Applications of Logic to Analysis

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A topological space $\mathcal{X}$ is a *Polish space* if $\mathcal{X}$ is separable and metrizable by a complete distance function.

Examples: $\mathbb{R}$, ω^k, the *Baire space* $\mathcal{N} = \omega^\omega$, $(C(K), \|.\|_\infty)$, K compact and separable Banach spaces. We will denote Polish spaces by $\mathcal{X}$, $\mathcal{Y}$.

If $P \subseteq \mathcal{X}$ we will write $P(x)$ instead of $x \in P$.

Borel sets $\underset{\sim}{\boldsymbol{B}}$.

Analytic Sets. A set $P \subseteq \mathcal{X}$ is *analytic* or $\underset{\sim}{\boldsymbol{\Sigma}}^1_1$ if there is a closed set $Q \subseteq \mathcal{X} \times \mathcal{N}$

$$P(x) \Longleftrightarrow \exists \alpha \in \mathcal{N}\ Q(x, \alpha)$$

Coanalytic sets. A set is *coanalytic* or $\underset{\sim}{\boldsymbol{\Pi}}^1_1$ if it is the complement of an analytic set.

$$\underset{\sim}{\boldsymbol{\Delta}}^1_1 = \underset{\sim}{\boldsymbol{\Sigma}}^1_1 \cap \underset{\sim}{\boldsymbol{\Pi}}^1_1.$$

Suslin's Theorem.

$$\underset{\sim}{\boldsymbol{B}} = \underset{\sim}{\boldsymbol{\Delta}}^1_1$$

Effective Theory.

Suppose that $\mathcal{X}$ is a Polish space, d is compatible distance function for $\mathcal{X}$ and $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{X}$. Define the relation $P_<$ of ω^4 as follows $P_<(i,j,k,m) \iff d(x_i,x_j) < \frac{k}{m+1}$. Similarly we define the relation $P_\leq$.

The sequence $(x_n)_{n\in\mathbb{N}}$ is a *recursive presentation* of $\mathcal{X}$, if
(1) it is a dense sequence and
(2) the relations $P_<$ and $P_\leq$ are recursive.

The spaces $\mathbb{R}$, $\mathcal{N}$ and ω^k admit a recursive presentation i.e., they are *recursively presented*. Some other examples: $\mathbb{R}\times\omega$, $\mathbb{R}\times\mathcal{N}$. However not all Polish spaces are recursively presented.

Every Polish space admits an ε-recursive presentation for some suitable ε.

Without loss of generality we will deal with recursively presented Polish spaces.

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$N(\mathcal{X}, s) =$ the ball with center $x_{(s)_0}$ and radius $\frac{(s)_1}{(s)_2+1}$.

A set $P \subseteq \mathcal{X}$ is *semirecursive* if $P = \bigcup_{i \in \mathbb{N}} N(\mathcal{X}, \alpha(i))$ where α is a recursive function from ω to ω.

$\Sigma^0_1 =$ all semirecursive sets
$\rightsquigarrow$ effective open sets.

$\Pi^0_1 =$ the complements of semirecursive sets
$\rightsquigarrow$ effective closed sets.

$\Sigma^1_1 =$ projections of Π^0_1 sets
$\rightsquigarrow$ effective analytic sets.

$\Pi^1_1 =$ the complements of Σ^1_1 sets
$\rightsquigarrow$ effective coanalytic sets.

$\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1 =$ effective Borel sets (Kleene).

Similarly one defines the *relativized* pointclasses with respect to some parameter $y \in \mathcal{Y}$ which will be denoted by $\Delta^1_1(y)$ for example.

A set A is Borel exactly when A is in $\Delta^1_1(\varepsilon)$ for some $\varepsilon \in \mathcal{N}$.

A function $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if the set $R^f \subseteq \mathcal{X} \times \omega$, $R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$ is open.

A function $f : \mathcal{X} \to \mathcal{Y}$ is Σ^0_1-*recursive* if and only if the set $R^f \subseteq \mathcal{X} \times \omega$, $R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$, is Σ^0_1.

A point $x \in \mathcal{X}$ *is in* Δ^1_1 if the relation $U \subseteq \omega$ which is defined by

$$U(s) \iff x \in N(\mathcal{X}, s)$$

is Δ^1_1.

As before one gets the class of points which are in $\Delta^1_1(y)$.

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Points in Δ^1_1 are important because:

(a) they preserve the complexity of Δ^1_1 under projections i.e., if $P \subseteq \mathcal{X} \times \mathcal{Y}$ is in Δ^1_1 and for all x the section $P_x = \{y \in \mathcal{Y} \mid P(x, y)\}$ is either empty or contains a member in $\Delta^1_1(x)$ then the projection $\{x \in \mathcal{X} \mid (\exists y \in \mathcal{Y})P(x, y)\}$ is also in Δ^1_1. Similarly for Π^1_1 instead of Δ^1_1.

(b) they provide uniformizing functions for Δ^1_1 sets i.e., if $P \subseteq \mathcal{X} \times \mathcal{Y}$ is in Δ^1_1 and for all x either P_x is empty or contains a member in $\Delta^1_1(x)$ then there is a Δ^1_1 recursive function $f : \mathcal{X} \to \mathcal{Y}$ such that for all x for which $P_x \neq \emptyset$ we have that $P(x, f(x))$. Similarly for Π^1_1 in case $P_x \neq \emptyset$ for all $x \in \mathcal{X}$.

Example. The projection of a convex Borel set $R \subseteq \mathbb{R} \times \mathbb{R}$ is also a Borel set.

Proof. R is $\Delta^1_1(\varepsilon)$ for some ε; each non-empty section R_x is a convex subset of $\mathbb{R}$ and so it is either a singleton or it contains a rational number. In either case R_x contains a member in $\Delta^1_1(\varepsilon, x)$. From (b) pr$R$ is in $\Delta^1_1(\varepsilon)$.

Theorem (VG). Let X be a Banach space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X which is weakly convergent to $x \in X$. Then x is in $\Delta^1_1((x_n))$, (in the parameters in which the related spaces are recursively presented).

Corollary (VG). Let X be a separable Banach space. Then the set

$$P = \{ (y_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}} \mid \text{the sequence } (y_i)_{i\in\mathbb{N}} \text{ is weakly convergent}\}$$

is a coanalytic subset of $X^{\mathbb{N}}$.
Let Q be a coanalytic subset of $X^{\mathbb{N}} \times X$. Then the set

$$\begin{aligned} P_Q \quad = \quad & \{ (y_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}} \mid \text{the sequence } (y_i)_{i\in\mathbb{N}} \text{ is weakly convergent} \\ & \text{to some } y \text{ and } Q((y_i)_{i\in\mathbb{N}}, y) \} \end{aligned}$$

is a coanalytic subset of $X^{\mathbb{N}}$.

Theorem (Erdös-Magidor). Let X be a Banach space and $(e_i)_{i\in\mathbb{N}}$ be a bounded sequence in X. Then there is a subsequence $(e_{k_i})_{i\in\mathbb{N}}$ such that: either (a) every subsequence of $(e_{k_i})_{i\in\mathbb{N}}$ is Cesàro summable with respect to the norm and all being summed to the same limit; or (b) *no* subsequence of $(e_{k_i})_{i\in\mathbb{N}}$ is Cesàro summable.

Theorem (VG). Let X be a Banach space, $(e_i)_{i\in\mathbb{N}}$ be a bounded sequence in X and let $Q \subseteq X^{\mathbb{N}} \times X$ be a coanalytic set. Then there is a subsequence $(e_i)_{i\in L}$ of $(e_i)_{i\in\mathbb{N}}$ for which: either
(a) there is some $e \in X$ such that every subsequence $(e_i)_{i\in H}$ of $(e_i)_{i\in L}$ is weakly Cesàro summable to e and $Q((e_i)_{i\in H}, e)$; or
(b) for every subsequence $(e_i)_{i\in H}$ of $(e_i)_{i\in L}$ and every $e \in X$ with $Q((e_i)_{i\in H}, e)$ the sequence $(e_i)_{i\in H}$ is not weakly Cesàro summable to e.

Debs gave the following effective version of a known theorem of Bourgain, Fremlin and Talagrand.

Theorem (Debs). Let $\mathcal{X}$ be a recursively presented Polish space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions from $\mathcal{X}$ to $\mathbb{R}$ which satisfies conditions (1) the sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded; (2) every cluster point of $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathcal{X}}$ with the product topology is a Borel-measurable function; (3) the sequence $(f_n)_{n\in\mathbb{N}}$ is $\Delta^1_1(\alpha)$-recursive.

A set A is Borel exactly when A is in $\Delta_1^1(\varepsilon)$ for some $\varepsilon \in \mathcal{N}$.

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A function $f : \mathcal{X} \to \mathcal{Y}$ is *Δ_1^1-recursive* if and only if the set $R^f \subseteq \mathcal{X} \times \omega$, $R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$, is Δ_1^1.

A point $x \in \mathcal{X}$ *is in Δ_1^1* if the relation $U \subseteq \omega$ which is defined by

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Theorem (VG). Let X be a Banach space and $(e_i)_{i\in\mathbb{N}}$ a bounded sequence in X for which every subsequence $(e_i)_{i\in L}$ has a further subsequence $(e_i)_{i\in H}$ which is weakly Cesàro summable. Then

(1) every subsequence of $(e_i)_{i\in\mathbb{N}}$ has a weakly *convergent* subsequence and

(2) there is a Borel-measurable function $f : [\mathbb{N}]^\omega \to [\mathbb{N}]^\omega$ such that for all subsequences $(e_i)_{i\in L}$ the sequence $(e_i)_{i\in f(L)}$ is a weakly convergent subsequence of $(e_i)_{i\in L}$.

Thank you!