A recursive theoretic view to the decomposability conjecture

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Lusin's Problem and the Jayne-Rogers Theorem

Problem (Lusin)

Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a Borel-measurable function between Polish spaces. Can we partition $\mathcal{X}$ into countably many sets over which the function f is continuous?

The answer is negative. (Sierpinski, Keldis, Laczkovich, Motto Ros and many more.)

Theorem (Jayne-Rogers)

Suppose that X is analytic space, Y is separable metrizable and $f : X \to Y$ satisfies the property $f^{-1}\underset{\sim}{\mathbf{\Sigma}}^0_2 \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_2$. Then there exists a sequence of closed sets $(F_i)_{i\in\omega}$ such that $X = \cup_i F_i$ and $f \restriction F_i$ is continuous for all $i \in \omega$.

Notation. Suppose that $f : X \to Y$ is a function between topological spaces. We write $f \in \mathrm{dec}(\underset{\sim}{\boldsymbol{\Sigma}}^0_m)$ if there exists a sequence $(X_i)_{i\in\omega}$ of subsets of X such that $X = \cup_i X_i$ and $f \upharpoonright X_i$ is $\underset{\sim}{\boldsymbol{\Sigma}}^0_m$-measurable for all $i \in \omega$. We also write $f \in \mathrm{dec}_n(\underset{\sim}{\boldsymbol{\Sigma}}^0_m)$ if the preceding sets X_i can be chosen to be $\underset{\sim}{\boldsymbol{\Pi}}^0_n$ subsets of X.

The Jayne-Rogers Theorem now reads:

$$f^{-1}\underset{\sim}{\boldsymbol{\Sigma}}^0_2 \subseteq \underset{\sim}{\boldsymbol{\Sigma}}^0_2 \implies f \in \mathrm{dec}_1(\underset{\sim}{\boldsymbol{\Sigma}}^0_1)$$

for every function between an analytic space and a separable metrizable space.

Many mathematicians worked towards extending/generalising the Jayne-Rogers Theorem including: Andretta, Duparc, Motto Ros, Pauly, Pawlikowski-Sabok, Semmes, Solecki and Zapletal.

The decomposability conjecture

Conjecture (Andretta, Motto Ros, Pawlikowski-Sabok et al.)

Suppose that X is analytic and Y is separable metrizable. For every function $f : X \to Y$ and every $n \geq 2$ it holds

$$f^{-1}\utilde{\boldsymbol{\Sigma}}^0_n \subseteq \utilde{\boldsymbol{\Sigma}}^0_n \implies f \in \mathrm{dec}_{n-1}(\utilde{\boldsymbol{\Sigma}}^0_1).$$

More generally for all $2 \leq m \leq n$ we have

$$f^{-1}\utilde{\boldsymbol{\Sigma}}^0_m \subseteq \utilde{\boldsymbol{\Sigma}}^0_n \implies f \in \mathrm{dec}_{n-1}(\utilde{\boldsymbol{\Sigma}}^0_{n-m+1}).$$

Related results

Theorem (Semmes)

For a function $f : \omega^\omega \to \omega^\omega$ we have the following:

$$f^{-1}\utilde{\boldsymbol{\Sigma}}^0_2 \subseteq \utilde{\boldsymbol{\Sigma}}^0_3 \implies f \in \mathrm{dec}_2(\utilde{\boldsymbol{\Sigma}}^0_2) \quad m = 2, n = 3$$
$$f^{-1}\utilde{\boldsymbol{\Sigma}}^0_3 \subseteq \utilde{\boldsymbol{\Sigma}}^0_3 \implies f \in \mathrm{dec}_2(\utilde{\boldsymbol{\Sigma}}^0_1) \quad m = n = 3.$$

Theorem (Motto Ros, Pawlikowski-Sabok)

Suppose that X is analytic, Y is separable metrizable and $f : X \to Y$ satisfies $f^{-1}\utilde{\boldsymbol{\Sigma}}^0_n \subseteq \utilde{\boldsymbol{\Sigma}}^0_n$ for some $n \geq 3$. Then $f \in \mathrm{dec}(\utilde{\boldsymbol{\Sigma}}^0_1)$.
If moreover f is $\utilde{\boldsymbol{\Sigma}}^0_{n-1}$-measurable, or more generally has $\utilde{\boldsymbol{\Sigma}}^0_n$ graph then $f \in \mathrm{dec}_{n-1}(\utilde{\boldsymbol{\Sigma}}^0_1)$.

Kihara's approach

The idea. (Kihara, Pauly) We ask that condition $f^{-1}\underset{\sim}{\mathbf{\Sigma}}^0_m \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_n$ is witnessed by a continuous function in the codes of the sets.

Universal sets. Suppose that X is separable metrizable and that $(N(X,s))_s$ is an enumeration of a basis of its topology. We define by recursion the set $G^X_n \subseteq \omega^\omega \times X$ as follows

$$\begin{aligned} G^X_1(\varepsilon, x) &\iff (\exists k)[x \in N(X, \varepsilon(k))] \\ G^X_{n+1}(\varepsilon, x) &\iff (\exists k)\neg G_n((\varepsilon)_k, x). \end{aligned}$$

Then G^X_n is universal for $\underset{\sim}{\mathbf{\Sigma}}^0_n \upharpoonright X$.

Suppose that $f : X \to Y$ satisfies $f^{-1}\utilde{\mathbf{\Sigma}}^0_m \subseteq \utilde{\mathbf{\Sigma}}^0_n$. It is easy to see that for every $\utilde{\mathbf{\Sigma}}^0_m$ set $P \subseteq \omega \times Y$ the set

$$Q(i,x) \iff P(i,f(x))$$

is $\utilde{\mathbf{\Sigma}}^0_n$.

Definition

For a function $f : A \subseteq X \to Y$ we say that condition $f^{-1}\utilde{\mathbf{\Sigma}}^0_m \subseteq \utilde{\mathbf{\Sigma}}^0_n$ holds $\utilde{\mathbf{\Gamma}}$-*uniformly* in the codes if there exists a $\utilde{\mathbf{\Gamma}}$-measurable function $u : \omega^\omega \to \omega^\omega$ such that for all $\varepsilon, i, x \in A$ we have that

$$G_m^{\omega \times Y}(\varepsilon, i, f(x)) \iff G_n^{\omega \times X}(u(\varepsilon), i, x).$$

Theorem (Kihara)

Suppose that for a function $f : \omega^\omega \to \omega^\omega$ condition $f^{-1}\underset{\sim}{\mathbf{\Sigma}}^0_m \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_n$ holds $\underset{\sim}{\mathbf{\Sigma}}^0_1$-uniformly for some m, n with $3 \leq m \leq n < 2m - 1$. Then $f \in \mathrm{dec}_{n-1}(\underset{\sim}{\mathbf{\Sigma}}^0_{n-m+1})$.

Extensions. Quasi-Polish spaces of finite small inductive dimension, computable functions (Kihara).

In this talk we will give an overview of the proof of the preceding result. Moreover we present some new results.

What's new

Proposition (G.-Kihara)

Suppose that $\mathcal{X}, \mathcal{Y}$ are Polish, $A \subseteq \mathcal{X}$ is analytic and $f : A \to \mathcal{Y}$ satisfies $f^{-1}\boldsymbol{\Sigma}^0_m \subseteq \boldsymbol{\Sigma}^0_n$ for some $m \geq n \geq 1$. Then condition $f^{-1}\boldsymbol{\Sigma}^0_m \subseteq \boldsymbol{\Sigma}^0_n$ holds Borel-uniformly.

Theorem (G.-Kihara)

Suppose that $A \subseteq \omega^\omega$ is analytic and $f : A \to \omega^\omega$ satisfies $f^{-1}\boldsymbol{\Sigma}^0_m \subseteq \boldsymbol{\Sigma}^0_n$ for some $n \geq m \geq 2$. Then $f \in \mathrm{dec}_n(\boldsymbol{\Sigma}^0_{n-m+1})$.

If moreover $m \geq 3$ and f is $\boldsymbol{\Sigma}^0_{n-1}$-measurable then $f \in \mathrm{dec}_{n-1}(\boldsymbol{\Sigma}^0_{n-m+1})$.

The same assertion holds for spaces of finite small inductive dimension.

Turing degrees

Recursive functions. We consider the family of recursive functions on some subset of ω to ω. Intuitively these are the functions $f : \omega \rightharpoonup \omega$ for which there is a computer program P with input and output natural numbers such that for all $x \in \mathrm{Domain}(f)$: (a) the program P terminates on the input x and (b) for all $y \in \omega$, $f(x) = y$ iff P on the input x outputs y, i.e., P computes f. This family can enumerated in canonical way (Kleene). We denote by $\{e\}$ the e-th recursive function.

Relativization. For every $B \subseteq \omega$ we can define the family of B-recursive functions. Intuitively these are the functions for which the preceding program has as extra input arbitrary finite initial segments of the characteristic function of B. We denote by $\{e\}^B$ the e-th B-recursive function.

A set $A \subseteq \omega$ is recursive if its characterstic function can be computed by a recursive function, i.e., if there exists some e such that $\{e\}$ is a function from ω to $\{0, 1\}$ and for all n, $n \in A \iff \{e\}(n) = 1$.

Similarly $A \subseteq \omega$ is B-recursive if there exists e such that $\{e\}^B : \omega \to \{0, 1\}$ and $n \in A \iff \{e\}^B(n) = 1$ for all $n \in \omega$.

We write $A \leq_{\mathrm{T}} B$ iff A is B-recursive and $A =_{\mathrm{T}} B$ iff $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$.

We have that $\emptyset \leq_{\mathrm{T}} A$ for all A, and A is recursive exactly when $A =_{\mathrm{T}} \emptyset$.

The relation $\leq_{\mathrm{T}}$ is reflexive and transitive, so that $=_{\mathrm{T}}$ is an equivalence relation. The equivalence class of A under $=_{\mathrm{T}}$ is the Turing degree of A. The set of Turing degrees forms an upper semilattice.

Some effective descriptive set theory

The notion of a recursive set extends to recursively presented Polish spaces and it can be considered as the effective analogue of clopen sets.

In fact this analogy applies to all pointclasses $\undertilde{\mathbf{\Sigma}}^0_\xi$, $\undertilde{\mathbf{\Sigma}}^1_n$, i.e., one can define the lightface Σ^0_ξ, Σ^1_n pointclasses, as well as the relativized ones $\Sigma^0_\xi(x)$, $\Sigma^1_n(x)$. In fact every set belonging to $\undertilde{\mathbf{\Gamma}}$ is a member of $\Gamma(\varepsilon)$ for some $\varepsilon \in \omega^\omega$.

This analogy applies to functions as well, i.e., one defines Γ-recursive functions, which is the effective analogue of $\undertilde{\mathbf{\Gamma}}$-measurability. Every $\undertilde{\mathbf{\Gamma}}$-measurable function is $\Gamma(\varepsilon)$-recursive for some $\varepsilon \in \omega^\omega$.

The preceding universal sets $G^{\mathcal{X}}_n$ belong to the lightface Σ^0_n class, whenever $\mathcal{X}$ is recursively presented. Moreover $P \subseteq \mathcal{X}$ is Σ^0_n if it is the ε-section of $G^{\mathcal{X}}_n$ for some *recursive* function $\varepsilon \in \omega^\omega$.

Turing jumps

The Turing jump of a set $A \subseteq \omega$ is the set

$$A' = \{n \in \omega \mid \{n\}^A(n) \text{ is defined}\}.$$

Some properties:

(a) $A <_{\mathrm{T}} A'$, in particular the set $\emptyset'$ is not recursive.

(b) $A \leq_{\mathrm{T}} B \implies A' \leq_{\mathrm{T}} B'$.

(c) The relation $P(n, B) \iff n \in B'$ is Σ^0_1, hence B' is a $\Sigma^0_1(B)$ set.

(d) If A is $\Sigma^0_1(B)$ then $A \leq_{\mathrm{T}} B'$.

(e) The function $f : 2^\omega \to 2^\omega : f(A) = A'$ is injective and $\utilde{\boldsymbol{\Sigma}}^0_2$-measurable. Moreover $f \notin \mathrm{dec}(\utilde{\boldsymbol{\Sigma}}^0_1)$ (Kihara).

By iterating we define the *n-th Turing jump* $A^{(n+1)} = (A^{(n)})'$. By $A^{(0)}$ we mean A. The set $B^{(n)}$ is $\Sigma^0_n(B)$ and for every $A \in \Sigma^0_n(B)$ we have $A \leq_{\mathrm{T}} B^{(n)}$ for all $n \geq 1$. Moreover the function $A \mapsto A^{(n)}$ is $\underset{\sim}{\mathbf{\Sigma}}^0_{n+1}$-measurable.

It is clear that if $A \leq_{\mathrm{T}} B^{(k)}$ then $A^{(m)} \leq_{\mathrm{T}} B^{(k+m)}$. The converse is not correct. Well ... almost!

The Cancellation Lemma (Kihara)

Suppose that $A, B \subseteq \omega$ and $1 \leq m \leq n$ are given. If

$$(A \oplus C)^{(m)} \leq_{\mathrm{T}} (B \oplus C)^{(n)}$$

for all $C \subseteq \omega$, then $A \leq_{\mathrm{T}} B^{(n-m)}$.

The same assertion holds if we replace A and B with $A \oplus X$ and $B \oplus X$ repsectively for some fixed $X \subseteq \omega$, (relativized version).

The preceding applies a deep result on Turing degrees by Shore and Slaman.

Applications to the decomposability problem

Suppose that we are given a function $f : \omega^\omega \to \omega^\omega$ such that condition $f^{-1}\boldsymbol{\Sigma}^0_m \subseteq \boldsymbol{\Sigma}^0_n$ holds $\boldsymbol{\Sigma}^0_1$-uniformly for some n, m with $3 \leq m \leq n < 2m - 1$ and let $u : \omega^\omega \to \omega^\omega$ be the continuous witnessing function. Without loss of generality we may assume that u is recursive and so $u(\alpha) \leq_{\mathrm{T}} \alpha$ for all α. (We view ω^ω as a subset of 2^ω.)

Condition $A \leq_{\mathrm{T}} B$ is witnessed by some recursive function $\{e\}^B$. Let us write $A \leq_{\mathrm{T},e} B$ in the latter case. The idea is to show that $f(x) \leq_{\mathrm{T}} x^{(n-m)}$ holds for all x. Then we define

$$X_e = \{x \in \omega^\omega \mid f(x) \leq_{\mathrm{T},e} x^{(n-m)}\}.$$

It follows that $\omega^\omega = \cup_e X_e$ and the restriction of f on each X_e is a $\boldsymbol{\Sigma}^0_{n-m+1}$-measurable function.

We define $P \subseteq \mathcal{Z} = \omega^\omega \times \omega \times \omega^\omega$ by $P(C, i, y) \iff i \in (y \oplus C)^{(m)}$. Then P is a Σ^0_m set and so there exists some recursive ε such that P is the ε-section of $G^{\mathcal{Z}}_m$. For some recursive function S we have that

$$P(C, i, y) \iff G^{\mathcal{Z}}_m(\varepsilon, C, i, y) \iff G^{\omega\times\omega^\omega}_m(S(\varepsilon, C), i, y),$$

and so from the key property of u it follows

$$P(C, i, f(x)) \iff G^{\omega\times\omega^\omega}_n(u(S(\varepsilon, C)), i, x).$$

In other words the set $(f(x) \oplus C)^{(m)}$ is $\Sigma^0_n(u(S(\varepsilon, C)), x)$, and so

$$(f(x) \oplus C)^{(m)} \leq_{\mathrm{T}} (x \oplus u(S(\varepsilon, C)))^{(n)} \leq_{\mathrm{T}} (x \oplus C)^{(n)},$$

for all C, x. By applying the Cancellation Lemma it follows that $f(x) \leq_{\mathrm{T}} x^{(n-m)}$ for all x.

Decomposing the domain in a nice way. The method is similar to the one of Motto Ros and Pawlikowski-Sabok. We use the following result by Kuratowski: every $\boldsymbol{\Sigma}^0_k$-measurable function on a set A can be extended to a $\boldsymbol{\Sigma}^0_k$-measurable function on a $\boldsymbol{\Pi}^0_{k+1}$ set B, with $A \subseteq B \subseteq \overline{A}$.

At some point we have to deal with a set of the form $P = \{x \mid R(x, f(x))\}$, where R is $\boldsymbol{\Pi}^0_{n-m+1}$ - R computes the graph of a $\boldsymbol{\Sigma}^0_{n-m+1}$-measurable function. Since $n < 2m - 1$ it follows that $n - m + 1 \leq m - 1$ and so R is $\boldsymbol{\Pi}^0_{m-1} \subseteq \boldsymbol{\Sigma}^0_m$. Using the fact that $f^{-1}\boldsymbol{\Sigma}^0_m \subseteq \boldsymbol{\Sigma}^0_n$ holds $\boldsymbol{\Sigma}^0_1$-uniformly it follows that the set P is $\boldsymbol{\Sigma}^0_n$.

One can also notice that the graph of f is a $\boldsymbol{\Sigma}^0_n$ set - but this does not seem to help us here unless $n = m$.

This completes the proof of Kihara's Theorem about the decomposability problem.

The section problem

We proceed to the review of the new results - but first we need some preliminaries.

Theorem (Louveau)

Suppose that $\mathcal{X}, \mathcal{Y}$ are Polish spaces, ξ is a countable ordinal and that $P \subseteq \mathcal{X} \times \mathcal{Y}$ is Borel. If for all x the set P_x is $\undertilde{\boldsymbol{\Sigma}}^0_\xi$ then the topology of $\mathcal{X}$ is refined to a Polish topology $\mathcal{T}_\infty$, which has the same Borel sets, and P is $\undertilde{\boldsymbol{\Sigma}}^0_\xi$ in $(\mathcal{X}, \mathcal{T}_\infty) \times \mathcal{Y}$.

Louveau Separation

The heart of the proof of the preceding result lies in the following.

Theorem (Louveau)

Suppose that $\mathcal{X}$ is a recursively presented metric space and that A, B are disjoint Σ^1_1 subsets of $\mathcal{X}$. If A is separated from B by a $\underset{\sim}{\mathbf{\Pi}}^0_\xi$ set, then A is separated from B by a $\Pi^0_\xi(\varepsilon)$ set for some $\varepsilon \in \Delta^1_1$.

In particular if A is $\underset{\sim}{\mathbf{\Pi}}^0_\xi$ and Δ^1_1, then it is $\Pi^0_\xi(\varepsilon)$ for some $\varepsilon \in \Delta^1_1$.

Let's see how Louveau Separation solves the Section Problem.

Let be $P \subseteq \mathcal{X} \times \mathcal{Y}$ be Borel such that for all x the set P_x is $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$. Without loss of generality we may assume that P is Δ^1_1, so that every section P_x is $\Delta^1_1(x)$. Then from Louveau Separation P_x is $\Sigma^0_\xi(\varepsilon)$ for some $\varepsilon \in \Delta^1_1(x)$. It follows that every section P_x is the β-section of some suitably chosen universal set G_ξ for some $\beta \in \Delta^1_1(x)$.

A well-known result of effective descriptive set theory states that if $Q \subseteq \mathcal{X} \times \mathcal{Z}$ is Π^1_1 and for all $x \in \mathcal{X}$ there exists $z \in \Delta^1_1(x)$ such that $Q(x, z)$, then there exists a Borel measurable function $f : \mathcal{X} \to \mathcal{Z}$ such that $Q(x, f(x))$ for all x.

We apply the preceding to the Π^1_1 set

$$Q(x, \beta) \iff (\forall y)[P(x, y) \iff G_\xi(\beta, y)]$$

and voilà: there exists a Borel measurable function $f : \mathcal{X} \to \omega^\omega$ such that $P(x, y) \iff G_\xi(f(x), y)$. The topology $\mathcal{T}_\infty$ is the one which turns f continuous.

Using Louveau Separation and the preceding ideas one can prove the Proposition given in the Introduction:

Proposition (G.-Kihara)

Suppose that $A \subseteq \omega^\omega$ is analytic and $f : A \to \omega^\omega$ satisfies $f^{-1}\mathbf{\Sigma}^0_m \subseteq \mathbf{\Sigma}^0_n$ for some $m \geq n \geq 1$. Then condition $f^{-1}\mathbf{\Sigma}^0_m \subseteq \mathbf{\Sigma}^0_n$ holds Borel-uniformly.

We take the simple case where $A = \omega^\omega$. For every α the set

$$\{(i, x) \mid G_m^{\omega^\omega \times \omega \times \omega^\omega}(\alpha, i, f(x))\}$$

$\mathbf{\Sigma}^0_n$ and (without loss of generality) is $\Delta^1_1(\alpha)$. Hence it is a $\Sigma^0_n(\varepsilon)$ set for some $\varepsilon \in \Delta^1_1(\alpha)$. So we can find a Borel-measurable function $u : \omega^\omega \to \omega^\omega$ such that

$$G_m^{\omega^\omega \times \omega \times \omega^\omega}(\alpha, i, f(x)) \iff G_n^{\omega^\omega \times \omega \times \omega^\omega}(u(\alpha), i, x).$$

Now let us move to the Theorem in the Introduction:

Theorem (G.-Kihara)

Suppose that $A \subseteq \omega^\omega$ is analytic and $f : A \to \omega^\omega$ satisfies $f^{-1}\underset{\sim}{\boldsymbol{\Sigma}}^0_m \subseteq \underset{\sim}{\boldsymbol{\Sigma}}^0_n$ for some $n \geq m \geq 2$. Then $f \in \mathrm{dec}_n(\underset{\sim}{\boldsymbol{\Sigma}}^0_{n-m+1})$.

If moreover $m \geq 3$ and f is $\underset{\sim}{\boldsymbol{\Sigma}}^0_{n-1}$-measurable then $f \in \mathrm{dec}_{n-1}(\underset{\sim}{\boldsymbol{\Sigma}}^0_{n-m+1})$.

Comment. The preceding estimation $f \in \mathrm{dec}_{n-1}(\underset{\sim}{\boldsymbol{\Sigma}}^0_{n-m+1})$ reduces the Decomposability Conjecture (in the Baire space) to the cases $2 = m \leq n$.

By repeating the arguments in the proof of Kihara's Theorem we get that

$$(f(x) \oplus C)^{(m)} \leq_{\mathrm{T}} (x \oplus C^{(\xi)})^{(n)}$$

for some recursive ordinal ξ (where f is Δ^1_1-recursive).

The Cancellation Lemma - Extended Version

Suppose that $A, B \subseteq \omega$, $1 \leq m \leq n$ and $\xi < \omega_1^{\mathrm{CK}}$ are given. If

$$(A \oplus C)^{(m)} \leq_{\mathrm{T}} (B \oplus C^{(\xi)})^{(n)}$$

for all $C \subseteq \omega$, then $A \leq_{\mathrm{T}} (B \oplus \emptyset^{(\xi)})^{(n-m)}$. The same assertion holds if we replace A and B with $A \oplus X$ and $B \oplus X$ respectively for some fixed $X \subseteq \omega$, (relativized version).

We proceed as before and we get $f(x) \leq_{\mathrm{T}} (x \oplus \emptyset^{(\xi)})^{(n-m)}$ for all x. Now we take the set

$$X_e = \{x \mid f(x) \leq_{\mathrm{T},e} (x \oplus \emptyset^{(\xi)})^{(n-m)}\}.$$

It is not hard to see that X_e is $\mathbf{\Pi}^0_n$, and that if $m \geq 3$ and f is $\mathbf{\Sigma}^0_{n-1}$-measurable then X_e is in fact $\mathbf{\Pi}^0_{n-1}$.

Thank you for your attention!