Some uniformity aspects of the class of analytic sets

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CCC 2015 Kochel

Uniformity functions

The problem is to witness a property uniformly using a function of some certain complexity.

Example. Suppose that $P \subseteq \mathbb{R} \times \mathbb{R}$ is such that for all $x \in \mathbb{R}$ the section $P_x := \{y \in \mathbb{R} \mid (x, y) \in P\}$ is non-empty. Find a function $u : \mathbb{R} \to \mathbb{R}$ such that for all $(x, u(x)) \in P$ for all $x \in \mathbb{R}$. How complex can u be?
In descriptive set theory there are two main ways for obtaining uniformity functions.

1. Give a constructive proof to the theorem that we are interested in. This typically results to recursive/continuous uniformity functions.
2. Ensure the existence of a "definable" witness. This typically results to Borel-measurable functions (Louveau).

A standard example of the first method (constructive proof) is the Suslin-Kleene Theorem that we will mention in the sequel, and which has the consequence that

$$\Delta^1_1 = \mathrm{HYP}.$$

A classical application of the second method is the following result of Louveau: if $P \subseteq \mathcal{X} \times \mathcal{Y}$ is Borel and such that every section P_x is a $\underset{\sim}{\mathbf{\Sigma}}^0_n$ subset of $\mathcal{Y}$, then there is a Polish topology $\mathcal{T}'$ on $\mathcal{X}$, which refines the original one and P is a $\underset{\sim}{\mathbf{\Sigma}}^0_n$ subset of $(\mathcal{X}, \mathcal{T}') \times \mathcal{Y}$.
Another (recent) application is

Theorem (G.-Kihara)

Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is such that for all $A \in \underset{\sim}{\mathbf{\Sigma}}^0_m$ the preimage $f^{-1}[A]$ is $\underset{\sim}{\mathbf{\Sigma}}^0_n$. Then there is a Borel-measurable function $u : \mathcal{N} \to \mathcal{N}$ such that if α is a "$\underset{\sim}{\mathbf{\Sigma}}^0_m$-code" for A then $u(\alpha)$ is a $\underset{\sim}{\mathbf{\Sigma}}^0_n$-code for $f^{-1}[A]$.

The latter result has an important application to a still open problem in descriptive set theory, the Decomposability Conjecture. (G.-Kihara)

In this talk we deal with the first method. More specifically we will present the uniform version of a special separation theorem for analytic sets and give some constructive consequences. We will also deal with another structural property of analytic sets, namely the Baire property.

Notation

Underlying spaces: Polish spaces, i.e., complete separable metric spaces, $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \dots$. We will also assume that our Polish spaces admit a recursive presentation. The Baire space $\mathcal{N}$ is the space ω^ω with the product topology. This is a Polish space. We denote its members with α, β, γ etc.

Notation. We will write $P(x)$ instead of $x \in P$. By $\neg P(x)$ we mean that $x \notin P$. Given $P \subseteq X \times Y$ we define

$$\exists^Y P = \{x \in X \mid \text{there is } y \text{ s.t. } P(x,y)\}$$

$$P_x = \{y \in Y \mid P(x,y)\}, \quad x \in X.$$

Given $\alpha \in \mathcal{N}$ we denote by α^* the function $(t \mapsto \alpha(t+1))$. Given also $n \in \omega$ we denote by $(\alpha)_n$ the n-th component of α, which comes by some fixed recursive injection from ω^2 to ω.

Borel and Luzin pointclasses

We consider the following classes of sets in Polish spaces:

(Borel pointclasses of finite order)

$$\utilde{\boldsymbol{\Sigma}}^0_1 = \text{all open sets}$$

$$\utilde{\boldsymbol{\Pi}}^0_1 = \text{complements of } \utilde{\boldsymbol{\Sigma}}^0_1 = \text{ all closed sets}$$

$$\utilde{\boldsymbol{\Sigma}}^0_{n+1} = \text{all countable unions of } \utilde{\boldsymbol{\Pi}}^0_n \text{ sets}$$

$$\utilde{\boldsymbol{\Pi}}^0_{n+1} = \text{all complements of } \utilde{\boldsymbol{\Sigma}}^0_{n+1} \text{ sets}$$

(Luzin pointclasses)

$$\utilde{\boldsymbol{\Sigma}}^1_1 = \exists^{\mathcal{N}} \utilde{\boldsymbol{\Pi}}^0_1 \quad \text{(analytic sets)}$$

$$\utilde{\boldsymbol{\Pi}}^1_1 = \text{all complements of } \utilde{\boldsymbol{\Sigma}}^1_1 \quad \text{(coanalytic sets)}$$

$$\utilde{\boldsymbol{\Sigma}}^1_{n+1} = \exists^{\mathcal{N}} \utilde{\boldsymbol{\Pi}}^1_n$$

$$\utilde{\boldsymbol{\Pi}}^0_{n+1} = \text{all complements of } \utilde{\boldsymbol{\Sigma}}^1_{n+1} \text{ sets}$$

Universal sets

A set $G \subseteq \mathcal{N} \times \mathcal{X}$ parametrizes $\underset{\sim}{\mathbf{\Gamma}} \upharpoonright \mathcal{X}$ if for all $P \subseteq \mathcal{X}$ we have that

$$P \in \underset{\sim}{\mathbf{\Gamma}} \iff \text{exists } \alpha \in \mathcal{N} \text{ such that } P = \{x \mid (\alpha, x) \in G\} = G_\alpha.$$

Any α as above is called a $\underset{\sim}{\mathbf{\Gamma}}$-code of P.

By $\underset{\sim}{\mathbf{\Gamma}} \upharpoonright \mathcal{X}$ we mean the family of all subsets of $\mathcal{X}$, which belong in $\underset{\sim}{\mathbf{\Gamma}}$.

The set G is universal for $\underset{\sim}{\mathbf{\Gamma}} \upharpoonright \mathcal{X}$ if G is in $\underset{\sim}{\mathbf{\Gamma}}$ and parametrizes $\underset{\sim}{\mathbf{\Gamma}} \upharpoonright \mathcal{X}$.

Universal sets for the classical pointclasses

Open codes. For every Polish $\mathcal{X}$ we fix a basis $\{N(\mathcal{X}, s) \mid s \in \omega\}$ of its topology, we also include the empty set, and we define $\mathrm{U}^{\mathcal{X}} \subseteq \mathcal{N} \times \mathcal{X}$ by

$$\mathrm{U}^{\mathcal{X}}(\alpha, x) \iff (\exists n)[x \in N(\mathcal{X}, \alpha(n))].$$

Then $\mathrm{U}^{\mathcal{X}}$ is universal for $\utilde{\boldsymbol{\Sigma}}^0_1 \upharpoonright \mathcal{X}$.

Closed codes. For every $\mathcal{X}$ we define $\mathrm{F}^{\mathcal{X}} \subseteq \mathcal{N} \times \mathcal{X}$ by

$$\mathrm{F}^{\mathcal{X}}(\alpha, x) \iff \neg \mathrm{U}^{\mathcal{X}}(\alpha, x).$$

Then $\mathrm{F}^{\mathcal{X}}$ is universal for $\utilde{\boldsymbol{\Pi}}^0_1 \upharpoonright \mathcal{X}$.

$\utilde{\boldsymbol{\Sigma}}^0_n$-codes. For every $\mathcal{X}$ we define $\mathrm{H}^{\mathcal{X}}_n \subseteq \mathcal{N} \times \mathcal{X}$ by induction on $n \geq 1$,

$$\mathrm{H}^{\mathcal{X}}_1 = \mathrm{U}^{\mathcal{X}}$$

$$\mathrm{H}^{\mathcal{X}}_{n+1}(\alpha, x) \iff (\exists i) \neg \mathrm{H}^{\mathcal{X}}_n((\alpha)_i, x).$$

Analytic and $\underset{\sim}{\mathbf{\Sigma}}^1_n$ codes. For every $\mathcal{X}$ and every $n \geq 1$ we define the sets $\mathrm{G}^{\mathcal{X}}_n \subseteq \mathcal{N} \times \mathcal{X}$ as follows

$$\begin{aligned} \mathrm{G}^{\mathcal{X}}_1(\alpha, x) &\iff (\exists \gamma \in \mathcal{N}) \mathrm{F}^{\mathcal{X} \times \mathcal{N}}(\alpha, x, \gamma) \\ \mathrm{G}^{\mathcal{X}}_{n+1}(\alpha, x) &\iff (\exists \gamma \in \mathcal{N}) \neg \mathrm{G}^{\mathcal{X} \times \mathcal{N}}_n(\alpha, x, \gamma). \end{aligned}$$

Remark. If $\underset{\sim}{\mathbf{\Gamma}}$ is one of the previous pointclasses, then every $\alpha \in \mathcal{N}$ is a $\underset{\sim}{\mathbf{\Gamma}}$-code of some (perhaps empty) set in $\underset{\sim}{\mathbf{\Gamma}}$.

The Kleene pointclasses

We assume that whenever $\mathcal{X}$ is a recursive Polish space then the family $\{N(\mathcal{X}, s) \mid s \in \omega\}$ that we chose before comes from its recursive presentation. The Kleene pointclasses are defined as follows

$$\Sigma^0_1 = \{\mathrm{U}^{\mathcal{X}}_\alpha \mid \alpha \text{ is recursive}\} = \text{all recursive sections of } \mathrm{U}^{\mathcal{X}},$$

where $\mathcal{X}$ above ranges over all recursive Polish spaces. Similarly one defines the classes Σ^0_{n+1}, Σ^1_n and (by taking complements) Π^0_n, Π^1_n, where $n \geq 1$.

The preceding notions relativize with respect to some oracle $\varepsilon \in \mathcal{N}$, so that we get the pointclasses $\Sigma^0_n(\varepsilon)$ etc.

Borel codes (Louveau - Moschovakis)

We denote by $\{\alpha\}$ the largest partial function from ω to $\mathcal{N}$ whose graph is *computed* [correction] by $\mathrm{U}^{\omega\times\omega}$, i.e.,

$$\{\alpha\}(n)\downarrow \iff (\exists!\beta)(\forall s)[\beta \in N(\mathcal{X}, s) \longleftrightarrow \mathrm{U}^{\omega\times\omega}(\alpha, n, s)]$$
$$\{\alpha\}(n)\downarrow \Longrightarrow \{\alpha\}(n) = \text{ the unique } \beta \text{ as above.}$$

Define the sets $\mathrm{BC}_\xi \subseteq \mathcal{N}$, $\xi < \omega_1$ recursively

$$\alpha \in \mathrm{BC}_0 \iff \alpha(0) = 0,$$
$$\alpha \in \mathrm{BC}_\xi \iff \alpha(0) = 1 \ \& \ (\forall n)(\exists \zeta < \xi)[\{\alpha^*\}(n) \in \mathrm{BC}_\zeta].$$

The set of Borel codes is

$$\mathrm{BC} = \cup_{\xi<\omega_1} \mathrm{BC}_\xi.$$

This is a Π^1_1 set and not Borel. In particular not all α's are Borel codes.

For $\alpha \in \mathrm{BC}$ we put

$$|\alpha|_{\mathrm{BC}} = \text{the least } \xi \text{ such that } \alpha \in \mathrm{BC}_\xi.$$

Given a Polish space $\mathcal{X}$ we define the functions $\pi^{\mathcal{X}}_\xi : \mathrm{BC}_\xi \to \utilde{\boldsymbol{\Sigma}}^0_\xi \upharpoonright \mathcal{X}$ by recursion,

$$\pi^{\mathcal{X}}_1(\alpha) = \cup_n N(\mathcal{X}, \{\alpha^*\}(n)(1))$$
$$\pi^{\mathcal{X}}_\xi(\alpha) = \cup_n \mathcal{X} \setminus \pi^{\mathcal{X}}_{|\{\alpha^*\}(n)|_{\mathrm{BC}}}(\{\alpha^*\}(n)), \quad (1 < \xi < \omega_1).$$

An easy induction shows that for all $1 \leq \zeta \leq \xi$ we have that $\mathrm{BC}_\zeta \subseteq \mathrm{BC}_\xi$ and $\pi^{\mathcal{X}}_\xi \upharpoonright \mathrm{BC}_\zeta = \pi^{\mathcal{X}}_\zeta$. We now define

$$\pi^{\mathcal{X}} : \mathrm{BC} \to \mathrm{Borel}(\mathcal{X}) : \pi^{\mathcal{X}} = \cup_\xi \pi^{\mathcal{X}}_\xi.$$

Hyperarithmetical sets

For every countable ordinal ξ we define the pointclass

$$\Sigma^0_\xi = \{\pi^{\mathcal{X}}(\alpha) \mid \alpha \text{ is a recursive member of } \mathrm{BC}_\xi\},$$

where $\mathcal{X}$ ranges over all recursive Polish spaces. The induced hierarchy stabilizes at the ω_1^{CK} level.
The pointclass HYP of hyperarithmetical sets is defined by

$$\mathrm{HYP} = \cup_{1 \leq \xi < \omega_1^{\mathrm{CK}}} \Sigma^0_\xi.$$

Let us put $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.

Theorem (Kleene)

For every $A \subseteq \omega$ we have that

$$A \in \Delta^1_1 \iff A \in \mathrm{HYP}.$$

The Suslin-Luzin separation

We denote the class $\underset{\sim}{\mathbf{\Sigma}}^1_1 \cap \underset{\sim}{\mathbf{\Pi}}^1_1$ by $\underset{\sim}{\mathbf{\Delta}}^1_1$, (bi-analytic sets). It is easy to verify that every Borel set is $\underset{\sim}{\mathbf{\Delta}}^1_1$. The converse is also true.

Theorem (Suslin)

In every Polish space it holds $\underset{\sim}{\mathbf{\Delta}}^1_1 =$ Borel.

The preceding theorem is extended to

Theorem (Luzin Separation)

For all Polish spaces $\mathcal{X}$ and all disjoint analytic sets $A, B \subseteq \mathcal{X}$ there is a Borel set $C \subseteq \mathcal{X}$ such that

$$A \subseteq C \quad \textit{and} \quad C \cap B = \emptyset.$$

The Suslin-Kleene Theorem

The Luzin Separation Theorem has (also) a “constructive" proof. This yields the following.

Theorem (Suslin-Kleene)

For every recursive Polish space $\mathcal{X}$ there is a recursive function $u : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ such that for all $\alpha, \beta \in \mathcal{N}$ if the analytic sets A and B encoded by α and β are disjoint, then $u(\alpha, \beta)$ is a Borel code of a set C with

$$A \subseteq C \quad \text{and} \quad C \cap B = \emptyset.$$

This has the following application.

Theorem (Kleene - Louveau - Moschovakis)

In every recursive Polish space it holds

$$\Delta^1_1 = \mathrm{HYP}.$$

A few words about the proof of the Suslin-Kleene Theorem

Let A, B be non-empty disjoint analytic subsets of $\mathcal{N}$, and let T and S be trees of pairs such that

$$\begin{aligned} x \in A &\iff (\exists\alpha)(\forall t)[(x(0), \alpha(0), \dots x(t), \alpha(t)) \in T] \\ x \in B &\iff (\exists\beta)(\forall t)[(x(0), \beta(0), \dots x(t), \beta(t)) \in S]. \end{aligned}$$

We then define the tree J of triples by

$$(u, a, b) \in J \iff (u, a) \in T \;\&\; (u, b) \in S$$

where $u, a, b \in \omega^{<\omega}$ of the same length.
An infinite branch in J would provide some $x \in A \cap B$ contradicting that $A \cap B = \emptyset$. Hence the tree J is well-founded.

One defines by bar recursion on J a family $(C_\sigma)_{\sigma\in J}$ of subsets of $\mathcal{N}$ such that for all $\sigma = (u, a, b) \in J$ we have:

(a) C_σ is Borel,

(b) C_σ separates $\mathrm{pr}[T_{(u,a)}]$ from $\mathrm{pr}[S_{(u,b)}]$.
From this it follows that $C := C_\emptyset$ is Borel which separates $A = \mathrm{pr}[T_\emptyset]$ from $B = \mathrm{pr}[S_\emptyset]$.

The definition of C_σ is further refined as follows. We define a family $(D^\sigma_{(t,n,s,m)})_{t,n,s,m}$ of Borel sets such that for all (t, n, s, m) the set $D^\sigma_{(t,n,s,m)}$ separates $\mathrm{pr}[T_{(u,a)\hat{}(t,n)}]$ from $\mathrm{pr}[S_{(u,b)\hat{}(s,m)}]$.
Then it is easy to see that the set

$$C_\sigma := \cup_{(t,n)} \cap_{(s,m)} D^\sigma_{(t,n,s,m)}$$

separates $\mathrm{pr}[T_{(u,a)}]$ from $\mathrm{pr}[S_{(u,b)}]$.

If σ is terminal in J then $D^\sigma_{(t,n,s,m)}$ is one of the following sets: $\emptyset$, $\mathcal{N}$, $\{x \in \mathcal{N} \mid x(i) = j\}$.

Dyck Separation

We consider the following subsets of 2^ω,

$$U_n := \{x \in 2^\omega \mid x(n) = 1\}.$$

The family of all positive sets is the least family which contains $\{U_n \mid n \in \omega\}$ and is closed under countable unions and intersections. The family of semi-positive sets is the least family which contains $\{U_n \mid n \in \omega\} \cup \{\emptyset, 2^\omega\}$ and is closed under countable unions and intersections.

Every $x \in 2^\omega$ can be identified with the subsets of the naturals $\{n \in \omega \mid x(n) = 1\}$. We say that a set $A \subseteq 2^\omega$ is monotone if for all $x \in A$ and all $y \in 2^\omega$ with $x \subseteq y$ it holds $y \in A$.

It is not difficult to see that every semi-positive set is monotone. The converse is also true. This is a corollary to:

Theorem (Dyck Separation)

Let $A, B \subseteq 2^\omega$ be disjoint analytic sets. If A is monotone then there is a semi-positive Borel set C such that $A \subseteq C$ and $C \cap B = \emptyset$.

We can give a “constructive" proof to the latter in the style of the Suslin-Kleene Theorem.

The idea is to define the tree J of quadruples of length n by

$$(u, a, v, b) \in J \iff$$
$$(u, a) \in T \ \& \ (v, b) \in S \ \& \ (\forall i < n)[u(i) = 1 \longrightarrow v(i) = 1].$$

Then J is well-founded. The definition of the sets C_σ proceeds similarly. At the terminal nodes of J we can choose sets of the form U_n, $\emptyset$, 2^ω.

The uniform Dyck Theorem

Theorem (G.)

There exists a recursive function u : $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$ such that whenever α, β are codes of disjoint analytic sets A, B respectively with A being monotone, then u(α, β) is a Borel code of a semi-positive set C such that A $\subseteq$ C and C $\cap$ B $= \emptyset$.

Question. Is there a constructive consequence to the preceding result in the style $\mathrm{HYP} = \Delta^1_1$?

The answer is affirmative but first we need to introduce the effective semi-positive sets.

First we introduce the following hierarchy of $\underset{\sim}{\boldsymbol{SP}}$ = the family of all semi-positive sets,

$$V_0 = \emptyset, \quad V_1 = 2^\omega, \quad V_{n+2} := U_n = \{x \in 2^\omega \mid x(n) = 1\}, \quad n \in \omega;$$

$$\underset{\sim}{\boldsymbol{SP}}_0 = \{V_n \mid n \in \omega\};$$

$$\underset{\sim}{\boldsymbol{SP}}_\xi = \{\cup_{i\in\omega} \cap_{j\in\omega} A_{ij} \mid \text{for all } i, j \text{ there is } \xi_{ij} < \xi \text{ such that } A_{ij} \in \underset{\sim}{\boldsymbol{SP}}_{\xi_{ij}}\},$$
$$\text{where } 1 \leq \xi < \omega_1.$$

We also define for $\alpha \in \mathcal{N}$,

$$\alpha \in \mathrm{SPC}_0 \iff \alpha(0) = 0$$
$$\alpha \in \mathrm{SPC}_\xi \iff \alpha(0) = 1 \;\&\; (\forall i, j)(\exists \eta < \xi)[\{\alpha^*\}(\langle i, j\rangle) \in \mathrm{SPC}_\eta],$$

for all $\xi < \omega_1$, and

$$\mathrm{SPC} = \cup_{\xi<\omega_1} \mathrm{SPC}_\xi.$$

The members of SPC will be called semi-positive codes.
Given $\alpha \in \mathrm{SPC}$ we put

$$|\alpha|_{\mathrm{SP}} = \text{the least } \xi < \omega_1 \text{ such that } \alpha \in \mathrm{SPC}_\xi.$$

The coding τ_ξ of the family $\underset{\sim}{\boldsymbol{SP}}_\xi$ is as usual defined by recursion on ξ,

$$\tau_0 : \mathrm{SPC}_0 \twoheadrightarrow \underset{\sim}{\boldsymbol{SP}}_0 : \tau_0(\alpha) = V_{\alpha^*(1)}$$

$$\tau_\xi : \mathrm{SPC}_\xi \twoheadrightarrow \underset{\sim}{\boldsymbol{SP}}_\xi : \tau_\xi(\alpha) = \cup_i \cap_j \tau_{|\{\alpha^*\}(\langle i,j\rangle)|_{\mathrm{SP}}}(\{\alpha^*\}(\langle i,j\rangle)).$$

The analogous (to the coding BC) properties hold in this setting. The function

$$\tau := \cup_{\xi<\omega_1} \tau_\xi : \mathrm{SPC} \twoheadrightarrow \underset{\sim}{\boldsymbol{SP}}$$

defines a coding of the family $\underset{\sim}{\boldsymbol{SP}}$.
A set $A \subseteq 2^\omega$ is effective semi-positive if it is of the form $\tau(\alpha)$ for some recursive $\alpha \in \mathrm{SPC}$. (In this case we necessarily have that $|\alpha|_{\mathrm{SP}} < \omega_1^{\mathrm{CK}}$.)

The constructive consequence

Theorem (G. Uniform Dyck Separation for semi-positive codes)

There exists a recursive function $u : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ such that whenever α, β are codes of disjoint analytic sets A, B respectively with A being monotone, then $u(\alpha, \beta)$ is a semi-positive code of a set C such that $A \subseteq C$ and $C \cap B = \emptyset$.

As a consequence to this we get

Corollary (G.)

It holds

$$\begin{aligned} \cup_{\xi < \omega_1^{\mathrm{CK}}} SP_\xi &= \Delta^1_1 \cap \{A \subseteq 2^\omega \mid A \text{ is semi-positive}\} \\ &= \Delta^1_1 \cap \{A \subseteq 2^\omega \mid A \text{ is monotone}\}. \end{aligned}$$

A set $P \subseteq \mathcal{X}$ has the Baire property or simpler P has the BP if there exists an open set U such that the symmetric difference

$$P \triangle U := (P \setminus U) \cup (U \setminus P)$$

is meager $=$ countable union of nowhere dense sets. The family all subsets of $\mathcal{X}$ which have the BP is a σ-algebra, which contains all open subsets of $\mathcal{X}$. Hence it contains the family of all Borel subsets of $\mathcal{X}$ ($=$ the least σ-algebra which contains all open subsets of $\mathcal{X}$).
Every $\undertilde{\mathbf{\Sigma}}^1_1$ set has the BP, and under some determinacy assumptions every $\undertilde{\mathbf{\Sigma}}^1_n$ set has the BP as well.

The Baire property holds almost uniformly

For convenience we denote in the sequel the α-sections of some set $P \subseteq \mathcal{N} \times \mathcal{X}$ by $P(\alpha)$.

Proposition (G. Axiom of Projective Determinacy for $n > 1$)

For every Polish space $\mathcal{X}$ and every $n \in \omega$ there exists a continuous function $u_n^{\mathcal{X}} : \mathcal{N} \to \mathcal{N}$ such that for almost all $\alpha \in \mathcal{N}$ the set

$$\mathrm{G}_n^{\mathcal{X}}(\alpha) \triangle \mathrm{U}^{\mathcal{X}}(u_n^{\mathcal{X}}(\alpha))$$

is meager.

It is really almost

We cannot improve upon the "for almost all" part.

Theorem (G.)

For every $n \geq 1$ there is no $\underset{\sim}{\mathbf{\Delta}}^1_n$-measurable function $u : \mathcal{N} \to \mathcal{N}$ such that the set

$$\mathrm{G}^{\mathcal{N}}_n(\alpha) \triangle \mathrm{U}^{\mathcal{N}}(u(\alpha))$$

is meager for all $\alpha \in \mathcal{N}$.

Idea of the proof

Construct an open $V \subseteq \mathcal{N}$, which has "complex" $\underset{\sim}{\mathbf{\Sigma}}^1_n$-codes.

Some plans for the future

1. Give constructive proofs to other separation-type results, (e.g. Preiss).
2. Find Borel-measurable uniformity functions using definable points.

Problem. Suppose that X is a recursive Banach space and that K is a non-empty Δ^1_1 weakly compact subset of X. Does K contain a hyperarithmetical member?

An affirmative answer would provide a Borel-measurable uniformity function dealing with the fixed point property in Banach spaces.

Thank you for your attention!