Choices that we face: The good, the bad and the recursive

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Clarification:

Recursive is beautiful!

The Question

Suppose that $P \subseteq X \times Y$ is such that for all $x \in X$ the section $P_x := \{y \in Y \mid (x, y) \in P\}$ is non-empty, where X, Y are arbitrary sets.

We want to find a function $u : X \to Y$ such that for all $(x, u(x)) \in P$ for all $x \in \mathbb{R}$ (uniformity function).

The question is how "good" can such a function u be? For example if P is of some certain complexity can u be of the same complexity?

Some definitions

In the sequle we deal with complete separable metric spaces, which are computable, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ .... The Baire space is $\mathbb{N}^{\mathbb{N}}$.

A set $P \subseteq \mathcal{Z}$ is Σ^1_1 if there is a Π^0_1 (i.e., the complement is effectively open) set $F \subseteq \mathcal{Z} \times \mathbb{N}^{\mathbb{N}}$ such that $P = \mathrm{proj}_{\mathcal{Z}} F$; and P is $\Sigma^1_1(x)$, where $y \in \mathcal{X}$, if it is the x-section of a Π^0_1 set,

$$P = \{z \in \mathcal{Z} \mid (\exists \beta \in \mathbb{N}^{\mathbb{N}})[(x, z, \beta) \in F]\}.$$

The class $\utilde{\mathbf{\Sigma}}^1_1$ of analytic sets, is the union $\cup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \Sigma^1_1(\alpha)$.

By taking complements we obtain the classes Π^1_1, $\Pi^1_1(x)$, and $\utilde{\mathbf{\Pi}}^1_1$ (coanalytic sets).

Moreover we define $\Delta^1_1(x) = \Sigma^1_1(x) \cap \Pi^1_1(x)$ and $\utilde{\mathbf{\Delta}}^1_1 = \utilde{\mathbf{\Sigma}}^1_1 \cap \utilde{\mathbf{\Pi}}^1_1$.
By the Suslin Theorem it holds

$$\utilde{\mathbf{\Delta}}^1_1 = \text{Borel sets}.$$

To each space $\mathcal{Y}$ we can associate a countable basis $(N(\mathcal{Y}, s))_{s \in \mathbb{N}}$ in a natural way. Given a class of sets Γ and a function $f : \mathcal{X} \to \mathcal{Y}$ we say that f is Γ-recursive if the set

$$\{(x, s) \in \mathcal{X} \times \mathbb{N} \mid f(x) \in N(\mathcal{Y}, s)\}$$

is in Γ.

Examples. (1) If Γ is the class of open sets then Γ-recursive is the same as continuous.

(2) If Γ is Σ^0_1 (i.e., effectively open) then Γ-recursive is the same as computable a.k.a. recursive.

(3) If Γ is the class of Borel sets then Γ-recursive is the same as Borel-measurable.

A point $y \in \mathcal{Y}$ is a $\Delta^1_1(x)$-point, where $x \in \mathcal{X}$, if the set

$$\{s \in \mathbb{N} \mid y \in N(\mathcal{Y}, s)\}$$

is both a $\Sigma^1_1(x)$ and a $\Pi^1_1(x)$ subset of $\mathbb{N}$.

Theorem (Kleene). $\Delta^1_1(\alpha) = \mathrm{HYP}(\alpha)$, i.e.,
when $\mathcal{X} \subseteq \mathbb{N}^\mathbb{N}$, and $\alpha \in \mathbb{N}^\mathbb{N}$ the condition $y \in \Delta^1_1(\alpha)$ is equivalent to $y \leq_{\mathrm{T}} \alpha^{(\xi)} :=$ the ξ-th Turing jump of α, for some α-recursive ordinal ξ.

Good and Bad (informal definition)

A bad choice is one, which does not provide any additional information on the uniformity function u. This is usually done by invoking the Axiom of Choice.

A good choice is one, where the complexity does not exceed the complexity of the set that we start with. *Warning:* The u does not have to be recursive. E.g.

(Classical version) In the conclusion we have that the given set $P \subseteq \mathcal{X} \times \mathcal{Y}$ which satisfies $P_x \neq \emptyset$ for all $x \in \mathcal{X}$ is *Borel*, and it admits a *Borel-measurable* uniformity function.

(Effective version) In the conclusion we have that given set $P \subseteq \mathcal{X} \times \mathcal{Y}$ which satisfies $P_x \neq \emptyset$ for all $x \in \mathcal{X}$ is $\Delta^1_1(\alpha)$, and it admits a $\Delta^1_1(\alpha)$-recursive uniformity function, where $\alpha \in \mathbb{N}^{\mathbb{N}}$.

An example:

Theorem (Luzin-Novikov Uniformization)

If $P \subseteq \mathcal{X} \times \mathcal{Y}$ is Borel and each section P_x is countable then it admits a Borel-measurable uniformity function u.

Actually its effective version is also true:

Theorem (Harrison - Louveau)

If $P \subseteq \mathcal{X} \times \mathcal{Y}$ is $\Delta^1_1(\alpha)$ and each section P_x is countable then it admits a $\Delta^1_1(\alpha)$-measurable uniformity function u.

Not good does not have to be bad

The uniformity function may still provide some useful information:

Theorem (von Neumann Uniformization)

Let $P \subseteq \mathcal{X} \times \mathcal{Y}$ be $\utilde{\mathbf{\Sigma}}^1_1$. Then P admits a uniformity function $u : proj_{\mathcal{X}} \to \mathcal{Y}$ whose graph belongs to the σ-algebra generated by $\utilde{\mathbf{\Sigma}}^1_1$ sets.

Given that $\utilde{\mathbf{\Sigma}}^1_1$ sets are absolutely measurable (Luzin) it follows that every analytic set has an absolutely measurable uniformity function.

In descriptive set theory there are two main ways for obtaining uniformity functions.

1. Ensure the a point $y \in P_x$, which is “definable” from x. This typically results to Borel-measurable functions (Louveau).
2. Give a constructive proof to the theorem that we are interested in. This typically results to recursive/continuous uniformity functions.

In this talk we present examples of the preceding ways and some applications.

Why finding Δ^1_1-points is useful

Theorem (Louveau)

Suppose that $P \subseteq \mathcal{X} \times \mathcal{Y}$ is $\Delta^1_1(z)$ and for all $x \in \mathrm{proj}_{\mathcal{X}} P$ there is $y \in \Delta^1_1(z, x)$ such that $(x, y) \in P$. Then the projection $\mathrm{proj}_{\mathcal{X}} P$ is a $\Delta^1_1(z)$ set and P admits a $\Delta^1_1(z)$-recursive uniformity function.

It is a consequence of the effective Perfect Set Theorem (Harrison) that countable non-empty $\Delta^1_1(w)$ sets consist of $\Delta^1_1(w)$-points. By combining the latter with the preceding result of Louveau we obtain the effective version of the Lusin-Novikov uniformization that we mentioned previously.

A few more examples: Polish group actions

A Polish group is a triple $(G, *, \mathcal{T})$ such that $(G, *)$ is a group, $(G, \mathcal{T})$ is a Polish space, and the function $(x, y) \in G \times G \mapsto x * y^{-1}$ is $\mathcal{T}$-continuous.

An action of a group $(G, *)$ on a set X is a function $\cdot : G \times X \to X$ such that $e_G \cdot x = x$ and $(g_1 * g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $g_1, g_2 \in G$ and all $x \in X$.

A Polish G-space is a tuple $(\mathcal{X}, \mathcal{T}_{\mathcal{X}}, \cdot, G, *, \mathcal{T}_G)$ such that $(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$ is a Polish space, $(G, *, \mathcal{T}_G)$ is a Polish group, and $\cdot$ is a *continuous* action of G on $\mathcal{X}$.

This induces the equivalence relation E_G on $\mathcal{X}$:

$$xE_Gy \iff (\exists g \in G)[y = g \cdot x].$$

The orbit of $x \in \mathcal{X}$ is the equivalence class $[x]_{E_G}$.The relation E_G is a $\utilde{\boldsymbol{\Sigma}}^1_1$ subset of $\mathcal{X} \times \mathcal{X}$ in general. However...

Theorem (Miller)

In every Polish G-space every orbit is a Borel set.

Proposition (G.)

In every recursive Polish G-space it holds:

$$xE_G y \iff (\exists g \in \mathrm{W}^x, y)[y = g \cdot x],$$

where W^x is essentially the hyperjump of x.

Corollary (G.)

For every recursive Polish G-space $\mathcal{X}$ and all $x \in \mathcal{X}$, we have the following:

1. *The equivalence class $[x]_{E_G}$ is a $\Delta^1_1(\mathrm{W}^x)$ set.*
2. *There is a $\Delta^1_1(\mathrm{W}^x)$-recursive function $f_x : [x]_{E_G} \to G$ such that for all $y \in [x]_{E_G}$ it holds $y = f_x(y) \cdot x$.*

The preceding proposition is also an essentiall tool to proving the following.

Theorem (G.)

For every recursive Polish G-space $\mathcal{X}$ the restriction of E_G on

$$\{x \in \mathcal{X} \mid \omega_1^x = \omega_1^{\mathrm{CK}}\} \times \mathcal{X}$$

is a Borel set, where ω_1^x is the least non x-recursive ordinal and ω_1^{CK} the least non-recursive ordinal.

A few more examples: The Decomposability Conjecture

The Decomposability Conjecture states that for every function $f : \mathcal{X} \to \mathcal{Y}$ with the property that $f^{-1}\utilde{\mathbf{\Sigma}}^0_m \subseteq \utilde{\mathbf{\Sigma}}^0_n$, where $2 \leq m \leq n$ we can find a sequence $(X_i)_{i\in\mathbb{N}}$ of $\utilde{\mathbf{\Pi}}^0_{n-1}$ sets, which cover $\mathcal{X}$ and such that the restriction of f on each X_i is a $\utilde{\mathbf{\Sigma}}^0_{n-1}$-measurable function.

Theorem (Kihara)

The Decomposability Conjecture is correct for functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and for $3 \leq m \leq n < 2m - 1$ as long as the property $f^{-1}\utilde{\mathbf{\Sigma}}^0_m \subseteq \utilde{\mathbf{\Sigma}}^0_n$ is realized by a continuous uniformity function in the codes.

Theorem (G. - Kihara)

For every $f : \mathcal{X} \to \mathcal{Y}$, if the property $f^{-1}\underset{\sim}{\mathbf{\Sigma}}^0_m \subseteq \underset{\sim}{\mathbf{\Sigma}}^0_n$ holds, then the latter can be realized by a Borel uniformity function in the codes.

The preceding result has allowed us to make significant progress on the Decomposability Conjecture.

Intermediate steps

Recall:

Theorem (Louveau)

Suppose that $P \subseteq \mathcal{X} \times \mathcal{Y}$ is $\Delta^1_1(z)$ and for all $x \in \mathrm{proj}_{\mathcal{X}} P$ there is $y \in \Delta^1_1(z, x)$ such that $(x, y) \in P$. Then the projection $\mathrm{proj}_{\mathcal{X}} P$ is a $\Delta^1_1(z)$ set and P admits a $\Delta^1_1(z)$-recursive uniformity function.

Question. How about the intermediate steps of the preceding result? What applications can we find?

The Suslin Kleene Theorem

Theorem (Suslin-Kleene)

For every recursive Polish space $\mathcal{X}$ there is a recursive function $u : \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ if the analytic sets A and B encoded by α and β are disjoint, then $u(\alpha, \beta)$ is a Borel code of a set C with

$$A \subseteq C \quad \text{and} \quad C \cap B = \emptyset.$$

This has the following application.

Theorem (Kleene - Louveau - Moschovakis)

For every space $\mathcal{X}$ we have

$$\Delta^1_1 \cap \mathcal{X} = \mathrm{HYP} \cap \mathcal{X}.$$

Dyck Separation

We consider the following subsets of $2^{\mathbb{N}}$,

$$U_n := \{x \in 2^{\mathbb{N}} \mid x(n) = 1\}.$$

The family of all positive sets is the least family which contains $\{U_n \mid n \in \mathbb{N}\}$ and is closed under countable unions and intersections. The family of semi-positive sets is the least family which contains $\{U_n \mid n \in \mathbb{N}\} \cup \{\emptyset, 2^{\mathbb{N}}\}$ and is closed under countable unions and intersections.

Every $x \in 2^{\mathbb{N}}$ can be identified with the subsets of the naturals $\{n \in \omega \mid x(n) = 1\}$. We say that a set $A \subseteq 2^{\mathbb{N}}$ is monotone if for all $x \in A$ and all $y \in 2^{\mathbb{N}}$ with $x \subseteq y$ it holds $y \in A$.

It is not difficult to see that every semi-positive set is monotone. The converse is also true. This is a corollary to:

Theorem (Dyck Separation)

Let $A, B \subseteq 2^{\mathbb{N}}$ be disjoint analytic sets. If A is monotone then there is a semi-positive Borel set C such that $A \subseteq C$ and $C \cap B = \emptyset$.

Theorem (G.)

There exists a recursive function $u : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that whenever α, β are codes of disjoint analytic sets A, B respectively with A being monotone, then $u(\alpha, \beta)$ is a Borel code of a semi-positive set C such that $A \subseteq C$ and $C \cap B = \emptyset$.

The constructive consequence

Question. Is there a constructive consequence to the preceding result in the style $\mathrm{HYP} = \Delta^1_1$?

To do this one needs to introduce the notion of a semi-positive code, with the help of which one defines the family of all effective semi-positive sets $\cup_{\xi<\omega_1^{\mathrm{CK}}} SP_\xi$.

The preceding result (Uniform Dyck Separation) remains valid if we replace “Borel code” with “semi-positive code”. We then get

Corollary (G.)

It holds

$$\begin{aligned}\cup_{\xi<\omega_1^{\mathrm{CK}}} SP_\xi &= \Delta^1_1 \cap \{A \subseteq 2^{\mathbb{N}} \mid A \text{ is semi-positive}\}\\ &= \Delta^1_1 \cap \{A \subseteq 2^{\mathbb{N}} \mid A \text{ is monotone}\}.\end{aligned}$$

Preiss Separation

A Borel subset of $\mathbb{R}^n$ is convexly generated if it belongs to the smallest class of sets containing the the compact convex sets and is closed under countable intersections as well as increasing countable unions.

Theorem (Preiss Separation)

Let $A, B \subseteq \mathbb{R}^n$ be disjoint analytic sets with A convex. Then there is a convexly generated Borel set C, which separates A from B.

Corollary (Preiss)

For every Borel $B \subseteq \mathbb{R}^n$ we have that B is convex exactly when B is convexly generated.

Conjecture (almost proved)

The Preiss Separation Theorem is realized by a recursive uniformity function, which in turn gives the analogous constructive consequence.

Thank you for your attention!