}^{\mathcal{X}}(\alpha) \times \mathcal{X}$ is Borel.
The same assertion holds for $\mathrm{M}^{\mathcal{X}}(\varepsilon)$ in the place of $\mathrm{M}^{\mathcal{X}}(\alpha)$ for any $\alpha \leq_{\mathrm{h}} \varepsilon$.

The preceding result is related to the following.

Theorem (Becker)

For every Polish G-space $(\mathcal{X}, G, \cdot)$ there exists a transfinite sequence $(A_\xi)_{\xi<\omega_1}$ of disjoint Borel sets such that (among other things) we have $\mathcal{X} = \cup_{\xi<\omega_1} A_\xi$ and that $E_G \cap A_\xi \times A_\xi$ if Borel for all $\xi < \omega_1$.

Actually it follows from some remarks in Becker-Kechris, that the sets A_ξ can be chosen in such a way that A_ξ is comeager for the first ξ for which $A_\xi \neq \emptyset$. To see this we assume that $(\mathcal{X}, G, \cdot)$ is recursive, and following Becker-Kechris we set

$$\omega_1^{G \cdot x} = \inf\{\omega_1^{g \cdot x} \mid g \in G\},$$

then one can choose $A_\xi := \{x \in \mathcal{X} \mid \omega_1^{G \cdot x} = \xi\}$. Clearly $A_\xi = \emptyset$ for all $\xi < \omega_1^{\mathrm{CK}}$. The set $A_{\omega_1^{\mathrm{CK}}}$ is Σ_1^1 and contains all Δ_1^1 points of $\mathcal{X}$, hence it is comeager (Thomason-Hinman).

Corollary

Suppose that $(\mathcal{X}, G, \cdot)$ *is a Polish G-space, and assume that* E_G *is not Borel. Then there exists some* $\alpha \in \omega^\omega$ *such that for all* $\varepsilon \in \omega^\omega$ *with* $\alpha \leq_{\mathrm{h}} \varepsilon$ *the set is* $E_G \cap \mathrm{M}^{\mathcal{X}}(\varepsilon)^c \times \mathrm{M}^{\mathcal{X}}(\varepsilon)^c$ *is (analytic) non Borel.*

Proof.

Let α be as in the preceding theorem and $\alpha \leq_{\mathrm{h}} \varepsilon$. E_G is Borel on $\mathrm{M}^{\mathcal{X}}(\varepsilon) \times \mathcal{X}$ and hence also on $(\mathrm{M}^{\mathcal{X}}(\varepsilon) \times \mathcal{X}) \cup (\mathcal{X} \times \mathrm{M}^{\mathcal{X}}(\varepsilon))$. So if E_G is Borel on $\mathrm{M}^{\mathcal{X}}(\varepsilon)^c \times \mathrm{M}^{\mathcal{X}}(\varepsilon)^c$ then E_G is Borel. $\dashv$

Now let us go back to

Theorem

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The same assertion holds for $\mathrm{M}^{\mathcal{X}}(\varepsilon)$ in the place of $\mathrm{M}^{\mathcal{X}}(\alpha)$ for any $\alpha \leq_{\mathrm{h}} \varepsilon$.

We describe another proof. Let $F(G)$ be the Effros-Borel space of closed subsets of G and $d : F(G) \to G$ be Borel-measurable such that $d(F) \in F$ for all $F \neq \emptyset$. By G_x we denote the stabilizer of $x \in G$: $\{g \in G \mid g \cdot x = x\}$. Then by unraveling the proof of the preceding result of Miller (orbits are Borel sets) we have that

$$\begin{aligned} y \in G \cdot x &\iff (\exists g)[d(gG_x) = g \;\&\; y = g \cdot x] \\ &\iff (\exists! g))[d(gG_x) = g \;\&\; y = g \cdot x]. \end{aligned}$$

By choosing the function the function $d : F(G) \to G$ accordingly, the whole thing boils down to estimating the complexity of the relation

$$P(g, x, s) \iff g \cdot G_x \cap U_s \neq \emptyset,$$

for some fixed Suslin scheme $(U_s)_{s \in \omega^{<\omega}}$ on G.

From the Kleene Basis Theorem (assuming that everything is recursive) and *after some computations [correction]* it follows

$$g \cdot G_x \cap U_s \neq \emptyset \iff (\exists h \leq_{\mathrm{T}} \mathrm{W}^x)[h \cdot x = x \;\&\; g \cdot h \in U_s].$$

We set $C(\beta, g, x, s) \iff (\exists h \leq_{\mathrm{T}} \beta)[h \cdot x = x \;\&\; g \cdot h \in U_s]$, then C is $\underset{\sim}{\Sigma}^0_n$ for some small n, and

$$g \cdot G_x \cap U_s \neq \emptyset \iff C(\mathrm{W}^x, g, x, s).$$

At the end we find some $\underset{\sim}{\Sigma}^0_m$ relation B such that

$$y \in G \cdot x \iff (\exists g) B(\mathrm{W}^x, g, x, y) \iff (\exists ! g) B(\mathrm{W}^x, g, x, y).$$

The function $x \mapsto \mathrm{W}^x$ is Borel-measurable on $\mathrm{M}^{\mathcal{X}}$, and so there exists a Borel set A such that for all $x \in \mathrm{M}^{\mathcal{X}}$ and all g, y we have that

$$B(\mathrm{W}^x, g, x, y) \iff A(x, g, y).$$

Hence

$$y \in G \cdot x \iff (\exists ! g) A(x, g, y)$$

for all $x \in \mathrm{M}^{\mathcal{X}}$ and all $y \in \mathcal{X}$. It follows that

$$(x, y) \in E_G \cap \mathrm{M}^{\mathcal{X}} \times \mathcal{X} \iff x \in \mathrm{M}^{\mathcal{X}} \ \& \ (\exists ! g) A(x, g, y),$$

which shows that the set $E_G \cap \mathrm{M}^{\mathcal{X}} \times \mathcal{X}$ is coanalytic.

Bounding the Church-Kleene ordinals

Suppose that $(\mathcal{X}, G, \cdot)$ is a recursive G-Polish space. We make the following assumption about the group action

$$\text{for all } x \in \mathcal{X} \text{ there exists } y \in \mathrm{M}^{\mathcal{X}} \text{ such that } y \in G \cdot x.$$

We consider the Borel equivalence relation $F = E_G \cap \mathrm{M}^{\mathcal{X}} \times \mathrm{M}^{\mathcal{X}}$ on $\mathrm{M}^{\mathcal{X}}$ and the (well-defined) injective function

$$\rho : \mathcal{X}/E_G \to \mathrm{M}^{\mathcal{X}}/F : \rho(C) = C \cap \mathrm{M}^{\mathcal{X}}.$$

F satisfies Silver's Dichotomy. If F has only countably many equivalence classes, i.e., the quotient $\mathrm{M}^{\mathcal{X}}/F$ is countable, it follows from the injectiveness of ρ that $\mathcal{X}/E_G$ is countable as well. If F has perfectly many classes, i.e., there exists a Cantor set $C \subseteq \mathrm{M}^{\mathcal{X}}$ such that $[y_1]_F \neq [y_2]_F$ for all $y_1 \neq y_2$ in C, then we also have that $[y_1]_{E_G} \neq [y_2]_{E_G}$ for all $y_1 \neq y_2$ in C. Thus such an E_G has either countably many or perfectly many equivalence classes.

Now let us assume again the preceding condition, i.e., $\omega_1^{G\cdot x} = \omega_1^{\mathrm{CK}}$ for all $x \in \mathcal{X}$.
How about the Glimm-Effros dichotomy? The set $\mathrm{M}^{\mathcal{X}}$ is Σ_1^1, so it is open in the Gandy-Harrington topology, and the arguments in the effective proof of the Harrington-Kechris-Louveau Theorem seem to go through. However…

Theorem (Sami). a) Every orbit is a $G \cdot x$ is a $\undertilde{\mathbf{\Pi}}^0_{\omega_1^{G\cdot x}+2}$ set.

b) If there exists some $\xi < \omega_1$ such that every orbit is $G \cdot x$ is a $\undertilde{\mathbf{\Pi}}^0_{\xi}$ set, then E_G is Borel; from which it follows that:

If $\omega_1^{G\cdot x} = \omega_1^{\mathrm{CK}}$ for all $x \in \mathcal{X}$ then E_G is Borel (see also Becker-Kechris). So the Silver and the Glimm-Effros dichotomy do hold for E_G.

Question. What can be said if $\omega_1^{G\cdot x} = \omega_1^{f(x)}$ for some Borel function f? (Or: $\omega_1^{G\cdot x} = \omega_1^{(\alpha,x)}$ for some fixed α.)

Thank you for your attention!